HYPERSYMPLECTIC STRUCTURES ON COURANT ALGEBROIDS

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ABSTRACT. We introduce the notion of hypersymplectic structure on a Courant algebroid and we prove the existence of a one-to-one correspondence between hypersymplectic and hyperkähler structures. This correspondence provides a simple way to define a hyperkähler structure on a Courant algebroid. We show that hypersymplectic structures on Courant algebroids encompass hypersymplectic structures with torsion on Lie algebroids. In the latter, the torsion existing at the Lie algebroid level is incorporated in the Courant structure. Cases of hypersymplectic structures on Courant algebroids which are doubles of Lie, quasi-Lie and proto-Lie bialgebroids are investigated.

1. INTRODUCTION

In the past years, hyperstructures on Courant algebroids received the attention of several authors. Namely, we mention Bursztyn *et al.* [6] who discussed hyperkähler structures and Stiénon [14] for the case of hypercomplex structures. In the present article we introduce and study hypersymplectic structures and, more generally, ε-hypersymplectic structures on Courant algebroids, a notion that encompasses hypercomplex structures and hyperkähler structures. In fact, a very interesting feature of hypersymplectic structures on Courant algebroids is that they are in a one-to-one correspondence with hyperkähler structures. An important point to notice, which is a direct consequence of the existence of this correspondence, is that one gets a simpler way to define a hyperkähler structure because, contrary to the case of hyperkähler structures, our definition of hypersymplectic structure does not require the existence *ab initio* of a pseudo-metric on the Courant algebroid. As we shall see in Section 6, the pseudo-metric is constructed out of the given endomorphisms.

The basic example of a Courant algebroid is the vector bundle $A \oplus A^*$ equipped with the Dorfman bracket (or its skew-symmetrization, called the Courant bracket). This Courant structure on $A \oplus A^*$ is the double of a Lie bialgebroid structure on (A, A^*), where A^* is the null Lie algebroid. Doubles of more general cases of Lie bialgebroids structures on (A, A^*) are also Courant structures on $A \oplus A^*$. More generally, doubles of quasi-Lie bialgebroids and of proto-Lie bialgebroids structures on (A, A^*) determine Courant structures on $A \oplus A^*$ [10, 13]. In the Lie algebroid setting, and inspired by hypersymplectic structures on manifolds, defined by Xu in [16], we have introduced in [3] the notion of hypersymplectic structure (see also [1]). A hypersymplectic structure on a Lie algebroid A is a triple $(\omega_1, \omega_2, \omega_3)$ of symplectic forms on A, such that the squares of the transition morphisms, endomorphisms of

A constructed out of the 2-forms ω_i and their inverses, are equal to id_A or to $-\mathrm{id}_A$. The extension of the theory of hypersymplectic structures to Courant algebroids is not straightforward, since the notion of symplectic section on a Courant algebroid is not known. However, inspired by the generalized complex geometry *à la Hitchin* [9], where a 2-form on a manifold M gives rise to an endomorphism of $TM \oplus T^*M$, we replace the morphism $\omega^\flat$ associated to a 2-form ω on A by an endomorphism of $A \oplus A^*$. More precisely, the idea which is behind our results is to associate to each triple $(\omega_1, \omega_2, \omega_3)$ of non-degenerate 2-forms on a Lie algebroid A, with inverse $(\pi_1, \pi_2, \pi_3) \in (\Gamma(\wedge^2 A))^3$, a triple of endomorphisms $\mathcal{S}_i = \begin{bmatrix} 0 & \varepsilon_i \, \pi_i^\sharp \\ \omega_i^\flat & 0 \end{bmatrix}$, $\varepsilon_i = \pm 1, i = 1, 2, 3$, of the vector bundle $A \oplus A^*$, and to introduce an appropriate notion of hypersymplectic structure on a Courant algebroid in such a way that the following holds: $(\omega_1, \omega_2, \omega_3)$ is a hypersymplectic structure on a Lie algebroid (A, μ) if and only if $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is a hypersymplectic structure on the Courant algebroid $(A \oplus A^*, \mu)$. However, while considering the vector bundle $A \oplus A^*$ equipped with a Courant structure which is of type $\mu + \psi$, with ψ a trivector on A, we realize that the previous equivalence fails. In fact, the structure that has to be considered on A is a hypersymplectic structure with torsion, a notion that we study in a separate article [5] and which is related to hyperkähler structures with torsion. Hyperkähler structures with torsion on manifolds, also known as HKT structures, first appeared in [7] in relation with sigma models in string theory and, since then, HKT and other geometries with torsion caught the interest of many physicists and mathematicians.

In this article most results are established in the more general pre-Courant framework, and hold without any change in the Courant algebroid case. The exception is when we deal with hypersymplectic structures with torsion on Lie algebroids (Section 9), where an additional condition must be considered when we pass from pre-Courant to Courant algebroids. We show that, although hypersymplectic structures and hypersymplectic structures with torsion on Lie algebroids are different in nature, when we look at them in the pre-Courant algebroid setting, they become of the same type. Roughly speaking, when one goes from Lie to pre-Courant algebroids, the torsion carried by the hypersymplectic structure on the Lie algebroid passes to the pre-Courant structure itself. More precisely, we prove that having a hypersymplectic structure $(\omega_1, \omega_2, \omega_3)$ on A, with or without torsion, is equivalent to having a hypersymplectic structure $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ on $A \oplus A^*$, with a suitable pre-Courant structure. More involved situations are those where, besides the structures considered on A, the vector bundle A^* itself is endowed with a hypersymplectic structure, with or without torsion, determined by (π_1, π_2, π_3). We also prove that, under some conditions, this is equivalent to $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ being a hypersymplectic structure on $A \oplus A^*$.

This article contains an introduction and eight sections. Since many of the computations are performed using the big bracket, Section 2 contains a brief review of the supergeometric setting as well as the main notions concerning the Courant and pre-Courant algebroid definitions. Section 3 is devoted to some properties of the Nijenhuis torsion on pre-Courant algebroids, that are used in the remaining sections. To the best of our knowledge some of these properties (Propositions 3.1 and 3.2) are new. In Section 4 we introduce the notion of ε-hypersymplectic structure on a pre-Courant algebroid (Definition 4.1) and we explore the properties of the morphisms induced by this structure. Sections 5 and 6 treat the case $\varepsilon_1 \varepsilon_2 \varepsilon_3 = -1$. The main result of Section 5 is that the transition morphisms $\mathcal{T}_i$ are Nijenhuis (Theorem 5.3).

We also show that $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is a hypersymplectic structure on a pre-Courant algebroid (E, Θ) if and only if it is hypersymplectic for the pre-Courant structure on E deformed by $\mathcal{T}_i$ or by $\mathcal{S}_i$ (Theorem 5.5). In Section 6 we prove that there exists a one-to-one correspondence between hypersymplectic and hyperkähler structures on a pre-Courant algebroid (Theorem 6.8). Moreover, we show how the transition morphisms $\mathcal{T}_i$ can play the role of the morphisms $\mathcal{S}_i$ in order to define a new hypersymplectic structure on the pre-Courant algebroid (Theorem 6.9). Sections 7, 8 and 9 are devoted to examples of hypersymplectic structures on $A \oplus A^*$, equipped with several (pre-)Courant structures. First we treat the simplest case in Section 7. We prove that $(\omega_1, \omega_2, \omega_3)$ is a hypersymplectic structure on a Lie algebroid (A, μ) if and only if $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is a hypersymplectic structure on the Courant algebroid $(A \oplus A^*, \mu)$ (Theorem 7.2). This case is illustrated with an example involving the Heisenberg Lie group. In Section 8 the Courant structure on $A \oplus A^*$ is the double of a Lie bialgebroid $((A, A^*), \mu, \gamma)$ and we prove that $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is a hypersymplectic structure on $(A \oplus A^*, \mu + \gamma)$ if and only if $(\omega_1, \omega_2, \omega_3)$ is a hypersymplectic structure on (A, μ) and (π_1, π_2, π_3) is a hypersymplectic structure on (A^*, γ) (Theorem 8.1). The particular case of a triangular Lie bialgebroid is also considered (Corollary 8.2). The class of examples we give in Section 9 deals with the notion of hypersymplectic structure with torsion on a Lie algebroid. This is a structure that generalizes the hypersymplectic case, where the non-degenerate 2-forms ω_i are not closed but satisfy the condition $N_1 d\omega_1 = N_2 d\omega_2 = N_3 d\omega_3$, with N_i's the transition morphisms. We show that having a hypersymplectic structure with torsion $(\omega_1, \omega_2, \omega_3)$ on a Lie algebroid (A, μ) is equivalent to $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ being a hypersymplectic structure on the pre-Courant algebroid $(A \oplus A^*, \mu + \psi)$, with $\psi \in \Gamma(\wedge^3 A)$ (Proposition 9.3). The corresponding result on the Courant algebroid $(A \oplus A^*, \mu + \psi)$, which is the double of the quasi-Lie bialgebroid $((A, A^*), \mu, 0, \psi)$, requires the bivectors π_i to be weak-Poisson with respect to μ, that is, $[\pi_i, \pi_i]$ has to be a cocycle with respect to μ (Theorem 9.4). Finally, we treat the general case where both Lie algebroids (A, μ) and (A^*, γ) are equipped with hypersymplectic structures with torsion. We show that this is equivalent to having a hypersymplectic structure on the pre-Courant algebroid $(A \oplus A^*, \mu + \gamma + \psi + \phi)$, with $\psi \in \Gamma(\wedge^3 A)$ and $\phi \in \Gamma(\wedge^3 A^*)$ (Proposition 9.5). As before, the corresponding result on the Courant algebroid $(A \oplus A^*, \mu + \gamma + \psi + \phi)$, which is the double of the proto-Lie bialgebroid $((A, A^*), \mu, \gamma, \psi, \phi)$, requires additional conditions on the ω_i's and on the π_i's (Theorem 9.6).

2. Preliminaries on Courant algebroids

We begin this section by introducing the supergeometric setting, following the same approach as in [15, 12] (see also [1]). Given a vector bundle $A \to M$, we denote by $A[m]$ the graded manifold obtained by shifting the fibre degree by m. The graded manifold $T^*[2]A[1]$ is equipped with a canonical symplectic structure which induces a Poisson bracket on its algebra of functions $\mathcal{F} := C^\infty(T^*[2]A[1])$. This Poisson bracket is sometimes called the *big bracket* (see [10]).

Let us describe locally this Poisson algebra. Fix local coordinates $x_i, p^i, \xi_a, \theta^a$, $i \in \{1, \ldots, n\}, a \in \{1, \ldots, d\}$, in $T^*[2]A[1]$, where x_i, ξ_a are local coordinates on $A[1]$ and p^i, θ^a are their associated moment coordinates. In these local coordinates, the Poisson bracket is given by

$$\{p^i, x_i\} = \{\theta^a, \xi_a\} = 1, \quad i = 1, \ldots, n, \ a = 1, \ldots, d,$$

while all the remaining brackets vanish.

The Poisson algebra of functions $\mathcal{F}$ is endowed with an $(\mathbb{N} \times \mathbb{N})$-valued bidegree. We define this bidegree (locally but it is well defined globally, see [15, 12]) as follows: the coordinates on the base manifold M, x_i, $i \in \{1, \ldots, n\}$, have bidegree $(0,0)$, while the coordinates on the fibres, ξ_a, $a \in \{1, \ldots, d\}$, have bidegree $(0,1)$ and their associated moment coordinates, p^i and θ^a, have bidegree $(1,1)$ and $(1,0)$, respectively. We denote by $\mathcal{F}^{k,l}$ the space of functions of bidegree (k,l) and we verify that the big bracket has bidegree $(-1,-1)$, i.e.,

$$\{\mathcal{F}^{k_1,l_1}, \mathcal{F}^{k_2,l_2}\} \subset \mathcal{F}^{k_1+k_2-1,l_1+l_2-1}.$$

This construction is a particular case of a more general one [12] in which we consider a vector bundle E equipped with a fibrewise non-degenerate symmetric bilinear form $\langle ., . \rangle$. In this more general setting, we consider the graded symplectic manifold $\mathcal{E} := p^*(T^*[2]E[1])$, which is the pull-back of $T^*[2]E[1]$ by the map $p : E[1] \to E[1] \oplus E^*[1]$ defined by $X \mapsto (X, \frac{1}{2}\langle X, .\rangle)$. We denote by $\mathcal{F}_E$ the graded algebra of functions on $\mathcal{E}$, i.e., $\mathcal{F}_E := C^\infty(\mathcal{E})$. The algebra $\mathcal{F}_E$ is equipped with the canonical Poisson bracket, denoted by $\{.,.\}$, which has degree -2. Notice that $\mathcal{F}_E^0 = C^\infty(M)$ and $\mathcal{F}_E^1 = \Gamma(E)$. Under these identifications, the Poisson bracket of functions of degrees 0 and 1 is given by

$$\{f, g\} = 0, \quad \{f, X\} = 0 \quad \text{and} \quad \{X, Y\} = \langle X, Y\rangle,$$

for all $X, Y \in \Gamma(E)$ and $f, g \in C^\infty(M)$.

When $E := A \oplus A^*$ (with A a vector bundle over M) and when $\langle ., .\rangle$ is the usual symmetric bilinear form:

$$(1) \qquad \langle X + \alpha, Y + \beta\rangle = \alpha(Y) + \beta(X), \quad \forall X, Y \in \Gamma(A), \alpha, \beta \in \Gamma(A^*),$$

the algebras $\mathcal{F} = C^\infty(T^*[2]A[1])$ and $\mathcal{F}_{A \oplus A^*}$ are isomorphic Poisson algebras [12] and the two constructions above coincide.

Definition 2.1. [2] *A* pre-Courant *structure on* $(E, \langle \cdot, \cdot \rangle)$ *is a pair* $(\rho, [\cdot, \cdot])$, *where* $\rho : E \to TM$ *is a morphism of vector bundles called the* anchor, *and* $[\cdot, \cdot] : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ *is a* $\mathbb{R}$*-bilinear (not necessarily skew-symmetric) bracket, called the* Dorfman bracket, *satisfying the relations*

$$(2) \qquad\qquad \rho(X) \cdot \langle Y, Z\rangle = \langle [X, Y], Z\rangle + \langle Y, [X, Z]\rangle$$

and

$$(3) \qquad\qquad \rho(X) \cdot \langle Y, Z\rangle = \langle X, [Y, Z] + [Z, Y]\rangle,$$

for all $X, Y, Z \in \Gamma(E)$.

From (2) and (3), we obtain the Leibniz rule [10]

$$[X, fY] = f[X, Y] + (\rho(X).f)Y,$$

for all $X, Y \in \Gamma(E)$ and $f \in C^\infty(M)$. If a pre-Courant structure $(\rho, [\cdot, \cdot])$ satisfies the Jacobi identity,

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]],$$

for all $X, Y, Z \in \Gamma(E)$, then the pair $(\rho, [\cdot, \cdot])$ is called a *Courant* structure on $(E, \langle \cdot, \cdot \rangle)$.

There is a one-to-one correspondence between pre-Courant structures on $(E, \langle \cdot, \cdot \rangle)$ and elements in $\mathcal{F}_E^3$. The anchor and Dorfman bracket associated to a given $\Theta \in \mathcal{F}_E^3$ are defined, for all $X, Y \in \Gamma(E)$ and $f \in C^\infty(M)$, by the derived bracket expressions

$$(4) \qquad \rho(X) \cdot f = \{\{X, \Theta\}, f\} \quad \text{and} \quad [X, Y] = \{\{X, \Theta\}, Y\}.$$

The next theorem shows how a Courant structure can be defined in the super-geometric setting.

Theorem 2.2. [12] *There is a $1 - 1$ correspondence between Courant structures on $(E, \langle ., . \rangle)$ and functions $\Theta \in \mathcal{F}_E^3$ such that $\{\Theta, \Theta\} = 0$.*

If Θ is a (pre-)Courant structure on $(E, \langle \cdot, \cdot \rangle)$, then the triple $(E, \langle \cdot, \cdot \rangle, \Theta)$ is called a *(pre-)Courant algebroid*. For the sake of simplicity, we often denote a (pre-)Courant algebroid by the pair (E, Θ) instead of the triple $(E, \langle \cdot, \cdot \rangle, \Theta)$.

When $E = A \oplus A^*$ and $\langle \cdot, \cdot \rangle$ is the usual symmetric bilinear form (1), a pre-Courant structure $\Theta \in \mathcal{F}_E^3$ can be decomposed using the bidegrees:

$$\Theta = \mu + \gamma + \phi + \psi,$$

with $\mu \in \mathcal{F}_{A \oplus A^*}^{1,2}, \gamma \in \mathcal{F}_{A \oplus A^*}^{2,1}, \phi \in \mathcal{F}_{A \oplus A^*}^{0,3} = \Gamma(\wedge^3 A^*)$ and $\psi \in \mathcal{F}_{A \oplus A^*}^{3,0} = \Gamma(\wedge^3 A)$. We recall from [13] that, when $\gamma = \phi = \psi = 0$, Θ is a Courant structure on $(A \oplus A^*, \langle \cdot, \cdot \rangle)$ *if and only if* (A, μ) is a Lie algebroid; the anchor and the bracket of the Lie algebroid (A, μ) are given by (4), where a section X of A is identified with $X \oplus 0 \in \Gamma(A \oplus A^*)$. When $\phi = \psi = 0$, Θ is a Courant structure on $(A \oplus A^*, \langle \cdot, \cdot \rangle)$ *if and only if* $((A, A^*), \mu, \gamma)$ is a Lie bialgebroid and when $\phi = 0$ (resp. $\psi = 0$), Θ is a Courant structure on $(A \oplus A^*, \langle \cdot, \cdot \rangle)$ *if and only if* $((A, A^*), \mu, \gamma, \psi)$ (resp. $((A^*, A), \gamma, \mu, \phi))$ is a quasi-Lie bialgebroid. In the more general case, $\Theta = \mu + \gamma + \phi + \psi$ is a Courant structure *if and only if* $((A, A^*), \mu, \gamma, \psi, \phi)$ is a proto-Lie bialgebroid.

3. Nijenhuis torsion on pre-Courant algebroids

Let $(E, \langle \cdot, \cdot \rangle, \Theta)$ be a pre-Courant algebroid with anchor and Dorfman bracket defined by (4). Given an endomorphism $\mathcal{I} : E \to E$, the transpose morphism $\mathcal{I}^* : E^* \simeq E \to E^* \simeq E$ is defined by $\langle \mathcal{I}^* u, v \rangle = \langle u, \mathcal{I} v \rangle$ for all $u, v \in E$. The morphism $\mathcal{I}$ is *orthogonal* if $\mathcal{I} \circ \mathcal{I}^* = \mathrm{id}_E$. If $\mathcal{I} = \mathcal{I}^*$ (resp. $\mathcal{I} = -\mathcal{I}^*$), the morphism $\mathcal{I}$ is said to be *symmetric* (resp. *skew-symmetric*).

Given a skew-symmetric endomorphism $\mathcal{I} : E \to E$, we define a *deformed* pre-Courant algebroid structure $(\rho_{\mathcal{I}}, [\cdot, \cdot]_{\mathcal{I}})$ on E by setting

$$\begin{cases} \rho_{\mathcal{I}} = \rho \circ \mathcal{I} \\ [X, Y]_{\mathcal{I}} = [\mathcal{I} X, Y] + [X, \mathcal{I} Y] - \mathcal{I}[X, Y], \quad \forall X, Y \in \Gamma(E). \end{cases}$$

The deformed pre-Courant structure $(\rho_{\mathcal{I}}, [\cdot, \cdot]_{\mathcal{I}})$ corresponds to the function $\Theta_{\mathcal{I}} := \{\mathcal{I}, \Theta\} \in \mathcal{F}_E^3$, (via (4)). The deformation of $\Theta_{\mathcal{I}}$ by a skew-symmetric morphism $\mathcal{J}$ is denoted by $\Theta_{\mathcal{I}, \mathcal{J}}$, i.e. $\Theta_{\mathcal{I}, \mathcal{J}} = \{\mathcal{J}, \{\mathcal{I}, \Theta\}\}$. The concomitant $C_\Theta(\mathcal{I}, \mathcal{J})$ is the element of $\mathcal{F}_E^3$ defined as [2]:

$$(5) \qquad\qquad C_\Theta(\mathcal{I}, \mathcal{J}) = \Theta_{\mathcal{I}, \mathcal{J}} + \Theta_{\mathcal{J}, \mathcal{I}}.$$

Recall that a skew-symmetric endomorphism $\mathcal{I} : E \to E$ on a pre-Courant algebroid $(E, \langle \cdot, \cdot \rangle, \Theta)$ is a *Nijenhuis morphism* if its Nijenhuis torsion $\widetilde{\mathcal{C}}_\Theta \mathcal{I}$ vanishes, where

$$(6) \qquad\qquad \widetilde{\mathcal{C}}_\Theta \mathcal{I}(X, Y) = [\mathcal{I} X, \mathcal{I} Y] - \mathcal{I}([X, Y]_{\mathcal{I}}),$$

for all $X, Y \in \Gamma(E)$.

When $\mathcal{I}$ is skew-symmetric and satisfies $\mathcal{I}^2 = \lambda \, \mathrm{id}_E$, for some $\lambda \in \mathbb{R}$, we have [8, 1]

$$(7) \qquad\qquad \widetilde{\mathcal{C}}_\Theta \mathcal{I} = \frac{1}{2}(\Theta_{\mathcal{I}, \mathcal{I}} - \lambda \Theta).$$

If $\mathcal{I}^2 = -\mathrm{id}_E$ (resp. $\mathcal{I}^2 = \mathrm{id}_E$) then $\mathcal{I}$ is said to be an *almost complex* (resp. *almost para-complex*) *structure*. If moreover $\widetilde{\mathfrak{C}}_\Theta \mathcal{I} = 0$, then $\mathcal{I}$ is a *complex* (resp. *para-complex*) *structure*.

Next, we prove some results that will be used in Section 5.

Proposition 3.1. *Let $\mathcal{I}$ be a skew-symmetric endomorphism on a pre-Courant algebroid (E, Θ), such that $\mathcal{I}^2 = \lambda \, \mathrm{id}_E$, for some $\lambda \in \mathbb{R}\backslash\{0\}$. Then, $\widetilde{\mathfrak{C}}_\Theta \mathcal{I} = 0 \Leftrightarrow \widetilde{\mathfrak{C}}_{\Theta_\mathcal{I}} \mathcal{I} = 0$.*

Proof. The statement $\widetilde{\mathfrak{C}}_\Theta \mathcal{I} = 0 \Rightarrow \widetilde{\mathfrak{C}}_{\Theta_\mathcal{I}} \mathcal{I} = 0$ follows directly from the equality

$$(8) \qquad \widetilde{\mathfrak{C}}_{\Theta_\mathcal{I}} \mathcal{I}(X, Y) = \widetilde{\mathfrak{C}}_\Theta \mathcal{I}(\mathcal{I}X, Y) + \widetilde{\mathfrak{C}}_\Theta \mathcal{I}(X, \mathcal{I}Y) - \mathcal{I}\left(\widetilde{\mathfrak{C}}_\Theta \mathcal{I}(X, Y)\right),$$

that holds for all sections X, Y of E (see [2]). Let us now suppose that $\widetilde{\mathfrak{C}}_{\Theta_\mathcal{I}} \mathcal{I} = 0$. Evaluating (8) on pairs of sections of E of type $(\mathcal{I}X, Y)$ and of type $(X, \mathcal{I}Y)$, we obtain

$$(9) \qquad \mathcal{I}\left(\widetilde{\mathfrak{C}}_\Theta \mathcal{I}(\mathcal{I}X, Y)\right) = \lambda \widetilde{\mathfrak{C}}_\Theta \mathcal{I}(X, Y) + \widetilde{\mathfrak{C}}_\Theta \mathcal{I}(\mathcal{I}X, \mathcal{I}Y)$$

and

$$(10) \qquad \mathcal{I}\left(\widetilde{\mathfrak{C}}_\Theta \mathcal{I}(X, \mathcal{I}Y)\right) = \lambda \widetilde{\mathfrak{C}}_\Theta \mathcal{I}(X, Y) + \widetilde{\mathfrak{C}}_\Theta \mathcal{I}(\mathcal{I}X, \mathcal{I}Y),$$

respectively. Equations (9) and (10) yield

$$\mathcal{I}\left(\widetilde{\mathfrak{C}}_\Theta \mathcal{I}(\mathcal{I}X, Y)\right) = \mathcal{I}\left(\widetilde{\mathfrak{C}}_\Theta \mathcal{I}(X, \mathcal{I}Y)\right),$$

which implies

$$(11) \qquad \widetilde{\mathfrak{C}}_\Theta \mathcal{I}(\mathcal{I}X, Y) = \widetilde{\mathfrak{C}}_\Theta \mathcal{I}(X, \mathcal{I}Y).$$

Substituting (11) in (8), we get

$$(12) \qquad 2\widetilde{\mathfrak{C}}_\Theta \mathcal{I}(\mathcal{I}X, Y) = \mathcal{I}\left(\widetilde{\mathfrak{C}}_\Theta \mathcal{I}(X, Y)\right).$$

Applying $\mathcal{I}$ to both sides of (12), we get

$$(13) \qquad 2\mathcal{I}\left(\widetilde{\mathfrak{C}}_\Theta \mathcal{I}(\mathcal{I}X, Y)\right) = \lambda \widetilde{\mathfrak{C}}_\Theta \mathcal{I}(X, Y).$$

On the other hand, replacing X by $\mathcal{I}X$ in (12), we obtain

$$(14) \qquad 2\lambda \widetilde{\mathfrak{C}}_\Theta \mathcal{I}(X, Y) = \mathcal{I}\left(\widetilde{\mathfrak{C}}_\Theta \mathcal{I}(\mathcal{I}X, Y)\right).$$

From (13) and (14) we conclude that $\widetilde{\mathfrak{C}}_\Theta \mathcal{I} = 0$, as claimed. $\qquad \square$

Proposition 3.2. *Let $\mathcal{I}$ and $\mathcal{J}$ be two anti-commuting skew-symmetric endomorphisms on a pre-Courant algebroid (E, Θ), such that $C_\Theta(\mathcal{I}, \mathcal{J}) = 0$ and $\mathcal{I}^2 = \lambda \, \mathrm{id}_E$, for some $\lambda \in \mathbb{R}\backslash\{0\}$. Then, $\widetilde{\mathfrak{C}}_\Theta \mathcal{J} = 0 \Leftrightarrow \widetilde{\mathfrak{C}}_{\Theta_\mathcal{I}} \mathcal{J} = 0$.*

Proof. The statement $\widetilde{\mathfrak{C}}_\Theta \mathcal{J} = 0 \Rightarrow \widetilde{\mathfrak{C}}_{\Theta_\mathcal{I}} \mathcal{J} = 0$ is an immediate consequence of Proposition 4.18 in [2]. In fact, when $C_\Theta(\mathcal{I}, \mathcal{J}) = 0$, that proposition gives

$$(15) \qquad \widetilde{\mathfrak{C}}_{\Theta_\mathcal{I}} \mathcal{J}(X, Y) = -\widetilde{\mathfrak{C}}_\Theta \mathcal{J}(IX, Y) - \widetilde{\mathfrak{C}}_\Theta \mathcal{J}(X, \mathcal{I}Y) - \mathcal{I}\left(\widetilde{\mathfrak{C}}_\Theta \mathcal{J}(X, Y)\right),$$

for all sections X, Y of E. Let us now suppose that $\widetilde{\mathfrak{C}}_{\Theta_\mathcal{I}} \mathcal{J} = 0$. Evaluating (15) on pairs of sections of E of type $(\mathcal{I}X, Y)$ and of type $(X, \mathcal{I}Y)$ and doing computations similar to those in the proof of Proposition 3.1, we obtain

$$(16) \qquad \widetilde{\mathfrak{C}}_\Theta \mathcal{J}(\mathcal{I}X, Y) = \widetilde{\mathfrak{C}}_\Theta \mathcal{J}(X, \mathcal{I}Y).$$

Substituting (16) in (15), we get

$$(17) \qquad 2\widetilde{\mathfrak{C}}_\Theta \mathcal{J}(\mathcal{I}X, Y) = -\mathcal{I}\left(\widetilde{\mathfrak{C}}_\Theta \mathcal{J}(X, Y)\right).$$

Applying $\mathcal{I}$ to both sides of (17), yields

$$(18) \qquad 2\mathcal{I}\left(\widetilde{\mathfrak{C}}_\Theta \mathcal{J}(\mathcal{I}X, Y)\right) = \lambda \widetilde{\mathfrak{C}}_\Theta \mathcal{J}(X, Y).$$

On the other hand, replacing X by $\mathcal{I}X$ in (17), we obtain

$$(19) \qquad 2\lambda \widetilde{\mathfrak{G}}_\Theta \mathcal{J}(X,Y) = \mathcal{I}\left(\widetilde{\mathfrak{G}}_\Theta \mathcal{J}(\mathcal{I}X,Y)\right).$$

From (18) and (19) we conclude that $\widetilde{\mathfrak{G}}_\Theta \mathcal{J} = 0$, as pretended. □

Proposition 3.3. *Let $\mathcal{I}$ and $\mathcal{J}$ be two anti-commuting skew-symmetric endomorphisms on a pre-Courant algebroid (E,Θ), with vanishing Nijenhuis torsion. Then,*

i) $C_\Theta(\mathcal{I}, \mathcal{IJ})(X,Y) = \mathcal{I}\big(C_\Theta(\mathcal{I},\mathcal{J})(X,Y)\big)$, for all sections $X, Y \in \Gamma(E)$.

Moreover, if $\mathcal{I}^2 = \lambda_\mathcal{I}\,\mathrm{id}_E$ and $\mathcal{J}^2 = \lambda_\mathcal{J}\,\mathrm{id}_E$, for some $\lambda_\mathcal{I}, \lambda_\mathcal{J} \in \mathbb{R}\backslash\{0\}$, then

ii) $C_\Theta(\mathcal{I}, \mathcal{J}) = 0$.

Proof. Part i) is directly obtained from Proposition 3.13 in [2].

ii) From $\mathcal{I}^2 = \lambda_\mathcal{I}\,\mathrm{id}_E$ we have $\mathcal{J} = \lambda_\mathcal{I}^{-1}\mathcal{I}(\mathcal{IJ})$ and, for all $X, Y \in \Gamma(E)$,

$$(20) \qquad C_\Theta(\mathcal{I},\mathcal{J})(X,Y) = \lambda_\mathcal{I}^{-1} C_\Theta(\mathcal{I},\mathcal{I}(\mathcal{IJ}))(X,Y) = \lambda_\mathcal{I}^{-1}\mathcal{I}(C_\Theta(\mathcal{IJ},\mathcal{I})(X,Y)),$$

where, in the last equality, we used (i) and the fact that $C_\Theta(\cdot,\cdot)$ is symmetric. Using $\mathcal{J}^2 = \lambda_\mathcal{J}\,\mathrm{id}_E$ and (i), Equation (20) becomes

$$C_\Theta(\mathcal{I},\mathcal{J})(X,Y) = \lambda_\mathcal{I}^{-1}\lambda_\mathcal{J}^{-1}\mathcal{I}(C_\Theta(\mathcal{IJ},\mathcal{IJ}(\mathcal{J}))(X,Y))$$

$$= \lambda_\mathcal{I}^{-1}\lambda_\mathcal{J}^{-1}\mathcal{I}(\mathcal{IJ})(C_\Theta(\mathcal{IJ},\mathcal{J})(X,Y)) = -\lambda_\mathcal{J}^{-1}\mathcal{J}(C_\Theta(\mathcal{JI},\mathcal{J})(X,Y)),$$

where we used the fact that $\mathcal{IJ} = -\mathcal{JI}$ in the last equality. Finally, applying (i) once more, we get

$$C_\Theta(\mathcal{I},\mathcal{J})(X,Y) = -\lambda_\mathcal{J}^{-1}\mathcal{J}^2(C_\Theta(\mathcal{I},\mathcal{J})(X,Y)) = -C_\Theta(\mathcal{I},\mathcal{J})(X,Y).$$

Therefore,

$$C_\Theta(\mathcal{I},\mathcal{J})(X,Y) = 0,$$

for all $X, Y \in \Gamma(E)$.

□

Let us recall a result from [2] that will be used later.

Proposition 3.4. *Let $\mathcal{I}$ and $\mathcal{J}$ be two anti-commuting endomorphisms on a pre-Courant algebroid (E,Θ). Then, for all sections X and Y of E,*

$$(21) \quad 2\,\widetilde{\mathfrak{G}}_\Theta(\mathcal{I}\circ\mathcal{J})(X,Y) = \Bigg(\widetilde{\mathfrak{G}}_\Theta\mathcal{I}(\mathcal{J}X,\mathcal{J}Y) - \mathcal{J}\left(\widetilde{\mathfrak{G}}_\Theta\mathcal{I}(\mathcal{J}X,Y) + \widetilde{\mathfrak{G}}_\Theta\mathcal{I}(X,\mathcal{J}Y)\right) -$$

$$- \mathcal{J}^2(\widetilde{\mathfrak{G}}_\Theta\mathcal{I}(X,Y))\Bigg) + \underset{\mathcal{I},\mathcal{J}}{\circlearrowright},$$

where $\underset{\mathcal{I},\mathcal{J}}{\circlearrowright}$ stands for permutation of $\mathcal{I}$ and $\mathcal{J}$. In particular, if $\mathcal{I}$ and $\mathcal{J}$ have vanishing Nijenhuis torsion then so has $\mathcal{I}\circ\mathcal{J}$.

4. Hypersymplectic structures on pre-Courant algebroids

In this section we introduce the notion of an $\boldsymbol{\varepsilon}$-*hypersymplectic structure* on a pre-Courant algebroid (E,Θ) and study the main relations and properties of the induced morphisms.

In the rest of the paper, when $\mathcal{I}$ and $\mathcal{J}$ are endomorphisms of E, the composition $\mathcal{I}\circ\mathcal{J}$ will be denoted simply by $\mathcal{IJ}$. Also, we consider $1, 2$ and 3 as the representative elements of the equivalence classes of $\mathbb{Z}_3$, i.e., $\mathbb{Z}_3 := \{[1],[2],[3]\}$. Although we omit the brackets, and write i instead of $[i]$, the indices in the corresponding computations must be thought of as elements in $\mathbb{Z}_3 := \mathbb{Z}/3\mathbb{Z}$.

Definition 4.1. *An $\boldsymbol{\varepsilon}$-hypersymplectic structure on a pre-Courant algebroid (E, Θ) is a triple $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ of skew-symmetric endomorphisms $\mathcal{S}_i : E \to E$, $i = 1, 2, 3$, such that*

i) $\mathcal{S}_i^2 = \varepsilon_i \operatorname{id}_E$,
ii) $\mathcal{S}_i \mathcal{S}_j = \varepsilon_1 \varepsilon_2 \varepsilon_3 \mathcal{S}_j \mathcal{S}_i$, $i \neq j \in \{1, 2, 3\}$,
iii) $\Theta_{\mathcal{S}_i, \mathcal{S}_i} = \varepsilon_i \Theta$,

where the parameters $\varepsilon_i = \pm 1$ form the triple $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$.

From conditions i) and iii) of Definition 4.1, and using formula (7), we immediately obtain the following proposition.

Proposition 4.2. *Let $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ be an $\boldsymbol{\varepsilon}$-hypersymplectic structure on a pre-Courant algebroid (E, Θ). Then, $\mathcal{S}_1, \mathcal{S}_2$ and $\mathcal{S}_3$ are Nijenhuis morphisms.*

Remark 4.3. *Notice that according to Proposition 4.2, condition iii) in Definition 4.1 can be replaced by*

iii') $\widetilde{\mathcal{C}}_\Theta \mathcal{S}_i = 0$, $i = 1, 2, 3$.

Given an $\boldsymbol{\varepsilon}$-hypersymplectic structure $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ on (E, Θ), let us define the morphisms $\mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{T}_3$ by setting

$$(22) \qquad \mathcal{T}_i := \varepsilon_{i-1} \mathcal{S}_{i-1} \mathcal{S}_{i+1}.$$

The morphisms $\mathcal{T}_i$, $i = 1, 2, 3$, can be seen as transition maps between the morphisms $\mathcal{S}_j$, $j = 1, 2, 3$. In fact we have, for all $i \in \mathbb{Z}_3$,

$$\mathcal{S}_{i-1} \mathcal{T}_i = \mathcal{S}_{i+1}.$$

The picture in Figure 1 helps to visualize these relations. For example, the bottom triangle shows that $\mathcal{S}_2 \mathcal{T}_3 = \mathcal{S}_1$ and $\varepsilon_1 \mathcal{T}_3 \mathcal{S}_1 = \varepsilon_2 \mathcal{S}_2$. For the latter equality we use the fact that the inverse of morphism $\mathcal{S}_i$ is $\varepsilon_i \mathcal{S}_i$.
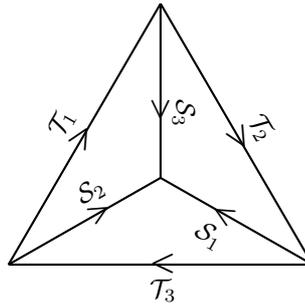
FIGURE 1.

Proposition 4.4. *Let $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ be an $\boldsymbol{\varepsilon}$-hypersymplectic structure on a pre-Courant algebroid (E, Θ). The morphisms $\mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{T}_3$ satisfy the following relations for all $i = 1, 2, 3$:*

i) $\mathcal{T}_i^* = \varepsilon_1 \varepsilon_2 \varepsilon_3 \mathcal{T}_i$;
ii) $\mathcal{T}_i^2 = \varepsilon_i \operatorname{id}_E$;
iii) $\mathcal{T}_{i-1} \mathcal{T}_{i+1} = \varepsilon_1 \varepsilon_2 \varepsilon_3 \mathcal{T}_{i+1} \mathcal{T}_{i-1} = \varepsilon_i \mathcal{T}_i$;
iv) $\mathcal{T}_3 \mathcal{T}_2 \mathcal{T}_1 = \varepsilon_1 \varepsilon_2 \varepsilon_3 \mathcal{T}_1 \mathcal{T}_2 \mathcal{T}_3 = \operatorname{id}_E$.

Proof. Using conditions i) and ii) of Definition 4.1 and also equation (22), we have:

i) $\mathcal{T}_i^* = (\varepsilon_{i-1}\mathcal{S}_{i-1}\mathcal{S}_{i+1})^* = \varepsilon_{i-1}\mathcal{S}_{i+1}\mathcal{S}_{i-1} = \varepsilon_i\varepsilon_{i+1}\mathcal{S}_{i-1}\mathcal{S}_{i+1} = \varepsilon_1\varepsilon_2\varepsilon_3\mathcal{T}_i$,

where we also used the fact that the endomorphisms $\mathcal{S}_i$ are skew-symmetric;

ii) $\mathcal{T}_i^2 = \mathcal{S}_{i-1}\mathcal{S}_{i+1}\mathcal{S}_{i-1}\mathcal{S}_{i+1} = \varepsilon_1\varepsilon_2\varepsilon_3\mathcal{S}_{i-1}^2\mathcal{S}_{i+1}^2 = \varepsilon_i\,\mathrm{id}_E$;

iii) $\mathcal{T}_{i-1}\mathcal{T}_{i+1} = \varepsilon_{i+1}\varepsilon_i\mathcal{S}_{i+1}\mathcal{S}_i\mathcal{S}_i\mathcal{S}_{i-1} = \varepsilon_{i+1}\mathcal{S}_{i+1}\mathcal{S}_{i-1} = \varepsilon_i\varepsilon_{i-1}\mathcal{S}_{i-1}\mathcal{S}_{i+1} = \varepsilon_i\mathcal{T}_i$.

This proves one part of the statement and we use it to prove the second equality of the statement. In fact, from $(\mathcal{T}_{i-1}\mathcal{T}_{i+1})^2 = (\varepsilon_i\mathcal{T}_i)^2 = \varepsilon_i\,\mathrm{id}_E$ and using item ii), we have

$$\mathcal{T}_{i-1}\mathcal{T}_{i+1} = \varepsilon_i(\mathcal{T}_{i-1}\mathcal{T}_{i+1})^{-1} = \varepsilon_i(\mathcal{T}_{i+1})^{-1}(\mathcal{T}_{i-1})^{-1} = \varepsilon_1\varepsilon_2\varepsilon_3\mathcal{T}_{i+1}\mathcal{T}_{i-1}.$$

iv) By item iii), $\mathcal{T}_3\mathcal{T}_2 = \varepsilon_1\mathcal{T}_1$; then, using item ii),

$$\mathcal{T}_3\mathcal{T}_2\mathcal{T}_1 = \varepsilon_1\mathcal{T}_1^2 = \mathrm{id}_E.$$

Furthermore, using item iii) three times, we can change the order of $\mathcal{T}_i$'s in the product $\mathcal{T}_3\mathcal{T}_2\mathcal{T}_1$ to get

$$\mathcal{T}_3\mathcal{T}_2\mathcal{T}_1 = (\varepsilon_1\varepsilon_2\varepsilon_3)^3\mathcal{T}_1\mathcal{T}_2\mathcal{T}_3 = \varepsilon_1\varepsilon_2\varepsilon_3\mathcal{T}_1\mathcal{T}_2\mathcal{T}_3.$$

$\square$

Remark 4.5. *In the particular case where $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = -1$ and (E,Θ) is a Courant algebroid, the triple $(\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3)$ is an* almost hypercomplex structure *on (E,Θ) in the terminology of* [14]*.*

Given an $\boldsymbol{\varepsilon}$-hypersymplectic structure $(\mathcal{S}_1,\mathcal{S}_2,\mathcal{S}_3)$ on a pre-Courant algebroid (E,Θ), we may define an endomorphism $\mathcal{G}: E \to E$ by setting, for all $i = 1,2,3$,

$$\mathcal{G} := \mathcal{S}_{i+1}\mathcal{S}_i\mathcal{S}_{i-1}. \tag{23}$$

Notice that $\mathcal{G}$ is well defined by (23). In fact, since $\mathcal{S}_i\mathcal{S}_j = \varepsilon_1\varepsilon_2\varepsilon_3\mathcal{S}_j\mathcal{S}_i$, for $i \neq j$, we obviously have $\mathcal{G} = \mathcal{S}_3\mathcal{S}_2\mathcal{S}_1 = \mathcal{S}_1\mathcal{S}_3\mathcal{S}_2 = \mathcal{S}_2\mathcal{S}_1\mathcal{S}_3$.

Proposition 4.6. *Let $(\mathcal{S}_1,\mathcal{S}_2,\mathcal{S}_3)$ be an $\boldsymbol{\varepsilon}$-hypersymplectic structure on a pre-Courant algebroid (E,Θ). Then, the morphism $\mathcal{G}$, given by (23), satisfies the following properties:*

i) $\mathcal{G}^* = -\varepsilon_1\varepsilon_2\varepsilon_3\mathcal{G}$;

ii) $\mathcal{G}^2 = \mathrm{id}_E$.

Proof. i) An easy computation using the skew-symmetry of each $\mathcal{S}_i$ and condition ii) in Definition 4.1, gives

$$\mathcal{G}^* = (\mathcal{S}_{i+1}\mathcal{S}_i\mathcal{S}_{i-1})^* = -\mathcal{S}_{i-1}\mathcal{S}_i\mathcal{S}_{i+1} = -\varepsilon_1\varepsilon_2\varepsilon_3\mathcal{S}_{i-1}\mathcal{S}_{i+1}\mathcal{S}_i = -\varepsilon_1\varepsilon_2\varepsilon_3\mathcal{G}.$$

ii) The proof is immediate using properties of $\mathcal{S}_i$ from Definition 4.1:

$$\mathcal{G}^2 = (\mathcal{S}_3\mathcal{S}_2\mathcal{S}_1)^2 = \varepsilon_1\varepsilon_2\varepsilon_3\,\mathcal{S}_3^2\mathcal{S}_2^2\mathcal{S}_1^2 = \mathrm{id}_E.$$

$\square$

The next proposition shows that, for each i, the morphisms $\mathcal{G}$, $\mathcal{S}_i$ and $\mathcal{T}_i$ commute pairwise and each one is obtained out of the other two.

Proposition 4.7. *Let $(\mathcal{S}_1,\mathcal{S}_2,\mathcal{S}_3)$ be an $\boldsymbol{\varepsilon}$-hypersymplectic structure on a pre-Courant algebroid (E,Θ). The morphisms $\mathcal{S}_i,\mathcal{T}_i$ and $\mathcal{G}, i = 1,2,3$, satisfy the following relations:*

i) $\mathcal{T}_i\mathcal{S}_i = \mathcal{S}_i\mathcal{T}_i = \varepsilon_{i-1}\mathcal{G}$;

ii) $\mathcal{G}\mathcal{S}_i = \mathcal{S}_i\mathcal{G} = \varepsilon_{i-1}\varepsilon_i\mathcal{T}_i$;

iii) $\mathcal{G}\mathcal{T}_i = \mathcal{T}_i\mathcal{G} = \varepsilon_{i-1}\varepsilon_i\mathcal{S}_i$.

Moreover, for all $i \neq j \in \{1,2,3\}$,

iv) $\mathcal{S}_j \mathcal{T}_i = \varepsilon_1 \varepsilon_2 \varepsilon_3 \mathcal{T}_i \mathcal{S}_j = \begin{cases} \mathcal{S}_{i+1}, & j = i-1 \\ \varepsilon_i \mathcal{S}_{i-1}, & j = i+1. \end{cases}$

Proof. i) Using (22) and the condition ii) of Definition 4.1 twice, we get

$$\mathcal{T}_i \mathcal{S}_i = \varepsilon_{i-1} \mathcal{S}_{i-1} \mathcal{S}_{i+1} \mathcal{S}_i = \varepsilon_{i-1} \mathcal{S}_i \mathcal{S}_{i-1} \mathcal{S}_{i+1} = \mathcal{S}_i \mathcal{T}_i.$$

On the other hand, from (22) and (23) we have

$$\mathcal{T}_i \mathcal{S}_i = \varepsilon_{i-1} \mathcal{S}_{i-1} \mathcal{S}_{i+1} \mathcal{S}_i = \varepsilon_{i-1} \mathcal{G}.$$

ii) From item i) we have $\mathcal{T}_i \mathcal{S}_i = \varepsilon_{i-1} \mathcal{G}$ and composing with $\mathcal{S}_i$, on the right, we get $\mathcal{T}_i (\mathcal{S}_i)^2 = \varepsilon_{i-1} \mathcal{G} \mathcal{S}_i$ or, equivalently, $\varepsilon_{i-1}\varepsilon_i \mathcal{T}_i = \mathcal{G} \mathcal{S}_i$. For the other equality, we start with $\mathcal{S}_i \mathcal{T}_i = \varepsilon_{i-1} \mathcal{G}$ and compose with $\mathcal{S}_i$, on the left, to obtain $(\mathcal{S}_i)^2 \mathcal{T}_i = \varepsilon_{i-1} \mathcal{S}_i \mathcal{G}$; so that $\varepsilon_{i-1}\varepsilon_i \mathcal{T}_i = \mathcal{S}_i \mathcal{G}$.

iii) Analogous to the proof of item ii), but composing with $\mathcal{T}_i$ instead of $\mathcal{S}_i$.

iv) Let us prove the case $j = i-1$. Using (22) and the condition i) of Definition 4.1, we have

$$\mathcal{S}_{i-1} \mathcal{T}_i = \varepsilon_{i-1} \mathcal{S}_{i-1}^2 \mathcal{S}_{i+1} = \mathcal{S}_{i+1}.$$

Moreover, by (22) and conditions i) and ii) of Definition 4.1, we get

$$\mathcal{T}_i \mathcal{S}_{i-1} = \varepsilon_{i-1} \mathcal{S}_{i-1} \mathcal{S}_{i+1} \mathcal{S}_{i-1} = \varepsilon_i \varepsilon_{i+1} \mathcal{S}_{i-1}^2 \mathcal{S}_{i+1} = \varepsilon_1 \varepsilon_2 \varepsilon_3 \mathcal{S}_{i+1},$$

which completes the proof of the statement. The case $j = i+1$ is analogous. □

The relations between $\mathcal{S}_i, \mathcal{T}_j$ and $\mathcal{G}$, for all $i,j \in \{1,2,3\}$, may be visualized in Figure 2.
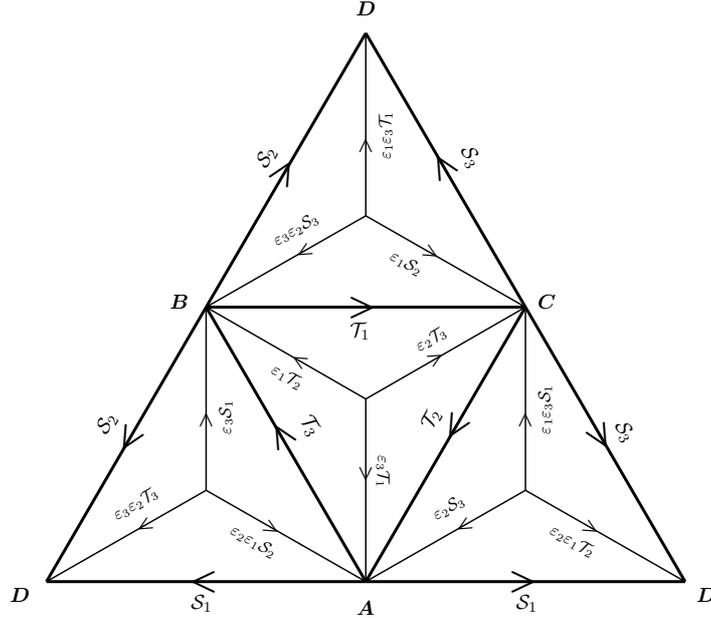

FIGURE 2.

This is to be understood as the pattern for a tetrahedron $ABCD$. The metric $\mathcal{G}$ does not appear in Figure 2. But in Figure 3, after building the tetrahedron, $\mathcal{G}$ appears as the altitude of the tetrahedron $ABCD$.
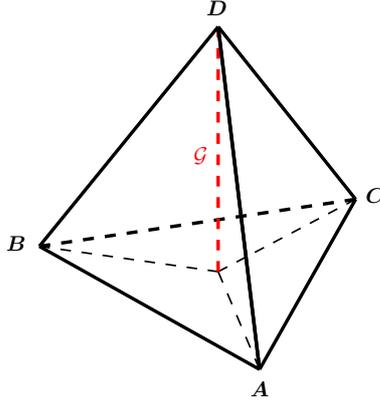
FIGURE 3.

Remark 4.8. *When $\mathcal{G} = \mathrm{id}_E$, we have $\varepsilon_i \mathcal{S}_i = \varepsilon_{i-1} \mathcal{T}_i$ and there is an identification between the upper edges of the tetrahedron $ABCD$ and their projections onto the face ABC (see Figure 3). In other words, in this case the tetrahedron degenerates into a (flat) triangle.*

The next proposition shows the behaviour of the bilinear map $\langle .,. \rangle$ under the endomorphisms $\mathcal{G}$ and $\mathcal{T}_i$, $i = 1, 2, 3$.

Proposition 4.9. *Let $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ be an ε-hypersymplectic structure on a pre-Courant algebroid $(E, \langle .,. \rangle, \Theta)$. The maps $\mathcal{G}$ and $\mathcal{T}_i$, $i = 1, 2, 3$, satisfy*

$$\langle \mathcal{G}\mathcal{T}_i(X), \mathcal{T}_i(Y) \rangle = \varepsilon_{i-1} \varepsilon_{i+1} \langle \mathcal{G}(X), Y \rangle,$$

for all $X, Y \in \Gamma(E)$.

Proof. Using Proposition 4.4 i) and ii) and Proposition 4.7 iii) we have:

$$\langle \mathcal{G}\mathcal{T}_i(X), \mathcal{T}_i(Y) \rangle = \varepsilon_1 \varepsilon_2 \varepsilon_3 \langle \mathcal{T}_i \mathcal{G}\mathcal{T}_i(X), Y \rangle = \varepsilon_1 \varepsilon_2 \varepsilon_3 \langle \mathcal{G}\mathcal{T}_i^2(X), Y \rangle = \varepsilon_{i-1} \varepsilon_{i+1} \langle \mathcal{G}(X), Y \rangle.$$

$\square$

5. Hypersymplectic structures on deformed pre-Courant algebroids

The results of Section 4 (Propositions 4.4, 4.6 and 4.7) show that the value of the parameter $\varepsilon_1 \varepsilon_2 \varepsilon_3 = \pm 1$ is determinant for some basic properties of the morphisms $\mathcal{T}_i$, $\mathcal{S}_j$ and $\mathcal{G}$, $i, j \in \{1, 2, 3\}$, and for the relations between them. In this section we consider an ε-hypersymplectic structure $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ on a pre-Courant algebroid (E, Θ) such that $\varepsilon_1 \varepsilon_2 \varepsilon_3 = -1$. We prove that the $\mathcal{T}_i$'s are Nijenhuis morphisms and we show that we may deform the pre-Courant structure Θ by $\mathcal{S}_i$ or by $\mathcal{T}_i$, without losing the property of $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ being hypersymplectic.

Definition 5.1. *Let $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ be an ε-hypersymplectic structure on a pre-Courant algebroid (E, Θ), such that $\varepsilon_1 \varepsilon_2 \varepsilon_3 = -1$.*

- *If $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = -1$, then $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is said to be a* hypersymplectic *structure on (E, Θ).*
- *Otherwise, $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is said to be a* para-hypersymplectic *structure on (E, Θ).*

Note that all para-hypersymplectic structures satisfy, eventually after a cyclic permutation of the indices, $\varepsilon_1 = \varepsilon_2 = 1$ and $\varepsilon_3 = -1$. In the sequel, every para-hypersymplectic structure will be considered of such form.

Example 5.2. *Any* hypercomplex structure (I, J, K) on a Courant algebroid (E, Θ), *in the sense of* [14], *is a hypersymplectic structure on (E, Θ). In fact, if we set $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3) := (I, J, K)$, then $\mathcal{T}_i = \mathcal{S}_i$, $i = 1, 2, 3$, and $\mathcal{G} = \mathrm{id}_E$ (see Remark 4.8).*

As a direct consequence of Propositions 4.2, 3.4 and 4.4 ii), we get the following:

Theorem 5.3. *Let $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ be a (para-)hypersymplectic structure on a pre-Courant algebroid (E, Θ). Then, for each $i = 1, 2, 3$, the transition morphism $\mathcal{T}_i$ is a Nijenhuis morphism. Moreover,*

- *i) if $\varepsilon_i = -1$, $\mathcal{T}_i$ is a complex structure;*
- *ii) if $\varepsilon_i = 1$, $\mathcal{T}_i$ is a para-complex structure.*

In the case where $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = -1$ and (E, Θ) is a Courant algebroid, the triple $(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$ is a hypercomplex structure on (E, Θ) in the sense of [14], see Remark 4.5.

In [2] the notion of *Nijenhuis pair* on a pre-Courant algebroid (E, Θ) was introduced as a pair $(\mathcal{I}, \mathcal{J})$ of anti-commuting Nijenhuis morphisms such that $C_\Theta(\mathcal{I}, \mathcal{J}) = 0$.

Proposition 3.3 ii) shows that when two Nijenhuis morphisms $\mathcal{I}$ and $\mathcal{J}$ are (para-)complex structures, i.e., $\lambda_\mathcal{I} = \pm 1$ and $\lambda_\mathcal{J} = \pm 1$, it is enough that they anti-commute to form a Nijenhuis pair. This is the case when we have a pre-Courant algebroid equipped with a (para-)hypersymplectic structure, as stated in the next proposition.

Proposition 5.4. *Let $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ be a (para-)hypersymplectic structure on a pre-Courant algebroid (E, Θ). Then, $(\mathcal{S}_i, \mathcal{S}_j), (\mathcal{T}_i, \mathcal{T}_j)$ and $(\mathcal{S}_i, \mathcal{T}_j)$ are Nijenhuis pairs, for all $i, j \in \{1, 2, 3\}, i \neq j$.*

Next we show that when a triple $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is a (para-)hypersymplectic structure on a pre-Courant algebroid (E, Θ), it is also a (para-)hypersymplectic structure on the pre-Courant algebroid deformed by $\mathcal{T}_i$ or by $\mathcal{S}_i$.

Theorem 5.5. *Let (E, Θ) be a pre-Courant algebroid. The following assertions are equivalent:*

- *i) $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is a (para-)hypersymplectic structure on (E, Θ);*
- *ii) $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is a (para-)hypersymplectic structure on $(E, \Theta_{\mathcal{S}_i})$;*
- *iii) $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is a (para-)hypersymplectic structure on $(E, \Theta_{\mathcal{T}_j})$,*

$i, j \in \{1, 2, 3\}$, *where $\mathcal{T}_j$ is defined by (22).*

Proof. Let us prove $(i) \Leftrightarrow (ii)$. By Remark 4.3, it is enough to show that $\widetilde{\mathfrak{c}}_\Theta \mathcal{S}_j = 0 \Leftrightarrow \widetilde{\mathfrak{c}}_{\Theta_{\mathcal{S}_i}} \mathcal{S}_j = 0$, $i, j \in \{1, 2, 3\}$. We consider two cases:

- a) The case $i = j$ is a direct consequence of Proposition 3.1;
- b) For the case $i \neq j$, we use Propositions 5.4 and 3.2.

To prove the equivalence $(i) \Leftrightarrow (iii)$ it is enough to notice that $\mathcal{T}_j = \varepsilon_{j-1}\mathcal{S}_{j-1}\mathcal{S}_{j+1}$ and to use twice the equivalence $(i) \Leftrightarrow (ii)$. $\qquad\qquad\qquad\qquad\qquad\square$

It is known that the deformation Θ_I of a Courant structure Θ by a Nijenhuis morphism $\mathcal{I}$ is a Courant structure [8]. So, if the pre-Courant algebroid (E, Θ) in Theorem 5.5 is in particular a Courant algebroid, then $(E, \Theta_{\mathcal{S}_i})$ and $(E, \Theta_{\mathcal{T}_j})$, $i, j \in \{1, 2, 3\}$, are also Courant algebroids.

6. One-to-one correspondence

In this section we keep considering an $\boldsymbol{\varepsilon}$-hypersymplectic structure $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ on a pre-Courant algebroid such that $\varepsilon_1\varepsilon_2\varepsilon_3 = -1$. We define hyperkähler structures on pre-Courant algebroids and prove a one-to-one correspondence between hypersymplectic and hyperkähler structures. We show how we may exchange the roles of the morphisms $\mathcal{S}_i$ and $\mathcal{T}_i$ to get a set of equivalent structures; these structures are summarized in a diagram at the end of the section.

Given a pre-Courant algebroid, we may define, in a natural way, a notion of (pseudo-)metric.

Definition 6.1. *A* pseudo-metric *on a pre-Courant algebroid* $(E, \langle ., . \rangle, \Theta)$ *is a symmetric and orthogonal bundle automorphism* $G : E \to E$. *If moreover* G *is* positive definite, *that is,* $\langle G(e), e \rangle > 0$, *for all non vanishing sections* $e \in \Gamma(E)$, *then* G *is said to be a* metric *on* $(E, \langle ., . \rangle, \Theta)$.

Remark 6.2. *In Definition 6.1, because* G *is symmetric, the orthogonality condition* $(GG^* = \mathrm{id}_E)$ *can be replaced by an almost para-complex condition* $(G^2 = \mathrm{id}_E)$.

The next proposition follows directly from Proposition 4.6 and Remark 6.2.

Proposition 6.3. *Let* $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ *be a (para-)hypersymplectic structure on a pre-Courant algebroid* $(E, \langle ., . \rangle, \Theta)$. *Then, the morphism* $\mathcal{G}$ *given by equation (23) is a pseudo-metric on* E.

Next, we define the notions of hermitian and para-hermitian pair on a pre-Courant algebroid.

Definition 6.4. *A* hermitian *(resp.,* para-hermitian*) pair*[1] *on a pre-Courant algebroid* $(E, \langle ., . \rangle, \Theta)$ *is a pair* $(\mathcal{J}, G)$ *where* $\mathcal{J}$ *is a complex (resp., para-complex) structure and* G *is a pseudo-metric such that, for all* $X, Y \in \Gamma(E)$,

$$(24) \quad \langle G(\mathcal{J}X), \mathcal{J}Y \rangle = \langle G(X), Y \rangle, \qquad (resp., \ \langle G(\mathcal{J}X), \mathcal{J}Y \rangle = -\langle G(X), Y \rangle).$$

Remark 6.5. *In Definition 6.4, if* $\mathcal{J}$ *is a skew-symmetric complex (resp. para-complex) structure then condition (24) is equivalent to* $G\mathcal{J} = \mathcal{J}G$.

As a direct consequence of Proposition 4.9, we have the following:

Proposition 6.6. *Let* $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ *be a (para-)hypersymplectic structure on a pre-Courant algebroid* $(E, \langle ., . \rangle, \Theta)$.

    i) If $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ *is a hypersymplectic structure, then* $(\mathcal{T}_i, \mathcal{G})$ *is a hermitian pair, for all* $i = 1, 2, 3$.

    ii) If $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ *is a para-hypersymplectic structure, then* $(\mathcal{T}_1, \mathcal{G})$ *and* $(\mathcal{T}_2, \mathcal{G})$ *are para-hermitian pairs while* $(\mathcal{T}_3, \mathcal{G})$ *is a hermitian pair.*

---

[1]Rigourously, we should say *pseudo-hermitian* and *para-pseudo-hermitian* but, in order to simplify the terminology, we omit the prefix "pseudo".

Let us define (para-)hyperkähler structures on a pre-Courant algebroid[2] and see how they are related to (para-)hypersymplectic structures.

Definition 6.7. *A quadruple $(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{G})$ of endomorphisms on a pre-Courant algebroid $(E, \langle ., . \rangle, \Theta)$ is a* hyperkähler *(resp.,* para-hyperkähler*) if:*

   i) $\mathcal{G}$ is a pseudo-metric;
   ii) $\mathcal{T}_1$ and $\mathcal{T}_2$ are anti-commuting complex (resp., para-complex) endomorphisms and $\mathcal{T}_3 = \mathcal{T}_1 \mathcal{T}_2$;
   iii) $(\mathcal{G}, \mathcal{T}_j)_{j=1,2}$ are hermitian (resp., para-hermitian) pairs;
   iv) $\widetilde{\mho}_\Theta(\mathcal{G}\mathcal{T}_j) = 0, \ j = 1, 2, 3.$

Notice that, when $(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{G})$ is a (para-)hyperkähler structure, $(\mathcal{G}, \mathcal{T}_3)$ is a hermitian pair and the morphisms $\mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{T}_3$ pairwise anti-commute.

Theorem 6.8. *The triple $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is a hypersymplectic (resp., para-hypersymplectic) structure on a pre-Courant algebroid (E, Θ) if and only if $(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{G})$ is a hyperkähler (resp., para-hyperkähler) structure on (E, Θ), with $\mathcal{S}_i = \varepsilon_i \varepsilon_{i-1} \mathcal{G} \mathcal{T}_i$, $i = 1, 2, 3$.*

Proof. If $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is a (para-)hypersymplectic structure on (E, Θ) then we easily conclude that $(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{G})$ is a (para-)hyperkähler structure on (E, Θ).

Let us now assume that $(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{G})$ is a (para-)hyperkähler structure on (E, Θ). Then, we have

$$\mathcal{S}_i^2 = \mathcal{G} \mathcal{T}_i \mathcal{G} \mathcal{T}_i = \mathcal{G}^2 \mathcal{T}_i^2 = \mathcal{T}_i^2 = \varepsilon_i \mathrm{id}_E,$$

where we used the fact that $\mathcal{G}$ and $\mathcal{T}_i$ commute and $\mathcal{G}^2 = \mathrm{id}_E$ (see Remarks 6.5 and 6.2). Moreover,

$$\mathcal{S}_i \mathcal{S}_{i+1} = \varepsilon_i \varepsilon_{i-1} \varepsilon_{i+1} \varepsilon_i \mathcal{G} \mathcal{T}_i \mathcal{G} \mathcal{T}_{i+1} = \varepsilon_{i-1} \varepsilon_{i+1} \mathcal{T}_i \mathcal{T}_{i+1} = -\varepsilon_{i-1} \varepsilon_{i+1} \mathcal{T}_{i+1} \mathcal{T}_i = -\mathcal{S}_{i+1} \mathcal{S}_i.$$

Finally, because $\mathcal{S}_i^2 = \varepsilon_i \mathrm{id}_E$ and $\widetilde{\mho}_\Theta \mathcal{S}_i = 0$ (see item iv) of Definition 6.7) we conclude that $\Theta_{\mathcal{S}_i, \mathcal{S}_i} = \varepsilon_i \Theta$. Therefore, $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is a (para-)hypersymplectic structure on (E, Θ). $\qquad\square$

Next, we see that the tetrahedron model (see Figure 3), besides being an efficient way to summarize all the algebraic relations between the morphisms of a (para-)hypersymplectic structure, is an accurate representation that enables us to discover new relations. In fact, the next theorem shows that the symmetries of the tetrahedron are symmetries of the (para-)hypersymplectic structures on pre-Courant algebroids. These symmetries do not exist for (para-)hypersymplectic structures on Lie algebroids (see definition in [3]).

Theorem 6.9. *The triple $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is a hypersymplectic (resp., para-hypersymplectic) structure on a pre-Courant algebroid (E, Θ) if and only if $(\mathcal{S}_1, \mathcal{T}_2, \mathcal{T}_3)$ is a hypersymplectic (resp., para-hypersymplectic) structure on (E, Θ). Furthermore, both (para-)hypersymplectic structures determine equal or opposite pseudo-metrics.*

Proof. If $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is a (para-)hypersymplectic structure then, previous definitions and results yield,

$$\begin{cases} \mathcal{S}_i^2 = \mathcal{T}_i^2 = \varepsilon_i \mathrm{id}_E; \\ \mathcal{S}_i \mathcal{S}_j + \mathcal{S}_j \mathcal{S}_i = \mathcal{T}_i \mathcal{T}_j + \mathcal{T}_j \mathcal{T}_i = \mathcal{S}_i \mathcal{T}_j + \mathcal{T}_j \mathcal{S}_i = 0; \\ \widetilde{\mho}_\Theta \mathcal{S}_i = \widetilde{\mho}_\Theta \mathcal{T}_i = 0. \end{cases}$$

----

[2]In [6], hyperkähler structures on Courant algebroids are called generalized hyper-Kähler structures.

Thus, $(\mathcal{S}_1, \mathcal{T}_2, \mathcal{T}_3)$ is a (para-)hypersymplectic structure.

Now, let us assume that $(\mathcal{S}_1, \mathcal{T}_2, \mathcal{T}_3)$ is a (para-)hypersymplectic structure. In this case, the transition morphisms are $\varepsilon_1\varepsilon_3\mathcal{T}_1, \varepsilon_1\varepsilon_3\mathcal{S}_2$ and $\varepsilon_1\varepsilon_3\mathcal{S}_3$ and, using the first part of the proof, we conclude that $(\mathcal{S}_1, \varepsilon_1\varepsilon_3\mathcal{S}_2, \varepsilon_1\varepsilon_3\mathcal{S}_3)$ is a (para-)hypersymplectic structure. Therefore, $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is a (para-)hypersymplectic structure.

Finally, because $\mathcal{S}_3\mathcal{S}_2\mathcal{S}_1 = \varepsilon_1\varepsilon_3\mathcal{T}_3\mathcal{T}_2\mathcal{S}_1$, the pseudo-metrics induced by both (para-)hypersymplectic structures are equal or opposite. $\square$

Applying successively Theorems 6.8 and 6.9, we conclude that one (para-)hypersymplectic structure $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ on (E, Θ) induces several (para-)hypersymplectic and (para-)hyperkähler structures on (E, Θ), as the next diagram shows.

$$
\begin{array}{ccc}
(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3) \text{ (para-)hypersymplectic} & \xleftrightarrow{Thm\ 6.8} & (\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{G}) \text{ (para-)hyperkähler} \\
\Big\updownarrow {\scriptstyle Thm\ 6.9} & & \Big\updownarrow \\
(\mathcal{S}_1, \mathcal{T}_2, \mathcal{T}_3) \text{ (para-)hypersymplectic} & \xleftrightarrow{Thm\ 6.8} & (\mathcal{T}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{G}) \text{ (para-)hyperkähler} \\
\Big\updownarrow {\scriptstyle Thm\ 6.9} & & \Big\updownarrow \\
(\mathcal{T}_1, \mathcal{S}_2, \mathcal{T}_3) \text{ (para-)hypersymplectic} & \xleftrightarrow{Thm\ 6.8} & (\mathcal{S}_1, \mathcal{T}_2, \mathcal{S}_3, \mathcal{G}) \text{ (para-)hyperkähler} \\
\Big\updownarrow {\scriptstyle Thm\ 6.9} & & \Big\updownarrow \\
(\mathcal{T}_1, \mathcal{T}_2, \mathcal{S}_3) \text{ (para-)hypersymplectic} & \xleftrightarrow{Thm\ 6.8} & (\mathcal{S}_1, \mathcal{S}_2, \mathcal{T}_3, \mathcal{G}) \text{ (para-)hyperkähler}
\end{array}
$$

Remark 6.10. *In Section 5 we pointed out that a hypersymplectic structure on a Courant algebroid determines a hypercomplex structure, in the sense of [14], which is formed by the associated transition morphisms. As it is shown in the diagram above, when $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is a hypersymplectic structure on a Courant algebroid (E, Θ), the triples $(\mathcal{S}_1, \mathcal{T}_2, \mathcal{T}_3)$, $(\mathcal{T}_1, \mathcal{S}_2, \mathcal{T}_3)$ and $(\mathcal{T}_1, \mathcal{T}_2, \mathcal{S}_3)$ are also hypersymplectic structures. Therefore, $(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$, $(\mathcal{T}_1, \mathcal{S}_2, \mathcal{S}_3)$, $(\mathcal{S}_1, \mathcal{T}_2, \mathcal{S}_3)$ and $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{T}_3)$ are hypercomplex structures on (E, Θ).*

Example 6.11. *Given a hypersymplectic structure $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ on a Courant algebroid (E, Θ), the triples $(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$, $(\mathcal{T}_1, \mathcal{S}_2, \mathcal{S}_3)$, $(\mathcal{S}_1, \mathcal{T}_2, \mathcal{S}_3)$ and $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{T}_3)$ are hypersymplectic structures on (E, Θ) (see Remark 6.10 and Example 5.2).*

7. HYPERSYMPLECTIC STRUCTURES ON LIE ALGEBROIDS

The purpose of this section is to present a first example of an $\boldsymbol{\varepsilon}$-hypersymplectic structure on a Courant algebroid, which is constructed out of an $\boldsymbol{\varepsilon}$-hypersymplectic structure on a Lie algebroid. First, we recall the definition and some properties of the latter [1, 3].

An $\boldsymbol{\varepsilon}$-*hypersymplectic structure* on a Lie algebroid (A, μ) is a triple $(\omega_1, \omega_2, \omega_3)$ of symplectic forms with inverse Poisson bivectors (π_1, π_2, π_3) such that the transition endomorphisms N_1, N_2 and N_3 on A, defined by

$$(25) \qquad\qquad N_i := \pi_{i-1}^{\#} \circ \omega_{i+1}^{\flat}, \quad i \in \mathbb{Z}_3,$$

satisfy

$$(26) \qquad\qquad N_i^2 = \varepsilon_i \mathrm{id}_A, \quad i = 1, 2, 3.$$

An important property of the transition morphisms N_i, $i = 1, 2, 3$, is that they are Nijenhuis morphisms.

Having an ε-hypersymplectic structure on a Lie algebroid (A, μ), we define $g \in \Gamma(\bigotimes^2 A^*)$ by setting, for all $X, Y \in \Gamma(A)$,

$$g(X, Y) := \langle g^\flat X, Y \rangle,$$

where $g^\flat : A \longrightarrow A^*$ is given by

$$(27) \qquad\qquad g^\flat := \varepsilon_3 \varepsilon_2 \; \omega_3{}^\flat \circ \pi_1{}^\sharp \circ \omega_2{}^\flat.$$

The definition of $g^\flat$ is not affected by a circular permutation of the indices in equation (27), that is,

$$(28) \qquad\qquad g^\flat = \varepsilon_{i-1} \varepsilon_{i+1} \; \omega_{i-1}{}^\flat \circ \pi_i{}^\sharp \circ \omega_{i+1}{}^\flat,$$

for all $i \in \mathbb{Z}_3$. Moreover, we have

$$(g^\flat)^* = -\varepsilon_1 \varepsilon_2 \varepsilon_3 \; g^\flat,$$

which means that g is symmetric or skew-symmetric, depending on the sign of the product $\varepsilon_1 \varepsilon_2 \varepsilon_3$. When $\varepsilon_1 \varepsilon_2 \varepsilon_3 = -1$, the morphism $g^\flat$ defined by (27) determines a *pseudo-metric* on A.

Let (A, μ) be a Lie algebroid and consider the Courant algebroid $(A \oplus A^*, \mu)$. If we take a triple $(\omega_1, \omega_2, \omega_3)$ of 2-forms and a triple (π_1, π_2, π_3) of bivectors on A, we may define the skew-symmetric bundle endomorphisms $\mathcal{S}_i : A \oplus A^* \to A \oplus A^*$, $i = 1, 2, 3$,

$$(29) \qquad\qquad \mathcal{S}_i := \begin{bmatrix} 0 & \varepsilon_i \, \pi_i^\sharp \\ \omega_i^\flat & 0 \end{bmatrix}.$$

In order to simplify the notation and if there is no risk of confusion, we shall omit the symbols $\sharp$ and $\flat$ and denote the morphisms $\omega_i^\flat$ and $\pi_i^\sharp$ by ω_i and π_i, respectively. Moreover, in the supergeometric setting, we have

$$\mathcal{S}_i(X + \alpha) = \{X + \alpha, \omega_i + \varepsilon_i \pi_i\},$$

for all $X + \alpha \in A \oplus A^*$.

Lemma 7.1. *Let ω_1, ω_2 and ω_3 be 2-forms on a vector bundle A over M and π_1, π_2 and π_3 bivectors on A. Consider the vector bundle morphisms N_1, N_2 and N_3 on A, given by (25), and the bundle endomorphisms $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ on $A \oplus A^*$, given by (29). Then, for all $i = 1, 2, 3$,*

i) $\mathcal{S}_i^2 = \varepsilon_i \mathrm{id}_{A \oplus A^*} \Leftrightarrow \pi_i \circ \omega_i = \mathrm{id}_A$,
ii) $\mathcal{S}_{i-1} \mathcal{S}_{i+1} = \varepsilon_1 \varepsilon_2 \varepsilon_3 \, \mathcal{S}_{i+1} \mathcal{S}_{i-1} \Leftrightarrow N_i^2 = \varepsilon_i \mathrm{id}_A$.

Proof. A simple computation gives i). To prove ii), we notice that $N_i^2 = \varepsilon_i \, \mathrm{id}_A$ is equivalent to

$$\omega_{i+1} \circ \pi_{i-1} = \varepsilon_i \, \omega_{i-1} \circ \pi_{i+1},$$

for all $i \in \mathbb{Z}_3$ (see [3]). On the other hand, we have

$$\mathcal{S}_{i-1} \mathcal{S}_{i+1} = \begin{bmatrix} \varepsilon_{i-1} \, \pi_{i-1} \circ \omega_{i+1} & 0 \\ 0 & \varepsilon_{i+1} \, \omega_{i-1} \circ \pi_{i+1} \end{bmatrix}$$

and

$$\varepsilon_1\varepsilon_2\varepsilon_3 \mathcal{S}_{i+1}\mathcal{S}_{i-1} = \varepsilon_1\varepsilon_2\varepsilon_3 \left[\begin{array}{cc} \varepsilon_{i+1}\,\pi_{i+1}\circ\omega_{i-1} & 0 \\ 0 & \varepsilon_{i-1}\,\omega_{i+1}\circ\pi_{i-1} \end{array} \right]$$

$$= \left[\begin{array}{cc} \varepsilon_{i-1}\varepsilon_i\,\pi_{i+1}\circ\omega_{i-1} & 0 \\ 0 & \varepsilon_{i+1}\varepsilon_i\,\omega_{i+1}\circ\pi_{i-1} \end{array} \right].$$

So, $\mathcal{S}_{i-1}\mathcal{S}_{i+1} = \varepsilon_1\varepsilon_2\varepsilon_3 \mathcal{S}_{i+1}\mathcal{S}_{i-1}$ if and only if $\omega_{i+1}\circ\pi_{i-1} = \varepsilon_i\,\omega_{i-1}\circ\pi_{i+1}$ and this completes the proof. $\qquad\square$

Theorem 7.2. *A triple* $(\omega_1,\omega_2,\omega_3)$*, with inverse* (π_1,π_2,π_3)*, is an* $\boldsymbol{\varepsilon}$*-hypersymplectic structure on a Lie algebroid* (A,μ) *if and only if the triple* $(\mathcal{S}_1,\mathcal{S}_2,\mathcal{S}_3)$ *is an* $\boldsymbol{\varepsilon}$*-hypersymplectic structure on the Courant algebroid* $(A\oplus A^*,\mu)$*, with* $\mathcal{S}_i$*,* $i=1,2,3$*, given by (29).*

Proof. Suppose that $(\omega_1,\omega_2,\omega_3)$ is an $\boldsymbol{\varepsilon}$-hypersymplectic structure on a Lie algebroid (A,μ) and π_i is the inverse of ω_i, $i=1,2,3$. According to Definition 4.1 and Lemma 7.1, we only have to check that $\mu_{\mathcal{S}_i,\mathcal{S}_i} = \varepsilon_i\mu$, for $i=1,2,3$. Recalling that π is a Poisson bivector if and only if $\{\pi,\{\pi,\mu\}\}=0$ and ω is a closed 2-form if and only if $\{\mu,\omega\}=0$, a simple computation gives:

$$\begin{aligned} \{\mathcal{S}_i,\{\mathcal{S}_i,\mu\}\} &= \{\omega_i+\varepsilon_i\pi_i,\{\omega_i+\varepsilon_i\pi_i,\mu\}\} = \varepsilon_i\{\omega_i,\{\pi_i,\mu\}\} \\ &= \varepsilon_i\,\mu, \end{aligned}$$

where we used, in the last equality, the formula

$$(30) \qquad\qquad \{\mathrm{id}_A,\chi\} = (q-p)\chi, \quad \chi\in\mathcal{F}_{A\oplus A^*}^{(p,q)}.$$

Conversely, assume that the endomorphisms $\mathcal{S}_i = \left[\begin{array}{cc} 0 & \varepsilon_i\,\pi_i \\ \omega_i & 0 \end{array} \right]$, $i=1,2,3$, form an $\boldsymbol{\varepsilon}$-hypersymplectic structure on the Courant algebroid $(A\oplus A^*,\mu)$. Using again Lemma 7.1, we only have to prove that the non-degenerate 2-forms ω_i are symplectic. From

$$\{\omega_i+\varepsilon_i\pi_i,\{\omega_i+\varepsilon_i\pi_i,\mu\}\} = \varepsilon_i\mu,$$

we get $\{\pi_i,\{\pi_i,\mu\}\}=0$, which means that π_i is a Poisson bivector on (A,μ). But π_i being a Poisson bivector on (A,μ) is equivalent to ω_i being a symplectic form on (A,μ). $\qquad\square$

Under the conditions of Theorem 7.2, the transition morphisms of the $\boldsymbol{\varepsilon}$-hypersymplectic structure $(\mathcal{S}_1,\mathcal{S}_2,\mathcal{S}_3)$ on $(A\oplus A^*,\mu)$, defined by (22), are given by

$$\mathcal{T}_i = \left[\begin{array}{cc} N_i & 0 \\ 0 & \varepsilon_1\varepsilon_2\varepsilon_3\,N_i^* \end{array} \right], \quad i=1,2,3,$$

where N_i is the transition morphism of the $\boldsymbol{\varepsilon}$-hypersymplectic structure $(\omega_1,\omega_2,\omega_3)$ on the Lie algebroid (A,μ), see (25). The endomorphism $\mathcal{G}: A\oplus A^* \to A\oplus A^*$ defined by (23) is given by

$$\mathcal{G} = \left[\begin{array}{cc} 0 & (g^\flat)^{-1} \\ g^\flat & 0 \end{array} \right],$$

where $g^\flat : A \to A^*$ is defined by (27).

Example 7.3 (Heisenberg Lie group). *Consider the Heisenberg Lie group* H^3 *and* $\{e^1,e^2,e^3\}$ *a basis of* 1*-forms on* H^3 *defined by*

$$e^1 := \mathrm{d}x, \qquad e^2 := \mathrm{d}y \qquad and \qquad e^3 := \mathrm{d}z - \frac{1}{2}x\,\mathrm{d}y + \frac{1}{2}y\,\mathrm{d}x,$$

where x, y, z are global coordinate functions on H^3.

We define a hypersymplectic structure $(\omega_1, \omega_2, \omega_3)$ on the Lie algebroid $T(H^3 \times I)$, where $I \subseteq \mathbb{R}$, by setting (see [4])

$$
\omega_1{}^\flat := \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & e^t \\ 1 & 0 & 0 & 0 \\ 0 & -e^t & 0 & 0 \end{bmatrix}, \quad
\omega_2{}^\flat := \begin{bmatrix} 0 & 0 & 0 & -e^t \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ e^t & 0 & 0 & 0 \end{bmatrix} \quad and \quad
\omega_3{}^\flat := \begin{bmatrix} 0 & -e^t & 0 & 0 \\ e^t & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix},
$$

on the basis $\{e_1, e_2, e_3, \partial_t\}$, t being the coordinate on $I \subset \mathbb{R}$.

Then, using Theorem 7.2, we define a hypersymplectic structure $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ on the Courant algebroid $T(H^3 \times I) \oplus T^*(H^3 \times I)$, by setting

$$
\mathcal{S}_1 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-t} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -e^{-t} & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^t & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -e^t & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad
\mathcal{S}_2 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -e^{-t} \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-t} & 0 & 0 & 0 \\ 0 & 0 & 0 & -e^t & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ e^t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

$$
and \quad \mathcal{S}_3 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -e^{-t} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-t} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -e^t & 0 & 0 & 0 & 0 & 0 & 0 \\ e^t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.
$$

The transition morphisms are

$$
\mathcal{T}_1 := \begin{bmatrix} 0 & 0 & -e^{-t} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ e^t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -e^t & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & e^{-t} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}, \quad
\mathcal{T}_2 := \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -e^{-t} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^t & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -e^t & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-t} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}
$$

$$
and \quad \mathcal{T}_3 := \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -e^t & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-t} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -e^{-t} \\ 0 & 0 & 0 & 0 & 0 & 0 & e^t & 0 \end{bmatrix}.
$$

The metric is given by

$$\mathcal{G} := \begin{bmatrix} 0 & 0 & 0 & 0 & e^{-t} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-t} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{t} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-t} \\ e^{t} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{t} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-t} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{t} & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Changing a couple of signs, we may also define a para-hypersymplectic structure on the Lie algebroid $T(H^3 \times I)$ by setting

$$\omega_1'^{\flat} := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{t} \\ -1 & 0 & 0 & 0 \\ 0 & -e^{t} & 0 & 0 \end{bmatrix}, \quad \omega_2'^{\flat} := \begin{bmatrix} 0 & 0 & 0 & -e^{t} \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ e^{t} & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \omega_3'^{\flat} := -\omega_3^{\flat}.$$

The corresponding para-hypersymplectic structure on the Courant algebroid $T(H^3 \times I) \oplus T^*(H^3 \times I)$ is the triple $(\mathcal{S}_1', \mathcal{S}_2', \mathcal{S}_3')$ given by

$$\mathcal{S}_1' := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -e^{-t} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-t} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{t} & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -e^{t} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{S}_2' := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-t} \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -e^{-t} & 0 & 0 & 0 \\ 0 & 0 & 0 & -e^{t} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ e^{t} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{and} \quad \mathcal{S}_3' := -\mathcal{S}_3.$$

The transition morphisms and the metric associated to $(\mathcal{S}_1', \mathcal{S}_2', \mathcal{S}_3')$ (defined, respectively, by (22) and (23)) can be easily computed.

8. Hypersymplectic structures on Lie bialgebroids

In this section we present a class of examples of ε-hypersymplectic structures on a Courant algebroid $(A \oplus A^*, \mu + \gamma)$, which is the double of a Lie bialgebroid $((A, A^*), \mu, \gamma)$.

Having in mind that a bivector π on A can be seen as a 2-form on A^*, through the identification $A = (A^*)^*$, we have the following result.

Theorem 8.1. *Let $((A, A^*), \mu, \gamma)$ be a Lie bialgebroid and $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ be a triple of bundle endomorphisms of $A \oplus A^*$, with $\mathcal{S}_i$ given by (29). The following assertions are equivalent:*

  i) *$(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is an ε-hypersymplectic structure on the Courant algebroid $(A \oplus A^*, \mu + \gamma)$*
  ii) *$(\omega_1, \omega_2, \omega_3)$ is an ε-hypersymplectic structure on the Lie algebroid (A, μ), (π_1, π_2, π_3) is an ε-hypersymplectic structure on the Lie algebroid (A^*, γ) and π_i is the inverse of ω_i, $i = 1, 2, 3$.*

Proof. We use Lemma 7.1 noticing that $\pi_i \circ \omega_i = \mathrm{id}_A \Leftrightarrow \omega_i \circ \pi_i = \mathrm{id}_{A^*}$ and $N_i^2 = \varepsilon_i \mathrm{id}_A \Leftrightarrow (N_i^*)^2 = \varepsilon_i \mathrm{id}_{A^*}$, $i = 1, 2, 3$, so that conditions i) and ii) of Definition 4.1 are

satisfied if and only if π_i and ω_i are inverses of each other and (26) holds. Moreover, using the bidegrees of $\mathcal{F}_{A \oplus A^*}^3$, we have

$$(31) \qquad \{\mathcal{S}_i, \{\mathcal{S}_i, \mu + \gamma\}\} = \varepsilon_i(\mu + \gamma) \Leftrightarrow \begin{cases} \{\pi_i, \{\pi_i, \mu\}\} = 0 \\ \{\omega_i, \{\omega_i, \gamma\}\} = 0 \\ \{\omega_i, \{\pi_i, \mu\}\} + \{\pi_i, \{\omega_i, \mu\}\} = \mu \\ \{\omega_i, \{\pi_i, \gamma\}\} + \{\pi_i, \{\omega_i, \gamma\}\} = \gamma. \end{cases}$$

The first equation on the right-hand side of (31) means that π_i is a Poisson bivector on (A, μ), which is equivalent to ω_i being a symplectic form on (A, μ). The second equation on the right-hand side of (31) means that ω_i, seen as a bivector on A^*, is Poisson on the Lie algebroid (A^*, γ), which is equivalent to saying that π_i is symplectic on (A^*, γ). Concerning the third and fourth equations on the right-hand side of (31), an easy computation using (30) shows that they are equivalent, respectively, to the first and second equations. □

It is well known that a Poisson bivector π on (A, μ) determines a Lie algebroid structure on A^*; we denote by μ_π this induced structure. In [4] we proved that if $(\omega_1, \omega_2, \omega_3)$ is an ε-hypersymplectic structure on a Lie algebroid (A, μ) and π_i is the inverse of ω_i, $i = 1, 2, 3$, then the triple (π_1, π_2, π_3) is an ε-hypersymplectic structure on the Lie algebroid (A^*, μ_{π_i}). So, given an ε-hypersymplectic structure $(\omega_1, \omega_2, \omega_3)$ on a Lie algebroid (A, μ), Theorem 8.1 yields that the triple $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is an ε-hypersymplectic structure on the Courant algebroid $(A \oplus A^*, \mu + \mu_{\pi_i})$. Conversely, if $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is an ε-hypersymplectic structure on the Courant algebroid $(A \oplus A^*, \mu + \mu_{\pi_i})$ then, by Theorem 8.1, $(\omega_1, \omega_2, \omega_3)$ is an ε-hypersymplectic structure on the Lie algebroid (A, μ).

Thus, we have proved:

Corollary 8.2. *The triple $(\omega_1, \omega_2, \omega_3)$, with inverse (π_1, π_2, π_3), is an ε-hypersymplectic structure on the Lie algebroid (A, μ) if and only if $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is an ε-hypersymplectic structure on the Courant algebroid $(A \oplus A^*, \mu + \mu_{\pi_i})$, with $\mathcal{S}_i$ given by (29), $i = 1, 2, 3$.*

9. HYPERSYMPLECTIC STRUCTURES WITH TORSION ON LIE ALGEBROIDS

In this section we study a class of examples of hypersymplectic structures on pre-Courant algebroids determined by some structures on Lie algebroids which are called *hypersymplectic with torsion*. These are introduced and discussed in [5] and may be considered as being equivalent to hyperkähler structures with torsion, also known as HKT structures [7]. The hypersymplectic structures with torsion on Lie algebroids provide examples of hypersymplectic structures (without torsion) on Courant algebroids which are doubles of quasi-Lie bialgebroids and even in the more general case where the Courant structure is the double of a proto-Lie bialgebroid.

We give the definition of a hypersymplectic structure with torsion on a Lie algebroid (A, μ), which is a particular case of an ε-hypersymplectic structure with torsion considered in [5].

Let ω_1, ω_2 and ω_3 be nondegenerate 2-forms on a Lie algebroid (A, μ), with inverses π_1, π_2 and $\pi_3 \in \Gamma(\wedge^2 A)$, respectively, and consider the transition morphisms $N_1, N_2, N_3 : A \to A$ given by (25).

Definition 9.1. *The triple* $(\omega_1, \omega_2, \omega_3)$ *is a* hypersymplectic structure with torsion *on the Lie algebroid* (A, μ) *if*

$$(32) \qquad N_i{}^2 = -\mathrm{id}_A, \quad i = 1, 2, 3, \quad \text{and} \quad N_1 \mathrm{d}\omega_1 = N_2 \mathrm{d}\omega_2 = N_3 \mathrm{d}\omega_3,$$

where $N_i \mathrm{d}\omega_i(X, Y, Z) = \mathrm{d}\omega_i(N_i X, N_i Y, N_i Z)$, *for all* $X, Y, Z \in \Gamma(A)$ *and* d *stands for the differential of the Lie algebroid* (A, μ).

When the non-degenerate 2-forms ω_1, ω_2 and ω_3 are closed, then they are symplectic forms and the right hand side of (32) is trivially satisfied. In this case, the triple $(\omega_1, \omega_2, \omega_3)$ is a *hypersymplectic structure* (without torsion) on (A, μ), that is, an ε-hypersymplectic structure with $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = -1$ (see Section 7).

The next lemma will be useful in what follows (see also [11]).

Lemma 9.2. *Let* $((A, A^*), \mu, \gamma)$ *be a Lie bialgebroid,* $\psi \in \Gamma(\wedge^3 A)$, $\phi \in \Gamma(\wedge^3 A^*)$, $\pi \in \Gamma(\wedge^2 A)$ *and* $\omega \in \Gamma(\wedge^2 A^*)$, *with* π *and* ω *inverse of each other. Then,*

   i) $\{\pi, \{\pi, \mu\}\} = 2\,\psi \Leftrightarrow 2\,\{\pi, \{\omega, \mu\}\} = \{\omega, \{\omega, \psi\}\}$;
   ii) $\{\omega, \{\omega, \gamma\}\} = 2\,\phi \Leftrightarrow 2\,\{\omega, \{\pi, \gamma\}\} = \{\pi, \{\pi, \phi\}\}$.

Proof. i) Let us assume that $\{\pi, \{\pi, \mu\}\} = 2\,\psi$. Then,

$$\{\omega, \{\pi, \{\pi, \mu\}\}\} = 2\,\{\omega, \psi\}$$

and the Jacobi identity together with (30) gives

$$\{\pi, \{\pi, \{\omega, \mu\}\}\} = 2\,\{\omega, \psi\}.$$

Thus,

$$\{\omega, \{\pi, \{\pi, \{\omega, \mu\}\}\}\} = 2\,\{\omega, \{\omega, \psi\}\}$$

or, equivalently,

$$(33) \qquad \{\pi, \{\omega, \mu\}\} + \{\pi, \{\omega, \{\pi, \{\omega, \mu\}\}\}\} = 2\,\{\omega, \{\omega, \psi\}\}.$$

Finally, (33) gives

$$2\{\pi, \{\omega, \mu\}\} = \{\omega, \{\omega, \psi\}\}.$$

Now, we assume that $\{\omega, \{\omega, \psi\}\} = 2\,\{\pi, \{\omega, \mu\}\}$. Then,

$$\{\pi, \{\omega, \{\omega, \psi\}\}\} = 2\,\{\pi, \{\pi, \{\omega, \mu\}\}\}$$

which is equivalent to

$$\{\omega, \psi\} + \{\omega, \{\pi, \{\omega, \psi\}\}\} = 2\,\{\pi, \{\pi, \{\omega, \mu\}\}\}.$$

Thus,

$$(34) \qquad \{\pi, \{\omega, \psi\}\} + \{\pi, \{\omega, \{\pi, \{\omega, \psi\}\}\}\} = 2\,\{\pi, \{\pi, \{\pi, \{\omega, \mu\}\}\}\}.$$

From (34) we get, applying the Jacobi identity and (30) several times,

$$3\psi + 3\{\pi, \{\omega, \psi\}\} = -2\,\{\pi, \{\pi, \mu\}\} + 2\,\{\pi, \{\pi, \{\omega, \{\pi, \mu\}\}\}\}$$
$$\Leftrightarrow 6\psi = \{\pi, \{\omega, \{\pi, \{\pi, \mu\}\}\}\} \Leftrightarrow 2\psi = \{\pi, \{\pi, \mu\}\}.$$

ii) The proof is similar to case i).

$\square$

Now, we have to mention that the definition of hypersymplectic structure with torsion on a Lie algebroid can be given using the inverses of the non-degenerate 2-forms ω_i. More precisely,

$(\omega_1, \omega_2, \omega_3)$ *is a hypersymplectic structure with torsion on* (A, μ) *if and only if*

$$(35) \qquad N_i^2 = -\mathrm{id}_A \quad and \quad [\pi_1, \pi_1] = [\pi_2, \pi_2] = [\pi_3, \pi_3],$$

where $[.,.]$ is the Schouten-Nijenhuis bracket of multivectors on A. The equivalence of the two definitions is proved in [5].

The next proposition gives a first example of a hypersymplectic structure on a pre-Courant algebroid, which is constructed out of a hypersymplectic structure with torsion on a Lie algebroid.

Proposition 9.3. *Let* $(\omega_1, \omega_2, \omega_3)$ *be a triple of* 2-*forms and* (π_1, π_2, π_3) *be a triple of bivectors on a Lie algebroid* (A, μ). *Consider the triple* $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ *of endomorphisms of* $A \oplus A^*$, *with* $\mathcal{S}_i$ *given by* (29). *The following assertions are equivalent:*

i) $(\omega_1, \omega_2, \omega_3)$ *is a hypersymplectic structure with torsion on the Lie algebroid* (A, μ) *and* π_i *is the inverse of* ω_i, $i = 1, 2, 3$;

ii) $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ *is a hypersymplectic structure on the pre-Courant algebroid* $(A \oplus A^*, \mu + \psi)$, *for some* $\psi \in \Gamma(\wedge^3 A)$.

Proof. From Lemma 7.1, conditions i) and ii) of Definition 4.1 are satisfied while for condition iii), we have

$$\{\mathcal{S}_i, \{\mathcal{S}_i, \mu + \psi\}\} = -\mu - \psi \Leftrightarrow \begin{cases} \{\omega_i, \{\pi_i, \mu\}\} + \{\pi_i, \{\omega_i, \mu\}\} - \{\omega_i, \{\omega_i, \psi\}\} = \mu \\ -\{\pi_i, \{\pi_i, \mu\}\} + \{\pi_i, \{\omega_i, \psi\}\} = \psi \end{cases}$$

$$\Leftrightarrow \begin{cases} 2\{\pi_i, \{\omega_i, \mu\}\} = \{\omega_i, \{\omega_i, \psi\}\} \\ \{\pi_i, \{\pi_i, \mu\}\} = 2\psi \end{cases}$$

$$(36) \qquad\qquad \Leftrightarrow \{\pi_i, \{\pi_i, \mu\}\} = 2\psi,$$

where the latter equivalence is given by Lemma 9.2. Equation (36) exhibits the appropriate definition of ψ in order to satisfy condition iii) of Definition 4.1.

Let us now assume that $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is a hypersymplectic structure on a pre-Courant algebroid $(A \oplus A^*, \mu + \psi)$, with $\psi \in \Gamma(\wedge^3 A)$. From (36), the 3-vector ψ is given by $\psi = -\frac{1}{2}[\pi_i, \pi_i]$, $i \in \{1, 2, 3\}$. Thus, $[\pi_1, \pi_1] = [\pi_2, \pi_2] = [\pi_3, \pi_3]$ and, from Lemma 7.1 and (35), we get that $(\omega_1, \omega_2, \omega_3)$ is a hypersymplectic structure with torsion on (A, μ). $\qquad \square$

Notice that in the assertion ii) of Proposition 9.3, since $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is a hypersymplectic structure for the pre-Courant structure $\mu + \psi$, condition $(\mu + \psi)_{\mathcal{S}_k, \mathcal{S}_k} = -\mu - \psi$ holds and implies that ψ has to be of the form $-\frac{1}{2}[\pi_k, \pi_k]$, for any $k \in \{1, 2, 3\}$.

If we aim to have a Courant structure on $A \oplus A^*$, in the statement ii) of Proposition 9.3, we have to require the bivectors to be weak-Poisson[3] as shown in the next theorem.

Theorem 9.4. *Let* $(\omega_1, \omega_2, \omega_3)$ *be a triple of* 2-*forms and* (π_1, π_2, π_3) *be a triple of bivectors on a Lie algebroid* (A, μ). *Consider the triple* $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ *of endomorphisms of* $A \oplus A^*$, *with* $\mathcal{S}_i$ *given by* (29). *The following assertions are equivalent:*

i) $(\omega_1, \omega_2, \omega_3)$ *is a hypersymplectic structure with torsion on the Lie algebroid* (A, μ), π_i *is the inverse of* ω_i *and* π_i *is weak-Poisson,* $i = 1, 2, 3$;

ii) $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ *is a hypersymplectic structure on the Courant algebroid* $(A \oplus A^*, \mu + \psi)$, *for some* $\psi \in \Gamma(\wedge^3 A)$.

---

[3]A bivector π on a Lie algebroid (A, μ) is *weak-Poisson* if $\{\mu, \{\{\pi, \mu\}, \pi\}\} = 0$ or, equivalently, $\{\mu, [\pi, \pi]\} = 0$.

Proof. In addition to the proof of Proposition 9.3, it is enough to notice that, since (A, μ) is a Lie algebroid,

$$\mu + \psi \text{ is Courant} \Leftrightarrow \{\mu + \psi, \mu + \psi\} = 0$$
$$\Leftrightarrow \{\mu, \psi\} = 0 \Leftrightarrow \pi_i \text{ is weak-Poisson, } i = 1, 2, 3.$$

$\square$

In the next proposition we show that having a Lie bialgebroid (A, A^*) equipped with a hypersymplectic structure with torsion on A and a hypersymplectic structure with torsion on A^* is equivalent to having a hypersymplectic structure (without torsion) on $A \oplus A^*$ equipped with a pre-Courant structure.

Proposition 9.5. *Let* $((A, A^*), \mu, \gamma)$ *be a Lie bialgebroid,* $(\omega_1, \omega_2, \omega_3)$ *be a triple of 2-forms and* (π_1, π_2, π_3) *be a triple of bivectors on* A. *Consider the triple* $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ *of endomorphisms of* $A \oplus A^*$, *with* $\mathcal{S}_i$ *given by* (29). *The following assertions are equivalent:*

- *i)* $(\omega_1, \omega_2, \omega_3)$ *is a hypersymplectic structure with torsion on the Lie algebroid* (A, μ) *and* (π_1, π_2, π_3) *is a hypersymplectic structure with torsion on the Lie algebroid* (A^*, γ), *with* π_i *the inverse of* ω_i, $i = 1, 2, 3$;
- *ii)* $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ *is a hypersymplectic structure on the pre-Courant algebroid* $(A \oplus A^*, \mu + \gamma + \psi + \phi)$, *for some* $\psi \in \Gamma(\wedge^3 A)$ *and* $\phi \in \Gamma(\wedge^3 A^*)$.

Proof. Let us assume that $(\omega_1, \omega_2, \omega_3)$ is a hypersymplectic structure with torsion on (A, μ), (π_1, π_2, π_3) is a hypersymplectic structure with torsion on (A^*, γ) and π_i is the inverse of ω_i, $i = 1, 2, 3$. From Lemma 7.1, conditions i) and ii) of Definition 4.1 are satisfied while, for condition iii), we have:

$$\{\mathcal{S}_i, \{\mathcal{S}_i, \mu + \gamma + \psi + \phi\}\} = -\mu - \gamma - \psi - \phi$$

$$\Leftrightarrow \begin{cases} \{\omega_i, \{\omega_i, \psi\}\} - \{\omega_i, \{\pi_i, \mu\}\} - \{\pi_i, \{\omega_i, \mu\}\} = -\mu \\ -\{\omega_i, \{\{\pi_i, \gamma\}\} - \{\pi_i, \{\omega_i, \gamma\}\} + \{\pi_i, \{\pi_i, \phi\}\} = -\gamma \\ \{\omega_i, \{\omega_i, \gamma\}\} - \{\omega_i, \{\pi_i, \phi\}\} = -\phi \\ \{\pi_i, \{\pi_i, \mu\}\} - \{\pi_i, \{\omega_i, \psi\}\} = -\psi \end{cases}$$

$$\Leftrightarrow \begin{cases} \{\omega_i, \{\omega_i, \psi\}\} = 2\{\pi_i, \{\omega_i, \mu\}\} \\ \{\pi_i, \{\pi_i, \phi\}\} = 2\{\omega_i, \{\pi_i, \gamma\}\} \\ \{\omega_i, \{\omega_i, \gamma\}\} = 2\phi \\ \{\pi_i, \{\pi_i, \mu\}\} = 2\psi \end{cases}$$

$$(37) \qquad \Leftrightarrow \begin{cases} \{\omega_i, \{\omega_i, \gamma\}\} = 2\phi \\ \{\pi_i, \{\pi_i, \mu\}\} = 2\psi, \end{cases}$$

where the latter equivalence is given by Lemma 9.2. Equation (37) gives the appropriate definition of ψ and ϕ in order to satisfy condition iii) of Definition 4.1.

Now, we assume that $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is a hypersymplectic structure on the pre-Courant algebroid $(A \oplus A^*, \mu + \gamma + \psi + \phi)$. Using Lemma 7.1, we conclude that π_i is the inverse of ω_i and $N_i^2 = -\mathrm{id}_A$, $i = 1, 2, 3$. Moreover, from (37), we get $\psi = -\frac{1}{2}\{\pi_i, \{\mu, \pi_i\}\}$ and $\phi = -\frac{1}{2}\{\omega_i, \{\gamma, \omega_i\}\}$, for $i = 1, 2, 3$. Thus, $(\omega_1, \omega_2, \omega_3)$ is a hypersymplectic structure with torsion on the Lie algebroid (A, μ) and (π_1, π_2, π_3) is a hypersymplectic structure with torsion on the Lie algebroid (A^*, γ) (see (35)). $\square$

The assumption of $((A, A^*), \mu, \gamma)$ being a Lie bialgebroid, in Proposition 9.5, can be weakened. In fact, it is enough to require that (A, μ) and (A^*, γ) are Lie algebroids, without assuming that $\{\mu, \gamma\} = 0$.

In the statement of Proposition 9.5, if we intend to replace the pre-Courant algebroid structure on $A \oplus A^*$ by a Courant algebroid structure, we need to impose some additional conditions on ϕ and ψ, as shown in the next theorem.

Theorem 9.6. *Let $((A, A^*), \mu, \gamma)$ be a Lie bialgebroid, $(\omega_1, \omega_2, \omega_3)$ be a triple of 2-forms and (π_1, π_2, π_3) be a triple of bivectors. Consider the triple $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ of endomorphisms of $A \oplus A^*$, with $\mathcal{S}_i$ given by (29). The following assertions are equivalent:*

  i) $(\omega_1, \omega_2, \omega_3)$ is a hypersymplectic structure with torsion on the Lie algebroid (A, μ), (π_1, π_2, π_3) is a hypersymplectic structure with torsion on the Lie algebroid (A^*, γ), with π_k the inverse of ω_k, $k = 1, 2, 3$, and $\{\gamma, \phi\} = \{\mu, \psi\} = \{\psi, \phi\} = 0$, where $\psi = -\frac{1}{2}\{\pi_i, \{\mu, \pi_i\}\}$ and $\phi = -\frac{1}{2}\{\omega_j, \{\gamma, \omega_j\}\}$, for any $i, j \in \{1, 2, 3\}$;

  ii) $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is a hypersymplectic structure on the Courant algebroid $(A \oplus A^*, \mu + \gamma + \psi + \phi)$, for some $\psi \in \Gamma(\wedge^3 A)$ and $\phi \in \Gamma(\wedge^3 A^*)$.

Proof. First notice that, using the fact that $((A, A^*), \mu, \gamma)$ is a Lie bialgebroid, we have the following equivalences:

$$\mu + \gamma + \psi + \phi \text{ is Courant} \Leftrightarrow \{\mu + \gamma + \psi + \phi, \ \mu + \gamma + \psi + \phi\} = 0$$

$$\Leftrightarrow \begin{cases} \{\mu, \mu\} = -2\{\gamma, \phi\} \\ \{\gamma, \gamma\} = -2\{\mu, \psi\} \\ \{\mu, \gamma\} = -\{\psi, \phi\} \\ \{\gamma, \psi\} = 0 \\ \{\mu, \phi\} = 0 \end{cases}$$

(38) $$\Leftrightarrow \{\gamma, \phi\} = \{\mu, \psi\} = \{\psi, \phi\} = \{\gamma, \psi\} = \{\mu, \phi\} = 0.$$

Let us assume assertion ii), then $\mu + \gamma + \psi + \phi$ is a Courant structure and condition (38) is satisfied. In particular, $\{\gamma, \phi\} = \{\mu, \psi\} = \{\psi, \phi\} = 0$. Furthermore, the same computations we have done in the proof of Proposition 9.5 (see (37)), yield $\psi = -\frac{1}{2}\{\pi_i, \{\mu, \pi_i\}\}$ and $\phi = -\frac{1}{2}\{\omega_i, \{\gamma, \omega_i\}\}$. The remaining part of assertion i) is a consequence of Proposition 9.5.

Assuming now assertion i), let us prove ii). Taking into account Proposition 9.5, we only need to prove that $\mu + \gamma + \psi + \phi$ is a Courant structure, i.e., that (38) is satisfied. Because part of (38) holds by assumption, we only need to prove $\{\gamma, \psi\} = \{\mu, \phi\} = 0$. We shall prove one equality, the other is similar. We have

$$\begin{aligned} \{\gamma, \psi\} &= -\frac{1}{2}\{\gamma, \{\pi_i, \{\mu, \pi_i\}\}\} \\ &= -\frac{1}{2}\{\{\gamma, \pi_i\}, \{\mu, \pi_i\}\} - \frac{1}{2}\{\pi_i, \{\gamma, \{\mu, \pi_i\}\}\} \\ &= -\frac{1}{2}\{\{\gamma, \pi_i\}, \{\mu, \pi_i\}\} - \frac{1}{2}\{\pi_i, \{\mu, \{\gamma, \pi_i\}\}\} = 0, \end{aligned}$$

where we used the Jacobi identity of $\{.,.\}$ and the fact that $((A, A^*), \mu, \gamma)$ is a Lie bialgebroid (in particular that $\{\mu, \gamma\} = 0$). $\qquad\square$

If we take $\phi = 0$ in Theorem 9.6, then the Lie algebroid (A^*, γ) is equipped with a hypersymplectic structure (without torsion) determined by (π_1, π_2, π_3). So,

Theorem 9.6 shows that having a Lie bialgebroid (A, A^*) equipped with a hypersymplectic structure with torsion on A and a hypersymplectic structure on A^* is equivalent to having a hypersymplectic structure on the Courant algebroid $(A \oplus A^*, \mu + \gamma + \psi)$, which is the double of the quasi-Lie bialgebroid $((A, A^*), \mu, \gamma, \psi)$.

Acknowledgments

References

[1] P. Antunes, *Crochets de Poisson gradués et applications: structures compatibles et généralisations des structures hyperkählériennes*, Ph.D thesis, École Polytechnique, Palaiseau, France, 2010.

[2] P. Antunes, C. Laurent-Gengoux and J. M. Nunes da Costa, Hierarchies and compatibility on Courant algebroids, *Pac. J. Math.* **261** (2013), no. 1, 1–32.

[3] P. Antunes and J. M. Nunes da Costa, Hyperstructures on Lie algebroids, *Rev. in Math. Phys.* **25** (2013), no. 10, 1343003 (19 pages).

[4] P. Antunes and J. M. Nunes da Costa, Induced hypersymplectic and hyperkähler structures on the dual of a Lie algebroid, *Int. J. Geom. Meth. Mod. Phys.* **11** (2014), 1460030 (9 pages).

[5] P. Antunes and J. M. Nunes da Costa, Hyperstructures with torsion on Lie algebroids, preprint arXiv1501.00887.

[6] H. Bursztyn, G. Cavalcanti and M. Gualtieri, Generalized Kähler and hyper-Kähler quotients, in *Poisson geometry in mathematics and physics*, Contemp. Math., 450, 61–77, Amer. Math. Soc., Providence, RI, (2008).

[7] P.S. Howe and G. Papadopoulos, Twistor spaces for hyper-Kähler manifolds with torsion, *Phys. Lett. B* **379** (1996), 80–86.

[8] J. Grabowski, Courant-Nijenhuis tensors and generalized geometries, in *Groups, geometry and physics*, Monogr. Real Acad. Ci. Exact. Fís.- Quím. Nat. Zaragoza, 29, 2006, pp. 101–112.

[9] N. Hitchin, Generalized Calabi-Yau manifolds, *Q. J. Math.* **54** (2003), no. 3, 281–308.

[10] Y. Kosmann-Schwarzbach, Quasi, twisted, and all that ... in Poisson geometry and Lie algebroid theory, in *The Breadth of symplectic and Poisson geometry*, J.E. Marsden and T. Ratiu, eds., Progr. Math., 232, Birkhäuser, Boston, MA, 2005, pp. 363–389.

[11] Y. Kosmann-Schwarzbach, Poisson and symplectic functions in Lie algebroid theory, in *Higher Structures in Geometry and Physics*, A. Cattaneo, A. Giaquinto and Ping Xu, eds., Progr. Math. 287, Birkhäuser, Boston, 2011, pp. 243–268.

[12] D. Roytenberg, On the structure of graded symplectic supermanifolds and Courant algebroids, in *Quantization, Poisson brackets and beyond*, T. Voronov, ed., Contemp. Math., 315, Amer. Math. Soc., Providence, RI, 2002, pp. 169–185.

[13] D. Roytenberg, Quasi-Lie bialgebroids and twisted Poisson manifolds, *Lett. Math. Phys.* **61** (2002), no. 2, 123–137.

[14] M. Stiénon, Hypercomplex structures on Courant algebroids, *C. R. Acad. Sci. Paris* **347** (2009), 545–550.

[15] T. Voronov, Graded manifolds and Drinfeld doubles for Lie bialgebroids, in *Quantization, Poisson brackets and beyond*, T. Voronov, ed., Contemp. Math. 315, Amer. Math. Soc., Providence, RI, 2002, pp. 131–168.

[16] P. Xu, Hyper-Lie Poisson structures, *Ann. Sci. École Norm. Sup.* **30** (1997), no. 3, 279–302.

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