GRADED PSEUDO-H-RINGS

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Abstract. Consider a pseudo-H-space E endowed with a separately continuous biadditive associative multiplication which induces a grading on E with respect to an abelian group G. We call such a space a graded pseudo-H-ring and we show that it has the form $E = cl(U + \sum_j I_j)$ with U a closed subspace of E_1 (the summand associated to the unit element in G), and any I_j runs over a well described closed graded ideal of E, satisfying $I_jI_k = 0$ if $j \neq k$. We also give a context in which graded simplicity of E is characterized. Moreover, the second Wedderburn-type theorem is given for certain graded pseudo-H-rings.

1. Introduction and Preliminaries

In this paper we start considering the notion of a *pseudo-H-space*, that is, a real or complex vector space E equipped with a a family $(\langle\cdot,\cdot\rangle_\alpha)_{\alpha\in I}$ of positive semi-definite (pseudo)-inner products. We endow E with the initial topology with respect to the family of seminorms $(p_\alpha)_{\alpha\in I}$, where $p_\alpha(x) := \sqrt{\langle x,x\rangle_\alpha}$ for $\alpha \in I$ and $x \in E$. Thus E becomes a locally convex space (see [5, p. 456, Definition 3.1]).

We assume that every *pseudo-H-space* E is Hausdorff and complete. The former condition may be stated as follows: if $x \in E$ satisfies $p_\alpha(x) = 0$ for each $\alpha \in I$ then $x = 0$. The latter condition means that each Cauchy net in E is convergent. A *pseudo-H-ring* is a pseudo-H-space endowed with a biadditive associative multiplication separately continuous, i.e., the endomorphisms of E given by $x \mapsto xy$ and $x \mapsto yx$ are continuous for all $y \in E$. For any subset S of E we shall denote its closure by $cl(S)$.

Example 1.1. Let I and J be two arbitrary nonempty sets of elements. Consider the set $\mathbb{C}^{(I\times J)\times(I\times J)}$ of all complex-valued functions a on $(I\times J)\times(I\times J)$ such that

(i) $a((i,j),(l,m)) = 0$ when $j \neq m$ and
(ii) $\sum\limits_{i,k\in I; j\in J} |a((i,j),(k,j))|^2 \in \mathbb{R}_+$.

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The latter, endowed with "point-wise" defined operations becomes a vector space and an algebra with "*matrix*" multiplication

$$(ab)((i,j),(l,m)) = \sum_{(k,s)\in I\times J} a((i,j),(k,s))b((k,s),(l,m)),$$

for all $a, b \in \mathbb{C}^{(I\times J)\times(I\times J)}$. Take a family of real numbers $(t_\alpha)_{\alpha\in\Lambda}$, such that $t_\alpha \geq 1$. For each $\alpha \in \Lambda$, the mapping $\langle\cdot,\cdot\rangle_\alpha : \mathbb{C}^{(I\times J)\times(I\times J)} \times \mathbb{C}^{(I\times J)\times(I\times J)} \to \mathbb{C}$ given by

$$\langle a, b\rangle_\alpha = t_\alpha \sum_{(i,j),(l,m)\in I\times J} a((i,j),(l,m))\bar{b}((i,j),(l,m))$$

defines a pseudo-inner product on $\mathbb{C}^{(I\times J)\times(I\times J)}$, where "$-$" denotes complex conjugation. Thus $E := (\mathbb{C}^{(I\times J)\times(I\times J)}, (\langle\cdot,\cdot\rangle_\alpha)_{\alpha\in\Lambda})$ becomes a locally convex pseudo-H-ring.

Two elements x, y in a pseudo-H-space E are called *orthogonal* if $\langle x, y\rangle_\alpha = 0$ for all $\alpha \in I$. The *orthogonal* set $S^\perp$ of a non-empty subset S in E is defined by

$$S^\perp := \{x \in E : \langle x, y\rangle_\alpha = 0 \textit{ for all } y \in S \textit{ and all } \alpha \in I\}.$$

It is a closed linear subspace of E. The symbol $\oplus^\perp$ shall denote *orthogonal* direct sum, that is, a direct sum of mutually orthogonal linear subspaces.

Definition 1.2. Let E be a pseudo-H-ring, over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and let G be an abelian group. We say that E is a *graded pseudo-H-ring* (with respect to G) if

$$E = cl(\bigoplus_{g\in G}^{\perp} E_g),$$

where E_g is a closed linear subspace satisfying $E_gE_h \subset E_{gh}$ (denoting by juxtaposition the product both in E and G), for any $g, h \in G$. We define the *support* of the grading to be the set $\Sigma := \{g \in G \setminus \{1\} : E_g \neq 0\}$.

Graded Hilbert spaces, and therefore graded classical H^*-algebras, and graded $l_2(\mathcal{G})$ algebras, where $\mathcal{G}$ is a compact topological group, are examples of graded pseudo-H-rings (see [1, 4, 10]). Let us also endow the family of pseudo-H-rings in Example 1.1 of different gradings.

Example 1.3. Consider the pseudo-H-ring $E = (\mathbb{C}^{(I\times J)\times(I\times J)}, (\langle\cdot,\cdot\rangle_\alpha)_{\alpha\in\Lambda})$ of Example 1.1. Let us fix an arbitrary abelian group G. For any $((i,j),(k,j))$, $i, k \in I$ and $j \in J$, denote $a_{((i,j),(k,j))} : (I\times J)\times(I\times J) \to \mathbb{C}$ by

$$a_{((i,j),(k,j))}((l,m),(n,s)) := \begin{cases} 1, & \text{if } ((l,m),(n,s)) = ((i,j),(k,j)); \\ 0, & \text{otherwise.} \end{cases}$$

the element units in E. We have that any function

$$\phi : I \times J \to G$$

gives rise to a G-grading on E given by

$$\mathbb{C}a_{((i,j),(k,j))} \subset E_g \text{ if and only if } g = \phi(i,j)^{-1}\phi(k,j).$$

Indeed, taking into account $a_{((i,j),(k,j))}a_{((m,l),(n,l))} = 0$ for $(k,j) \neq (m,l)$, and

$$\phi(i,j)^{-1}\phi(k,j)\phi(k,j)^{-1}\phi(n,j) = \phi(i,j)^{-1}\phi(n,j),$$

the above condition clearly defines the grading $E = cl(\bigoplus_{g\in G}^{\perp} E_g)$ with

$$E_g = \bigoplus^{\perp} \mathbb{C}a_{((i,j),(k,j))}, \tag{1.1}$$

where the orthogonal direct sum is taken over all $i,k \in I; j \in J$ with

$$\phi(i,j)^{-1}\phi(k,j) = g.$$

Let E be a graded pseudo-H-ring. A *graded pseudo-H-subring* F of E is a linear subspace with $FF \subset F$ and which is decomposed as $F = cl(\bigoplus_{g\in G}^{\perp} F_g)$ where $F_g := F \cap E_g$. A *graded ideal* I of E is a graded pseudo-H-subring satisfying $IE \subset I$ and $EI \subset I$. A graded pseudo-H-ring E shall be called *graded simple* if its product is nonzero and its only graded ideals are (0) and E.

In this work we study graded pseudo-H-rings E. In Section 2, we give a particular decomposition of E as $E = cl(U + \sum_j I_j)$ with U a closed subspace of E_1 (the summand associated to the unit element in G), and any I_j a well described closed graded ideal of E, satisfying $I_jI_k = 0$ if $j \neq k$;. Then, in Section 3 we give a context in which graded simplicity of E is characterized. Moreover, a second Wedderburn-type theorem is given for certain graded pseudo-H-rings.

In the next lemma, by a *topological ring* we mean a topological vector space which is a ring, such that the ring multiplication is separately continuous.

Lemma 1.4. *Let E be a topological ring.*

(i) *If A, B, C are subsets of E with $AB \subseteq C$, then $cl(A)cl(B) \subseteq cl(C)$.*

If, in addition E is Hausdorff and complete, then the following hold:

(ii) *If A and B are orthogonal closed subspaces of E, then $A \oplus B$ is closed.*

(iii) *If A and B are orthogonal subspaces with A closed, then $cl(A \oplus B) = A \oplus cl(B)$.*

Proof. (i) See [10, p. 6, Lemma 1.5].

(ii) Let A and B be orthogonal closed subspaces and let $(x_\lambda)_{\lambda\in\Lambda}$ be a net in $A \oplus B$ converging to x_0. For each $\lambda \in \Lambda$ we may write $x_\lambda = a_\lambda + b_\lambda$ for unique elements a_λ and b_λ belonging to A and B respectively. For any $\alpha \in I$, and as A and B are orthogonal, we have $\langle x_\lambda, x_\lambda\rangle_\alpha = \langle a_\lambda, a_\lambda\rangle_\alpha + \langle b_\lambda, b_\lambda\rangle_\alpha$. We deduce that $\langle a_\lambda, a_\lambda\rangle_\alpha \leq \langle x_\lambda, x_\lambda\rangle_\alpha$ and that $\langle b_\lambda, b_\lambda\rangle_\alpha \leq \langle x_\lambda, x_\lambda\rangle_\alpha$. Then $(a_\lambda)_{\lambda\in\Lambda}$ and $(b_\lambda)_{\lambda\in\Lambda}$ are Cauchy nets in A and B and, by the completeness of E, they converge to $a_0 \in A$ and $b_0 \in B$ respectively. As E is Hausdorff we have $x_0 = a_0 + b_0$ and we are done.

(iii) We start by noticing that $A \oplus cl(B) \subseteq cl(A \oplus B)$. By (ii) above, we get that $cl(A \oplus B) \subseteq A \oplus cl(B)$. It is left to prove that the sum $A \oplus cl(B)$ is indeed orthogonal. Take $a \in A$ and $c \in cl(B)$. Fix $\alpha \in I$. For any $\epsilon\rangle 0$ choose $b \in B$ such that $p_\alpha(c-b)\langle\epsilon$. Then we have

$$\langle a, c\rangle_\alpha = \langle a, c-b+b\rangle_\alpha = \langle a, c-b\rangle_\alpha \leq p_\alpha(a)p_\alpha(c-b)\langle\epsilon p_\alpha(a).$$

As this is true for each $\epsilon\rangle 0$ we get that $\langle a, c\rangle_\alpha = 0$. So, by (ii), $A \oplus cl(B)$ is closed and hence, $A \oplus cl(B) \subseteq A \oplus B \subseteq A \oplus cl(B)$. Thus, the assertion follows. □

2. Decompositions

From now on, E denotes a graded pseudo-H-ring over $\mathbb{R}$ or $\mathbb{C}$ and

$$E = cl(\bigoplus_{g\in G}^{\perp} E_g) = E_1 \oplus cl(\bigoplus_{g\in\Sigma}^{\perp} E_g)$$

the corresponding grading, with support Σ, and with respect to an abelian (multiplicative) group G. Let us denote by $\Sigma^{-1} := \{h^{-1} : h \in \Sigma\} \subset G$.

Definition 2.1. Let g, h be elements in Σ. We shall say that g is *connected* to h if there exist $g_1, g_2 ..., g_n \in \Sigma \cup \Sigma^{-1}$ such that

(i) $g_1 = g$.
(ii) $\{g_1, g_1g_2, ..., g_1g_2\cdots g_{n-1}\} \subset \Sigma \cup \Sigma^{-1}$.
(iii) $g_1g_2\cdots g_{n-1}g_n \in \{h, h^{-1}\}$.

We shall also say that $\{g_1, ..., g_n\}$ is a *connection* from g to h.

The next result shows that connectioness is an equivalence relation.

Proposition 2.2. *Let E be a graded pseudo-H-ring with support Σ. Then, the relation $\sim$ in Σ, defined by $g \sim h$ if and only if g is connected to h, is an equivalence one.*

Proof. Clearly the set $\{g\}$ is a connection from g to itself and so the relation is reflexive.

If $g \sim h$ then there exists a connection $\{g_1, g_2, ..., g_n\}$ from g to h:

$$\{g_1g_2, g_1g_2g_3, ..., g_1g_2\cdots g_{n-1}\} \subset \Sigma \cup \Sigma^{-1},$$

where $g_1g_2\cdots g_n \in \{h, h^{-1}\}$. Hence, we have two possibilities. In the first one $g_1g_2\cdots g_n = h$, and in the second one $g_1g_2\cdots g_n = h^{-1}$. Now observe that the set

$$\{h, g_n^{-1}, g_{n-1}^{-1}, ..., g_2^{-1}\}$$

gives us a connection from h to g for the first possibility and $\{h, g_n, g_{n-1}, ..., g_2\}$ for the second one. Hence $\sim$ is symmetric.

Finally, suppose that $g \sim h$ and $h \sim k$, and write $\{g_1, g_2, ..., g_n\}$ for a connection from g to h and $\{h_1, h_2, ..., h_m\}$ for a connection from h to k. If $h \notin \{k, k^{-1}\}$, then $m \geq 1$ and so $\{g_1, g_2, ..., g_n, h_2, ..., h_m\}$ (resp. $\{g_1, g_2, ..., g_n, h_2^{-1}, ..., h_m^{-1}\}$) is a connection from g to k if $g_1g_2\cdots g_n = h$ (resp. $g_1g_2\cdots g_n = h^{-1}$). If $h \in \{k, k^{-1}\}$ then, $\{g_1, g_2, ..., g_n\}$ is a connection from g to k. Therefore $g \sim k$ and this completes the assertion. □

By the above proposition we can consider the quotient set

$$\Sigma/\sim = \{[g] : g \in \Sigma\},$$

where $[g]$ denotes the set of elements of Σ which are connected to g. By the definition of $\sim$, it is clear that if $h \in [g]$ and $h^{-1} \in \Sigma$ then $h^{-1} \in [g]$.

Our next goal in this section is to associate a graded ideal $E_{[g]}$ to any $[g]$. Fix $g \in \Sigma$, we start by defining the set $E_{1,[g]} \subset E_1$ as follows

$$E_{1,[g]} := span_{\mathbb{K}}\{E_h E_{h^{-1}} : h \in [g]\} \subset E_1.$$

Next, we define

$$V_{[g]} := \bigoplus_{h\in[g]}^{\perp} E_h$$

Finally, we denote by $E_{[g]}$ the following closed linear subspace of E,

$$E_{[g]} := cl(E_{1,[g]} \oplus^{\perp} V_{[g]}).$$

Proposition 2.3. *For any $g \in \Sigma$, the linear subspace $E_{[g]}$ is a graded pseudo H-subring of E.*

Proof. We have

$$(E_{1,[g]} \oplus^{\perp} V_{[g]})(E_{1,[g]} \oplus^{\perp} V_{[g]}) \subset E_{1,[g]}E_{1,[g]} + E_{1,[g]}V_{[g]} + V_{[g]}E_{1,[g]} + V_{[g]}V_{[g]}. \quad (2.1)$$

Let us consider the last summand $V_{[g]}V_{[g]}$ in (2.1). Given $h, k \in [g]$ such that $E_h E_k \neq 0$, if $k = h^{-1}$ then, clearly $E_h E_k = E_h E_{h^{-1}} \subset E_{1,[g]}$. Suppose that $k \neq h^{-1}$ and consider a connection $\{g_1, ..., g_n\}$ from g to h. Since $E_h E_k \neq 0$ implies $hk \in \Sigma$, we get that $\{g_1, ..., g_n, k\}$ is a connection from g to hk, in case $g_1 \cdots g_n = h$ and $\{g_1 \cdots g_n, k^{-1}\}$, the respective one, in case $g_1 \cdots g_n = h^{-1}$. So $hk \in [g]$ and thus $E_h E_k \subset E_{hk} \subset V_{[g]}$. Therefore, $(\bigoplus_{h\in[g]}^{\perp} E_h)(\bigoplus_{h\in[g]}^{\perp} E_h) \subset E_{1,[g]} \oplus^{\perp} V_{[g]}$, that is,

$$V_{[g]}V_{[g]} \subset E_{1,[g]} \oplus^{\perp} V_{[g]}. \quad (2.2)$$

Consider now the first summand $E_{1,[g]}E_{1,[g]}$ in (2.1). By associativity, given $h, k \in [g]$, we have $(E_h E_{h^{-1}})(E_k E_{k^{-1}}) \subset (E_h E_{h^{-1}}) \cap (E_k E_{k^{-1}}) \subset E_{1,[g]}$. Hence,

$$E_{1,[g]}E_{1,[g]} \subset E_{1,[g]}. \quad (2.3)$$

Similarly, we show

$$E_{1,[g]}V_{[g]} + V_{[g]}E_{1,[g]} \subset V_{[g]}. \quad (2.4)$$

From the relations (2.1), (2.2),(2.3) and (2.4), we get

$$(E_{1,[g]} \oplus^{\perp} V_{[g]})(E_{1,[g]} \oplus^{\perp} V_{[g]}) \subset E_{1,[g]} \oplus^{\perp} V_{[g]}.$$

Finally, Lemma 1.4-(i) completes the proof. $\square$

Lemma 2.4. *If $[g] \neq [h]$ for some $g, h \in \Sigma$ then $E_{[g]}E_{[h]} = 0$.*

Proof. We have

$$(E_{1,[g]} \oplus^{\perp} V_{[g]})(E_{1,[h]} \oplus^{\perp} V_{[h]}) \subset$$

$$E_{1,[g]}E_{1,[h]} + E_{1,[g]}V_{[h]} + V_{[g]}E_{1,[h]} + V_{[g]}V_{[h]}. \quad (2.5)$$

Consider the above last summand $V_{[g]}V_{[h]}$ and suppose there exist $g_1 \in [g]$ and $h_1 \in [h]$ such that $E_{g_1}E_{h_1} \neq 0$. Since $g_1 \neq h_1^{-1}$, then $g_1h_1 \in \Sigma$. So $\{g_1, h_1, g_1^{-1}\}$ is a connection between g_1 and h_1. By the transitivity of the connection relation, we have $h \in [g]$, that is a contradiction. Hence $E_{g_1}E_{h_1} = 0$ and thus

$$V_{[g]}V_{[h]} = 0. \tag{2.6}$$

Consider now the first summand $E_{1,[g]}E_{1,[h]}$ of (2.5) and suppose there exist $g_1 \in [g]$ and $h_1 \in [h]$ so that $(E_{g_1}E_{g_1^{-1}})(E_{h_1}E_{h_1^{-1}}) \neq 0$. We have $E_{g_1}(E_{g_1^{-1}}E_{h_1})E_{h_1^{-1}} \neq 0$ and so $E_{g_1^{-1}}E_{h_1} \neq 0$, that contradicts (2.6). Hence $E_{1,[g]}E_{1,[h]} = 0$. Arguing in a similar way, we also get

$$E_{1,[g]}V_{[h]} + V_{[g]}E_{1,[h]} = 0.$$

From (2.5) we get

$$(E_{1,[g]} \oplus^{\perp} V_{[g]})(E_{1,[h]} \oplus^{\perp} V_{[h]}) = 0.$$

Applying Lemma 1.4, we finally get $E_{[g]}E_{[h]} = 0$. □

Theorem 2.5. *In any pseudo-H-ring E the following assertions hold.*

(i) *For any $g \in \Sigma$, the linear subspace*

$$E_{[g]} = cl(E_{1,[g]} \oplus^{\perp} V_{[g]})$$

of E associated to $[g]$ is a graded ideal of E.

(ii) *If E is graded simple, then there exists a connection between any two elements of Σ and $E_1 = cl(span_{\mathbb{K}}\{E_gE_{g^{-1}} : g \in \Sigma\})$.*

Proof. (i) We first observe that by the grading (see Definition 1.2)

$$E_hE_{(h)^{-1}}E_1 \subset E_hE_{(h)^{-1}} \tag{2.7}$$

and

$$E_hE_1 \subset E_h. \tag{2.8}$$

Let us prove that $E_{[g]}E_1 \subset E_{[g]}$. From (2.7), we obtain $E_{1,[g]}E_1 \subset E_{1,[g]}$, and taking into account (2.8), we get $V_{[g]}E_1 \subset V_{[g]}$. Therefore, $(E_{1,[g]}\oplus^{\perp}V_{[g]})E_1 \subset E_{1,[g]}\oplus^{\perp}V_{[g]}$. Taking closure, by Lemma 1.4-(i) and the fact E_1 is closed, we have

$$E_{[g]}E_1 \subset E_{[g]}.$$

Taking into account the above observation, Proposition 2.3 and Lemma 2.4, we have

$$E_{[g]}(E_1 \oplus^{\perp} (\bigoplus_{h\in[g]}^{\perp} E_h) \oplus^{\perp} (\bigoplus_{k\notin[g]}^{\perp} E_k)) \subset E_{[g]}.$$

Hence, Lemma 1.4 and the equality

$$E = cl(E_1 \oplus^{\perp} (\bigoplus_{h\in[g]}^{\perp} E_h) \oplus^{\perp} (\bigoplus_{k\notin[g]}^{\perp} E_k))$$

finally give $E_{[g]}E \subset E_{[g]}$.

In a similar way, we get $EE_{[g]} \subset E_{[g]}$ and so $E_{[g]}$ is a graded ideal of E.

(ii) The graded simplicity of E implies $E_{[g]} = E$. From here, it is easy to get $[g] = \Sigma$ and $E_1 = cl(span_{\mathbb{K}}\{E_gE_{g^{-1}} : g \in \Sigma\})$. □

Theorem 2.6. *Let E be a pseudo-H-ring. Then for an orthogonal complement U of*

$$cl(span_{\mathbb{K}}\{E_g E_{g^{-1}} : g \in \Sigma\})$$

in E_1, we have

$$E = cl(U + \sum_{[g]\in\Sigma/\sim} E_{[g]})$$

where any $E_{[g]}$ is one of the (closed) graded ideals of E described in Theorem 2.5-(i), satisfying $E_{[g]}E_{[h]} = 0$ if $[g] \neq [h]$.

Proof. By Proposition 2.2, we can consider the quotient set $\Sigma/\sim := \{[g] : g \in \Sigma\}$. For any $[g] \in \Sigma/\sim$ we know that $E_{[g]}$ is well defined and, by Theorem 2.5-(i), it is a graded ideal of E. We also have $E_1 \oplus^{\perp} (\bigoplus^{\perp}_{g\in\Sigma} E_g) = U + \sum_{[g]\in\Sigma/\sim} E_{[g]}$ and so

$$E = cl(U + \sum_{[g]\in\Sigma/\sim} E_{[g]}).$$

By applying Proposition 2.3-(ii), we get $E_{[g]}E_{[h]} = 0$ if $[g] \neq [h]$. □

The linear subspace E_1 of E, associated to $1 \in G$, plays a special role in any graded pseudo-H-ring $E = E_1 \oplus^{\perp} cl(\bigoplus^{\perp}_{g\in\Sigma} E_g)$. Hence, in order to obtain deeper structural descriptions of E we have to consider graded pseudo-H-rings in which E_1 and the (pseudo-)inner products $\{\langle , \rangle_\alpha\}_{\alpha\in I}$ of E are compatible in a sense. From here, we introduce the following notion motivated by the compatibility condition between the inner product, the involution and the multiplication which characterize a classical H^*-algebra ([1]) and its generalizations like Ambrose algebras ([6, 7, 8]).

Definition 2.7. We say that a graded pseudo-H-ring $(E, (\langle\cdot,\cdot\rangle_\alpha)_{\alpha\in I})$ has a *coherent 1-homogeneous space* if $E_1 = cl(span_{\mathbb{K}}\{E_g E_{g^{-1}} : g \in \Sigma\})$ and the following relation holds

$$\langle E_g E_{g^{-1}}, E_h E_{h^{-1}}\rangle_\alpha = \langle E_g, E_h E_{h^{-1}} E_g\rangle_\alpha$$

for any $g, h \in G$ and $\alpha \in I$.

Graded classical H^*-algebras are examples of graded pseudo-H-ring with coherent 1-homogeneous spaces. The graded pseudo-H-rings in Example 3.3 below are also examples of graded pseudo-H-rings having coherent 1-homogeneous spaces.

Theorem 2.8. *Let E be a pseudo-H-ring. If E has a coherent 1-homogeneous space, then*

$$E = cl(\bigoplus^{\perp}_{[g]\in\Sigma/\sim} E_{[g]}).$$

Namely, E is the topological orthogonal direct sum of the (closed) graded ideals given in Theorem 2.5.

Proof. Taking into account Theorem 2.6, we clearly have from the fact

$$E_1 = cl(span_{\mathbb{K}}\{E_g E_{g^{-1}} : g \in \Sigma\})$$

that $E = cl(\sum\limits_{[g]\in\Sigma/\sim} E_{[g]})$. Since E_1 is coherent, and $E_{[g]}E_{[h]} = 0$, for $[g] \neq [h]$ (Lemma 2.4), we get

$$\langle E_{g_1}E_{g_1^{-1}}, E_{g_2}E_{g_2^{-1}}\rangle_\alpha = \langle E_{g_1}, E_{g_2}(E_{g_2^{-1}}E_{g_1})\rangle_\alpha = 0$$

for any $g_1 \in [g], g_2 \in [h]$ and $\alpha \in I$. Hence the direct sum $\bigoplus\limits_{[g]\in\Sigma/\sim}^{\perp} E_{1,[g]}$ is orthogonal. So, since $E_{[g]} = cl(E_{1,[g]} \oplus^{\perp} (\bigoplus\limits_{h\in[g]}^{\perp} E_h))$, we get the orthogonal direct character of the sum of the ideals $E_{[g]}$, $[g] \in \Sigma/\sim$. □

3. The graded simple components

In this section, we study when the components in the decompositions given in Theorems 2.6 and 2.8 are graded simple. We begin by introducing the key notions of Σ-multiplicativity and maximal length in the context of graded pseudo-H-rings, in a similar way to that for graded associative algebras, graded Lie algebras, graded Poisson algebras and so on. For these notions and examples see [2, 3, 9].

Definition 3.1. It is said that a graded pseudo-H-ring E is of *maximal length* if $E_1 \neq 0$ and $\dim E_g = 1$ for any $g \in \Sigma$.

Definition 3.2. We say that a graded pseudo-H-ring E is Σ-*multiplicative* if given $g \in \Sigma$ and $h \in \Sigma \cup \{1\}$ such that $gh \in \Sigma$, then $E_gE_h + E_hE_g \neq 0$.

We recall that Σ is called *symmetric* when $\Sigma = \Sigma^{-1}$ and that the *annihilator* of E is the set $\mathcal{A}nn(E) := \{v \in E : vE = 0 \text{ and } Ev = 0\}$. From now on Σ will be supposed to be symmetric.

Example 3.3. Consider the graded pseudo-H-ring $E = cl(\bigoplus\limits_{g\in G}^{\perp} E_g)$ where

$$E = (\mathbb{C}^{(I\times J)\times(I\times J)}, (\langle\cdot,\cdot\rangle_\alpha)_{\alpha\in\Lambda})$$

as in Example 1.3. Take $I = \mathbb{N}$, $J = \{1, 2, ..., r\}$ a finite set, $G = \mathbb{Q}^\times$, (the multiplicative rational group), and a family of r sequences of prime natural numbers $\{x_{n,t}\}_{n\in\mathbb{N}}$ where $t \in J$, such that $x_{n,t} \neq x_{m,s}$ when $(n,t) \neq (m,s)$. Define

$$\phi : \mathbb{N} \times J \to \mathbb{Q}^\times$$
$$(n,p) \mapsto x_{n,p}$$

Taking into account (1.1) it is easy to verify that for any $q \in \mathbb{Q}^\times$, $q \neq 1$, either $E_q = 0$ or $E_q = \mathbb{C}a_{((n,t),(m,t))}$ for (unique) $n, m \in \mathbb{N}$ and $t \in J$ such that $x_{n,t}^{-1}x_{m,t} = q$. In this case $E_{q^{-1}} = \mathbb{C}a_{((m,t),(n,t))}$ and thus we get that E is of maximal length and that its support is symmetric.

Since

$$E_1 = cl(\bigoplus_{n\in\mathbb{N};\, t\in J}^{\perp} \mathbb{C}a_{((n,t),(n,t))}) \neq 0$$

and $a_{((n,t),(n,t))} = a_{((n,t),(m,t))}a_{((m,t),(n,t))}$ for any $m \in \mathbb{N}$ with $m \neq n$, we also get that $E_1 = cl(\sum_{q\in\Sigma} E_q E_{q^{-1}})$ and so E_1 is coherent.

In order to verify that E is Σ-multiplicative, take $q \in \Sigma$ and $p \in \Sigma$ such that $qp \in \Sigma$. By the above we can write $q = x_{n,t}^{-1}x_{m,t}$ and $p = x_{r,v}^{-1}x_{s,v}$. From here, if $qp \in \Sigma$ then either $(m,t) = (r,v)$ or $(n,t) = (s,v)$. So, either $qp = x_{n,t}^{-1}x_{s,t}$ and thus $E_qE_p = \mathbb{C}a_{((n,t),(m,t))}\mathbb{C}a_{((m,t),(s,t))} = \mathbb{C}a_{((n,t),(s,t))} \neq 0$ or $pq = x_{r,t}^{-1}x_{m,t}$ and we have $E_pE_q = \mathbb{C}a_{((r,t),(n,t))}\mathbb{C}a_{((n,t),(m,t))} = \mathbb{C}a_{((r,t),(m,t))} \neq 0$. If $p = 1$, then clearly we have that $a_{((n,t),(m,t))}a_{((m,t),(m,t))} = a_{((n,t),(m,t))}$ and so $E_qE_1 \neq 0$. Thus E is Σ-multiplicative.

Theorem 3.4. *Let E be a Σ-multiplicative graded pseudo-H-ring of maximal length and with $\mathcal{A}nn(E) = 0$. Then E is graded simple if and only if its support has all of its elements connected and $E_1 = cl(span_{\mathbb{K}}\{E_gE_{g^{-1}} : g \in \Sigma\})$.*

Proof. For the first implication, see Theorem 2.5-(ii). To prove the converse, consider $I = cl(\bigoplus_{g\in G}^{\perp} I_g)$, where $I_g := I \cap E_g$, a nonzero graded ideal of E. We denote by

$$\Sigma_I := \{g \in \Sigma : I_g \neq 0\}.$$

By the maximal length of the grading, if $g \in \Sigma_I$ then $0 \neq I_g = I \cap E_g = E_g$ and so we can write $I = I_1 \oplus^{\perp} cl(\bigoplus_{g\in\Sigma_I}^{\perp} E_g)$ where $I_1 = I \cap E_1$.

Observe that $\Sigma_I \neq \emptyset$. Indeed, in the opposite case $0 \neq I \subset E_1$ and then

$$I(\bigoplus_{g\in\Sigma}^{\perp} E_g) \subset (\bigoplus_{g\in\Sigma}^{\perp} E_g) \cap E_1 = 0.$$

Therefore $I(\bigoplus_{g\in\Sigma}^{\perp} E_g) = 0$. In a similar way $(\bigoplus_{g\in\Sigma}^{\perp} E_g)I = 0$. Hence, by Lemma 1.4

$$I(cl(\bigoplus_{g\in\Sigma}^{\perp} E_g)) = (cl(\bigoplus_{g\in\Sigma}^{\perp} E_g))I = 0. \tag{3.1}$$

Thus, the associativity of the product gives $I(E_gE_{g^{-1}}) + (E_gE_{g^{-1}})I = 0$ and so, since $E_1 = cl(span_{\mathbb{K}}\{E_gE_{g^{-1}} : g \in \Sigma\})$, Lemma 1.4 implies

$$IE_1 + E_1I = 0. \tag{3.2}$$

From equations (3.1) and (3.2) we finally get $I \subset \mathcal{A}nn(E) = 0$, a contradiction.

By the above we can take $g_0 \in \Sigma_I$, so that

$$0 \neq E_{g_0} \subset I. \tag{3.3}$$

For any $h \in \Sigma$ with $h \notin \{g_0, {g_0}^{-1}\}$. Since g_0 and h are connected, there is a connection $\{g_1, g_2, ..., g_r\}$ between them, such that

$$g_1 = g_0; \quad g_1g_2, g_1g_2g_3, ..., g_1g_2g_3 \cdots g_{r-1} \in \Sigma \text{ and } g_1g_2g_3 \cdots g_r \in \{h, h^{-1}\}.$$

Consider $g_0 = g_1, g_2$ and g_1g_2. The Σ-multiplicativity and maximal length of E give $0 \neq E_{g_0}E_{g_2} + E_{g_2}E_{g_0} = E_{g_0g_2}$. Thus, using (3.3), we get $0 \neq E_{g_0g_2} \subset I$.

In a similar way, and employing the elements g_0g_2, g_3 and $g_0g_2g_3$ we have $0 \neq E_{g_0g_2g_3} \subset I$.

Continuing this process on the connection $\{g_1, ..., g_r\}$ we obtain that $0 \neq E_{g_0 g_2 g_3 \cdots g_r} \subset I$. Therefore, either $0 \neq E_h \subset I$ or $0 \neq E_{h^{-1}} \subset I$ for any $h \in \Sigma$. Since $E_1 = cl(span_{\mathbb{K}}\{E_g E_{g^{-1}} : g \in \Sigma\})$ we conclude

$$E_1 \subset I. \tag{3.4}$$

Finally, given any $g \in \Sigma$, the Σ-multiplicativity and maximal length of E together with (3.4) allow us to assert that

$$0 \neq E_g E_1 + E_1 E_g = E_g \subset I. \tag{3.5}$$

From (3.4) and (3.5) we clearly get $I = E$. Hence, E is graded simple. □

We state now our main theorem that it is the second Wedderburn-type theorem for certain graded pseudo-H-rings:

Theorem 3.5. *Let E be a Σ-multiplicative graded pseudo-H-ring of maximal length and with $\mathcal{A}nn(E) = 0$. If E_1 is coherent then E is the topological orthogonal direct sum of its minimal (closed) graded ideals $E_{[g]}, g \in G$. Moreover, each $E_{[g]}$ is a graded simple, graded pseudo-H-ring, such that the elements of its support are connected.*

Proof. By Theorem 2.8, $E = cl(\oplus^{\perp}_{[g] \in \Sigma/\sim} E_{[g]})$. Namely, E is the topological orthogonal direct sum of the ideals

$$E_{[g]} = cl(E_{1,[g]} \oplus^{\perp} V_{[g]}) = cl(span_{\mathbb{K}}\{E_h E_{(h)^{-1}} : h \in [g]\}) \oplus^{\perp} cl(\oplus^{\perp}_{h \in [g]} E_h),$$

(see also Theorem 2.5 and the notation before Proposition 2.3). We claim that the support, say $\sum_{E_{[g]}}$, of $E_{[g]}, g \in G$, has all of its elements connected. Indeed, since $[g] = [g^{-1}]$ and $E_{[g]} E_{[g]} \subset E_{[g]}$ (see Proposition 2.3-1), we easily deduce that $[g]$ has all of its elements $[g]$-connected (connected through elements in $[g]$). Besides, the Σ-multiplicativity of E implies that of $E_{[g]}, g \in G$. Clearly $E_{[g]}$ is of maximal length. Moreover, $\mathcal{A}nn(E_{[g]}) = \{0\}$ (the latter denotes the annihilator of $E_{[g]}$ in itself), this is a consequence of the fact that $E_{[g]} E_{[h]} = 0$ if $[g] \neq [h]$ (Theorem 2.6), and $\mathcal{A}nn(E) = \{0\}$. An application of Theorem 3.4 leads to the graded simpleness of $E_{[g]}$. Thus, we easily get that any of the ideals $E_{[g]}$ are minimal, as well and this finishes the proof. □

Example 3.6. Let us consider the pseudo-H-ring of Example 3.3. This is Σ-multiplicative of maximal length with symmetric support and it has a coherent 1-homogeneous subspace. It is easy to check that $\mathcal{A}nn(E) = 0$. Observe that given any $q, p \in \Sigma$ with $p \notin \{q, q^{-1}\}$, we can write $q = x^{-1}_{n,t} x_{m,t}$ with $n \neq m$ and $p = x^{-1}_{r,v} x_{s,v}$ with $r \neq s$.

Suppose $v = t$. By fixing some $u, v \in \mathbb{N}$ such that $u \notin \{n, m, r\}$ and $v \notin \{m, r, s, u\}$ we get that the set

$$\{q, x^{-1}_{u,t} x_{n,t}, x^{-1}_{m,t} x_{v,t}, x^{-1}_{r,t} x_{u,t}, x^{-1}_{v,t} x_{s,t}\}$$

is a connection from q to p.

However, if $v \neq t$, and since $\Sigma = \{x^{-1}_{n,t} x_{m,t} : n, m \in \mathbb{N}$ with $n \neq m$ and $t \in J\}$, there is not any connection from q to p. We have shown that the equivalence classes in $\Sigma/\sim$ are $[x_{n,t}^{-1}x_{m,t}] = \{x_{r,t}^{-1}x_{s,t} : r, s \in \mathbb{N} \text{ with } r \neq s\}$ and by applying the results in this section we can assert that, under the notation

$$E_t = cl((\sum_{n\in\mathbb{N}} \mathbb{C}a_{((n,t),(n,t))}) \oplus (\bigoplus_{n,m\in\mathbb{N};n\neq m} \mathbb{C}a_{((n,t),(m,t))}))$$

for any $t \in J$, any E_t is a graded simple, graded pseudo-H-ring having all of the elements of its support connected. Moreover, E decomposes as the topological orthogonal direct sum of these family of minimal graded ideals, namely:

$$E = cl(\bigoplus_{t=1}^{r}{}^{\perp} E_t).$$

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