THE MODULAR CLASS OF A LIE ALGEBROID COMORPHISM

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Abstract. We introduce the definition of modular class of a Lie algebroid comorphism and exploit some of its properties.

1. Introduction

The modular class of a Poisson manifold $M$ is an element of the first Poisson cohomology group $H_1^\pi(M)$, which measures the obstruction to the existence of a measure in $M$ invariant under all Hamiltonian diffeomorphisms ([9, 12]). This notion was extended to Lie algebroids by Evan, Lu and Weinstein [3] who showed that the modular class of the cotangent bundle of a Poisson manifold is twice the modular class of the Poisson structure. Grabowski, Marmo and Michor [6] introduced the modular class of a Lie algebroid morphism and this was more deeply studied by Kosmann-Schwarzbach, Laurent-Gengoux and Weinstein in [7] and [8]. In a recent paper [2], the notion of modular class of a Poisson map was given and some of its properties studied. Even more recently Grabowski [5] generalizes all these definitions introducing the modular class of skew algebroid relations. In this paper we exploit the definition of the modular class of a Lie algebroid comorphism, following the approach in [2].

2. The modular class of a Lie algebroid

Let $A \to M$ be a Lie algebroid over $M$, with anchor $\rho : A \to TM$ and Lie bracket $\lbrack \cdot, \cdot \rbrack : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$. We will denote by $\Omega^k(A) \equiv \Gamma(\wedge^k A^*)$ the $A$-forms and by $X^k(A) \equiv \Gamma(\wedge^k A)$ the $A$-multivector fields. Recall that the $A$-differential $d_A : \Omega^k(A) \to \Omega^{k+1}(A)$ is given by

\[
d_A \alpha(X_0, X_1, \ldots, X_n) = \sum_{k=1}^{n} (-1)^k \rho(X_i) \cdots \alpha(X_0, \cdots, \hat{X}_i, \ldots, X_n) + \sum_{0 \leq i < j \leq k} (-1)^{i+j} \alpha([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_n)\]

and turns $\Omega^*(A)$ into a complex whose cohomology is called the Lie algebroid cohomology and will be denoted by $H^*(A)$.

Example 2.1. In case $A = TM$, the Lie algebroid cohomology is the De Rham cohomology.

Example 2.2. For any Poisson manifold $(M, \pi)$ there is a natural Lie algebroid structure on its cotangent bundle $T^*M$: the anchor is $\rho = \pi^2$ and the Lie bracket on sections of $A = T^*M$, i.e., on one forms, is given by:

\[
\lbrack \alpha, \beta \rbrack = \mathcal{L}_{\pi^2 \alpha} \beta - \mathcal{L}_{\pi^2 \beta} \alpha - d \pi(\alpha, \beta).
\]

The Poisson cohomology of $(M, \pi)$ is just the Lie algebroid cohomology of $T^*M$. 

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A morphism between two Lie algebroids $A \to M$ and $B \to N$ is a vector bundle map $(\Phi, \phi)$

$$
\begin{array}{c}
A \\
\downarrow \Phi \\
M \\
\downarrow \phi \\
B \\
\downarrow \\
N
\end{array}
$$
such that the dual vector bundle map $\Phi^*: (\Omega^\bullet(B), d_B) \to (\Omega^\bullet(A), d_A)$ is a chain map.

The most basic example of a Lie algebroid morphism is the tangent map $T\phi$ of a smooth map $\phi: M \to N$.

A representation of a Lie algebroid $A$ is a vector bundle $E \to M$ together with a flat $A$-connection $\nabla$ (see, e.g., [4]). The usual operations $\oplus$ and $\otimes$ on vector bundles turn the space of representations $\text{Rep}(A)$ into a semiring. Given a morphism of Lie algebroids $(\Phi, \phi)$, there is a pullback operation on representations $E \mapsto \phi^! E$, which gives a morphism of rings $\phi^!: \text{Rep}(B) \to \text{Rep}(A)$.

For an orientable line bundle $L \in \text{Rep}(A)$ the only characteristic class can be obtained as follows: for any nowhere vanishing section $\mu \in \Gamma(L)$,

$$
\nabla_X \mu = \langle \alpha_\mu, X \rangle \mu, \quad \forall X \in \mathfrak{x}(A).
$$

The 1-form $\alpha_\mu \in \Omega^1(A)$ is $d_A$-closed and it is called the characteristic cocycle of the representation $L$. Its cohomology class is independent of the choice of section $\mu$ and defines the characteristic class of the representation $L$:

$$
\text{char}(L) := [\alpha_\mu] \in H^1(A).
$$

One checks easily that if $L, L_1, L_2 \in \text{Rep}(A)$, then:

$$
\text{char}(L^*) = - \text{char}(L), \quad \text{char}(L_1 \otimes L_2) = \text{char}(L_1) + \text{char}(L_2).
$$

Also, if $(\Phi, \phi): A \to B$ is a morphism of Lie algebroids, and $L \in \text{Rep}(B)$ then:

$$
\text{char}(\phi^! L) = \Phi^* \text{char}(L),
$$

where $\Phi^*: H^\bullet(B) \to H^\bullet(A)$ is the map induced by $\Phi$ at the level of cohomology. If $L$ is not orientable, then one defines its characteristic class to be the one half of the representation $L \otimes L$, so the formulas above still hold, for non-orientable line bundles.

Every Lie algebroid $A \to M$ has a canonical representation in the line bundle $L_A = \wedge^{\text{top}} A \otimes \wedge^{\text{top}} T^* M$:

$$
\nabla_X (\omega \otimes \mu) = \mathcal{L}_X \omega \otimes \mu + \omega \otimes \mathcal{L}_\mu(X) \mu.
$$

Then we set:

**Definition 2.3.** The modular cocycle of a Lie algebroid $A$ relative to a nowhere vanishing section $\omega \otimes \mu \in \Gamma(\wedge^{\text{top}} A \otimes \wedge^{\text{top}} T^* M)$ is the characteristic cocycle $\alpha_{\omega \otimes \mu}$ of the representation $L_A$. The modular class of $A$ is the characteristic class:

$$
\text{mod}(A) := [\alpha_{\omega \otimes \mu}] \in H^1(A).
$$

**Remark 2.4.** Notice that, if $\nu = f\mu$ is another section of $L_A$, for a nonvanishing function $f \in C^\infty(M)$, then

$$
(1) \quad \alpha_\nu = \alpha_\mu - d_A \ln f.
$$

**Example 2.5.** The modular class of a tangent bundle is trivial.
Example 2.6. Let \((M,\pi)\) be a Poisson manifold. The first Poisson cohomology space \(H^1_\pi(M)\), is the space of Poisson vector fields modulo the hamiltonian vector fields.

The Lie derivative of any volume form along hamiltonian vector fields leads to a unique vector field \(X_\mu \in \mathfrak{X}(M)\) such that:

\[
\mathcal{L}_{X_\mu} f = X_\mu(f)\mu.
\]

One calls \(X_\mu\) the **modular vector field** of the Poisson manifold \((M,\pi)\) relative to \(\mu\). The modular vector field \(X_\mu\) is Poisson and, if \(\nu = g\mu\) is another volume form, then:

\[
X_{g\mu} = X_\mu - \pi^\#(d \ln |g|).
\]

This lead to the definition of modular class of a Poisson manifold, which is due to Weinstein [12]:

The modular class of a Poisson manifold \((M,\pi)\) is the Poisson cohomology class

\[
\text{mod}(M) := [X_\mu] \in H^1_\pi(M).
\]

Note that \(\text{mod}(M) = 0\) if and only if we can find a volume form \(\mu\) invariant under all hamiltonian flows. Therefore the modular class is the obstruction to the existence of a volume form in \((M,\pi)\) invariant under all hamiltonian flows.

In fact, the modular class of the Poisson manifold \((M,\pi)\) and the modular class of the Lie algebroid \(T^*M\) just differ by a multiplicative factor:

\[
\text{mod}(T^*M) = 2 \text{mod}(M).
\]

3. **The modular class of a Lie algebroid morphism**

Let \(\Phi : A \to B\) be a morphism of Lie algebroids covering a map \(\phi : M \to N\). The induced morphism at the level of cohomology \(\Phi^* : H^*(B) \to H^*(A)\), in general, does not map the modular classes to each other. Therefore one sets ([8]):

**Definition 3.1.** The modular class of a Lie algebroid morphism \(\Phi : A \to B\) is the cohomology class defined by:

\[
\text{mod}(\Phi) := \text{mod}(A) - \Phi^* \text{mod}(B) \in H^1(A).
\]

**Proposition 3.2.** Let \(\Phi : A \to B\) and \(\Psi : B \to C\) be Lie algebroid morphisms, then:

\[
\text{mod}(\Psi \circ \Phi) = \text{mod}(\Phi) + \Phi^* \text{mod}(\Psi).
\]

The basic properties for characteristic classes show that the modular class of a Lie algebroid morphism \((\Phi,\phi) : A \to B\) can be seen as the characteristic class of a representation. Namely, one takes the canonical representations \(L_A \in \text{Rep}(A)\) and \(L_B \in \text{Rep}(B)\) and forms the representation \(L^\phi := L_A \otimes \phi^!(L_B)^*\). Then:

**Proposition 3.3.** Let \((\Phi,\phi) : A \to B\) be a Lie algebroid morphism. Then:

\[
\text{mod}(\Phi) = \text{char}(L^\phi).
\]

4. **The modular class of a Lie algebroid comorphism**

In this section we extend some of the results for Poisson maps in [2] to comorphisms between Lie algebroids. We begin with the definition of a Lie algebroid comorphism. Further details about comorphisms can be seen in [10, 1, 11, 13].
Definition 4.1. Let $A \to M$ and $B \to N$ be two Lie algebroids. A comorphism between $A$ and $B$ covering $\phi : M \to N$ is a vector bundle map $\Phi : \phi^*B \to A$ from the pullback vector bundle $\phi^*B$ to $A$, such that the following two conditions hold:

$$[\Phi X, \Phi Y] = \Phi [X, Y],$$

and

$$d\phi \circ \rho_A(\Phi X) = \rho_B(X),$$

for $X, Y \in \mathfrak{X}(B)$, where $\Phi : \mathfrak{X}(B) \to \mathfrak{X}(A)$ is the natural map induced by $\Phi$.

Equivalently, we may say that $(\Phi, \phi)$ is a Lie algebroid comorphism if and only if $\Phi^* : \mathfrak{A}^* \to \mathfrak{B}^*$ is a Poisson map for the natural linear Poisson structures on the dual Lie algebroids.

Proposition 4.2. Let $\Phi : \phi^*B \to A$ be a Lie algebroid comorphism. The pullback vector bundle $\phi^*B \to M$ carries a natural Lie algebroid structure characterized by:

$$[X^!, Y^!] = [X, Y]!$$

and

$$\rho(X^!) = \rho_A(\Phi X^!),$$

for $X, Y \in \mathfrak{X}(B)$, $X^! = X \circ \phi \in \Gamma(\phi^*B)$ and $Y^! = Y \circ \phi \in \Gamma(\phi^*B)$.

For this structure, the natural maps

\[
\begin{array}{ccc}
\Phi & \to & A \\
\downarrow^\phi & & \downarrow^j \\
B & & \\
\end{array}
\]

are Lie algebroid morphisms.

The modular class of a Lie algebroid comorphism is defined as follows:

Definition 4.3. Let $\Phi : \phi^*B \to A$ be a Lie algebroid comorphism between the Lie algebroids $A$ and $B$. The modular class of $\Phi$ is the cohomology class:

$$\text{mod}(\Phi) := \Phi^* \text{mod}(A) - j^* \text{mod}(B) \in H^1(\phi^*B).$$

Example 4.4. A Poisson map $\phi : M \to N$ defines a comorphism between cotangent bundles: $\Phi : \phi^*T^*N \to T^*M$ such that $\Phi(\alpha^!) = (d\phi)^*\alpha$, where $\alpha^! = \alpha \circ \phi \in \mathfrak{X}(\phi^*T^*N)$, for all $\alpha \in \Omega^1(N)$. The modular class of the Poisson map $\phi$ was defined in [2] and we see that it is one half the modular class of the comorphism $\Phi$ induced by $\phi$.

Notice that the map $j^* : \Omega^k(B) \to \Omega^k(\phi^*B)$ is simply defined by

$$j^*(\alpha) = \alpha \circ \phi, \quad \alpha \in \Omega^k(B).$$

Taking this into account we can give an explicit description of a representative of the modular class of a comorphism $\Phi$:

Proposition 4.5. Let $\Phi : \phi^*B \to A$ be a Lie algebroid comorphism over $\phi : M \to N$ and fix non-vanishing sections $\mu \in \Gamma(L_A)$, $\nu \in \Gamma(L_B)$. The modular class $\text{mod}(\Phi)$ is represented by:

$$\alpha_{\mu, \nu} = \Phi^*(\alpha_\mu) - \alpha_\nu \circ \phi,$$

where $\alpha_\mu$ and $\alpha_\nu$ are the modular cocycle of $A$ and $B$ relative to $\mu$ and $\nu$, respectively.

We will refer to $\alpha_{\mu, \nu}$ as the modular cocycle of $\Phi$ relative to $\mu$ and $\nu$. 
Corollary 4.6. The class mod(Φ) is the obstruction to the existence of modular cocycles α ∈ Ω^1(A) and β ∈ Ω^1(B), such that

Φ∗α = β ∘ φ.

Proof. The Lie algebroid morphism Φ has trivial modular class if its modular cocycles are exact in the Lie algebroid cohomology of φ^∗B, i.e., if for each µ ∈ Γ(L_A) and ν ∈ Γ(L_B),

α_{µ,ν} = d_{φ^∗B} f = Φ∗(d_A f), \quad \text{for some } f ∈ C^∞(M)

By definition α_{µ,ν} = Φ∗(α_µ) − α_ν ∘ φ, hence we have Φ∗(α_µ + d_A f) = α_ν ∘ φ, and taking into account equation (1), we conclude that α_µ + d_A f = −X_µ and Φ∗α_e = X_µ.

Corollary 4.7. Let Φ : φ^∗B → A be a comorphism between Lie algebroids. If there exists a Lie algebroid morphism ˆΦ : A → B making the diagram commutative

\[
\begin{array}{ccc}
φ^∗B & \xrightarrow{Φ} & A \\
\downarrow{j} & & \downarrow{ˆΦ} \\
B & & B
\end{array}
\]

then

mod Φ = mod ˆΦ.

Proof. Since j = ˆΦ ∘ Φ we have j∗ = Φ∗ ∘ ˆΦ∗ and

Φ∗ mod ˆΦ = Φ∗(mod A − ˆΦ∗ B) = Φ∗ mod A − j∗ mod B = mod Φ.

□

Proposition 4.8. Let Φ : A → B be a comorphism between Lie algebroids. There is a natural representation of φ^∗B on the line bundle L^φ := L_A ⊗ φ^∗L_B, and we have:

mod(Φ) = char(L^φ).

Proof. We define a representation of φ^∗B on the line bundle L_A by setting:

∇_{X^∗}(µ ⊗ ν) := [X^∗, µ]_A ⊗ ν + µ ⊗ L_{µ,X^∗} ν

and another representation on φ^∗L_B by setting:

∇_{X^∗}(µ^∗ ⊗ ν^∗) := [α, µ]_B ⊗ ν + µ^∗ ⊗ (L_{µ,X} ν)^∗,

for X ∈ X(B) and µ ⊗ ν ∈ Γ(L_A). The tensor product of the first representation with the dual of the second representation defines a representation of φ^∗B on the line bundle

L^φ := L_A ⊗ φ^∗L_B.

□

Let us consider two Lie algebroids morphisms Φ : φ^∗B → A and Ψ : ψ^∗C → B over φ : M → N and ψ : N → P, respectively. The restriction ˜Ψ = Ψ_{(ψ ∘ φ)^∗C} maps (ψ ∘ φ)^∗C to φ^∗B and defines a map at the cohomology level:


The function Ψ ∘ Φ : (ψ ∘ φ)^∗C → A defined by:

Ψ ∘ Φ(X_{ψ ∘ φ(m)}) = Φ( ˜Ψ(X_{ψ ∘ φ(m)})), \quad (∀m ∈ M),

is a Lie algebroid comorphism.
We also have the natural Lie algebroid morphism $\tilde{j} : (\psi \circ \phi)^* C \to \psi^* C$ that defines a map at the cohomology level

$$\tilde{j}^* : H^*(\psi^* C) \to H^*((\psi \circ \phi)^* C), \alpha \mapsto \alpha \circ \phi.$$

**Proposition 4.9.** Let $\Phi : \phi^* B \to A$ and $\Psi : \psi^* C \to B$ be Lie algebroid comorphisms. Then:

$$\text{mod}(\Psi \circ \Phi) = \tilde{j}^* \Psi^* \text{mod}(\Phi) + \tilde{j}^* \text{mod}(\Psi).$$

**Proof.** The following diagram commutes:

$$\begin{array}{c}
H^*(A) \\[-1.5em]
\Phi^* \downarrow \quad \quad \quad \quad \quad \Psi^* \downarrow \quad \quad \quad \quad \quad \tilde{j}^* \downarrow \\
H^*(B) \\ H^*((\psi \circ \phi)^* C) \\[-1.5em]
\tilde{j}^* \Psi^* \downarrow \quad \quad \quad \quad \quad j^* \downarrow \quad \quad \quad \quad \quad j^* \circ \phi \\
H^*((\psi \circ \phi)^* C) \\
\end{array}$$

Hence, we find:

$$\text{mod}(\Phi \circ \Psi) = (\Phi \circ \Psi)^* \text{mod}(A) - j^* \circ \Phi \text{mod}(B) + \tilde{j}^* \text{mod}(C)$$

$$= \tilde{j}^* \Phi^* \text{mod}(A) - \tilde{j}^* \circ j^* \text{mod}(C)$$

$$= \tilde{j}^* \Phi^* \text{mod}(A) - \tilde{j}^* \circ j^* \text{mod}(B) + \tilde{j}^* \circ j^* \text{mod}(C)$$

$$= \tilde{j}^* \Phi^* \text{mod}(A) - j^* \text{mod}(B) + \tilde{j}^* \text{mod}(B) - j^* \text{mod}(C)$$

$$= \tilde{j}^* \Psi^* \text{mod}(\Phi) + \tilde{j}^* \text{mod}(\Psi).$$

\[\square\]

5. Generalization to Dirac structures

The modular class of a Lie algebroid morphism and the modular class of a Lie algebroid comorphism fit together into the notion of modular class of a skew algebroid relation, given by Grabowski in [5]. As a particular case we have the modular class of a Dirac map but very few was said about this particular case. The study of these structures will be exposed in a future work.

**References**


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