TOWARDS THE $C^{p'}$-REGULARITY CONJECTURE IN HIGHER DIMENSIONS

DAMIÃO J. ARAÚJO, EDUARDO V. TEIXEIRA, AND JOSÉ MIGUEL URBANO

ABSTRACT. A longstanding conjecture in elliptic regularity theory inquires whether a $W^{1,p}$ function whose p-laplacian is bounded is locally of class $C^{1,\frac{1}{p-1}}$. While it is well known that such functions are of class $C^{1,\alpha}$ for some unknown $0 < \alpha < 1$, establishing the sharp estimate turns out to be a rather delicate problem. Quite recently, the authors managed to establish the conjecture in the plane. In this article, we address the conjecture in higher dimensions and confirm its validity in a number of other meaningful cases.

1. Introduction

In this paper, we treat the so called $C^{p'}$-regularity conjecture, which predicts that a weak solution of the p-Poisson equation

$$-\Delta_p u = f(x) \in L^\infty(B_1), \tag{1.1}$$

lies in $C^{1,\frac{1}{p-1}}(B_{1/2})$. Note that

$$1 + \frac{1}{p-1} = p'$$

is precisely the conjugate of p. The radially symmetric example

$$-\Delta_p \left(|x|^{p'}\right) = c_p, \tag{1.2}$$

shows that indeed $C^{1,\frac{1}{p-1}}$–regularity is the best one can hope for. This (naïve) example is actually quite intriguing. If the conjecture is confirmed, it means that among all functions whose p-laplacian is bounded, $|x|^{p'}$ is the least regular one. Perhaps even more counterintuitive is to rephrase it as follows: among all sources in L^∞, a nonzero constant produces the least regular solution of the p-Poisson equation.

Quite recently, in the companion paper [1], the authors established the $C^{p'}$-regularity conjecture in the plane (see also [7, 8]). The main result proven is the following:

Date: February 3, 2017.
2010 *Mathematics Subject Classification.* Primary 35B65. Secondary 35J60, 35J70.
Key words and phrases. Nonlinear pdes, regularity theory, sharp estimates.

Theorem 1. *Let $B_1 \subset \mathbb{R}^2$, and let $u \in W^{1,p}(B_1)$ be a weak solution of*

$$-\Delta_p u = f(x), \qquad p > 2,$$

with $f \in L^\infty(B_1)$. Then $u \in C^{p'}(B_{1/2})$ and

$$\|u\|_{C^{p'}(B_{1/2})} \leq C_p \left(\|f\|_{L^\infty(B_1)}^{\frac{1}{p-1}} + \|u\|_{L^p(B_1)} \right).$$

The strategy developed in [1] to prove the flatland version of the $C^{p'}$-regularity conjecture has two main ingredients. The first concerns explicit estimates for p-harmonic functions in the plane (see [2, 3, 4, 6]) and the second is a new oscillation estimate for solutions of degenerate elliptic equations. In this article, we explore and refit the latter as to tackle the conjecture in arbitrary dimensions. The new idea we explore is, roughly speaking, based on the observation that if some further information is known about the solutions, one may try to carry it over through the tangential path and it will ultimately enforce further rigidity on the limiting profiles. The general analysis we develop confirms the conjecture in a number of meaningful cases. In particular, we prove the conjecture is true for the class of radially symmetric solutions of the p-Poisson equation. We also show that if u is a solution of the p-Poisson equation with no saddle critical points, then u is locally $C^{p'}$-smooth.

The paper is organized as follows: In section 2 we describe the mathematical setup, introducing some notation and recalling a crucial lemma from [1]. The main general regularity theorem is announced and proven in section 3. Applications of the general analysis to several particular scenarios are discussed in section 4.

2. Mathematical setup

As usual, hereafter in this paper, $d \geq 1$ denotes the dimension of the Euclidian space $\mathbb{R}^d$. A property $\mathcal{P}$ is said to be closed under the C^1 topology if whenever u_n satisfies $\mathcal{P}$, for all n, and $u_n \to u$ in the C^1-topology, then u also satisfies $\mathcal{P}$. This is, in some sense, a rather weak condition. In particular, symmetry properties, bounds, prescribed pointwise values for the gradient, sign restrictions, etc..., are all closed properties under the C^1 topology.

Given a real number $p > 2$, and a C^1-closed property $\mathcal{P}$, we consider the functional set

$$\Xi_{\mathcal{P}}(p,d) := \left\{ h \in W^{1,p}(B_{1/2}) \;\middle|\; h \text{ satisfies } \mathcal{P} \text{ and } \Delta_p h = 0 \right\},$$

where Δ_p denotes the p-Laplace operator and the equation is interpreted in the weak sense. The key idea explored in this work is that functions in the *tangential* space $\Xi_{\mathcal{P}}(p,d)$ enjoy a richer regularity theory than mere p-harmonic functions.

In order to announce our main result, we need a definition.

Definition 1. Given a number $0 < \alpha < 1$ and $r \in (0, 1/2)$, we define

$$\omega_\alpha(r) := \sup \left\{ \frac{|h(x) - [h(0) + \nabla h(0) \cdot x]|}{\|h\|_{L^\infty(B_1)} \cdot r^{1+\alpha}} \;\Big|\; x \in B_r \text{ and } h \in \Xi_{\mathcal{P}}(p,d) \right\}.$$

Moreover, we set

$$\alpha_{\mathcal{P}}^M := \sup \left\{ \alpha \in (0,1) \;\Big|\; \inf_r \omega_\alpha(r) < 1 \right\}. \tag{2.1}$$

The above setup, together with an oscillation control, fosters a friendly platform to treat common issues related to sharp regularity estimates for the p-Poisson equation and this will be explored in section 4.

We conclude this section by revisiting a key ingredient used in [1]: if u is a normalized solution of

$$-\Delta_p u = f(x),$$

and $\|f\|_\infty \ll 1$, then one can find a C^1 corrector ξ, with $\|\xi\|_{C^1} \ll 1$, such that $u + \xi$ is p-harmonic. Here is the precise statement, in a more general format.

Lemma 1. *Let $u \in W^{1,p}(B_1)$ be a weak solution of $-\Delta_p u = f$ in B_1, with $\|u\|_\infty \leq 1$ and satisfying a C^1-closed property $\mathcal{P}$. Given $\epsilon > 0$, there exists $\delta = \delta(p,d,\epsilon) > 0$ such that if $\|f\|_\infty \leq \delta$ then we can find a corrector $\xi \in C^1(B_{1/2})$, with*

$$|\xi(x)| \leq \epsilon \quad \text{and} \quad |\nabla \xi(x)| \leq \epsilon \quad \text{in } B_{1/2}, \tag{2.2}$$

such that

$$u + \xi \in \Xi_{\mathcal{P}}(p,d). \tag{2.3}$$

Proof. Suppose the thesis of the lemma fails to hold. This means that there exists an $\epsilon_\star > 0$ for which one cannot find a corresponding $\delta > 0$. Thus, taking $\delta = 1/n$, we find a sequence of source functions f_n, with $\|f_n\|_\infty < 1/n$, and a sequence of functions $u_n \in W^{1,p}(B_1)$, with $\|u_n\|_\infty \leq 1$, each one of u_n satisfies the C^1-closed property $\mathcal{P}$ and solves

$$-\Delta_p u_n = f_n \quad \text{in } B_1, \tag{2.4}$$

in the week sense; however, for all n and any possible corrector $\xi \in C^1(B_1)$, with $u_n + \xi \in \Xi_{\mathcal{P}}(p,d)$, either

$$\sup_{B_{1/2}} |\xi| \geq \epsilon_\star \quad \text{or} \quad \sup_{B_{1/2}} |\nabla \xi| \geq \epsilon_\star. \tag{2.5}$$

By standard $C^{1,\alpha}$ estimates for solutions to (2.4), up to a subsequence, we can assume u_n converges in the $C^1(B_{2/3})$-topology to a function v. From the fact that $\|f_n\|_\infty < 1/n$ and using stability arguments, we deduce v is p-harmonic. Also, since $\mathcal{P}$ is a C^1-closed property, we conclude $v \in \mathcal{P}$. Finally, for $n \gg 1$, if we define $\xi_n := v - u_n$, we have $u_n + \xi_n \in \mathcal{P}$ and

$$\|\xi_n\|_{C^1(B_{1/2})} \leq \epsilon/100,$$

which is a contradiction. □

3. An improved regularity estimate

In this section, we prove that, under a given C^1-closed property $\mathcal{P}$, solutions of the p-Poisson equation are asymptotically as regular as functions in the tangential space $\Xi_{\mathcal{P}}(p,d)$. We shall argue by means of a geometric iterative process and hence we also need to request that $\mathcal{P}$ is closed under translations, dilations and scalings, i.e., for every $a>0$, $0<b<1$ and $c\in\mathbb{R}$,

$$v(x) \text{ satisfies } \mathcal{P} \implies av(bx)+c \ \text{ satisfies } \mathcal{P}. \tag{3.1}$$

We now state our main general regularity theorem.

Theorem 2. *Let $\mathcal{P}$ be a C^1-closed property satisfying (3.1) and $B_1\subset\mathbb{R}^d$. Let $u\in W^{1,p}(B_1)$ be a weak solution of*

$$-\Delta_p u = f(x), \qquad p>2,$$

with $f\in L^\infty(B_1)$, and satisfying $\mathcal{P}$. Then, given any number

$$\gamma \in \left(0,\frac{1}{p-1}\right] \cap \left(0,\alpha_{\mathcal{P}}^M\right),$$

there exists a constant $C_\gamma \geq 1$, depending universal parameters of the problem, such that

$$\sup_{x\in B_r}\Big|u(x)-[u(0)+\nabla u(0)\cdot x]\Big| \leq C_\gamma\left(\|f\|_{L^\infty(B_1)}^{\frac{1}{p-1}}+\|u\|_{L^p(B_1)}\right)r^{1+\gamma},$$

for all $0<r\ll 1$.

A related approach, via integral estimates rather than pointwise estimates, has been first developed in [7] and, later on, also reconsidered in [8].

The proof of Theorem 2 follows the underlying ideas in [1] but involves a few adjustments, and we carry it over for the reader's convenience. Fixed a number

$$\gamma \in \left(0,\frac{1}{p-1}\right] \cap \left(0,\alpha_{\mathcal{P}}^M\right),$$

denote by μ_γ the average between $\inf\limits_{r\in(0,\frac{1}{2})}\omega_\gamma(r)$ and 1, that is,

$$\mu_\gamma := \frac{1+\inf\limits_{r\in(0,\frac{1}{2})}\omega_\gamma(r)}{2} < 1. \tag{3.2}$$

The following result is the first step in the iteration.

Lemma 2. *There exists $0<\lambda_0<1/2$ and $\delta_0>0$ such that if $\|f\|_\infty\leq\delta_0$ and $u\in W^{1,p}(B_1)$ is a weak solution of $-\Delta_p u=f$ in B_1, with $\|u\|_\infty\leq 1$ and satisfying $\mathcal{P}$, then*

$$\sup_{x\in B_{\lambda_0}}\Big|u(x)-[u(0)+\nabla u(0)\cdot x]\Big| \leq {\lambda_0}^{1+\gamma}.$$

Proof. Take $\epsilon > 0$ to be fixed later, apply the previous lemma to find δ_0 and, under the smallness assumption on f, a respective corrector ξ satisfying (2.2) and (2.3). From construction, there exists $\lambda_0 < 1/2$, such that $\omega_\gamma(\lambda_0) < \mu_\gamma$, and, since $(u+\xi)$ belongs to the tangential space $\Xi_{\mathcal{P}}(p,d)$, we can estimate

$$\sup_{B_{\lambda_0}} |u+\xi| \leq \mu_\gamma(1+\epsilon)\lambda_0^{1+\gamma}.$$

We further estimate in B_{λ_0}:

$$\begin{aligned} |u(x) - [u(0) + \nabla u(0)\cdot x]| &\leq |(u+\xi)(x) - [(u+\xi)(0) + \nabla(u+\xi)(0)\cdot x]| \\ &\quad + |\xi(x)| + |\xi(0)| + |\nabla\xi(0)\cdot x| \\ &\leq \mu_\gamma(1+\epsilon)\lambda_0^{1+\gamma} + 3\epsilon. \end{aligned}$$

Finally, by continuity, we can choose ϵ universally small such that

$$\mu_\gamma(1+\epsilon)\lambda_0^{1+\gamma} + 3\epsilon = \lambda_0^{1+\gamma},$$

which determines the smallness assumption on $\|f\|_\infty$ – the constant $\delta_0 > 0$ in the statement of the current lemma – through the conclusion of Lemma 1, and the proof is complete. □

Corollary 1. *Under the assumptions of the previous lemma,*

$$\sup_{x\in B_{\lambda_0}} |u(x) - u(0)| \leq {\lambda_0}^{1+\gamma} + |\nabla u(0)|\lambda_0.$$

We now argue as in [1] and obtain the following new oscillation estimate:

Theorem 3. *Let $u \in W^{1,p}(B_1)$ be a weak solution of $-\Delta_p u = f$ in B_1, with $\|u\|_\infty \leq 1$ and $\|f\| \leq \delta_0$. Assume u satisfies a C^1-closed property $\mathcal{P}$ for which* (3.1) *holds. Then, there exists a constant $C > 1$, such that*

$$\sup_{x\in B_r} |u(x) - u(0)| \leq C r^{1+\gamma}\left(1 + |\nabla u(0)|\, r^{-\gamma}\right),$$

for all $0 < r \ll 1$.

Proof. Let the universal constants λ_0 and δ_0, obtained in Lemma 2, be fixed. The idea is now to re-apply Lemma 2 to the new function

$$v(x) = \frac{u(\lambda_0 x) - u(0)}{{\lambda_0}^{1+\gamma} + |\nabla u(0)|\lambda_0}, \qquad x \in B_1 \tag{3.3}$$

and iterate. We first need to check that v satisfies the hypothesis of the lemma. It is clear that $\|v\|_\infty \leq 1$, $v(0) = 0$, and

$$\nabla v(0) = \frac{\lambda_0}{{\lambda_0}^{1+\gamma} + |\nabla u(0)|\lambda_0}\,\nabla u(0).$$

Also, one easily estimates

$$|\Delta_p v| = \frac{\lambda_0^p}{\left({\lambda_0}^{1+\gamma} + |\nabla u(0)|\lambda_0\right)^{p-1}}\;|f(\lambda_0 x)| \leq \frac{\lambda_0^p}{{\lambda_0}^{(1+\gamma)(p-1)}}\;|f(\lambda_0 x)| \leq \delta_0.$$

In view of (3.1), v also satisfies $\mathcal{P}$. Applying then Lemma 2 to v and changing variables, we get

$$\sup_{x\in B_{\lambda_0}} |v(x)-v(0)| \le {\lambda_0}^{1+\gamma} + |\nabla v(0)|\lambda_0,$$

which, in terms of the original function u, reads

$$\sup_{x\in B_{\lambda_0}} \left|\frac{u(\lambda_0 x)-u(0)}{{\lambda_0}^{1+\gamma}+|\nabla u(0)|\lambda_0}\right| \le {\lambda_0}^{1+\gamma} + \left|\frac{\lambda_0}{{\lambda_0}^{1+\gamma}+|\nabla u(0)|\lambda_0}\,\nabla u(0)\right|\lambda_0,$$

or else, again by a change of variables,

$$\sup_{x\in B_{\lambda_0^2}} |u(x)-u(0)| \le {\lambda_0}^{1+\gamma}\left[{\lambda_0}^{1+\gamma}+|\nabla u(0)|\lambda_0\right] + |\nabla u(0)|\,\lambda_0^2.$$

This is the first step of an iteration process eventually leading to the recurrence law

$$a_{k+1} \le {\lambda_0}^{1+\gamma} a_k + |\nabla u(0)|\,\lambda_0^{k+1},$$

where

$$a_k := \sup_{x\in B_{\lambda_0^k}} |u(x)-u(0)| \qquad \text{and} \qquad b_k := \frac{a_k}{\lambda_0^{k(1+\gamma)}}.$$

We also obtain

$$b_{k+1} = \frac{a_{k+1}}{\lambda_0^{(k+1)(1+\gamma)}} \le \frac{{\lambda_0}^{1+\gamma}a_k + |\nabla u(0)|\,\lambda_0^{k+1}}{\lambda_0^{(k+1)(1+\gamma)}} = b_k + |\nabla u(0)|\,{\lambda_0}^{-(k+1)\gamma}.$$

We are now ready to conclude the proof. Given $0<r\ll\lambda_0$, let $k\in\mathbb{N}$ be such that

$$\lambda_0^{k+1} < r \le \lambda_0^k.$$

We estimate, exploring both inequalities above at appropriate instances,

$$\begin{aligned}
\sup_{x\in B_r}\frac{|u(x)-u(0)|}{r^{1+\gamma}} &\le \sup_{x\in B_{\lambda_0^k}}\frac{|u(x)-u(0)|}{(\lambda_0^{k+1})^{1+\gamma}} = \frac{b_k}{\lambda_0^{1+\gamma}}\\
&\le \frac{b_0 + |\nabla u(0)|\displaystyle\sum_{i=1}^{k}\lambda_0^{-\gamma i}}{\lambda_0^{1+\gamma}}\\
&= \frac{a_0 + |\nabla u(0)|\,\lambda_0^{-\gamma}\,\frac{\lambda_0^{-\gamma k}-1}{\lambda_0^{-\gamma}-1}}{\lambda_0^{1+\gamma}}\\
&\le 2 + C(\lambda_0,\gamma)\,|\nabla u(0)|\,r^{-\gamma}\\
&\le C\left(1+|\nabla u(0)|\,r^{-\gamma}\right).
\end{aligned}$$

We finally remark that, γ being fixed, the constant $\lambda_0>0$ is universal. □

This immediately gives the desired regularity near critical points of u. In fact, when $|\nabla u(0)| \le r^\gamma$, Theorem 3 gives

$$\begin{aligned} \sup_{x\in B_r} \Big| u(x) - [u(0) + \nabla u(0)\cdot x] \Big| &\le \sup_{x\in B_r} |u(x)-u(0)| + |\nabla u(0)|\, r \\ &\le (2C+1) r^{1+\gamma}. \end{aligned}$$

We are left to understand the behaviour of u around points where its gradient is large. The idea is that the operator is uniformly elliptic in that region and we have stronger estimates at our disposal. We make this idea precise in [1, section 5] and omit it here to avoid an unnecessary duplication.

4. Special scenarios

Theorem 2 is linked to the $C^{p'}$-regularity conjecture in the following way: "if $\mathcal{P}$ restricts the tangential space as to $\alpha_{\mathcal{P}}^M > \frac{1}{p-1}$, then any function whose p-laplacian is bounded and verifies $\mathcal{P}$ is of class $C^{p'}$, and the conjecture is verified for that class of functions."

4.1. **Low dimensions.** We start off by observing that p-harmonic functions in the real line are affine functions, thus in $1d$, $\alpha_M = 1$ and hence Theorem 2 provides a proof of the $C^{p'}$-regularity conjecture in the line — a result that could perhaps be established by softer tools. In any case, for the sake of completeness, we write this conclusion as a proposition.

Proposition 1. *The $C^{p'}$-regularity conjecture holds true in the real line.*

Such result becomes more appealing when applied to the analysis of d-dimensional problems which carry some sort of symmetry. By way of example, we mention here the theory of (degenerate) phase transition problems, namely entire solutions of

$$x\Delta_p u = (1-u^2)^p, \tag{4.1}$$

satisfying $\partial_{x_d} u > 0$ and $\lim\limits_{x_d\to\pm\infty} u(\cdot, x_d) = \pm 1$. Clearly, by monotonicity in the x_d-variable, any solution to (4.1) is smooth; however, as no uniform lower bound on $\partial_{x_d} u$ is granted, such smoothness could deteriorate. Notwithstanding, by a striking result from [10], if $d \le 8$ or $\{u = 0\}$ has at most linear growth at infinity, then level sets are hyperplanes and thus it follows by the results established in the current paper that we can locally bound the $C^{p'}$-norm of u uniformly, i.e., independently of the $\inf \partial_{x_d} u$.

Next, for flatland problems, $d = 2$, more can be said about the underlying regularity theory for the p-Poisson equation. In [1], the authors derive from explicit regularity estimates for quasiregular gradient maps established in [3], a non-sharp, but universal estimate for p-harmonic functions in the plane: if ϕ is the complex gradient of a p-harmonic function, then

$$\int_{B_r} |\nabla\phi|^2 \le (p-1)\,(2r)^{2\alpha(p)} \int_{B_{1/2}} |\nabla\phi|^2, \qquad 0 < r \le \frac{1}{2},$$

with

$$\alpha(p) = \frac{1}{2p}\left(-3 - \frac{1}{p-1} + \sqrt{33 + \frac{30}{p-1} + \frac{1}{(p-1)^2}}\right).$$

While this is not the best regularity estimate expected for p-harmonic functions in the plane, cf. [6], it is beyond the threshold value $\frac{1}{p-1}$. Hence, as proved in [1] (see also [7, 8]), any function whose p-laplacian is bounded in $2d$ is locally of class $C^{1,\frac{1}{p-1}}$.

Still in the plane, Evans and Savin proved in [5] (see also [9]) that infinity harmonic functions, i.e., viscosity solutions of

$$\Delta_\infty u := u_{x_i x_j} u_{x_i} u_{x_j} = 0,$$

are locally of class $C^{1,\gamma}$ for some $0 < \gamma < 1/3$. This result also connects to the $C^{p'}$-regularity conjecture. Indeed, within the framework setup in section 2, the Evans-Savin Theorem suggests the possibility of proving that $\alpha_\emptyset^M$ is bounded below, uniformly with respect to p. Once this is confirmed, the $C^{p'}$-regularity conjecture is solved for $p \gg 1$. We plan to come back to this issue in a forthcoming paper.

4.2. **Problems with symmetry.** Continuing our analysis, in view of the extremal example mentioned in (1.2), it is only natural to inquire about problems having radial symmetry. Taking full advantage of our general setup, the key observation is that the functional set

$$\Xi_{\text{rad}}(p,d) := \left\{u \in W^{1,p}(B_{1/2}) \;\middle|\; \Delta_p u = 0 \text{ and } u \text{ is bounded and radial}\right\}$$

contains only constants. Indeed, if $u(x) = \varphi(r)$, then

$$\Delta_p u = |\varphi'(r)|^{p-2}\left\{(p-1)\varphi''(r) + \frac{d-1}{r}\varphi'(r)\right\}.$$

Solving the homogeneous ODE, we obtain

$$\varphi(r) = \begin{cases} a + b\cdot r^{\frac{1-d}{p-1}+1} & \text{if} \quad p \neq d, \\ a + b\cdot \ln r & \text{if} \quad p = d, \end{cases}$$

for constants $a, b \in \mathbb{R}$. For $d \geq 2$, φ is C^1 at the origin if, and only if, $b = 0$. As a consequence, when restricted to the set of radially symmetric functions, one has

$$\alpha_{\text{rad}}^M = 1,$$

and therefore we are able to establish the following version of the $C^{p'}$-regularity conjecture for radially symmetric functions.

Theorem 4. *Let $u \in W^{1,p}(B_1)$ be a radially symmetric function whose p-laplacian is bounded. Then $u \in C^{p'}(B_{1/2})$, with universal estimates.*

4.3. **Problems with controlled singular set.** In many applications, the aimed sharp estimate expected from the $C^{p'}$-regularity conjecture needs only to be verified along the set of critical points. This is particularly meaningful in the theory of free boundary problems. In this section we pursue a general analysis which in particular gives pointwise $C^{p'}$ estimates at local maxima or local minima of solutions of the p-Poisson equation.

Definition 2. Given a positive number $\sigma > 0$, a function $\varphi\colon B_1 \to \mathbb{R}$ is said to be a σ-cap at 0 if

$$\sup_{B_r} |\varphi(x) - \varphi(0)| \leq C|x|^\sigma,$$

for a constant $C > 0$. The infimum among all constants is denoted by $[\varphi]_\sigma$.

Theorem 5. *Let u be a bounded solution of $-\Delta_p u = f(x)$ in B_1, with $f \in L^\infty(B_1)$. Let ξ_0 be an interior point and suppose u can be touched from below at ξ_0 by a p'-cap φ. Then u is precisely $C^{p'}$ continuous at ξ_0, that is,*

$$|u(x) - u(\xi_0)| \leq C|x - \xi_0|^{p'},$$

for a positive constant $C > 0$ that depends only on dimension, p, $\|u\|_{L^\infty(B_1)}$, $\|f\|_{L^\infty(B_1)}$ and $[\varphi]_{p'}$.

Theorem 5 has an analogous version for points that one can touch from above by a p'-cap. An immediate consequence is that solutions to p-Poisson equations are $C^{p'}$-regular at any local extremal point, where in fact one can touch u by a hyperplane. The proof of Theorem 5 is based on a flattening argument, which is interesting on its own.

Lemma 3. *Given $\eta > 0$, there exists $\delta > 0$, depending only on η and universal parameters, such that if $|v| \leq 1$ in B_1 and $-\Delta_p v = f(x)$ in B_1, then*

$$\|f\|_{L^\infty(B_1)} + \left(v(0) - \inf_{B_1} v \right) \leq \delta,$$

implies

$$\operatorname*{osc}_{B_{1/2}} v \leq \eta.$$

Proof. Suppose, for the sake of contradiction, that the thesis of the lemma fails to hold. This means we can find $\eta_0 > 0$, a sequence of positive numbers $\kappa_j \to 0$, and sequences of functions $(v_j)_{j\in\mathbb{N}}$, $(f_j)_{j\in\mathbb{N}}$ satisfying

a) $|v_j| \leq 1$, and $\|f_j\|_{L^\infty(B_1)} + \left(v_j(0) - \inf_{B_1} v_j \right) \leq \kappa_j$;
b) $-\Delta_p v_j = f_j(x)$ in B_1;
c) $\operatorname*{osc}_{B_{1/2}} v_j \geq \eta_0$.

By compactness, v_j converges in the C^1-topology to a function v_∞ in $B_{1/2}$ and from stability we conclude that v_∞ solves the homogeneous equation

$$-\Delta_p v_\infty = 0, \quad \text{in } B_{1/2}.$$

From a) we immediately conclude v_∞ attains its minimum value at 0, and hence, by the strong maximum principle of Vázquez, see [11], we conclude that $v_\infty \equiv v_\infty(0)$. This gives a contradiction with c) if we take $j \gg 1$. The lemma is proven. $\square$

Proof of Theorem 5. With no loss of generality, we can assume ξ_0 is the origin and $u(0) = 0$. Let $\varphi(x)$ be the p'-cap touching u at 0 from below. Define $v(x) = u(\lambda x)$. One simply checks that v verifies

$$-\Delta_p v(x) = \lambda^p f(\lambda x).$$

Owing to Lemma 3, we choose $\lambda > 0$ such that

$$2\lambda^p \|f\|_{L^\infty(B_1)} \leq \delta_\star, \tag{4.2}$$

where $\delta_\star > 0$ is the closeness number given by Lemma 3 when one takes $\eta = 2^{-p'}$. From uniform Lipchitz continuity,

$$|v(x)| = |u(\lambda x) - u(0)| \leq C(\|u\|_{L^\infty(B_1)}, \|f\|_{L^\infty(B_1)})\lambda.$$

Hence, selecting a smaller $\lambda > 0$ if necessary, we can assume $|v| \leq 1$, for all $x \in B_1$. Yet by uniform continuity,

$$\begin{aligned} \left(v(0) - \inf_{B_1} v\right) &= -\inf_{B_\lambda} u \\ &\leq \frac{\delta_\star}{2}, \end{aligned}$$

if $\lambda > 0$ is once more diminished, if necessary. Finally, v is touched from below by the p'-cap $\tilde{\varphi}(x) := \varphi(\lambda x)$. We can estimate

$$\sup_{B_r} \tilde{\varphi} \leq [\varphi]_{p'} \lambda^{p'} \cdot r^{p'}.$$

Thus, if we take $\lambda > 0$ even smaller, if necessary, we can further assume

$$[\tilde{\varphi}]_{p'} \leq \frac{\delta_\star}{2}.$$

Now, we aim to prove that for any $j \in \mathbb{N}$, there holds

$$\operatorname*{osc}_{B_{2^{-j}}} v(x) \leq 2^{-jp'}. \tag{4.3}$$

We argue by finite induction. By our previous selections, Lemma 3 gives the first step in the induction process. Suppose we have verified (4.3) for $j = 1, 2, \cdots, k$. Define

$$v_{k+1}(x) := 2^{kp'} \cdot v(2^{-k}x).$$

Initially, from the induction process we readily verify that $v_{k+1}(0) = 0$, $|v_{k+1}| \leq 1$ and

$$\begin{aligned} |\Delta_p v_{k+1}| &\leq \lambda^p \left|f(2^{-k}\lambda x)\right| \\ &\leq \frac{\delta_\star}{2}, \end{aligned} \tag{4.4}$$

by the decision made in (4.2). Also, from the p'-cap control from below, we can estimate

$$\begin{aligned}\left(v_{k+1}(0) - \inf_{B_1} v_{k+1}\right) &= -2^{kp'} \inf_{B_{2^{-k}}} v \\ &\leq -2^{kp'} \inf_{B_{2^{-k}}} \varphi \\ &\leq \frac{\delta_\star}{2}.\end{aligned}$$

We can now apply Lemma 3 to v_{k+1}, which yields

$$\operatorname*{osc}_{B_{1/2}} v_{k+1}(x) = 2^{kp'} \operatorname*{osc}_{B_{2^{-k}}} v(x) \leq 2^{-p'}, \tag{4.5}$$

and the induction chain is complete. Finally, given any $0 < r \ll 1$, let $k \in \mathbb{N}$ be such that

$$2^{-k-1} < r \leq 2^{-k}.$$

We estimate, defining $\rho = \lambda r$,

$$\begin{aligned}\operatorname*{osc}_{B_\rho} u &= \operatorname*{osc}_{B_r} v \\ &\leq \operatorname*{osc}_{B_{2^{-k}}} v \\ &\leq 2^{-kp'} \\ &\leq \left(\frac{r}{2}\right)^{p'} \\ &\leq \frac{1}{(2\lambda)^{p'}} \cdot \rho^{p'},\end{aligned}$$

and the theorem is proven. □

Acknowledgments. This work was developed in the framework of the Brazilian Program *Ciência sem Fronteiras*. D.J.A. supported by CNPq. E.V.T. partially supported by CNPq and Fapesp. J.M.U. partially supported by CMUC – UID/MAT/00324/2013, funded by the Portuguese Government through FCT/MCTES and co-funded by the European Regional Development Fund through the Partnership Agreement PT2020.

References

[1] D.J. Araújo, E.V. Teixeira, and J.M. Urbano *A proof of the $C^{p'}$-regularity conjecture in the plane*, submitted.

[2] G. Aronsson, P. Lindqvist, *On p-harmonic functions in the plane and their stream functions*, J. Differential Equations 74 (1988), 157–178.

[3] A. Baernstein II and L.V. Kovalev, *On Hölder regularity for elliptic equations of non-divergence type in the plane*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 4 (2005), 295–317.

[4] Bojarski, B.; Iwaniec, T. *p-harmonic equation and quasiregular mappings.* Partial differential equations (Warsaw, 1984), 25–38, Banach Center Publ., 19, PWN, Warsaw, 1987.

[5] L.C. Evans and O. Savin, *$C^{1,\alpha}$ regularity for infinity harmonic functions in two dimensions*, Calc. Var. Partial Differential Equations 32 (2008), 325–347.

[6] T. Iwaniec and J. Manfredi, *Regularity of p-harmonic functions on the plane*, Rev. Mat. Iberoamericana 5 (1989), 1–19.
[7] T. Kuusi and G. Mingione, *Universal potential estimates*, J. Funct. Anal. 262 (2012), 4205–4269.
[8] E. Lindgren and P. Lindqvist, *Regularity of the p-Poisson equation in the plane*, J. Anal. Math., to appear.
[9] O. Savin, *C^1 regularity for infinity harmonic functions in two dimensions*, Arch. Rational Mech. Anal. 76 (2005), 351–361.
[10] E. Valdinoci, B. Sciunzi and O. Savin, *Flat level set regularity of p-Laplace phase transitions*, Mem. Amer. Math. Soc. 182 (2006), no. 858, vi+144 pp.
[11] J.L. Vázquez, *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim. 12 (1984), 191–202.

Universidade Federal da Paraíba, Departmento de Matemática, 58.051-900 João Pessoa, PB-Brazil
E-mail address: djunio@gmail.com

Department of Mathematics, University of Central Florida Orlando, FL 32816, USA.
E-mail address: Eduardo.Teixeira@ucf.edu

CMUC, Department of Mathematics, University of Coimbra, 3001-501 Coimbra, Portugal
E-mail address: jmurb@mat.uc.pt