A PROOF OF THE $C^{p'}$-REGULARITY CONJECTURE IN THE PLANE

DAMIÃO J. ARAÚJO, EDUARDO V. TEIXEIRA, AND JOSÉ MIGUEL URBANO

Abstract. We establish a new oscillation estimate for solutions of nonlinear partial differential equations of elliptic, degenerate type. This new tool yields a precise control on the growth rate of solutions near their set of critical points, where ellipticity degenerates. As a consequence, we are able to prove the planar counterpart of the longstanding conjecture that solutions of the degenerate p-Poisson equation with a bounded source are locally of class $C^{p'} = C^{1,\frac{1}{p-1}}$; this regularity is optimal.

1. Introduction

In this paper we investigate sharp $C^{1,\alpha}$-regularity estimates for solutions of the degenerate elliptic equation, with a bounded source,

$$-\Delta_p u = f(x) \in L^\infty(B_1), \qquad p > 2. \tag{1.1}$$

Establishing optimal regularity estimates is quite often a delicate matter and, in particular, $f(x) \in L^\infty$ is known to be a borderline condition for regularity. In the linear, uniformly elliptic case $p = 2$, solutions of

$$-\Delta u = f(x) \in L^\infty(B_1)$$

are locally in $C^{1,\alpha}$, for every $\alpha \in (0,1)$, but may fail to be in $C^{1,1}$. Obtaining such an estimate in specific situations, like free boundary problems, often involves a deep and fine analysis.

In the degenerate setting $p > 2$, the smoothing effects of the operator are far less efficient. Nonetheless, it is well established, see for instance [8, 23], that a weak solution to (1.1) is locally of class $C^{1,\beta}$, for some exponent $\beta > 0$ depending on dimension and p. If p' denotes the conjugate of p, i.e.,

$$p + p' = pp',$$

the radial symmetric example

$$-\Delta_p \left(c_p |x|^{p'} \right) = 1$$

sets the limits to the optimal regularity and gives rise to the following well known open problem among experts in the field.

Conjecture ($C^{p'}$-regularity conjecture). *Solutions to* (1.1) *are locally of class* $C^{1,\frac{1}{p-1}} = C^{p'}$.

Date: June 14, 2017.

2010 *Mathematics Subject Classification.* Primary 35B65. Secondary 35J60, 35J70.

Key words and phrases. Nonlinear pdes, regularity theory, sharp estimates.

This problem touches very subtle issues in regularity theory. As mentioned above, the conjecture is not true in the linear, uniformly elliptic setting, $p = 2$, where merely $C^{1,\mathrm{LogLip}}$-estimates are possible. Notice further that a positive answer implies that $|x|^{p'}$ – a function whose p-laplacian is constant (real analytic) – is the least regular among all functions whose p-laplacian is bounded. This is, at first sight, counterintuitive.

We show in this paper that the conjecture holds true provided p-harmonic functions, which are the solutions of the homogeneous counterpart of (1.1), are locally uniformly of class $C^{1,\alpha}$, with

$$\alpha > \frac{1}{p-1}.$$

While this is still open in higher dimensions, it holds true in the plane, thus yielding a full proof of the conjecture in 2-d. The crucial estimate follows from results by Baernstein II and Kovalev in [5], exploiting the fact that the complex gradient of a p-harmonic function in the plane is a K-quasiregular gradient mapping. In a somewhat related issue, let us mention that, yet in the plane, Evans and Savin proved in [9] (see also [16]) that infinity harmonic functions, i.e., viscosity solutions of

$$\Delta_\infty u := u_{x_i x_j} u_{x_i} u_{x_j} = 0,$$

are locally of class $C^{1,\gamma}$ for some $0 < \gamma \ll 1$. Whether infinity harmonic functions are of class C^1 in higher dimensions is still a major open problem in the field.

We next state the main result of this paper.

Theorem 1. *Let $B_1 \subset \mathbb{R}^2$, and let $u \in W^{1,p}(B_1)$ be a weak solution of*

$$-\Delta_p u = f(x), \qquad p > 2,$$

with $f \in L^\infty(B_1)$. Then $u \in C^{p'}(B_{1/2})$ and

$$\|u\|_{C^{p'}(B_{1/2})} \leq C_p \left(\|f\|_{L^\infty(B_1)}^{\frac{1}{p-1}} + \|u\|_{L^p(B_1)} \right).$$

Similar results have been independently obtained by Lindgren and Lindqvist using somewhat different methods. In [14] (see also [13]), they show that planar solutions to (1.1) are locally of class $C^{p'-\epsilon}$, for $\epsilon > 0$. The lack of a global $C^{1,\alpha}$-estimate for p-harmonic functions in 2-d was the main reason precluding the passing from this asymptotic version to the optimal result.

Our approach is based on a new oscillation estimate (see Theorem 6), which is interesting on its own and reveals some essential nuances of the problem, disclosing, for example, the conjecture in higher dimensions in a number of relevant scenarios (see [3]). It gives a precise control on the oscillation of a solution to (1.1) in terms of the magnitude of its gradient,

$$\sup_{B_r} |u(x) - u(0)| \lesssim r^{p'} + |\nabla u(0)| r, \tag{1.2}$$

and yields, by geometric iteration, improved $C^{1,\alpha}$ regularity estimates. The insight to obtain such a refined control comes from the striking results in [19], where *improved* regularity estimates are obtained for degenerate equations precisely along their set of critical points, $\{\nabla u = 0\}$.

We are convinced the set of ideas and insights included in this paper are robust and versatile, and will foster future developments of the theory, far beyond the $C^{p'}$-regularity conjecture. For example, when implementing Caffarelli's geometric approach to the analysis of $C^{1,\alpha}$-estimates for degenerate problems, a key obstruction that frequently arises is that if u is a solution and ℓ is an affine function, then no PDE is *a priori* satisfied by $(u-\ell)$. That is the case, for example, in [11], where the Ishii-Lions method was employed to control affine perturbations of the solution u. Our new oscillation estimate provides a definitive tool for treating this common issue in regularity theory.

The paper is organized as follows. To render the paper reasonably self-contained, we gather in section 2 the results concerning the regularity of p-harmonic functions in the plane that will be used in the proof of Theorem 1. In section 3 we introduce C^1-small correctors that link the regularity theory for (1.1) to that of p-harmonic functions. The key, new oscillation estimate is delivered in section 4, and in section 5 we conclude the proof of the main theorem.

Throughout the paper, we say a constant is *universal* if it only depends on p.

2. Flatland regularity for p-harmonic functions

In this section we revisit the $C^{1,\alpha}$ regularity theory for p-harmonic functions, i.e., solutions to the homogeneous equation

$$-\Delta_p u = 0. \tag{2.1}$$

That p-harmonic functions are locally of class $C^{1,\alpha(d,p)}$, for some exponent $0 < \alpha(d,p) < 1$, that depends on the dimension d and the power exponent p, is known since the late 60's (see [24]). Away from the set of critical points

$$\mathcal{S}(u) := \{x \mid \nabla u(x) = 0\},$$

p-harmonic functions are C^∞-smooth; however $C^{1,\alpha(d,p)}_{\text{loc}}$ is in fact the best possible regularity class since, along $\mathcal{S}(u)$, the Hessian of a p-harmonic function may become unbounded. Very little, if anything, is known concerning the value of the optimal, sharp Hölder exponent $\alpha(d,p)$ when $d \geq 3$.

In the plane, however, a remarkable result in [12], due to Iwaniec and Manfredi, assures that any p-harmonic function is of class $C^{1,\alpha^*(2,p)}$, for

$$\alpha^*(2,p) = \frac{1}{6}\left(\frac{p}{p-1} + \sqrt{1 + \frac{14}{p-1} + \frac{1}{(p-1)^2}}\right), \tag{2.2}$$

and this regularity is optimal. The proof exploits the fact that the complex gradient of a p-harmonic function is a K-quasiregular mapping and uses a hodograph transformation to linearize the problem. Unfortunately, no explicit estimates, yielding a local universal control of the C^{1,α^*}-norm of u, are written down in [12]. Near singular points we would have to examine the hodograph inversion process and track down the precise dependence of the successive norms involved – a quite delicate issue [15].

But, gloriously, there is more. In the apparently unrelated paper [5], concerning non-divergence elliptic equations in the plane, Baernstein II and Kovalev show that K-quasiregular *gradient* mappings are of class $C^{1,\alpha}_{\text{loc}}$, for an exponent α depending only on K, and obtain uniform estimates. For the reader's convenience we revisit their proof applied to our case.

Let $u \in W^{1,p}_{loc}(B_1)$ be p-harmonic, $p > 2$. Its complex gradient $\phi = \partial u/\partial z$ turns out to be a $(p-1)$-quasiregular mapping. This means that $\phi \in W^{1,2}_{loc}(B_1)$, which follows from estimates in [6] (see also [4]), and that

$$\left|\frac{\partial\phi(z)}{\partial\overline{z}}\right| \leq \left(1-\frac{2}{p}\right)\left|\frac{\partial\phi(z)}{\partial z}\right|, \quad \text{a.e. in } B_1, \tag{2.3}$$

which follows from giving (2.1) the form of the complex equation

$$\frac{\partial\phi}{\partial\overline{z}} = \left(\frac{1}{p}-\frac{1}{2}\right)\left(\frac{\overline{\phi}}{\phi}\frac{\partial\phi}{\partial z}+\frac{\phi}{\overline{\phi}}\overline{\frac{\partial\phi}{\partial z}}\right).$$

Here $z = x+iy$ is the complex variable and the operators of complex differentiation are defined by

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right) \qquad \text{and} \qquad \frac{\partial}{\partial\overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right).$$

The complex gradient ϕ is also a gradient mapping, i.e.,

$$\operatorname{Im}\frac{\partial\phi}{\partial\overline{z}} = 0,$$

which holds since

$$\frac{\partial\phi}{\partial\overline{z}} = \frac{\partial^2 u}{\partial\overline{z}\partial z} = \frac{1}{4}\Delta u.$$

This fact can be used to significantly improve the lower bound on the Jacobian of ϕ (cf. [5, Lemma 2.1]) in

$$J_\phi = \det\nabla\phi \geq \frac{1}{p-1}\left|\nabla\phi\right|^2,$$

which follows from (2.3) alone. This is the crucial new ingredient to prove (see [5, Section 2, and in particular inequality (2.5)]) that

$$\int_{B_r}|\nabla\phi|^2 \leq (p-1)\,(2r)^{2\alpha(p)}\int_{B_{1/2}}|\nabla\phi|^2, \qquad 0<r\leq\frac{1}{2},$$

with

$$\alpha(p) = \frac{1}{2p}\left(-3-\frac{1}{p-1}+\sqrt{33+\frac{30}{p-1}+\frac{1}{(p-1)^2}}\right),$$

which gives, by Morrey's lemma [10, Lemma 12.2], that $\phi\in C^{0,\alpha(p)}_{\text{loc}}$ and

$$[\phi]_{C^{0,\alpha(p)}(B_{1/2})} \leq 2^{1+\alpha(p)}\sqrt{\frac{p-1}{\alpha(p)}}\|\nabla\phi\|_{L^2(B_{1/2})}.$$

Now, using standard estimates (see [14] for example), we obtain

$$[\phi]_{C^{0,\alpha(p)}(B_{1/2})} \leq C_p\|u\|_{L^\infty(B_1)},$$

for a universal constant C_p that only depends on p. Finally observe that, for any $p>2$, we indeed have

$$\alpha^*(2,p) > \alpha(p) > \frac{1}{p-1}. \tag{2.4}$$

We summarize these results in the following proposition, which will be crucial in the proof of our main result.

Proposition 2. *For any $p > 2$, there exists $0 < \tau_0 < \frac{p-2}{p-1}$ such that p-harmonic functions in $B_1 \subset \mathbb{R}^2$ are locally of class $C^{p'+\tau_0}$. Furthermore, if $u \in W^{1,p}(B_1) \cap C(B_1)$ is p-harmonic in the unit disk $B_1 \subset \mathbb{R}^2$ then there is a constant C_p, depending only on p, such that*

$$[\nabla u]_{C^{0,\frac{1}{p-1}+\tau_0}(B_{1/2})} \leq C_p \|u\|_{L^\infty(B_1)}. \tag{2.5}$$

In the next three sections we will provide a proof of the the $C^{p'}$-regularity conjecture in the plane, i.e. a proof of Theorem 1.

3. Existence of C^1-small correctors

In this section, we show that if u is a normalized solution of

$$-\Delta_p u = f(x),$$

and $\|f\|_\infty \ll 1$, then we can find a C^1 corrector ξ, with $\|\xi\|_{C^1} \ll 1$, such that $u + \xi$ is p-harmonic. This will allow us to frame the $C^{p'}$ conjecture into the formalism of the so called geometric tangential analysis, e.g. [7], [2, 1] and [17, 18, 19, 20, 21, 22]. Here is the precise statement.

Lemma 3. *Let $u \in W^{1,p}(B_1)$ be a weak solution of $-\Delta_p u = f$ in B_1, with $\|u\|_\infty \leq 1$. Given $\epsilon > 0$, there exists $\delta = \delta(p, d, \epsilon) > 0$ such that if $\|f\|_\infty \leq \delta$ then we can find a corrector $\xi \in C^1(B_{1/2})$, with*

$$|\xi(x)| \leq \epsilon \quad \text{and} \quad |\nabla \xi(x)| \leq \epsilon, \quad \text{in } B_{1/2} \tag{3.1}$$

such that

$$-\Delta_p(u + \xi) = 0 \quad \text{in } B_{1/2}. \tag{3.2}$$

Proof. Suppose the result does not hold. We can then find $\epsilon_0 > 0$ and sequences of functions (u_j) and (f_j) in $W^{1,p}(B_1)$ and $L^\infty(B_1)$, respectively, such that

$$-\Delta_p u_j = f_j \quad \text{in } B_1; \qquad \|u_j\|_\infty \leq 1; \qquad \|f_j\|_\infty \leq 1/j$$

but, nonetheless, for every $\xi \in C^1(B_{1/2})$ such that

$$-\Delta_p(u_j + \xi) = 0 \quad \text{in } B_{1/2},$$

we have either $|\xi(x_0)| > \epsilon_0$ or $|\nabla \xi(x_0)| > \epsilon_0$, for a certain $x_0 \in B_{1/2}$.

From classical estimates for the p-Poisson equation, we can extract a subsequence, such that, upon relabelling,

$$u_j \longrightarrow u_\infty$$

in $C^1(B_{1/2})$ as $j \to \infty$. Passing to the limit in the pde, we obtain

$$-\Delta_p u_\infty = 0 \quad \text{in } B_{1/2}, \qquad \text{with} \quad \|u_\infty\|_\infty \leq 1.$$

Now, let $\xi_j := u_\infty - u_j$. For $j_* \gg 1$, we have

$$-\Delta_p(u_{j_*} + \xi_{j_*}) = -\Delta_p u_\infty = 0 \quad \text{in } B_{1/2}$$

and

$$|\xi_{j_*}(x)| \leq \epsilon_0 \quad \text{and} \quad |\nabla \xi_{j_*}(x)| \leq \epsilon_0, \quad \forall x \in B_{1/2},$$

thus reaching a contradiction. □

We conclude this section by commenting that in order to prove Theorem 1 it is enough to establish it for normalized solutions with small RHS, i.e., with $\|f\|_\infty \le \delta_0$. Indeed, if u verifies $-\Delta_p u = f(x)$, with $f \in L^\infty$, then the function

$$v(x) := \frac{u(\theta x)}{\|u\|_\infty}$$

is obviously normalized and

$$-\Delta_p v = \frac{\theta^p}{\|u\|_\infty^{p-1}} f(\theta x).$$

Thus, choosing

$$\theta := \sqrt[p]{\frac{\delta_0 \|u\|_\infty^{p-1}}{\|f\|_\infty}},$$

v satisfies (1.1), with small RHS. Once Theorem 1 is proven for v, it immediately gives the corresponding estimate for u.

4. Analysis on the critical set

In this section, based on an iterative reasoning, we establish the main tool that allows us to prove the $C^{p'}$ conjecture in the plane. The following result is the first step in the iteration.

Lemma 4. *There exists $0 < \lambda_0 < 1/2$ and $\delta_0 > 0$ such that if $\|f\|_\infty \le \delta_0$ and $u \in W^{1,p}(B_1)$ is a weak solution of $-\Delta_p u = f$ in B_1, with $\|u\|_\infty \le 1$, then*

$$\sup_{x \in B_{\lambda_0}} \Big| u(x) - [u(0) + \nabla u(0) \cdot x] \Big| \le {\lambda_0}^{p'}.$$

Proof. Take $\epsilon > 0$ to be fixed later, apply the previous lemma to find δ_0 and, under the smallness assumption on f, a respective corrector ξ satisfying (3.1) and (3.2). As $(u+\xi)$ is p-harmonic in $B_{1/2}$ and, in view of Proposition 2, $(u+\xi) \in C^{p'+\tau_0}$, we can estimate in $B_{\lambda_0} \subset B_{1/2}$,

$$\begin{array}{rcl} |u(x) - [u(0) + \nabla u(0)\cdot x]| & \le & |(u+\xi)(x) - [(u+\xi)(0) + \nabla (u+\xi)(0) \cdot x]| \\ & & + |\xi(x)| + |\xi(0)| + |\nabla \xi(0) \cdot x| \\ & \le & C_p {\lambda_0}^{p'+\tau_0} + 3\epsilon. \end{array}$$

We are also using the smallness of the corrector, assured by Lemma 3. In order to complete the proof, we now make universal choices. Initially we choose $\lambda_0 \ll 1/2$ such that

$$C_p {\lambda_0}^{p'+\tau_0} < \frac{1}{2} {\lambda_0}^{p'}.$$

In the sequel, we take

$$\epsilon = \frac{1}{6} {\lambda_0}^{p'},$$

which determines the smallness assumption on $\|f\|_\infty$ – constant $\delta_0 > 0$ in the statement of this current lemma – through the conclusion of Lemma 3. Lemma 4 is proven. □

The conclusion of Lemma 4 does not, *per se*, allow an iteration since no obvious pde is satisfied by $u + \ell$, when ℓ is an affine function. Nonetheless, it provides the following information on the oscillation of u in B_{λ_0}.

Corollary 5. *Under the assumptions of the previous lemma,*

$$\sup_{x\in B_{\lambda_0}} |u(x)-u(0)| \le {\lambda_0}^{p'} + |\nabla u(0)|\lambda_0.$$

Proof. This is a immediate application of the triangle inequality. □

The idea is now to iterate Corollary 5 in dyadic balls, keeping a precise track on the magnitude of the influence of $|\nabla u(0)|$.

Theorem 6. *Under the same assumptions of Lemma 4, there exists a constant $C > 1$ depending only on p, such that*

$$\sup_{x\in B_r} |u(x)-u(0)| \le C r^{p'} \left(1 + |\nabla u(0)|\, r^{\frac{1}{1-p}}\right),$$

holds for all $r > 0$.

Proof. We proceed by geometric iteration. Consider the universal constants λ_0 and δ_0 obtained in the previous Lemma 4 and let

$$v(x) = \frac{u(\lambda_0 x)-u(0)}{{\lambda_0}^{p'} + |\nabla u(0)|\lambda_0}, \qquad x \in B_1.$$

We have $\|v\|_\infty \le 1$, $v(0) = 0$, and

$$\nabla v(0) = \frac{\lambda_0}{{\lambda_0}^{p'} + |\nabla u(0)|\lambda_0}\, \nabla u(0).$$

Also, we have

$$-\Delta_p v = \frac{\lambda_0^p}{\left({\lambda_0}^{p'} + |\nabla u(0)|\lambda_0\right)^{p-1}}\, f(\lambda_0 x) \le \frac{\lambda_0^p}{{\lambda_0}^{p'(p-1)}}\, |f(\lambda_0 x)| \le \delta_0,$$

which entitles v to Lemma 4. Thus

$$\sup_{x\in B_{\lambda_0}} |v(x)-v(0)| \le {\lambda_0}^{p'} + |\nabla v(0)|\lambda_0,$$

which reads

$$\sup_{x\in B_{\lambda_0}} \left|\frac{u(\lambda_0 x)-u(0)}{{\lambda_0}^{p'} + |\nabla u(0)|\lambda_0}\right| \le {\lambda_0}^{p'} + \left|\frac{\lambda_0}{{\lambda_0}^{p'} + |\nabla u(0)|\lambda_0}\, \nabla u(0)\right| \lambda_0,$$

and hence

$$\sup_{x\in B_{\lambda_0^2}} |u(x)-u(0)| \le {\lambda_0}^{p'}\left[{\lambda_0}^{p'} + |\nabla u(0)|\lambda_0\right] + |\nabla u(0)|\, \lambda_0^2.$$

In the sequel, we define

$$a_k := \sup_{x\in B_{\lambda_0^k}} |u(x)-u(0)|,$$

and set

$$b_k := \frac{a_k}{\lambda_0^{kp'}}.$$

Iterating the previous reasoning we obtain the recurrence law

$$a_{k+1} \le {\lambda_0}^{p'} a_k + |\nabla u(0)|\, \lambda_0^{k+1}.$$

Consequently, we estimate

$$b_{k+1} = \frac{a_{k+1}}{\lambda_0^{(k+1)p'}} \leq \frac{{\lambda_0}^{p'} a_k + |\nabla u(0)|\, \lambda_0^{k+1}}{\lambda_0^{(k+1)p'}} = b_k + |\nabla u(0)|\, \lambda_0^{-(k+1)(p'-1)}.$$

Now, given $0 < r \ll \lambda_0$, let $k \in \mathbb{N}$ be such that $\lambda_0^{k+1} < r \leq \lambda_0^k$. Then

$$\begin{aligned}
\sup_{x\in B_r} \frac{|u(x)-u(0)|}{r^{p'}} &\leq \sup_{x\in B_{\lambda_0^k}} \frac{|u(x)-u(0)|}{(\lambda_0^{k+1})^{p'}} = \frac{b_k}{\lambda_0^{p'}} \\
&\leq \frac{b_0 + |\nabla u(0)| \displaystyle\sum_{i=1}^{k} \left[\lambda_0^{-(p'-1)}\right]^i}{\lambda_0^{p'}} \\
&= \frac{a_0 + |\nabla u(0)|\, \lambda_0^{-(p'-1)} \, \frac{\lambda_0^{-(p'-1)k}-1}{\lambda_0^{-(p'-1)}-1}}{\lambda_0^{p'}} \\
&\leq 2\lambda_0^{-p'} + C(\lambda_0, p')\, |\nabla u(0)|\, r^{-(p'-1)} \\
&\leq C\left(1 + |\nabla u(0)|\, r^{\frac{1}{1-p}}\right),
\end{aligned}$$

as desired. Observe that λ_0 is a universal constant. □

In accordance to [19, Theorem 3], Theorem 6 provides the aimed regularity along the set of critical points of u, $|\nabla u|^{-1}(0)$. In fact, when $|\nabla u(0)| \leq r^{\frac{1}{p-1}}$, Theorem 6 gives

$$\begin{aligned}
\sup_{x\in B_r} \Big| u(x) - [u(0) + \nabla u(0)\cdot x] \Big| &\leq \sup_{x\in B_r} |u(x)-u(0)| + |\nabla u(0)|\, r \\
&\leq (C+1) r^{p'}.
\end{aligned}$$

In the next section we show how Theorem 6 can be used in its full strength to yield $C^{p'}$-regularity at any point, regardless of the value of $|\nabla u|$; it will be a softer analysis.

5. Analysis on the set of non-degenerate points

We now analyze the oscillation decay around points where the gradient is large. Recall our ultimate goal is to show that

$$\sup_{x\in B_r} \Big| u(x) - [u(0) + \nabla u(0)\cdot x] \Big| \leq C\, r^{p'}, \quad \forall\, 0 < r \ll 1.$$

For large values of $|\nabla u|$, the operator is uniformly elliptic and hence stronger estimates are available. Assume then $|\nabla u(0)| > r^{\frac{1}{p-1}}$, define $\mu := |\nabla u(0)|^{p-1}$ and take

$$w(x) := \frac{u(\mu x) - u(0)}{\mu^{p'}}.$$

Clearly

$$w(0) = 0, \quad |\nabla w(0)| = 1 \quad \text{and} \quad -\Delta_p w = f(\mu x) \in L^\infty.$$

Moreover, from Theorem 6, it follows that

$$\sup_{x\in B_1} |w(x)| = \sup_{x\in B_\mu} \frac{|u(x)-u(0)|}{\mu^{p'}} \leq C,$$

since $\mu^{\frac{1}{p-1}} = |\nabla u(0)|$. From classical $C^{1,\alpha}$ regularity estimates, there exists a radius ρ_0, depending only on the data, such that

$$|\nabla w(x)| \geq \frac{1}{2}, \quad \forall\, x \in B_{\rho_0}.$$

This implies that, in B_{ρ_0}, w solves a uniformly elliptic equation. In particular, we have

$$w \in C^{1,\beta}(B_{\rho_0}), \quad \text{for some } \ \frac{1}{p-1} \leq \beta < 1.$$

As an immediate consequence,

$$\sup_{x\in B_r} \Big| w(x) - \nabla w(0)\cdot x \Big| \leq C\, r^{1+\beta}, \quad \forall\, 0 < r < \frac{\rho_0}{2}$$

which, in terms of u, reads

$$\sup_{x\in B_r} \Big| \frac{u(\mu x) - u(0)}{\mu^{p'}} - \mu^{1-p'}\nabla u(0)\cdot x \Big| \leq C\, r^{1+\beta}.$$

Since $p' \leq 1+\beta$, we conclude

$$\sup_{x\in B_r} \Big| u(x) - [u(0) + \nabla u(0)\cdot x] \Big| \leq C\, r^{p'}, \quad \forall\, 0 < r < \mu\frac{\rho_0}{2}.$$

Finally, for $\mu\frac{\rho_0}{2} \leq r < \mu$, we have

$$\begin{aligned}
\sup_{x\in B_r} \Big| u(x) - [u(0) + \nabla u(0)\cdot x] \Big| &\leq \sup_{x\in B_\mu} \Big| u(x) - [u(0) + \nabla u(0)\cdot x] \Big| \\
&\leq \sup_{x\in B_\mu} |u(x) - u(0)| + |\nabla u(0)|\,\mu \\
&\leq (C+1)\mu^{p'} \\
&\leq C\left(\frac{2r}{\rho_0}\right)^{p'} \\
&= Cr^{p'}.
\end{aligned}$$

In view of the reduction discussed at the end of Section 3, the proof of Theorem 1 is complete. □

Acknowledgments. The authors would like thank Erik Lindgren and Juan Manfredi for their valuable comments and suggestions and Wenhui Shi for bringing [5] to our attention.

This work was developed in the framework of the Brazilian Program *Ciência sem Fronteiras*. The second and third authors thank the hospitality of ICMC–Instituto de Ciências Matemáticas e de Computação, from Universidade de São Paulo in São Carlos, where this work was initiated. D.A. supported by CNPq. E.V.T. partially supported by CNPq and Fapesp. J.M.U. partially supported by the Centre for Mathematics of the University of Coimbra – UID/MAT/00324/2013, funded by the Portuguese Government through FCT/MCTES and co-funded by the European Regional Development Fund through the Partnership Agreement PT2020.

References

[1] M. Amaral and E.V. Teixeira, *Free transmission problems*, Comm. Math. Phys. 337 (2015), 1465–1489.

[2] D.J. Araújo, G. Ricarte and E.V. Teixeira, *Geometric gradient estimates for solutions to degenerate elliptic equations*, Calc. Var. Partial Differential Equations 53 (2015), 605–625.

[3] D.J. Araújo, E.V. Teixeira and J.M. Urbano, *Towards the $C^{p'}$ regularity conjecture in higher dimensions*, to appear in Int. Math. Res. Not. IMRN.

[4] K. Astala, T. Iwaniec and G. Martin, *Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane*, Princeton Mathematical Series 48, Princeton University Press, Princeton, NJ, 2009.

[5] A. Baernstein II and L.V. Kovalev, *On Hölder regularity for elliptic equations of non-divergence type in the plane*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 4 (2005), 295–317.

[6] B. Bojarski and T. Iwaniec, *p-harmonic equation and quasiregular mappings*, in: Partial differential equations (Warsaw, 1984), 25–38, Banach Center Publ.19, PWN, Warsaw, 1987.

[7] L. Caffarelli, *Interior a priori estimates for solutions of fully nonlinear equations*, Ann. of Math. (2) 130 (1989), 189–213.

[8] E. DiBenedetto, *$C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations*, Nonlinear Anal. TMA 7 (1983), 827–850.

[9] L.C. Evans and O. Savin, *$C^{1,\alpha}$ regularity for infinity harmonic functions in two dimensions*, Calc. Var. Partial Differential Equations 32 (2008), 325–347.

[10] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd edition, Grundlehren der Mathematischen Wissenschaften 224. Springer-Verlag, Berlin, 1983.

[11] C. Imbert and L. Silvestre, *$C^{1,\alpha}$ regularity of solutions of some degenerate fully non-linear elliptic equations*, Adv. Math. 233 (2013), 196–206.

[12] T. Iwaniec and J. Manfredi, *Regularity of p-harmonic functions on the plane*, Rev. Mat. Iberoamericana 5 (1989), 1–19.

[13] T. Kuusi and G. Mingione, *Universal potential estimates*, J. Funct. Anal. 262 (2012), 4205–4269.

[14] E. Lindgren and P. Lindqvist, *Regularity of the p-Poisson equation in the plane*, J. Anal. Math., to appear.

[15] J. Manfredi, *Private communication*, 2015.

[16] O. Savin, *C^1 regularity for infinity harmonic functions in two dimensions*, Arch. Rational Mech. Anal. 76 (2005), 351–361.

[17] E.V. Teixeira, *Sharp regularity for general Poisson equations with borderline sources*, J. Math. Pures Appl. (9) 99 (2013), 150–164.

[18] E.V. Teixeira, *Universal moduli of continuity for solutions to fully nonlinear elliptic equations*, Arch. Rational Mech. Anal. 211 (2014), 911–927.

[19] E.V. Teixeira, *Regularity for quasilinear equations on degenerate singular set*, Math. Ann. 358 (2014), 241–256.

[20] E.V. Teixeira, *Hessian continuity at degenerate points in nonvariational elliptic problems*, Int. Math. Res. Not. IMRN. 2015, no. 16, 6893–6906.

[21] E.V. Teixeira, *Geometric regularity estimates for elliptic equations*, Contemporary Mathematics/American Mathematical Society, Mathematical Congress of the Americas, 185–201, Contemp. Math., 656, Amer. Math. Soc., Providence, RI, 2016.

[22] E.V. Teixeira and J.M. Urbano, *A geometric tangential approach to sharp regularity for degenerate evolution equations*, Anal. PDE 7 (2014), 733–744.

[23] P. Tolksdorf, *Regularity for a more general class of quasilinear elliptic equations*, J. Differential Equations 51 (1984), 126–150.

[24] N. Uraltseva, *Degenerate quasilinear elliptic systems*, Zap. Na. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 7 (1968), 184–222.

Universidade Federal da Paraíba, Department of Mathematics, João Pessoa 58.051-900, Brazil
E-mail address: araujo@mat.ufpb.br

University of Central Florida, Department of Mathematics, Orlando, FL, 32828
E-mail address: eduardo.teixeira@ucf.edu

CMUC, Department of Mathematics, University of Coimbra, 3001-501 Coimbra, Portugal
E-mail address: jmurb@mat.uc.pt