REGULARITY FOR ANISOTROPIC FULLY NONLINEAR INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT. We consider fully nonlinear integro-differential equations governed by kernels that have different homogeneities in different directions. We prove a nonlocal version of the ABP estimate, a Harnack inequality and the interior $C^{1,\gamma}$ regularity, extending the results of [4] to the anisotropic case.

KEY WORDS: fully nonlinear integro-differential equations, regularity theory, ABP estimate, Harnack inequality, anisotropy.
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1. INTRODUCTION

In this work we develop a regularity theory for elliptic fully nonlinear integro-differential equations of the type

$$Iu(x) := \inf_{\alpha} \sup_{\beta} L_{\alpha\beta} u(x) = 0, \tag{1.1}$$

where

$$L_{\alpha\beta} u(x) := \int_{\mathbb{R}^n} \left(u(x+y) - u(x) - \nabla u(x) \cdot y \chi_{B_1}(y) \right) K_{\alpha\beta}(y)\, dy, \tag{1.2}$$

and the kernels $K_{\alpha\beta}$ are symmetric, $K_{\alpha\beta}(y) = K_{\alpha\beta}(-y)$, and satisfy the anisotropic bounds

$$\frac{\lambda c_\sigma}{\sum_{i=1}^n |y_i|^{n+\sigma_i}} \leq K_{\alpha\beta}(y) \leq \frac{\Lambda c_\sigma}{\sum_{i=1}^n |y_i|^{n+\sigma_i}}, \quad \forall y \in \mathbb{R}^n, \tag{1.3}$$

for $0 < \lambda \leq \Lambda$, $0 < \sigma_i < 2$, and $c_\sigma = c(\sigma_1, \ldots, \sigma_n) > 0$ a normalization constant.

Equations of type (1.1) appear extensively in the context of stochastic control problems (see [12]), namely in competitive stochastic games with two or more players, which are allowed to choose from different strategies at every step in order to maximize the expected value of some function at the first exit point of a domain. Integral operators like (1.2) correspond to purely jump processes when diffusion and drift are neglected. The anisotropic setting we consider is bound to be of use in the context of financial mathematics, namely for Black-Scholes models that use certain jump-type processes instead of diffusions (cf. [11]).

The isotropic version of the problem, with (1.3) replaced by

$$\frac{\lambda\left(2-\sigma\right)}{\left|y\right|^{n+\sigma}}\leq K_{\alpha\beta}\left(y\right)\leq\frac{\Lambda\left(2-\sigma\right)}{\left|y\right|^{n+\sigma}},\quad\forall y\in\mathbb{R}^{n},\tag{1.4}$$

for $0<\sigma<2$, is studied in [4], exploring the analogy between ellipticity and the condition

$$M_{\mathcal{L}}^{-}v\left(x\right)\leq I\left(u+v\right)\left(x\right)-Iv\left(x\right)\leq M_{\mathcal{L}}^{+}v\left(x\right),\qquad\forall y\in\mathbb{R}^{n}.$$

Here, $\mathcal{L}$ is the class of operators $L_{\alpha\beta}$ whose kernels satisfy (1.4) and the operators

$$M_{\mathcal{L}}^{-}u\left(x\right):=\inf_{L\in\mathcal{L}}Lu\left(x\right)\quad\text{ and }\quad M_{\mathcal{L}}^{+}u\left(x\right):=\sup_{L\in\mathcal{L}}Lu\left(x\right)$$

correspond to the extremal Pucci operators in the theory of elliptic equations of second order. The non-variational approach to regularity theory for (sub and super) viscosity solutions of the isotropic version of equation (1.1) is a nonlocal version of the strategy used in [5] for second order fully nonlinear elliptic equations.

In the classical non-variational approach, the crucial step towards a regularity theory is the celebrated Aleksandrov–Bakel'man–Pucci estimate (ABP estimate, in short), which amounts to the bound

$$\sup_{B_1}u\leq C\left(n\right)\left(\int_{\{\Gamma=u\}\cap B_1}\left(f^{+}\right)^{n}\right)^{1/n},\tag{1.5}$$

for any viscosity subsolution u of the maximal Pucci equation with right-hand side $(-f)$ taking non-positive values outside the unit ball B_1 . Here, Γ is the concave envelope of u in B_3. The technical advantage of the ABP estimate stems from relating a pointwise estimate with an estimate in measure. More precisely, $u\left(0\right)\geq 1$ implies

$$1\leq C\|f\|_{L^{\infty}}\left|\{\Gamma=u\}\cap B_1\right|^{\frac{1}{n}}\leq C\|f\|_{L^{\infty}}\left|\{u\geq 0\}\cap B_1\right|^{\frac{1}{n}}.$$

In the nonlocal setting, the ABP estimate must be modified in face of the structural differences of the operator. In the isotropic case of [4], we have to replace (1.5) by the following two assertions, which still give access to the regularity theory:

i. u stays quadratically close to the tangent plane to Γ in a large portion of a neighbourhood around a contact point:

$$\left|\left\{y\in 8\sqrt{n}Q_j:u\left(y\right)\geq\Gamma\left(y\right)-\left(\max_{\overline{Q}_j}f^{+}\right)d_j^2\right\}\right|\geq\varsigma\left|Q_j\right|;$$

ii. Γ has quadratic growth and therefore

$$\left|\nabla\Gamma\left(\overline{Q}_j\right)\right|\leq C\left(\max_{\overline{Q}_j}f^{+}\right)^{n}\left|\overline{Q}_j\right|,$$

for a finite family of disjoint open cubes $\{Q_j\}$ with diameters $d_j \leq \frac{1}{8\sqrt{n}}$ such that

$$\{u = \Gamma\} \subset \bigcup_j \overline{Q}_j \quad \text{and} \quad \{u = \Gamma\} \cap \overline{Q}_j \neq \emptyset,$$

where $\nabla\Gamma$ stands for any element of the superdifferential of Γ, and the constants $\varsigma > 0$ and $C > 0$ only depend on dimension and the ellipticity constants.

Then, using i. and ii., we get from $u(0) \geq 1$,

$$\begin{aligned} 1 &\leq C\|f\|_{L^\infty} \left| \left\{ u \geq \Gamma - \frac{1}{64n}\|f\|_{L^\infty} \right\} \cap B_1 \right|^{\frac{1}{n}} \\ &\leq C\|f\|_{L^\infty} \left| \left\{ u \geq -\frac{1}{64n}\|f\|_{L^\infty} \right\} \cap B_1 \right|^{\frac{1}{n}}, \end{aligned} \tag{1.6}$$

which is still enough to complete a regularity theory. A covering lemma by open cubes Q_j that satisfy assertions i. and ii. is crucial in obtaining (1.6) in the nonlocal case, for which the classical inequality (1.5) does not hold (but see [8] for a nearly classical ABP-type estimate in this context).

To treat the anisotropic case we use the same strategy as in [4] but the anisotropic geometry driven by the kernels $K_{\alpha\beta}$ requires a refinement of the techniques (see also [9], where the bounds on the kernels can degenerate to be zero in some directions). We comment in the sequel on the main difficulties we came across and how to overcome them.

(1) *Assertion i.* At this step of the analysis, the challenge is to find the suitable geometry of the neighbourhoods of the contact points within which there is a (large) portion where a subsolution u stays quadratically close to the tangent plane to Γ and such that, in smaller neighbourhoods (with the same geometry), the concave envelope Γ has quadratic growth. A careful analysis of the anisotropic nonlocal version of inequality $M^+_{\mathcal{L}} u \geq -f$ satisfied by u at the contact points allows us to conclude that the appropriate geometry is the geometry determined by the level sets of the kernels $K_{\alpha\beta}$:

$$\Theta_r(x) := \left\{ (y_1, \ldots, y_n) \in \mathbb{R}^n : \sum_{i=1}^n |y_i - x_i|^{n+\sigma_i} < r \right\},$$

for $x \in \{\Gamma = u\} \cap B_1$. It is also here that we choose the appropriate normalisation constant:

$$c_\sigma = -1 + \frac{3}{n + \sigma_{\max}} + \sum_{\sigma_j \neq \sigma_{\max}} \frac{1}{n + \sigma_j}.$$

(2) *Assertion ii.* Given a positive number $h > 0$, a fine analysis allows us to conclude that if a concave function, for instance the concave envelope Γ, remains below its tangent plane translated by $-h$ in a (universally sufficiently small in measure) portion of a (sufficiently

large) annulus of the unit ball, for example $B_1 \setminus B_{\frac{1}{2}}$, then $\Gamma + h$ is above its tangent plane in the interior ball of the annulus, in this case $B_{\frac{1}{2}}$. In the anisotropic case, the difficulty is to extend this argument to the anisotropic balls Θ_r. Through the anisotropic transformation $T : \mathbb{R}^n \to \mathbb{R}^n$, defined by

$$Te_i := r^{\frac{1}{n+\sigma_i}} e_i,$$

and taking into account that the composition of a concave function with an affine function is still concave, we extend this fine analysis to ellipses. We then use the previous step and the symmetry of the anisotropic balls Θ_r with respect to x to conclude that Γ grows quadratically in such anisotropic balls.

(3) *Covering Lemma.* In [4], the Besicovitch Covering Lemma is used. Our covering is naturally made of n-dimensional rectangles $\mathcal{R}_j$ and we invoke a covering lemma from [6]. We stress that this covering lemma allows for a change of direction in the homogeneity degrees σ_i, but each σ_i must remain constant. Degenerate spatial changes of the homogeneities σ_i, arising for example in the context of spherical operators or other special weights, would require the use of a more general covering lemma like the one in [7]. In adapting our results to that case, the main difficulty lies in the use of the barriers and we plan to address this issue in a forthcoming paper.

With this at hand, we then use the natural anisotropic scaling to build an adequate barrier function and, together with the nonlocal anisotropic version of the ABP estimate, we prove a lemma that links a pointwise estimate with an estimate in measure, Lemma 5.1. This is the fundamental step towards a regularity theory. The iteration of Lemma 5.1 implies the decay of the distribution function $\lambda_u := |\{u > t\}|$ and the tool that makes this iteration possible is the so called Calderón -Zygmund decomposition. Since our scaling is anisotropic we need a Calderón -Zygmund decomposition for n-dimensional rectangles generated by our scaling. A fundamental device we use for that decomposition is the Lebesgue differentiation theorem for n-dimensional rectangles that satisfy the condition of Caffarelli-Calderón in [6]. Then we prove the Harnack inequality and, as a consequence, we obtain the interior C^γ regularity for a solution u of equation (1.1) and, under additional assumptions on the kernels $K_{\alpha\beta}$, interior $C^{1,\gamma}$ estimates.

We finally observe that the power of the estimates obtained in [4] is revealed as $\sigma \to 2$. In fact, since the estimates remain uniform in the degree σ, it was possible to obtain an interesting relation between the theory of integro-differential equations and that of elliptic differential equations through the natural limit:

$$\lim_{\sigma \to 2} \int_{\mathbb{R}^n} \frac{c_n (2-\sigma)}{|y|^{n+\sigma}} \left(u(x+y) + u(x-y) - 2u(x) \right) dy$$

$$= \lim_{\sigma \to 2} -(-\Delta)^{\frac{\sigma}{2}} u(x) = \Delta u(x),$$

where $c_n > 0$ is a constant. This contrasts with previous results in the literature on Harnack inequalities and Hölder estimates for integro-differential equations, with either analytical proofs [10] or probabilistic proofs [1, 2, 3, 13], whose estimates blow up as the order of the equation approaches 2.

We emphasize that our estimates are also stable as $\sigma_{\min} := \min\{\sigma_1, \dots, \sigma_n\}$ approaches 2. An heuristic analysis of the limiting behaviour of the operator against a second order polynomial seems to indicate that we get, as in the homogeneous case, a second order fully nonlinear PDE. In order to deduce what would be the limit of $L^{\sigma}_{\alpha\beta} u$ as $\sigma_{\min} \to 2$, it is enough to evaluate $L^2_{\alpha\beta} u = a_{ij} D_{ij} u$ on quadratic polynomials, for example on a function $u(x) \sim x_1^2$ near the origin. For instance, if the kernel $K_{\alpha\beta}$ behaves near the origin like

$$K_{\alpha\beta}(y) = \frac{c_\sigma}{\sum_{i=1}^n |y_i|^{n+\sigma_i}},$$

then

$$L_{\alpha\beta} u(0) = c_\sigma \int_{\mathbb{R}^n} \frac{u(y_1)}{|y_1|^{n+\sigma_1}} \cdot \frac{dy}{1 + \sum_{i=2}^n \frac{|y_i|^{n+\sigma_i}}{|y_1|^{n+\sigma_1}}}.$$

Denoting $\theta_i = 2 - \sigma_i$ and integrating on y_j, $j \neq 1$, we obtain an expression of the form

$$L_{\alpha\beta} u(0) = c_\sigma \int_{\mathbb{R}^n} \frac{u(y_1)}{|y_1|^{3-\gamma(\theta)}},$$

where, as $\theta \to 0$, γ behaves at first order like

$$\gamma(\theta) = \frac{3\theta_1 + \sum_{j=2}^n \theta_j}{n+2}.$$

For $L_{\alpha\beta}$ to converge as $\theta \to 0$, we must choose $c_\sigma \sim \sum \theta_j$. In that case, I^σ will converge to a standard Isaac's equation.

The paper is organised as follows. In section 2 we gather all the necessary tools for our analysis: the notion of viscosity solution for the problem (1.1), the extremal operators of Pucci type associated with the family of kernels $K_{\alpha\beta}$ and some notation. Section 3, where the nonlocal ABP estimate for a solution u of equation (1.1) is obtained, is the most important of the paper. Sections 4 and 5 are devoted to the proof of the Harnack inequality and its consequences.

2. Viscosity solutions and extremal operators

In this section we collect the technical properties of the operator I that we will use throughout the paper. Since $K_{\alpha\beta}$ is symmetric and positive, we have

$$L_{\alpha\beta} u(x) = PV \int_{\mathbb{R}^n} (u(x+y) - u(x)) K_{\alpha\beta}(y)\, dy$$

and

$$L_{\alpha\beta}u\left(x\right)=\frac{1}{2}\int_{\mathbb{R}^n}\left(u\left(x+y\right)-u\left(x-y\right)-2\left(x\right)\right)K_{\alpha\beta}\left(y\right)dy.$$

For convenience of notation, we denote

$$\delta\left(u,x,y\right):=u\left(x+y\right)+u\left(x-y\right)-2u\left(x\right)$$

and we can write

$$L_{\alpha\beta}=\int_{\mathbb{R}^n}\delta\left(u,x,y\right)K_{\alpha\beta}\left(y\right)dy,$$

for some kernel $K_{\alpha\beta}$.

We now define the adequate class of test functions for our operators.

Definition 2.1. A function ϕ is said to be $C^{1,1}$ at the point x, and we write $\phi\in C^{1,1}\left(x\right)$, if there is a vector $v\in\mathbb{R}^n$ and numbers $M,\eta_0>0$ such that

$$|\phi\left(x+y\right)-\phi\left(x\right)-v\cdot y|\leq M|y|^2,$$

for $|x|<\eta_0$. We say that a function ϕ is $C^{1,1}$ in a set Ω, and we denote $\phi\in C^{1,1}\left(\Omega\right)$, if the previous holds at every point, with a uniform constant M.

Remark 2.2. Let $u\in C^{1,1}\left(x\right)\cap L^{\infty}\left(\mathbb{R}^n\right)$ and $M>0$ and $\eta_0>0$ be as in definition 2.1. Then we estimate

$$L_{\alpha\beta}u\left(x\right)=PV\int_{\mathbb{R}^n}\delta\left(u,x,y\right)K_{\alpha\beta}\left(y\right)dy$$

$$\leq\left[4c_\sigma\Lambda|u|_{L^\infty(\mathbb{R}^n)}\int_{\mathbb{R}^n\setminus B_{\eta_0}}\frac{1}{\sum_{i=1}^n|y_i|^{n+\sigma_i}}dy+2Mc_\sigma\Lambda\int_{B_{\eta_0}}\frac{|y|^2}{\sum_{i=1}^n|y_i|^{n+\sigma_i}}dy\right]$$

$$\leq\left[4c_\sigma\Lambda|u|_{L^\infty}2^{\frac{n+2}{2}}\eta_0^{-(\sigma_{\max}-\sigma_{\min})}\int_{\mathbb{R}^n\setminus B_{\eta_0}}\frac{1}{|y|^{n+\sigma_{\min}}}dy+C\left(n,\Lambda,M,\eta_0\right)\right]$$

$$=\left[c_\sigma C\left(n,\Lambda,|u|_{L^\infty}\right)\frac{\eta_0^{-\sigma_{\max}}}{\sigma_{\min}}+C\left(n,\Lambda,M,\eta_0\right)\right]$$

and conclude that $Iu\left(x\right)\in\mathbb{R}$.

We now introduce the notion of viscosity subsolution (and supersolution) u in a domain Ω, with C^2 test functions that touch u from above or from below. We stress that u is allowed to have arbitrary discontinuities outside of Ω.

Definition 2.3. Let f be a bounded and continuous function in $\mathbb{R}^n$. A function $u:\mathbb{R}^n\to\mathbb{R}$, upper (lower) semicontinuous in $\overline{\Omega}$, is said to be a subsolution (supersolution) to equation $Iu=f$, and we write $Iu\geq f$ $(Iu\leq f)$, if whenever the following happen:

(1) $x_0\in\Omega$ is any point in Ω;

(2) $B_r\left(x_0\right)\subset\Omega$, for some $r>0$;

(3) $\phi \in C^2\left(\overline{B_r(x_0)}\right)$;

(4) $\phi(x_0) = u(x_0)$;

(5) $\phi(y) > u(y)$ $(\phi(y) < u(y))$ for every $y \in B_r(x_0) \setminus \{x_0\}$;

then, if we let

$$v := \begin{cases} \phi, & \text{in} \quad B_r(x_0) \\ u & \text{in} \quad \mathbb{R}^n \setminus B_r(x_0), \end{cases}$$

we have $Iv(x_0) \geq f(x_0)$ $(Iv(x_0) \leq f(x_0))$.

Remark 2.4. Functions which are $C^{1,1}$ at a contact point x can be used as test functions in the definition of viscosity solution (see Lemma 4.3 in [4]).

Next, we define the class of linear integro-differential operators that will be a fundamental tool for the regularity analysis. Let $\mathcal{L}_0$ be the collection of linear operators $L_{\alpha\beta}$. We define the maximal and minimal operator with respect to $\mathcal{L}_0$ as

$$M^+u(x) := \sup_{L \in \mathcal{L}_0} Lu(x)$$

and

$$M^-u(x) := \inf_{L \in \mathcal{L}_0} Lu(x).$$

By definition, if $M^+u(x) < \infty$ and $M^-u(x) < \infty$, we have the simple form

$$M^+u(x) = c_\sigma \int_{\mathbb{R}^n} \frac{\Lambda\delta^+ - \lambda\delta^-}{\sum_{i=1}^n |y_i|^{n+\sigma_i}} dy$$

and

$$M^-u(x) = c_\sigma \int_{\mathbb{R}^n} \frac{\lambda\delta^+ - \Lambda\delta^-}{\sum_{i=1}^n |y_i|^{n+\sigma_i}} dy.$$

Remark 2.5. As in [4], we could consider equation (1.1) for a more general class $\mathcal{L}$ satisfying

$$\int_{\mathbb{R}^n} \frac{|y|^2}{1+|y|^2} K(y)\, dy < \infty,$$

where $K(y) := \sup_{\alpha \in \mathcal{L}} K_\alpha(y)$ and $K_\alpha(y) = K_\alpha(-y)$.

The proofs of the results that we now present can be found in the sections 3, 4 and 5 of [4]. The first result ensures that if u can be touched from above, at a point x, with a paraboloid then $Iu(x)$ can be evaluated classically.

Lemma 2.6. *If we have a subsolution, $Iu \geq f$ in Ω, and ϕ is a C^2 function that touches u from above at a point $x \in \Omega$, then $Iu(x)$ is defined in the classical sense and $Iu(x) \geq f(x)$.*

Another important property of I is the continuity of $I\phi$ in Ω if $\phi \in C^{1,1}(\Omega)$.

Lemma 2.7. *Let v be a bounded function in $\mathbb{R}^n$ and $C^{1,1}$ in some open set Ω. Then Iv is continuous in Ω.*

The next lemma allows us to conclude that the difference between a subsolution of the maximal operator M^+ and a supersolution of the minimal operator M^- is a subsolution of the maximal operator.

Lemma 2.8. *Let Ω be a bounded open set and u and v be two bounded functions in $\mathbb{R}^n$ such that*

(1) *u is upper-semicontinuous and v is lower-semicontinuous in $\overline{\Omega}$;*

(2) *$Iu \geq f$ and $Iv \leq g$ in the viscosity sense in Ω for two continuous functions f and g.*

Then

$$M^+(u-v) \geq f-g \quad \text{in} \quad \Omega$$

in the viscosity sense.

We conclude this section introducing some notation that will be instrumental in the sequel. Given $r, s > 0$ and $x \in \mathbb{R}^n$, we will denote

$$E_{r,s}(x) := \left\{ (y_1, \ldots, y_n) \in \mathbb{R}^n : \sum_{i=1}^{n} \frac{(y_i - x_i)^2}{r^{\frac{2}{n+\sigma_i}}} < s^2 \right\}$$

and

$$R_{r,s}(x) := \left\{ (y_1, \ldots, y_n) \in \mathbb{R}^n : |y_i - x_i| < s^{\frac{1}{n+\sigma_{\min}}} r^{\frac{1}{n+\sigma_i}} \right\}.$$

Given the box $R_{r,s}$, we define the corresponding box $\tilde{R}_{r,s}$ by

$$\tilde{R}_{r,s}(x) := \left\{ (y_1, \ldots, y_n) \in \mathbb{R}^n : |y_i - x_i| < (sr)^{\frac{1}{n+\sigma_i}} \right\}.$$

If $\sigma_{\min} := \min\{\sigma_1, \ldots, \sigma_n\}$ we define

$$i_{\min} := \min\{j : \sigma_{\min} = \sigma_j\}.$$

Remark 2.9. Let $r > 0$. Hereafter, we will use the following relations:

(1) $E_{r,\frac{1}{2}} \subset \Theta_r \subset E_{r,\sqrt{n}}$;

(2) $\Theta_{2^{-\mathfrak{C}}r} \subset E_{r,\frac{1}{8}}$, for some natural number $\mathfrak{C} = \mathfrak{C}(n) > 0$;

(3) $R_{r,s} \subset \tilde{R}_{r,s}$, if $0 < s < 1$.

3. Nonlocal anisotropic ABP estimate

Let u be a non positive function outside the ball B_1. We define the concave envelope of u by

$$\Gamma(x) := \begin{cases} \min\{p(x) : \text{ for all planes } p \geq u^+ \text{ in } B_3\}, & \text{in} \quad B_3 \\ 0 & \text{in} \quad \mathbb{R}^n \setminus B_3. \end{cases}$$

Lemma 3.1. *Let $u \leq 0$ in $\mathbb{R}^n \setminus B_1$ and Γ be its concave envelope. Suppose $M^+u(x) \geq -f(x)$ in B_1. Let $\rho_0 = \rho_0(n) > 0$,*

$$r_k := \rho_0 2^{-\frac{1}{q_{\max}}} 2^{-\mathfrak{C}(n+\sigma_{\min})k},$$

where

$$q_i := -1 + \frac{3}{n+\sigma_i} + \sum_{j \neq i} \frac{1}{n+\sigma_j}$$

and $q_{\max} := \max\{q_1, \ldots, q_n\}$. Given $M > 0$, we define

$$W_k(x) := \Theta_{r_k} \setminus \Theta_{r_{k+1}} \cap$$

$$\cap \left\{ y : u(x+y) < u(x) + \langle y, \nabla\Gamma(x)\rangle - M \inf_{z \in \Theta_{r_k} \setminus \Theta_{r_{k+1}}} \langle Az, z\rangle \right\},$$

where the matrix $A = (a_{ij})$ is defined by

$$a_{ij} := \begin{cases} 1, & \text{if } \ i = j = i_{\min} \\ 0, & \text{if } \ i \neq j \\ 2^{\left(-\frac{1}{n+\sigma_{\min}} + \frac{1}{n+\sigma_j}\right)\frac{2}{q_{\max}}}, & \text{if } \ i = j \neq i_{\min}. \end{cases}$$

Then there exists a constant $C_0 > 0$, depending only on n, λ (but not σ_i), such that, for any $x \in \{u = \Gamma\}$ and any $M > 0$, there is a k such that

$$|W_k(x)| \leq C_0 \frac{f(x)}{M} \left|\Theta_{r_k} \setminus \Theta_{r_{k+1}}\right|. \tag{3.1}$$

Proof. Notice that u is touched by the plane

$$\Gamma(x) + \langle y - x, \nabla\Gamma(x)\rangle$$

from above at x. Then, from Lemma 2.6, $M^+u(x)$ is defined classically and we have

$$M^+u(x) = c_\sigma \int_{\mathbb{R}^n} \frac{\Lambda\delta^+ - \lambda\delta^-}{\sum_{i=1}^n |y_i|^{n+\sigma_i}} dy. \tag{3.2}$$

We will show that

$$\delta(y) := \delta(u, x, y) = u(x+y) + u(x-y) - 2u(x) \leq 0. \tag{3.3}$$

In fact, if both $x - y \in B_3$ and $x + y \in B_3$ then we conclude that $\delta(y) \leq 0$, since $u(x) = \Gamma(x) = p(x)$, for some plane p that remains above u in the whole ball B_3. Moreover, if either $x - y \notin B_3$ or $x + y \notin B_3$, then both $x - y$ and $x + y$ are not in B_1, and thus $u(x+y) \leq 0$ and $u(x-y) \leq 0$. Therefore, in any case the inequality (3.3) is proved. Combining (3.2) and (3.3), we find

$$\begin{aligned} -f(x) &\leq M^+u(x) \\ &= c_\sigma \int_{\Theta_{r_0}} \frac{-\lambda\delta^-}{\sum_{i=1}^n |y_i|^{n+\sigma_i}} dy, \end{aligned} \tag{3.4}$$

where $r_0 = \rho_0 2^{-\frac{1}{q_{\max}}}$. Since $x \in \{u = \Gamma\}$, we would like to emphasize that $y \in W_k(x)$ implies $-y \in W_k(x)$. Thus, we find

$$W_k(x) \subset \Theta_{r_k} \setminus \Theta_{r_{k+1}} \cap \left\{ y : -\delta(y) > 2M \inf_{z \in \Theta_{r_k} \setminus \Theta_{r_{k+1}}} \langle Az, z \rangle \right\}. \tag{3.5}$$

Using (3.4), we estimate

$$\begin{aligned} f(x) &\geq c(n,\lambda) \left[c_\sigma \sum_{k=1}^{\infty} \int_{\Theta_{r_k} \setminus \Theta_{r_{k+1}}} \frac{\delta^-}{\sum_{i=1}^n |y_i|^{n+\sigma_i}} dy \right] \\ &\geq c(n,\lambda) \sum_{k=1}^{\infty} \left[c_\sigma r_k^{-1} \int_{W_k} \delta^- dy \right]. \end{aligned} \tag{3.6}$$

Let us assume by contradiction that (3.1) is not valid. Then, using (3.5) and (3.6), we obtain

$$\begin{aligned} f(x) \geq c(n,\lambda) &\left[c_\sigma \sum_{k=1}^{\infty} \left(M 2^{\left(-\frac{2}{n+\sigma_{\min}}\right)\frac{1}{q_{\max}}} \sum_{i=1}^{n} 2^{-\frac{2(n+\sigma_{\min})}{n+\sigma_i}k} \right) \right. \\ &\left. \qquad \frac{C_0 f(x) r_k^{-1} \left| \Theta_{r_k} \setminus \Theta_{r_{k+1}} \right|}{M} \right] \\ \geq c_1 C_0 f(x) &\left[c_\sigma 2^{\left(-\frac{2}{n+\sigma_{\min}}\right)\frac{1}{q_{\max}}} \sum_{k=1}^{\infty} \sum_{i=1}^{n} \left(2^{-\frac{2(n+\sigma_{\min})}{n+\sigma_i}k} \left(r_k^{-1} \prod_{j=1}^{n} r_k^{\frac{1}{n+\sigma_j}} \right) \right) \right] \\ &= 2^{-1} c_2 C_0 f(x) \left[c_\sigma \sum_{i=1}^{n} \left(\sum_{k=1}^{\infty} 2^{-\mathfrak{C}(n+\sigma_{\min}) q_i k} \right) \right]. \end{aligned}$$

Then, we get

$$\begin{aligned} f(x) &\geq c_3 C_0 f(x) \left[\sum_{i=1}^{n} \left(c_\sigma \sum_{i=1}^{\infty} 2^{-\mathfrak{C}(n+\sigma_{\min}) q_i k} \right) \right] \\ &= c_3 C_0 f(x) \sum_{i=1}^{n} \frac{c_\sigma}{1 - 2^{-\mathfrak{C}(n+\sigma_{\min}) q_i}} \\ &\geq \frac{c_3 C_0 c_\sigma f(x)}{1 - 2^{-\mathfrak{C}(n+\sigma_{\min}) c_\sigma}}. \end{aligned}$$

Finally, since $\frac{c_\sigma}{1-2^{-\mathfrak{C}(n+\sigma_{\min})c_\sigma}}$ is bounded away from zero, for all $\sigma_i \in (0,2)$, we find

$$f(x) \geq c_4(n,\lambda) C_0 f(x),$$

which is a contradiction if C_0 is chosen large enough. □

Remark 3.2. In the proof of Lemma 3.1 we have used the matrix $A := (a_{ij})$ to control the term $2^{-\frac{1}{q_{\max}}}$, which can degenerate. This term corresponds to the factor $2^{-\frac{1}{2-\sigma}}$ in the isotropic nonlocal ABP estimate in [4]. We also emphasise that the matrix A is diagonal, has norm one and, if $\sigma_i = \sigma$, we obtain the matrix for the isotropic case $A = Id$.

The following result is a direct consequence of the arguments used in the proof of [4, Lemma 8.4].

Lemma 3.3. *Let Γ be a concave function in B_1 and $v \in \mathbb{R}^n$. Assume that, for a small $\varepsilon > 0$,*

$$\left|\left(B_1 \setminus B_{\frac{1}{2}}\right) \cap \{y : \Gamma(y) < \Gamma(0) + \langle T(y), v\rangle - h\}\right| \leq \varepsilon \left|B_1 \setminus B_{\frac{1}{2}}\right|,$$

where $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map. Then

$$\Gamma(y) \geq \Gamma(0) + \langle T(y), v\rangle - h$$

in the whole ball $B_{\frac{1}{2}}$.

Proof. Let $y \in B_{\frac{1}{2}}$. There exist $B_{\frac{1}{2}}(y_1) \subset B_1 \setminus B_{1/2}$ and $B_{\frac{1}{2}}(y_2) \subset B_1 \setminus B_{1/2}$ such that

$$L\left(B_{\frac{1}{2}}(y_1)\right) = B_{\frac{1}{2}}(y_2),$$

where $L : B_{\frac{1}{2}}(y_1) \to B_{\frac{1}{2}}(y_2)$ is the linear map

$$L(z) = 2y - z.$$

Geometrically, the balls $B_{\frac{1}{2}}(y_1)$ and $B_{\frac{1}{2}}(y_2)$ are symmetrical with respect to y. Then, if $\varepsilon > 0$ is sufficiently small, there will be two points $z_1 \in B_{\frac{1}{2}}(y_1)$ and $z_2 \in B_{\frac{1}{2}}(y_2)$ such that

(1) $y = \dfrac{z_1 + z_2}{2}$;

(2) $\Gamma(z_1) \geq \Gamma(0) + \langle T(z_1), v\rangle - h$;

(3) $\Gamma(z_2) \geq \Gamma(0) + \langle T(z_2), v\rangle - h$.

Hence, since T and $\langle \cdot, v\rangle$ are linear maps and Γ is a concave function, we obtain

$$\Gamma(y) \geq \Gamma(0) + \langle T(y), v\rangle - h.$$

□

Using Lemma 3.3, we will prove the version of Lemma 8.4 in [4] for our problem.

Lemma 3.4. *Let $r > 0$ and Γ be a concave function in $E_{r,\frac{1}{2}}$. There exists $\varepsilon_0 > 0$ such that if*

$$\left|E_{r,\frac{1}{2}} \setminus E_{r,\frac{1}{4}} \cap \{y : \Gamma(y) < \Gamma(0) + \langle y, \nabla\Gamma(0)\rangle - h\}\right| \leq \varepsilon \left|E_{r,\frac{1}{2}} \setminus E_{r,\frac{1}{4}}\right|,$$

for $0 < \varepsilon \leq \varepsilon_0$, then

$$\Gamma(y) \geq \Gamma(0) + \langle y, \nabla\Gamma(0)\rangle - h$$

in the whole set $E_{r,\frac{1}{4}}$.

Proof. Let $T:\mathbb{R}^n\to\mathbb{R}^n$ be the linear map defined by

$$Te_i=\frac{r^{\frac{1}{n+\sigma_i}}}{2}e_i,$$

where e_i denotes the i-th vector of the canonical basis of $\mathbb{R}^n$. If

$$A:=\left(B_1\setminus B_{\frac12}\right)\cap\left\{y:\tilde\Gamma\left(y\right)<\tilde\Gamma\left(0\right)+\left\langle T\left(y\right),\nabla\Gamma\left(0\right)\right\rangle-h\right\}$$

and

$$D:=E_{r,\frac12}\setminus E_{r,\frac14}\cap\left\{y:\Gamma\left(y\right)<\Gamma\left(0\right)+\left\langle y,\nabla\Gamma\left(0\right)\right\rangle-h\right\},$$

we have

$$A=T^{-1}\left(D\right),$$

where $\tilde\Gamma\left(x\right):=\Gamma\left(T\left(x\right)\right)$. Moreover,

$$B_1\setminus B_{\frac12}=T^{-1}\left(E_{r,\frac12}\setminus E_{r,\frac14}\right)\quad\text{and}\quad B_{\frac12}=T^{-1}\left(E_{r,\frac14}\right).$$

Then, taking into account that $\tilde\Gamma$ is concave, the lemma follows from Lemma 3.3. □

Corollary 3.5. *Let $\varepsilon_0>0$ be as in Lemma 3.4. Given $0<\varepsilon\le\varepsilon_0$, there exists a constant $C\left(n,\lambda,\varepsilon\right)>0$ such that for any function u satisfying the same hypothesis as in Lemma 3.1, there exist $r\in\left(0,\rho_0 2^{-\frac{1}{q_{\max}}}\right)$ and $k=k\left(x\right)$ such that*

$$\left|\Theta_r\setminus\Theta_{sr}\cap\left\{y:u\left(x+y\right)<u\left(x\right)+\left\langle y,\nabla\Gamma\left(x\right)\right\rangle-Cf\left(x\right)\sum_{i=1}^{n}r^{\frac{2}{n+\sigma_i}}\right\}\right|$$
$$\le\varepsilon_1\left|\Theta_r\setminus\Theta_{sr}\right| \tag{3.7}$$

and

$$\left|\nabla\Gamma\left(R_{a,s^{k+1}}\left(x\right)\right)\right|\le Cf\left(x\right)^n\left|R_{a,s^{k+1}}\left(x\right)\right|,$$

where $r=\rho_0 2^{-\frac{1}{q_{\max}}}2^{-\mathfrak{C}(n+\sigma_{\min})k}$, $a=\rho_0 2^{-\frac{1}{q_{\max}}}$ and $s=2^{-\mathfrak{C}(n+\sigma_{\min})}$.

Proof. Taking $M=\frac{C_0}{\varepsilon}f\left(x\right)$ in Lemma 3.1, we obtain (3.7) with $C_1:=\frac{C_0}{\varepsilon}$. Moreover, since $u\left(x\right)=\Gamma\left(x\right)$ and $u\left(x+y\right)\le\Gamma\left(x+y\right)$, for $y\in E_{r,\frac12}$, we have

$$E_{r,\frac12}\setminus E_{r,\frac14}\cap\left\{y:\Gamma\left(x+y\right)<u\left(x\right)+\left\langle y,\nabla\Gamma\left(x\right)\right\rangle-C_1f\left(x\right)\inf_{z\in\Theta_r\setminus\Theta_{sr}}\left\langle Az,z\right\rangle\right\}$$
$$\subset W_r\left(x\right)$$

where

$$W_r\left(x\right):=\Theta_r\setminus\Theta_{sr}\cap$$
$$\cap\left\{y:u\left(x+y\right)<u\left(x\right)+\left\langle y,\nabla\Gamma\left(x\right)\right\rangle-C_1f\left(x\right)\inf_{z\in\Theta_r\setminus\Theta_{sr}}\left\langle Az,z\right\rangle\right\}.$$

Then, from Lemma 3.4 and the concavity of Γ, we find

$$0\le F\left(y\right)\le 2C_1f\left(x\right)\inf_{z\in\Theta_r\setminus\Theta_{sr}}\left\langle Az,z\right\rangle\quad\text{in }E_{r,\frac14},$$

where

$$F(y) := \Gamma(x+y) - \Gamma(x) - \langle y, \nabla\Gamma(x)\rangle + C_1 f(x) \inf_{z\in\Theta_r\setminus\Theta_{sr}} \langle Az, z\rangle.$$

Notice that

$$\nabla F(x+y) = \nabla\Gamma(x+y) - \nabla\Gamma(x).$$

Then, since F is concave, we obtain

$$\begin{aligned} |\nabla\Gamma(x+y) - \nabla\Gamma(x)| &\le \frac{\|F\|_{L^\infty\left(E_{r,\frac{1}{4}}\right)}}{\operatorname{dist}\left(E_{r,\frac{1}{4}}, E_{r,\frac{1}{8}}\right)} \\ &\le \frac{C_1 f(x) \inf_{z\in\Theta_r\setminus\Theta_{sr}} \langle Az, z\rangle}{\operatorname{dist}\left(E_{r,\frac{1}{4}}, E_{r,\frac{1}{8}}\right)} \\ &\le C_2 f(x)\, r^{\frac{1}{n+\sigma_{\min}}}. \end{aligned}$$

Thus, we have

$$\nabla\Gamma\left(E_{r,\frac{1}{8}}\right) \subset B_{C_2 f(x) r^{\frac{1}{n+\sigma_{\min}}}}(\nabla\Gamma(x))$$

and obtain

$$\left|\nabla\Gamma\left(R_{a,s^{k+1}}\right)\right| \le \left|\nabla\Gamma\left(E_{sr,\frac{1}{8}}\right)\right| \le C_3 f(x)^n \left|R_{a,s^{k+1}}\right|.$$

Finally, taking $C = \max\{C_1, C_3\}$, the lemma is proven. □

The following covering lemma is a fundamental tool in our analysis.

Lemma 3.6 (Covering Lemma, [6, Lemma 3])**.** *Let S be a bounded subset of $\mathbb{R}^n$ such that for each $x \in S$ there exists an n-dimensional rectangle $\mathcal{R}(x)$, centered at x, such that:*

- *the edges of $\mathcal{R}(x)$ are parallel to the coordinate axes;*
- *the length of the edge of $\mathcal{R}(x)$ corresponding to the i-th axis is given by $h_i(t)$, where $t = t(x)$, $h_i(t)$ is an increasing function of the parameter $t \ge 0$, continuous at $t = 0$, and $h_i(0) = 0$.*

Then there exist points $\{x_k\}$ in S such that

(1) *$S \subset \bigcup_{k=1}^{\infty} \mathcal{R}(x_k)$;*

(2) *each $x \in S$ belongs to at most $C = C(n) > 0$ different rectangles.*

The Corollary 3.5 and the Covering Lemma 3.6 allow us to obtain a lower bound on the volume of the union of the level sets Θ_r where Γ and u detach quadratically from the corresponding tangent planes to Γ by the volume of the image of the gradient map, as in the standard ABP estimate.

Corollary 3.7. *For each $x \in \Sigma$, let $\Theta_r(x)$ be the level set obtained in Corollary 3.5. Then, we have*

$$C(\sup u)^n \leq \left| \bigcup_{x \in \Sigma} \Theta_r(x) \right|.$$

The nonlocal anisotropic version of the ABP estimate now reads as follows.

Theorem 3.8. *Let u and Γ be as in Lemma 3.1. There is a finite family of open rectangles $\{\mathcal{R}_j\}_{j \in \{1,\ldots,m\}}$ with diameters d_j such that the following hold:*

1. *Any two rectangles $\mathcal{R}_i$ and $\mathcal{R}_j$ in the family do not intersect.*
2. $\{u = \Gamma\} \subset \bigcup_{j=1}^m \overline{\mathcal{R}}_j$.
3. $\{u = \Gamma\} \cap \overline{\mathcal{R}}_j \neq \emptyset$ *for any* $\mathcal{R}_j$.
4. $d_j \leq \sqrt{\sum_{i=1}^n \left(\rho_0 2^{-\frac{1}{q_{\max}}}\right)^{\frac{2}{n+\sigma_i}}}$.
5. $\left|\nabla\Gamma\left(\overline{\mathcal{R}}_j\right)\right| \leq C\left(\max_{\overline{\mathcal{R}}_j} f^+\right)^n \left|\overline{\mathcal{R}}_j\right|$.
6. $\left|\left\{y \in C\tilde{\mathcal{R}}_j : u(y) \geq \Gamma(y) - C\left(\max_{\overline{\mathcal{R}}_j} f\right)\left(\tilde{d}_j\right)^2\right\}\right| \geq \varsigma\left|\tilde{\mathcal{R}}_j\right|$,

where $\tilde{d}_j$ is the diameter of the rectangle $\tilde{\mathcal{R}}_j$ corresponding to $\mathcal{R}_j$. The constants $\varsigma > 0$ and $C > 0$ depend only on n, λ and Λ.

Proof. We cover the ball B_1 with a tiling of rectangles of edges

$$\frac{\left(\rho_0 2^{-\frac{1}{q_{\max}}}\right)^{\frac{1}{n+\sigma_i}}}{2^{-\mathfrak{C}}}.$$

We discard all those that do not intersect $\{u = \Gamma\}$. Whenever a rectangle does not satisfy (5) and (6), we split its edges by $2^{n\mathfrak{C}}$ and discard those whose closure does not intersect $\{u = \Gamma\}$. Now we prove that all remaining rectangles satisfy (5) and (6) and that this process stops after a finite number of steps.

As in [4] we will argue by contradiction. Suppose the process is infinite. Thus, there is a sequence of nested rectangles $\mathcal{R}_j$ such that the intersection of their closures will be a point x_0. Moreover, since

$$\{u = \Gamma\} \cap \overline{\mathcal{R}}_j \neq \emptyset$$

and $\{u = \Gamma\}$ is closed, we have $x_0 \in \{u = \Gamma\}$. Let $0 < \varepsilon_1 < \varepsilon_0$, where ε_0 is as in Lemma 3.5. Then, there exist

$$r \in \left(0, \rho_0 2^{-\frac{1}{q_{\max}}}\right)$$

and $k_0 = k_0(x_0)$ such that

$$\left|\Theta_r \setminus \Theta_{sr} \cap \left\{y : u(x+y) < u(x) + \langle y, \nabla\Gamma(x)\rangle - Cf(x)\sum_{i=1}^{n} r^{\frac{2}{n+\sigma_i}}\right\}\right| \leq \varepsilon_1 |\Theta_r \setminus \Theta_{sr}| \tag{3.8}$$

and

$$\left|\nabla\Gamma\left(R_{a,s^{k_0+1}}(x_0)\right)\right| \leq Cf(x_0)^n \left|R_{a,s^{k_0+1}}(x_0)\right|, \tag{3.9}$$

where

$$r = \rho_0 2^{-\frac{1}{q_{\max}}} 2^{-\mathfrak{C}(n+\sigma_{\min})k_0}.$$

Let $\mathcal{R}_j$ be the largest rectangle in the family containing x_0 and contained in $R_{a,s^{k_0+1}}(x_0)$. Then $x_0 \in \mathcal{R}_j$ and $\mathcal{R}_j$ has edges l_i satisfying

$$2^{-\mathfrak{C}(k_0+2)}\left(\rho_0 2^{-\frac{1}{q_{\max}}}\right)^{\frac{1}{n+\sigma_i}} \leq l_i < 2^{-\mathfrak{C}(k_0+1)}\left(\rho_0 2^{-\frac{1}{q_{\max}}}\right)^{\frac{1}{n+\sigma_i}}.$$

Thus, we get

$$\mathcal{R}_j \subset R_{a,s^{k_0+1}} \quad \text{and} \quad \Theta_r \subset C\tilde{\mathcal{R}}_j,$$

for some $C = C(n) > 1$. Furthermore, since Γ is concave in B_2, we find

$$\Gamma(y) \leq u(x_0) + \langle y - x_0, \nabla\Gamma(x_0)\rangle$$

in B_2. Thus, denoting

$$A_j := \left\{y \in C\tilde{\mathcal{R}}_j : u(y) \geq \Gamma(y) - C\left(\max_{\overline{\mathcal{R}}_j} f\right)\left(\tilde{d}_j\right)^2\right\},$$

using (3.8), (3.9) and that l_i and $s^{-k_0}\left(\rho_0 2^{-\frac{1}{q_{\max}}}\right)^{\frac{1}{n+\sigma_i}}$ are comparable, we obtain

$$\begin{aligned}
|A_j| &\geq \left|\left\{y \in C\tilde{\mathcal{R}}_j : u(y) \geq u(x_0) + \langle y - x_0, \nabla\Gamma(x_0)\rangle \right.\right.\\
&\qquad \left.\left. -Cf(x_0)\sum_{i=1}^{n} r^{\frac{2}{n+\sigma_i}}\right\}\right| \\
&\geq (1-\varepsilon_1)|\Theta_r \setminus \Theta_{sr}| \\
&\geq \varsigma\left|\tilde{\mathcal{R}}_j\right|
\end{aligned}$$

and

$$\begin{aligned}
|\nabla\Gamma(\mathcal{R}_j)| &\leq \left|\nabla\Gamma\left(R_{a,s^{k_0+1}}(x_0)\right)\right| \\
&\leq Cf(x_0)^n\left|R_{a,s^{k_0+1}}(x_0)\right| \\
&= C_1 f(x_0)^n |\mathcal{R}_j|.
\end{aligned}$$

Then $\mathcal{R}_j$ would not be split and the process must stop, which is a contradiction. □

4. A barrier function

With the aim of localising the contact set of a solution u of the maximal equation, as in Lemma 3.1, we build a barrier function which is a supersolution of the minimal equation outside a small ellipse and is positive outside a large ellipse.

Lemma 4.1. *Given $R > 1$, there exist $p > 0$ and $\sigma_0 \in (0,2)$ such that the function*

$$f(x) = \min\left(2^p,\ |x|^{-p}\right)$$

satisfies

$$M^- f(x) \geq 0,$$

for $\sigma_0 < \sigma_{\min}$ and $1 \leq |x| \leq R$, where $p = p(n, \lambda, \Lambda, R)$, $\sigma_0 = \sigma_0(n, \lambda, \Lambda, R)$.

Proof. In the sequel we will use the following elementary inequalities:

$$(a_2 + a_1)^{-s} + (a_2 - a_1)^{-s} \geq 2a_2^{-s} + s(s+1)\, a_1^2 a_2^{-s-2} \tag{4.1}$$

and

$$(a_2 + a_1)^{-s} \geq a_2^{-s}\left(1 - s\frac{a_1}{a_2}\right). \tag{4.2}$$

where $0 < a_1 < a_2$ and $s > 0$. Taking into account the inequalities (4.1) and (4.2), we estimate, for $|y| < \frac{1}{2}$,

$$\begin{aligned}
\delta(f, e_1, y) &:= |e_1 + y|^{-p} + |e_1 - y|^{-p} - 2 \\
&= \left(1 + |y|^2 + 2y_1\right)^{-\frac{p}{2}} + \left(1 + |y|^2 - 2y_1\right)^{-\frac{p}{2}} - 2 \\
&\geq \left(1 + |y|^2\right)^{-\frac{p}{2}} + p(p+2)\, y_1^2 \left(1 + |y|^2\right)^{-\frac{p+4}{2}} - 2 \\
&\geq 2\left(1 - \frac{p}{2}|y|^2\right) + p(p+2)^2\, y_1^2 - p(p+4)\, \frac{(p+2)}{2} y_1^2 |y|^2 - 2 \\
&= p\left[-|y|^2 + (p+2)\, y_1^2 - (p+4)\, \frac{(p+2)}{2} y_1^2 |y|^2\right].
\end{aligned}$$

Given $1 \leq |x| \leq R$, there is a rotation $T_x : \mathbb{R}^n \to \mathbb{R}^n$ such that $x = |x| T e_1$. Thus, changing variables, we get

$$M^- f(x) = c_\sigma |x|^{n-p} \left|\det T_x\right| \left[\int_{\mathbb{R}^n} \frac{\lambda \delta^+(f, e_1, y) - \Lambda \delta^-(f, e_1, y)}{\sum_{i=1}^n \left|\left(|x| T_x y\right)_i\right|^{n+\sigma_i}} dy\right].$$

Then, we estimate

$$\begin{aligned}|x|^{p-n}M^{-}f(x) &= c_\sigma\int_{B_{1/4}(0)}\frac{\Lambda\delta^{+}(f,e_1,y)-\lambda\delta^{-}(f,e_1,y)}{\sum_{i=1}^{n}||x|(T_xy)_i|^{n+\sigma_i}}dy\\ &\quad+c_\sigma\int_{\mathbb{R}^n\setminus B_{1/4}(0)}\frac{\Lambda\delta^{+}(f,e_1,y)-\lambda\delta^{-}(f,e_1,y)}{\sum_{i=1}^{n}||x|(T_xy)_i|^{n+\sigma_i}}dy\\ &\geq c_\sigma\int_{B_{1/4}(0)}\frac{2p\lambda(p+2)y_1^2}{\sum_{i=1}^{n}||x|(T_xy)_i|^{n+\sigma_i}}dy\\ &\quad-c_\sigma\int_{B_{1/4}(0)}\frac{2p\Lambda|y|^2}{\sum_{i=1}^{n}||x|(Ty)_i|^{n+\sigma_i}}dy\\ &\quad-c_\sigma\int_{B_{1/4}(0)}\frac{\frac{1}{2}p(p+4)(p+2)|y|^4}{\sum_{i=1}^{n}||x|(T_xy)_i|^{n+\sigma_i}}dy\\ &\quad+c_\sigma\int_{\mathbb{R}^n\setminus B_{1/4}(0)}\frac{-\lambda 2^{p+1}}{\sum_{i=1}^{n}||x|(T_xy)_i|^{n+\sigma_i}|}dy\\ &:= I_1+I_2+I_3+I_4,\end{aligned}\tag{4.3}$$

where I_1, I_2, I_3 and I_4 represent the three terms on the right-hand side of the above inequality.

We estimate

$$\begin{aligned}p^{-1}I_1 &\geq n^{-1}c_\sigma\lambda(p+2)|x|^{-(n+2)}\int_{B_{1/4}(0)}\frac{y_1^2}{|y|^{n+\sigma_{\min}}}dy\\ &\geq R^{-(n+2)}n^{-1}\left[c_\sigma\lambda(p+2)\int_{\partial B_1}y_1^2d\nu(y)\right]\int_0^{\delta/4}t^{1-\sigma_{\min}}dt\\ &\geq C_3\frac{c_\sigma}{2-\sigma_{\min}}\left[(p+2)\int_{\partial B_1}y_1^2d\nu(y)\right]\left(\frac{1}{4}\right)^{2-\sigma_{\min}}\\ &\geq C_3c(n)\left[(p+2)\int_{\partial B_1}y_1^2d\nu(y)\right],\end{aligned}$$

where $C_3 = C_3(n,\lambda,\Lambda,R) > 0$. Moreover, if $C = C(n) > 0$ is a positive constant such that $B_{1/4}(0)\subset\Theta_C$, we have, for $|x|\geq 1$,

$$\begin{aligned}p^{-1}I_2 &\geq -C_4c_\sigma\int_{B_{1/4}(0)}\frac{|y|^2}{\sum_{i=1}^{n}|(T_xy)_i|^{n+\sigma_i}}dy\\ &= -C_4c_\sigma\left|\det T_x^{-1}\right|\int_{B_{1/4}(0)}\frac{|T_x^{-1}y|^2}{\sum_{i=1}^{n}|y_i|^{n+\sigma_i}}dy\\ &= -C_4c_\sigma\int_{B_{1/4}(0)}\frac{|y|^2}{\sum_{i=1}^{n}|y_i|^{n+\sigma_i}}dy\\ &\geq -C_4c_\sigma\int_{\Theta_C}\frac{|y|^2}{\sum_{i=1}^{n}|y_i|^{n+\sigma_i}}dy,\end{aligned}$$

where $C_4 = C_4(n, \lambda, \Lambda)$. We have also

$$c_\sigma \int_{\Theta_C} \frac{|y|^2}{\sum_{i=1}^n |y_i|^{n+\sigma_i}} dy = c_\sigma \sum_{k=1}^{\infty} \int_{\Theta_{r_k} \setminus \Theta_{r_{k+1}}} \frac{|y|^2}{\sum_{i=1}^n |y_i|^{n+\sigma_i}} dy \leq C_5,$$

where $r_k := C2^{-k}$ and $C_5 = C_5(n, \lambda, \Lambda)$. Moreover, using the elementary inequality

$$(a+b)^m \leq 2^m (a^m + b^m), \quad \text{for all, } a, b, m \in (0, \infty),$$

we get

$$\begin{aligned} I_3 &\geq -C_6 c_\sigma 2^{\frac{n+\sigma_{\max}}{2}} \int_{B_{\delta/4}(0)} \frac{|y|^4}{|y|^{n+\sigma_{\max}}} dy \\ &\geq -C_7 \frac{c_\sigma}{(4-\sigma_{\max})} \left(\frac{1}{4}\right)^{4-\sigma_{\max}} \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} I_4 &\geq -c_\sigma \left(\frac{1}{4}\right)^{-\sigma_{\max}+\sigma_{\min}} 2^{\frac{n+\sigma_{\min}}{2}} \int_{\mathbb{R}^n \setminus B_{1/4}(0)} \frac{\Lambda 2^{p+2}}{|y|^{n+\sigma_{\min}}} dy \\ &= -C_8 \left(\frac{1}{4}\right)^{-\sigma_{\max}} 2^{\frac{n+\sigma_{\min}}{2}} \frac{c_\sigma}{\sigma_{\min}} \\ &\geq -C_8 \left(\frac{1}{4}\right)^{-\sigma_{\max}} \frac{c_\sigma}{\sigma_{\min}}, \end{aligned} \tag{4.5}$$

for positive constants $C_7 = C_7(n, \lambda, \Lambda, p)$ and $C_8 = C_8(n, \lambda, \Lambda, p)$. Choosing $p = p(n, \lambda, \Lambda, R) > 0$ such that

$$C_3(p+2) \int_{\partial B_1} y_1^2 d\nu(y) - C_4 C_5 > 0$$

and combining (4.3), (4.4) and (4.5), there is a positive constant $\sigma_0 = \sigma_0(n, \lambda, \Lambda, R) < 2$ such that

$$|x|^{p-n} M^- f(x) \geq C_9 > 0,$$

for a positive constant $C_9 = C_9(n, \lambda, \Lambda, R)$ and $\sigma_0 < \sigma_{\min} < 2$. □

Corollary 4.2. *Given $r > 0$, $\sigma_0 \in (0, 2)$, $\sigma_0 < \sigma_{\min}$, and $R > 1$, there exist $s > 0$ and $p > 0$ such that the function*

$$f(x) = \min\left(s^{-p}, \ |x|^{-p}\right)$$

satisfies

$$M^- f(x) \geq 0,$$

for $1 \leq |x| \leq R$, where $p = p(n, \lambda, \Lambda, R)$ and $s = s(n, \lambda, \Lambda, \sigma_0, R)$.

Proof. Since $c_\sigma \geq c(n)(2 - \sigma_{\min})$ and

$$c_\sigma \int_{\Theta_C} \frac{|y|^2}{\sum_{i=1}^n |y_i|^{n+\sigma_i}} dy = c_\sigma \sum_{k=1}^{\infty} \int_{\Theta_{r_k} \setminus \Theta_{r_{k+1}}} \frac{|y|^2}{\sum_{i=1}^n |y_i|^{n+\sigma_i}} dy \leq C_1(n),$$

if $C = C(n) > 0$ and $r_k := C2^{-k}$, we can argue as in Corollary 9.2 in [4]. □

Corollary 4.3. *Given $r > 0$, $R > 1$ and $\sigma_0 \in (0, 2)$, there exist $s > 0$ and $p > 0$ such that the function*

$$g(x) = \min\left(s^{-p},\ |T_r^{-1}|^{-p}\right)$$

satisfies

$$M^{-}g(x) \geq 0$$

for $\sigma_0 < \sigma_{\min}$ and $x \in E_{r,R} \setminus E_{r,1}$, where $p = p(n, \lambda, \Lambda, R)$ and $s = s(n, \lambda, \Lambda, \sigma_0, R)$.

Proof. Considering the anisotropic scaling

$$g(x) = f\left(T_r^{-1}x\right), \quad x \in \mathbb{R}^n,$$

we have $T_r^{-1}(E_{r,R} \setminus E_{r,1}) = B_R \setminus B_r$. Furthermore, changing variables, we estimate

$$M^{-}g(x) = r^{-1}|\det T_r| M^{-}f\left(T_r^{-1}x\right) \geq 0,$$

for all $x \in E_{r,R} \setminus E_{r,1}$. □

Lemma 4.4. *Given $\sigma_0 \in (0, 2)$, there is a function $\Psi : \mathbb{R}^n \to \mathbb{R}$ satisfying*

1. *Ψ is continuous in $\mathbb{R}^n$;*
2. *$\Psi = 0$ for $x \in \mathbb{R}^n \setminus E_{\frac{1}{4},3\sqrt{n}}$;*
3. *$\Psi > 3$ for $x \in \mathcal{R}_{\frac{1}{4},3}$;*
4. *$M^{-}\Psi(x) > -\phi(x)$ for some positive function $\phi \in C_0\left(E_{\frac{1}{4},1}\right)$ for $\sigma_0 < \sigma_{\min}$.*

Proof. We define the function $\Psi : \mathbb{R}^n \to \mathbb{R}$ by

$$\Psi(x) = \tilde{c} \begin{cases} 0, & \text{in} \quad \mathbb{R}^n \setminus E_{\frac{1}{4},3\sqrt{n}} \\ |T_{\frac{1}{4}}^{-1}x|^{-p} - (3\sqrt{n})^{-p} & \text{in} \quad E_{\frac{1}{4},3\sqrt{n}} \setminus E_{\frac{1}{4},1} \\ q_{p,\sigma}, & \text{in} \quad E_{\frac{1}{4},1}, \end{cases}$$

where $q_{p,\sigma}$ is a quadratic function with different coefficients in different directions so that Ψ is $C^{1,1}$ across $E_{\frac{1}{4},1}$. Choose $\tilde{c} > 0$ such that $\Psi > 3$ in $\mathcal{R}_{\frac{1}{4},3}$. By Lemma 2.7,

$$M^{-}\Psi \in C\left(E_{\frac{1}{4},3\sqrt{n}}\right)$$

and, from Corollary 4.3, we get $M^{-}\Psi \geq 0$ in $\mathbb{R}^n \setminus E_{\frac{1}{4},1}$. The lemma is proved. □

5. Harnack inequality and regularity

The next lemma is the fundamental tool towards the proof of the Harnack inequality. It bridges the gap between a pointwise estimate and an estimate in measure.

Lemma 5.1. *Let* $0 < \sigma_0 < 2$. *If* $\sigma_{\min} \in (\sigma_0, 2)$, *then there exist constants* $\varepsilon_0 > 0$, $0 < \varsigma < 1$, *and* $M > 1$, *depending only* σ_0, λ, Λ *and* n, *such that if*

(1) $u \geq 0$ *in* $\mathbb{R}^n$;

(2) $u(0) \leq 1$;

(3) $M^- u \leq \varepsilon_0$ *in* $E_{\frac{(3\sqrt{n})^{n+2}}{4},1}$,

then

$$|\{u \leq M\} \cap Q_1| > \varsigma.$$

Proof. Let $v = \Psi - u$ and let Γ be the concave envelope of v in $E_{\frac{(3\sqrt{n})^{n+2}}{4},3}$. We have

$$M^+ v \geq M^- \Psi - M^- u \geq -\phi - \varepsilon_0 \quad \text{in} \quad E_{\frac{(3\sqrt{n})^{n+2}}{4},1}.$$

Applying Theorem 3.8 to v (anisotropically scaled), we obtain a family of rectangles $\mathcal{R}_j$ such that

$$\sup_{E_{\frac{(3\sqrt{n})^{n+2}}{4},1}} v \leq C \left| \nabla\Gamma \left(E_{\frac{(3\sqrt{n})^{n+2}}{4},1} \right) \right|^{\frac{1}{n}}.$$

Thus, by Theorem 3.8 and condition (3) in Lemma 4.4, we obtain

$$\begin{aligned}
\sup_{E_{\frac{(3\sqrt{n})^{n+2}}{4},1}} v &\leq C \left| \nabla\Gamma \left(E_{\frac{(3\sqrt{n})^{n+2}}{4},1} \right) \right|^{\frac{1}{n}} \\
&\leq C_1 \left(\sum_{i=1}^{n} |\nabla\Gamma(\mathcal{R}_j)| \right)^{\frac{1}{n}} \\
&\leq C_1 \left(\sum_{i=1}^{n} \left(\max_{\overline{\mathcal{R}}_j} (\phi + \varepsilon_0)^+ \right)^n |\mathcal{R}_j| \right)^{\frac{1}{n}} \\
&\leq C_1 \varepsilon_0 + \left(\sum_{i=1}^{n} \left(\max_{\overline{\mathcal{R}}_j} (\phi)^+ \right)^n |\mathcal{R}_j| \right)^{\frac{1}{n}}.
\end{aligned}$$

Furthermore, since $\Psi > 3$ in $E_{\frac{(3\sqrt{n})^{n+2}}{4},1} \supset \mathcal{R}_{\frac{1}{4},3}$ and $u(0) \leq 1$, we get

$$2 \leq C_1 \varepsilon_0 + \left(\sum_{i=1}^{n} \left(\max_{\overline{\mathcal{R}}_j} (\phi)^+ \right)^n |\mathcal{R}_j| \right)^{\frac{1}{n}}.$$

If $\varepsilon_0 > 0$ is small enough, we have

$$c \leq \left(\sum_{\mathcal{R}_j \cap E_{\frac{1}{4},1} \neq \emptyset} |\mathcal{R}_j| \right), \tag{5.1}$$

where we used that ϕ is supported in $E_{\frac{1}{4},1}$. We also have that the diameter of $\mathcal{R}_j$ is bounded by $\left(\rho_0 = \frac{1}{C}\right)^{\frac{1}{n+2}}$. Then, if $\mathcal{R}_j \cap E_{\frac{1}{4},1} \neq \emptyset$ we have $C\tilde{\mathcal{R}}_j \subset B_{\frac{1}{2}}$. By Theorem 3.8, we get

$$\begin{aligned} & \left| \left\{ y \in C\tilde{\mathcal{R}}_j : v(y) \geq \Gamma(y) - C\rho_0^{\frac{2}{n+2}} \right\} \right| \\ \geq & \left| \left\{ y \in C\tilde{\mathcal{R}}_j : v(y) \geq \Gamma(y) - Cd_j^2 \right\} \right| \\ \geq & \ \varsigma \left| \mathcal{R}_j \right|, \end{aligned} \tag{5.2}$$

where we used that $Cd_j^2 < C\rho_0^{\frac{2}{n+2}}$. For each rectangles $\mathcal{R}_j$ that intersects $E_{\frac{1}{4},1}$ we consider $C\tilde{\mathcal{R}}_j$. The family $\left\{ C\tilde{\mathcal{R}}_j \right\}$ is an open covering for $\bigcup_{i=1}^m \overline{\mathcal{R}}_j$. We consider a subcover with finite overlapping (Lemma 3.6) that also covers $\bigcup_{i=1}^m \overline{\mathcal{R}}_j$. Then, using (5.1) and (5.2) we obtain

$$\begin{aligned} & \left| \left\{ y \in B_{\frac{1}{2}} : v(y) \geq \Gamma(y) - C\rho_0^{\frac{2}{n+2}} \right\} \right| \\ \geq & \left| \bigcup_{j=1}^m \left\{ y \in C\tilde{\mathcal{R}}_j : v(y) \geq \Gamma(y) - C\rho_0^{\frac{2}{n+2}} \right\} \right| \\ \geq & \ C_1 \sum_{j=1}^m \left| \left\{ y \in C\tilde{\mathcal{R}}_j : v(y) \geq \Gamma(y) - C\rho_0^{\frac{2}{n+2}} \right\} \right| \\ \geq & \ C_1 c_1. \end{aligned}$$

We recall that $B_{\frac{1}{2}} \subset Q_1$ and $\Gamma \geq 0$. Hence, if $M := \sup_{B_{\frac{1}{2}}} \Psi + C\rho_0^{\frac{2}{n+2}}$, we have

$$\begin{aligned} \left| \{ y \in Q_1 : u(y) \leq M \} \right| & \geq \left| \left\{ y \in B_{\frac{1}{2}} : u(y) \leq M \right\} \right| \\ & \geq \left| \left\{ y \in B_{\frac{1}{2}} : v(y) \geq \Gamma(y) - C\rho_0^{\frac{2}{n+2}} \right\} \right| \\ & \geq c. \end{aligned}$$

□

The next lemma is crucial to iterate Lemma 5.1 and to obtain the L_ε decay of the distribution function $\lambda_u := |\{u > t\} \cap B_1|$. Since our scaling is anisotropic, the following Calderón-Zygmund decomposition is performed with boxes that satisfy the covering lemma of Caffarelli-Calderón (Lemma

3.6). We can then apply Lebesgue's differentiation theorem having these boxes as a differentiation basis.

If R is a dyadic rectangle different from Q_1, we say that R_{pred} is the predecessor of R if R is one of the 2^n rectangles obtained from dividing R_{pred}. We recall from section 3 that if R is a rectangle then $\tilde{R}$ is the rectangle corresponding to R.

Lemma 5.2 (Calderón-Zygmund). *Let $A \subset B \subset Q_1$ be measurable sets and $0 < \delta < 1$ be such that*

(1) $|A| \leq \delta$;

(2) *if R is a dyadic rectangle such that $\left|A \cap \tilde{R}\right| > \delta \left|\tilde{R}\right|$, then $\tilde{R}_{pred} \subset B$.*

Then

$$|A| \leq \delta C |B|,$$

where $C > 0$ is a constant depending only on n.

Proof. Just as in [5, Lemma 4.2.], using Lebesgue's differentiation theorem, we obtain a sequence of boxes R_j satisfying

(1) $|A \cap R_j| \leq \delta |R_j|$;

(2) $A \subset \bigcup_{j=1}^{\infty} R_j$.

Then, we have

$$|A| \leq \sum_{j=1}^{\infty} |A \cap R_j| \leq \delta \sum_{j=1}^{\infty} |R_j| \leq C\delta |B|,$$

where $C = C(n) > 0$ is the constant from Lemma 3.6. □

Lemma 5.3. *Let u be as in Lemma 5.1. Then*

$$\left|\left\{u > M^k\right\} \cap Q_1\right| \leq C(1-\varsigma)^k, \quad k = 1, \ldots,$$

where M and ς are as in Lemma 5.1. Thus, there exist positive universal constants d and ε such that

$$|\{u \geq t\} \cap Q_1| \leq d t^{-\varepsilon}, \quad \forall t > 0.$$

Using standard covering arguments we get the following theorem.

Theorem 5.4. *Let $u \geq 0$ in $\mathbb{R}^n$, $u(0) \leq 1$ and $M^- u \leq \varepsilon_0$ in B_2. Suppose that $\sigma_{\min} \geq \sigma_0$ for some $\sigma_0 > 0$. Then*

$$|\{u \geq t\} \cap B_1| \leq C t^{-\varepsilon}, \quad \forall t > 0,$$

where $C = C(n, \lambda, \Lambda, \sigma_0) > 0$ and $\varepsilon = \varepsilon(n, \lambda, \Lambda, \sigma_0) > 0$.

Remark 5.5. For each $s > 0$, we will denote $E^j_{r,s} := E_{r^{n+\sigma_j},s}$. Let $u \geq 0$ in $\mathbb{R}^n$ and $M^- u \leq C_0$ in $E^j_{r,2}$, with $0 < r \leq 1$. We consider the anisotropic scaling

$$v(x) = \frac{u(T_{j,r}x)}{u(0) + C_0 r^{\left[(n-1) - \sum_{i=1}^{n-1} \frac{n+\sigma_n}{n+\sigma_i}\right]} r^{\sigma_j}}, \quad x \in \mathbb{R}^n,$$

where $T_{j,r}:\mathbb{R}^n\to\mathbb{R}^n$ is defined by

$$T_{j,r}e_i := \begin{cases} re_j, & \text{for} \quad i=j \\ r^{\frac{n+\sigma_j}{n+\sigma_i}}e_i, & \text{for} \quad i\neq j. \end{cases}$$

We have $v\geq 0$ in $\mathbb{R}^n$, $v(0)\leq 1$ and $T_{j,r}(B_2)\subset E^j_{r,2}$. Moreover, changing variables, we estimate

$$M^- v(x) = \frac{r^{\sigma_j} r^{\left[(n-1)-\sum_{i=1}^{n-1}\frac{n+\sigma_j}{n+\sigma_i}\right]}}{u(0)+C_0 r^{\left[(n-1)-\sum_{i=1}^{n-1}\frac{n+\sigma_j}{n+\sigma_i}\right]} r^{\sigma_j}} M^- u(T_{j,r}x)\leq 1,$$

for all $x\in B_2$.

Then, using the anisotropic scaling $T_{j,r}$ and Theorem 5.4 we have the following scaled version.

Theorem 5.6 (Pointwise Estimate)**.** *Let $u\geq 0$ in $\mathbb{R}^n$ and $M^-u\leq C_0$ in $E^j_{r,2}$. Suppose that $\sigma_{\min}\geq\sigma_0$ for some $\sigma_0>0$. Then*

$$|\{u\geq t\}\cap E^j_{r,1}|\leq C|E^j_{r,1}|\left(u(0)+C_0 r^{\left[(n-1)-\sum_{i=1}^{n-1}\frac{n+\sigma_j}{n+\sigma_i}\right]}r^{\sigma_j}\right)^{\varepsilon}t^{-\varepsilon}\quad \forall t>0$$

where $C=C(n,\lambda,\Lambda,\sigma_0)>0$ and $\varepsilon=\varepsilon(n,\lambda,\Lambda,\sigma_0)>0$.

We are now ready to prove the Harnack inequality.

Theorem 5.7 (Harnack Inequality)**.** *Let $u\geq 0$ in $\mathbb{R}^n$, $M^-u\leq C_0$, and $M^+u\geq -C_0$ in B_2. Suppose that $\sigma_{\min}\geq\sigma_0$, for some $\sigma_0>0$. Then*

$$u\leq C(u(0)+C_0)\quad \text{in} \quad B_{\frac{1}{2}}.$$

Proof. Without loss of generality, we can suppose that $u(0)\leq 1$ and $C_0=1$. Let

$$\tau=\frac{n(n+\sigma_{\max})}{\varepsilon(n+\sigma_{\min})},$$

where $\varepsilon>0$ is as in Theorem 5.4. For each $\vartheta>0$, we define the function

$$f_\vartheta(x):=\vartheta(1-|x|)^{-\tau},\quad x\in B_1.$$

Let $t>0$ be such that $u\leq f_t$ in B_1. There is an $x_0\in B_1$ such that $u(x_0)=f_t(x_0)$. Let $d:=(1-|x_0|)$ be the distance from x_0 to ∂B_1.

If $\sigma_{\max}=\sigma_{i_0}$ and $E^{\max}_{r,s}(x_0):=E^{i_0}_{r,s}(x_0)$, for all $s>0$, we will estimate the portion of the ellipsoid $E^{\max}_{r,1}(x_0)$ covered by $\left\{u>\frac{u(x_0)}{2}\right\}$ and by $\left\{u<\frac{u(x_0)}{2}\right\}$. As in [4], we will prove that $t>0$ cannot be too large. Thus, since $\tau\leq\dfrac{2n}{\varepsilon}$, we conclude the proof of the theorem. By Theorem 5.4, we have

$$\left|\left\{u>\frac{u(x_0)}{2}\right\}\cap B_1\right|\leq C\left|\frac{2}{u(x_0)}\right|^{\varepsilon}=Ct^{-\varepsilon}d^n\leq C_1t^{-\varepsilon}\left(r^{\frac{n+\sigma_{\max}}{n+\sigma_{\min}}}\right)^n,$$

where $r = \frac{d}{2}$. Thus, we get

$$\left|\left\{u > \frac{u(x_0)}{2}\right\} \cap E^{\max}_{r,1}(x_0)\right| \leq C_1 t^{-\varepsilon} |E^{\max}_{r,1}|. \tag{5.3}$$

Now we will estimate $\left|\left\{u > \frac{u(x_0)}{2}\right\} \cap E^{\max}_{r,1}(x_0)\right|$, where $0 < \theta < 1$. Since

$$|x| \leq |x - x_0| + |x_0|, \quad \forall x \in \mathbb{R}^n,$$

we have

$$(1 - |x|) \geq \left[d - \frac{d\theta}{2}\right],$$

for $x \in B_{r\theta}(x_0)$. Hence, if $x \in B_{r\theta}(x_0)$, we get

$$u(x) \leq f_t(x) \leq t(1 - |x|)^{-\tau} \leq u(x_0)\left(1 - \frac{\theta}{2}\right)^{-\tau}.$$

Then, since $M^+ u \geq -1$, the function

$$v(x) = \left(1 - \frac{\theta}{2}\right)^{-\tau} u(x_0) - u(x)$$

satisfies

$$v \geq 0 \quad \text{in } B_{r\theta}(x_0) \quad \text{and} \quad M^- v \leq 1.$$

We will consider the function $w := v^+$. For $x \in \mathbb{R}^n$ we have

$$M^- w(x) = M^- v(x) + \big(M^- w(x) - M^- v(x)\big)$$

and

$$\begin{aligned} \frac{M^- w(x) - M^- v(x)}{c_\sigma} &= \lambda \int_{\mathbb{R}^n} \frac{\delta^+(w,x,y) - \delta^+(v,x,y)}{\sum_{i=1}^n |y_i|^{n+\sigma_i}} dy \\ &\quad + \Lambda \int_{\mathbb{R}^n} \frac{\delta^-(v,x,y) - \delta^-(w,x,y)}{\sum_{i=1}^n |y_i|^{n+\sigma_i}} dy \\ &= I_1 + I_2, \end{aligned}$$

where I_1 and I_2 represent the two terms in the right-hand side above. Using the elementary equality

$$v^+(x+y) = v(x+y) + v^-(x+y),$$

and denoting $\delta_w := \delta(w,x,y)$ and $\delta_v := \delta(v,x,y)$, we obtain

$$\delta_w^+ = \delta_v + v^-(x-y) + v^-(x+y).$$

Thus, taking in account that

$$\delta_w^+ \geq \delta_v^+ \quad \text{and} \quad \delta_v = \delta_v^+ - \delta_v^-,$$

we estimate

$$\begin{aligned} I_1 &= -\lambda \int_{\left\{\delta_w^+ > \delta_v^+\right\}} \frac{\delta_v^-}{\sum_{i=1}^n |y_i|^{n+\sigma_i}} dy \\ &\quad + \lambda \int_{\left\{\delta_w^+ > \delta_v^+\right\}} \frac{v^-(x+y) + v^-(x-y)}{\sum_{i=1}^n |y_i|^{n+\sigma_i}} dy \\ &\leq \Lambda \int_{\left\{\delta_w^+ > 0\right\}} \frac{v^-(x+y) + v^-(x-y)}{\sum_{i=1}^n |y_i|^{n+\sigma_i}} dy. \end{aligned} \tag{5.4}$$

Analogously, we get

$$\begin{aligned} I_2 &= \Lambda \int_{\left\{\delta_v^- > 0\right\} \cap \left\{\delta_w^- \neq \delta_v^-\right\}} \frac{\delta_v^- - \delta_w^-}{\sum_{i=1}^n |y_i|^{n+\sigma_i}} dy \\ &\quad + \Lambda \int_{\left\{\delta_v^- = 0\right\} \cap \left\{\delta_w^- \neq \delta_v^-\right\}} \frac{v^-(x+y) + v^-(x-y)}{\sum_{i=1}^n |y_i|^{n+\sigma_i}} dy \\ &\leq \Lambda \int_{\left\{\delta_v^- > 0\right\} \cap \left\{\delta_w^- \neq \delta_v^-\right\}} \frac{-\delta_v - \delta_v^-}{\sum_{i=1}^n |y_i|^{n+\sigma_i}} dy. \end{aligned} \tag{5.5}$$

We also have

$$\begin{aligned} -\delta_v^- - \delta_w^- &= 2v(x) - (v(x+y) + v(x-y)) - \delta_w^- \\ &= 2v(x) - \left[\left(v^+(x+y) + v^+(x-y)\right) \right. \\ &\quad \left. - \left(v^-(x+y) + v^-(x-y)\right)\right] \\ &= \left(-\delta_w - \delta_w^-\right) + v^-(x+y) + v^-(x-y) \\ &= -\delta_w^+ + v^-(x+y) + v^-(x-y). \end{aligned} \tag{5.6}$$

Then, from (5.6) and (5.5), we obtain

$$\begin{aligned} I_2 &\leq -\Lambda \int_{\left\{\delta_v^- > 0\right\} \cap \left\{\delta_w^- \neq \delta_v^-\right\}} \frac{\delta_w^+}{\sum_{i=1}^n |y_i|^{n+\sigma_i}} dy \\ &\quad + \Lambda \int_{\left\{\delta_v^- > 0\right\} \cap \left\{\delta_w^- \neq \delta_v^-\right\}} \frac{v^-(x+y) + v^-(x-y)}{\sum_{i=1}^n |y_i|^{n+\sigma_i}} dy \\ &\leq \Lambda \int_{\left\{\delta_w^- \geq 0\right\}} \frac{v^-(x+y) + v^-(x-y)}{\sum_{i=1}^n |y_i|^{n+\sigma_i}} dy. \end{aligned} \tag{5.7}$$

Hence, using (5.4), (5.7), and changing variables, we find

$$\begin{aligned} \frac{M^- w(x) - M^- v(x)}{c_\sigma} &\leq \Lambda \int_{\mathbb{R}^n} \frac{v^-(x+y) + v^-(x-y)}{\sum_{i=1}^n |y_i|^{n+\sigma_i}} dy \\ &= -2\Lambda \int_{\{v(x+y) < 0\}} \frac{v(x+y)}{\sum_{i=1}^n |y_i|^{n+\sigma_i}} dy. \end{aligned}$$

Moreover, if $x \in B_{\frac{r\theta}{2}}(x_0)$, we have

$$\frac{M^- w(x) - M^- v(x)}{c_\sigma} \leq 2\Lambda \int_{\mathbb{R}^n \setminus B_{r\theta}(x_0 - x)} \frac{-v(x+y)}{\sum_{i=1}^n |y_i|^{n+\sigma_i}} dy$$

$$\leq 2\Lambda \int_{\mathbb{R}^n \setminus B_{r\theta}(x_0 - x)} \frac{\left(u(x+y) - \left(1 - \frac{\theta}{2}\right)^{-\tau} u(x_0)\right)^+}{\sum_{i=1}^n |y_i|^{n+\sigma_i}} dy.$$

If $\iota > 0$ is the largest value such that $u(x) \geq \iota\left(1 - |4x|^2\right)$, then there is a point $x_1 \in B_{\frac{1}{4}}$ such that $u(x_1) = \left(1 - |4x_1|^2\right)$. Moreover, since $u(0) \leq 1$, we get $\iota \leq 1$. Then, we have

$$c_\sigma \int_{\mathbb{R}^n} \frac{\delta^-(u, x_1, y)}{\sum_{i=1}^n |y_i|^{n+\sigma_i}} dy \leq c_\sigma \int_{\mathbb{R}^n} \frac{\delta^-\left(\left(1 - |4x|^2\right), x_1, y\right)}{\sum_{i=1}^n |y_i|^{n+\sigma_i}} dy \leq C,$$

where the constant $C > 0$ is independent of σ_i. Moreover, since $M^- u(x_1) \leq 1$, we find

$$c_\sigma \int_{\mathbb{R}^n} \frac{\delta^+(u, x_1, y)}{\sum_{i=1}^n |y_i|^{n+\sigma_i}} dy \leq C.$$

Recall that $u(x_1 - y) \geq 0$ and $u(x_1) \leq 1$. Thus, we obtain

$$c_\sigma \int_{\mathbb{R}^n} \frac{(u(x_1 + y) - 2)^+}{\sum_{i=1}^n |y_i|^{n+\sigma_i}} dy \leq C.$$

Since $t > 0$ is large enough, we can suppose that $u(x_0) > 2$. Let

$$x \in E^{\max}_{\frac{r\theta}{2}, 1}(x_0) \subset B_{\frac{r\theta}{2}}(x_0)$$

and

$$y \in \mathbb{R}^n \setminus B_{r\theta}(x_0 - x) \subset \mathbb{R}^n \setminus E^{\max}_{\frac{r\theta}{2}, 1}(x_0 - x).$$

Then, we have the inequalities

$$\begin{aligned} & \sum_{i=1}^n |(y + x + x_1)_i|^{n+\sigma_i} \\ \leq\ & C\left(\sum_{i=1}^n |y_i|^{n+\sigma_i} + \sum_{i=1}^n |x_i|^{n+\sigma_i} + \sum_{i=1}^n |(x_1)_i|^{n+\sigma_i}\right) \\ \leq\ & C\sum_{i=1}^n |y_i|^{n+\sigma_i} + 2C \end{aligned}$$

and

$$\begin{aligned} |y_i| &\geq |(y - (x_0 - x))_i| - |(x_0 - x)_i| \\ &\geq \frac{(r\theta)^{\frac{n+\sigma_{\max}}{n+\sigma_i}}}{2}. \end{aligned}$$

Then, taking into account the obvious equalities

$$u(x+y)-\left(1-\frac{\theta}{2}\right)^{-\tau}u(x_0)=u(x+x_1+y-x_1)-\left(1-\frac{\theta}{2}\right)^{-\tau}u(x_0),$$

and

$$\frac{1}{\sum_{i=1}^{n}|y_i|^{n+\sigma_i}}=\left(\sum_{i=1}^{n}|\,(y+x+x_1)_i\,|^{n+\sigma_i}\right)^{-1}\frac{\sum_{i=1}^{n}|\,(y+x+x_1)_i\,|^{n+\sigma_i}}{\sum_{i=1}^{n}|y_i|^{n+\sigma_i}},$$

we estimate

$$2\Lambda\int_{\mathbb{R}^n\setminus B_{r\theta}(x_0-x)}\frac{\left(u(x+y)-\left(1-\frac{\theta}{2}\right)^{-\tau}u(x_0)\right)^{+}}{\sum_{i=1}^{n}|y_i|^{n+\sigma_i}}dy\le C_1(\theta r)^{-(n+\sigma_{\max})}.$$

Thus, we have

$$M^{-}w\le C_1(\theta r)^{-(n+\sigma_{\max})}\quad\text{in } E^{\max}_{\frac{r\theta}{2},1}(x_0).$$

Applying Theorem 5.6 to w in $E^{\max}_{\frac{r\theta}{2},1}(x_0)\subset B_{\frac{r\theta}{2}}(x_0-x)$ and using that

$$w(x_0)=\left(\left(1-\frac{\theta}{2}\right)^{-\tau}-1\right)u(x_0),$$

we get

$$\begin{aligned}
&\left|\left\{u>\frac{u(x_0)}{2}\right\}\cap E^{\max}_{\frac{r\theta}{2},\frac{1}{2}}\right|\\
=\;&\left|\left\{w>\left[\left(1-\frac{\theta}{2}\right)^{-\tau}-\frac{1}{2}\right]u(x_0)\right\}\cap E^{\max}_{\frac{r\theta}{2},\frac{1}{2}}\right|\\
\le\;& C\left|E^{\max}_{\frac{r\theta}{2},\frac{1}{2}}\right|\left[\left(\left(1-\frac{\theta}{2}\right)^{-\tau}-\frac{1}{2}\right)u(x_0)+C_1(r\theta)^{-n-C_2}\right]^{\varepsilon}\\
&\cdot\left[\left(\left(1-\frac{\theta}{2}\right)^{-\tau}-\frac{1}{2}\right)u(x_0)\right]^{-\varepsilon}\\
\le\;& C\left|E^{\max}_{\frac{r\theta}{2},\frac{1}{2}}\right|\left[\left(\left(1-\frac{\theta}{2}\right)^{-\tau}-\frac{1}{2}\right)u(x_0)+C_1(r\theta)^{-C(n)}\right]^{\varepsilon}\\
&\cdot\left[\left(\left(1-\frac{\theta}{2}\right)^{-\tau}-\frac{1}{2}\right)u(x_0)\right]^{-\varepsilon},
\end{aligned}\tag{5.8}$$

where

$$C_2=\left[\sum_{i=1}^{n-1}\frac{n+\sigma_{\max}}{n+\sigma_i}-(n-1)\right]$$

and where we have used that $0 < C_2 \leq C_1(n)$. Thus, using (5.8) and the elementary inequalities

$$\left[\left(\left(1-\frac{\theta}{2}\right)^{-\tau}-\frac{1}{2}\right)u(x_0)+C_1(r\theta)^{-C(n)}\right]^{\varepsilon}$$

$$\leq \left(\left(1-\frac{\theta}{2}\right)^{-\tau}-\frac{1}{2}\right)^{\varepsilon}u(x_0)^{\varepsilon}+C_1(r\theta)^{-C(n)\varepsilon}$$

and

$$\left(1-\frac{\theta}{2}\right)^{-\tau}-\frac{1}{2}\geq\left(1-\frac{\theta}{2}\right)^{-\frac{n}{\varepsilon}}-\frac{1}{2}\geq\frac{1}{2},$$

for $\theta > 0$ sufficiently small, and yet

$$C_3\theta^{-C(n)\varepsilon}r^{-C(n)\varepsilon}u(x_0)^{-\varepsilon}\left(\left(1-\frac{\theta}{2}\right)^{-\tau}-\frac{1}{2}\right)^{-\varepsilon}$$

$$\leq C_4\theta^{-C(n)\varepsilon}r^{-C(n)\varepsilon}u(x_0)^{-\varepsilon}\leq C_5\theta^{-C(n)\varepsilon}t^{-\varepsilon}d^{n\left[1-\tilde{C}\varepsilon\right]}\leq C_6\theta^{-C\varepsilon}t^{-\varepsilon},$$

we obtain

$$\left|\left\{u>\frac{u(x_0)}{2}\right\}\cap E^{\max}_{\frac{r\theta}{2},\frac{1}{2}}\right|\leq C\left|E^{\max}_{\frac{r\theta}{2},\frac{1}{2}}\right|\left[\left(\left(1-\frac{\theta}{2}\right)^{-\tau}-1\right)^{\varepsilon}+\theta^{-C\varepsilon}t^{-\varepsilon}\right].$$

Now we choose $\theta > 0$ sufficiently small such that

$$\begin{aligned} C\left|E^{\max}_{\frac{r\theta}{2},\frac{1}{2}}\right|\left[\left(1-\frac{\theta}{2}\right)^{-\tau}-1\right]^{\varepsilon} &\leq C\left|E^{\max}_{\frac{r\theta}{2},\frac{1}{2}}\right|\left[\left(1-\frac{\theta}{2}\right)^{-\frac{2n}{\varepsilon}}-1\right]^{\varepsilon}\\ &\leq \frac{1}{4}\left|E^{\max}_{\frac{r\theta}{2},\frac{1}{2}}\right|. \end{aligned}$$

Having fixed $\theta > 0$ (independently of t), we take $t > 0$ sufficiently large such that

$$C\left|E^{\max}_{\frac{r\theta}{2},\frac{1}{2}}\right|\theta^{-C\varepsilon}t^{-\varepsilon}\leq\frac{1}{4}\left|E^{\max}_{\frac{r\theta}{2},\frac{1}{2}}\right|.$$

Then, using (5.8), we find

$$\left|\left\{u>\frac{u(x_0)}{2}\right\}\cap E^{\max}_{\frac{r\theta}{2},\frac{1}{2}}\right|\leq\frac{1}{4}\left|E^{\max}_{\frac{r\theta}{2},\frac{1}{2}}\right|.$$

Hence, we have, for $t > 0$ large,

$$\begin{aligned} \left|\left\{u<\frac{u(x_0)}{2}\right\}\cap E^{\max}_{\frac{r\theta}{2},\frac{1}{2}}\right| &\geq c\theta^{1+\sum_{i=1}^{n-1}\frac{n+\sigma_{\max}}{n+\sigma_i}}\left|E^{\max}_{r,1}\right|\\ &\geq c_2\left|E^{\max}_{r,1}\right|, \end{aligned}$$

which is a contradiction to (5.3). □

As a consequence of the Harnack inequality we obtain the C^γ regularity.

Theorem 5.8 (C^{γ} estimates)**.** *Let u be a bounded function such that*

$$M^{-}u \leq C_0 \quad \text{and} \quad M^{+}u \geq -C_0 \text{ in } B_1.$$

If $(0,2) \ni \sigma_0 < \sigma_{\min}$, then there is a positive constant $0 < \gamma < 1$, that depends only n, λ, Λ and σ_0, such that $u \in C^{\gamma}\left(B_{1/2}\right)$ and

$$|u|_{C^{\gamma}\left(B_{1/2}\right)} \leq C\left(\sup_{\mathbb{R}^n}|u| + C_0\right),$$

for some constant $C > 0$.

The next result is a consequence of the arguments used in [4] and Theorem 5.8. As in [4], if we suppose a modulus of continuity of $K_{\alpha\beta}$ in measure, so as to make sure that faraway oscillations tend to cancel out, we obtain the interior $C^{1,\gamma}$ regularity for solutions of equation $Iu = 0$.

Theorem 5.9 ($C^{1,\gamma}$ estimates)**.** *Suppose that $0 < \sigma_0 < \sigma_{\min}$. There exists a constant $\tau_0 > 0$, that depends only on λ, Λ, n and σ_0, such that*

$$\int_{\mathbb{R}^n \setminus B^{\tau_0}} \frac{|K_{\alpha\beta}(y) - K_{\alpha\beta}(y-h)|}{|h|} dy \leq C_0, \qquad \text{whenever } |h| < \frac{\tau_0}{2}.$$

If u is a bounded function satisfying $Iu = 0$ in B_1, then there is a constant $0 < \gamma < 1$, that depends only n, λ, Λ and σ_0, such that $u \in C^{1,\gamma}\left(B_{1/2}\right)$ and

$$|u|_{C^{1,\gamma}\left(B_{1/2}\right)} \leq C \sup_{\mathbb{R}^n}|u|,$$

for some constant $C = C(n, \lambda, \Lambda, \sigma_0, C_0) > 0$.

Remark 5.10. We can also get C^{γ} and $C^{1,\gamma}$ estimates for truncated kernels, i.e., kernels that satisfy (1.3) only in a neighborhood of the origin. Let $\mathcal{L}$ be the class of operators $L_{\alpha\beta}$ such that the corresponding kernels $K_{\alpha\beta}$ have the form

$$K_{\alpha\beta}(y) = K_{\alpha\beta,1}(y) + K_{\alpha\beta,2}(y) \geq 0,$$

where

$$\frac{\lambda c_\sigma}{\sum_{i=1}^{n}|y_i|^{n+\sigma_i}} \leq K_{\alpha\beta,1}(y) \leq \frac{\Lambda c_\sigma}{\sum_{i=1}^{n}|y_i|^{n+\sigma_i}}$$

and $K_{\alpha\beta,2} \in L^1(\mathbb{R}^n)$ with $\|K_{\alpha\beta,2}\|_{L^1(\mathbb{R}^n)} \leq c_0$, for some constant $c_0 > 0$. The class $\mathcal{L}$ is larger than $\mathcal{L}_0$ but the extremal operators $M^{-}_{\mathcal{L}}$ and $M^{+}_{\mathcal{L}}$ are controlled by M^{+} and M^{-} plus the L^{∞} norm of u (see Lemma 14.1 and Corollary 14.2 in [4]). Thus the interior C^{γ} and $C^{1,\gamma}$ regularity follow.

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