PARALLEL KUSTIN–MILLER UNPROJECTION WITH AN APPLICATION TO CALABI–YAU GEOMETRY

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ABSTRACT. Kustin–Miller unprojection is a process by which one can construct interesting new Gorenstein rings starting from simpler ones. Geometrically, it inverts certain projections and appears in the constructions of explicit birational geometry. It is often desirable to perform not only one but a series of unprojections. The main aim of the present paper is to develop a theory, which we call *parallel Kustin–Miller unprojection*, that applies when all the unprojection ideals of a series of unprojections correspond to ideals already present in the initial ring. As an application, we explicitly construct 7 families of Calabi–Yau 3-folds of high codimensions.

1. INTRODUCTION

Following Miles Reid [22], unprojection aims to study graded rings in terms of simpler ones using adjunction. Geometrically, unprojection is an inverse of certain projections. The simplest type of unprojection is the Kustin–Miller unprojection (or type I), which is originally due to Andrew Kustin and Matthew Miller [16], and was later studied by Reid and the second author in a scheme-theoretic formulation [20, 21].

According to [21, Definition 1.2], Kustin–Miller unprojection is specified by the data of a Gorenstein local ring R and a codimension 1 ideal $I \subset R$ with the quotient ring R/I being Gorenstein. Under these assumptions $\operatorname{Hom}_R(I, R)$ is generated as an R-module by the inclusion $I \to R$ and an extra homomorphism φ. The Kustin–Miller unprojection ring of the pair $I \subset R$ is by definition the quotient

$$R_{\text{un}} = R[S]/(Sr - \varphi(r) \mid r \in I), \tag{1.1}$$

where S is a new variable. By [21] R_{un} is again a Gorenstein ring.

The main philosophy in using Kustin–Miller unprojection in projective geometry is to transform by means of a birational modification a projectively Gorenstein scheme into another projectively Gorenstein scheme whose invariants we control

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and which can be taken as a suitable ambient space for constructing varieties as complete intersections. Kustin–Miller unprojection has found many applications in algebraic geometry, for example in the birational geometry of Fano 3-folds [8, 9], in the construction of $K3$ surfaces and Fano 3-folds inside weighted projective spaces [1, 6], and in the study of Mori flips [7]. It has also been used in the geometry of Calabi–Yau 3-folds [4, 14, 15].

For some applications it is desirable to perform not only one but a series of Kustin–Miller unprojections. This idea is already present in the $\binom{n}{2}$ Pfaffians format introduced in [17] to construct Campedelli surfaces with torsion $\mathbb{Z}/6$. The main aim of the present paper is to develop a theory of parallel unprojection, which, under certain assumptions, enables one to perform explicitly in a single step a series of Kustin–Miller unprojections. Ideally, parallel Kustin–Miller unprojection would start from a Gorenstein positively graded ring R and a finite set $\{I_a \mid a \in \mathcal{L}\}$ of codimension 1 homogeneous ideals of R yielding Gorenstein quotients. However such a procedure does not exist in general, if no additional assumptions are imposed. One of the aims of this work is to find a set of simple sufficient conditions on the set of ideals guaranteeing the series of unprojections. In addition we give a relatively simple and symmetric description of the end product ring, which we denote by $R_{\mathcal{L}}$, as a quotient of the polynomial ring over R in $\#\mathcal{L}$ variables by an explicitly given ideal. Choose an ordering $\mathcal{L} = \{w_1, \dots, w_n\}$ of the elements of $\mathcal{L}$ and set $\mathcal{L}_i = \{w_1, \dots, w_i\}$. We show that for $i = 1, \dots, n-1$ there exists a graded ring $R_{\mathcal{L}_i}$, quotient of $R\big[y_u \mid u \in \mathcal{L}_i\big]$, such that the natural map $R \to R_{\mathcal{L}_i}$ is injective. Setting $R_{\mathcal{L}_0} = R$, we show that for $i = 0, \dots, n-1$ there exists a homogeneous ideal $J_{\mathcal{L}_i, w_{i+1}} \subset R_{\mathcal{L}_i}$ with quotient isomorphic to $R/I_{w_{i+1}}$ such that $R_{\mathcal{L}_{i+1}}$ is the Kustin–Miller unprojection of the pair $J_{\mathcal{L}_i, w_{i+1}} \subset R_{\mathcal{L}_i}$, thereby obtaining a series of unprojections

$$R = R_{\mathcal{L}_0} \to R_{\mathcal{L}_1} \to \cdots \to R_{\mathcal{L}_n} = R_{\mathcal{L}}.$$

We also study in more detail the case when R is a complete intersection inside a polynomial ring and obtain a Gorenstein format. Finally, as an application we produce 7 families of Calabi–Yau 3-folds embedded with high codimensions, including one with codimension 21. These families, together with some numerical invariants, are listed in Table I. We computed the Hodge numbers for Cases 3.1, 3.2 and 3.3 using the fact that in these cases X is birational to a nodal complete intersection, and for Case 1.1 using that X is a $(3,3)$-divisor inside $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$. For the remaining cases, we could not find similar descriptions and due to the high codimensions of the embeddings our efforts to compute the Hodge numbers by counting rational points over finite fields were unsuccessful.

Gavin Brown's online database [5] contains a large number of candidate $K3$ surfaces and Fano 3-folds of high codimension, which according to the philosophy of [2] may be related by a series of Kustin–Miller unprojections to varieties of low codimensions 1, 2 or 3. We believe that the ideas of the present work together with explicit equation and singularity calculations can establish the existence of many of them. The ideas and techniques of this paper are used in [19] where the authors describe a component of the moduli space of Burniat surfaces with $K^2 = 3$.

Case	Embedding	deg(X)	Sing(X)	Hodge numbers†
1.1	$X \subset \mathbb{P}^8$	18	$\emptyset$	$h^{1,1} = 2,\ h^{1,2} = 83$
1.2	$X \subset \mathbb{P}^{10}$	24	$\emptyset$	?
2.1	$X \subset \mathbb{P}(1^6, 2^3)$	12	$\emptyset$	?
2.2	$X \subset\ \mathbb{P}(1^8, 2^7)$	24	$\emptyset$	?
3.1	$X \subset \mathbb{P}(1^6, 3^5)$	$\frac{20}{3}$	$5 \times \frac{1}{3}(1, 1, 1)$	$h^{1,1} = 6,\ h^{1,2} = 36$
3.2	$X \subset \mathbb{P}(1^6, 3^9)$	12	$9 \times \frac{1}{3}(1, 1, 1)$	$h^{1,1} = 21,\ h^{1,2} = 12$
3.3	$X \subset \mathbb{P}(1^8, 3^{16})$	$\frac{64}{3}$	$16 \times \frac{1}{3}(1, 1, 1)$	$h^{1,1} = 27,\ h^{1,2} = 11$

† Hodge numbers of a crepant resolution of X.

Table I. List of families of Calabi–Yau 3-folds.

The structure of the paper is as follows. Section 2 introduces the setting of parallel Kustin–Miller unprojection. We give a set of sufficient conditions on a ring R and a set of ideals $\{I_a \mid a \in \mathcal{L}\}$ that enable the unprojection of the ideals. Then, in Definition 2.2, for each subset $\mathcal{M} \subset \mathcal{L}$ we define explicitly and in a symmetric way a new ring $R_{\mathcal{M}}$, which corresponds to the unprojection of the ideals indexed by $\mathcal{M}$. A key ingredient in this definition is the existence of $A_{ba} \in R$, for $a \neq b \in \mathcal{L}$, established in Proposition 2.1. The main result is Theorem 2.4, which shows that $R_{\mathcal{L}}$ is the end product of a series of Kustin–Miller unprojections. This theorem is proved in Subsection 2.3 using the technical results developed in Subsections 2.1 and 2.2.

Sections 3 and 4 contain examples and applications. In Section 3 we study in more detail a complete intersection case format. In particular, we describe explicitly the relations of the new ring and, in Proposition 3.6, we compute its canonical module and degree. In Subsection 4.1 we sketch the explicit construction, via parallel Kustin–Miller unprojection, of the 7 families of Calabi–Yau 3-folds of Table I. In Remark 4.3 we make some comments about their geometry leading to the computation of the Hodge numbers of the Calabi–Yau 3-folds in Case 1.1 and the Hodge numbers of a crepant resolution in cases 3.1, 3.2 and 3.3.

Finally, Subsection 4.2 contains a detailed treatment of the construction of one of the families, which consists of degree 12 Calabi–Yau 3-folds inside $\mathbb{P}(1^6, 3^9)$. For this case we start with a complete intersection of 2 cubics in $\mathbb{P}^8$ containing a configuration of 9 linear subspaces of dimension 5, any two of which intersect at most along a 3-dimensional subspace. We then use parallel Kustin–Miller unprojection with this initial data to produce a birationally equivalent 6-fold inside $\mathbb{P}(1^9, 3^9)$. The Calabi–Yau 3-fold is obtained by intersecting this 6-fold with 3 general degree 1 hypersurfaces. Most of the subsection is devoted to establishing the quasismoothness. It follows from the explicit nature of the construction that the only singularities of the Calabi–Yau 3-fold are 9 isolated quotient singularities of type $\frac{1}{3}(1, 1, 1)$.

2. Statement and proof of the main theorem

It the following we assume that $R = \oplus_{n\geq 0} R_n$ is a Gorenstein graded ring with R_0 a field, $\mathcal{L}$ is a nonempty finite indexing set, and for all $a \in \mathcal{L}$, I_a is a codimension 1 homogeneous ideal of R such that the quotient ring R/I_a is Gorenstein. Moreover, we fix graded R-module homomorphisms $\varphi_a \colon I_a \to R$, such that $\operatorname{Hom}_R(I_a, R)$ is generated as an R-module by $\{i_a, \varphi_a\}$, where i_a denotes the inclusion $I_a \to R$. We assume that for all $a \in \mathcal{L}$ the degree of the homomorphism φ_a is positive, that for distinct $a, b \in \mathcal{L}$ there exist a homogeneous element $C_{ab} \in R$ with $\deg C_{ab} = \deg \varphi_a$ such that

$$(\varphi_a + C_{ab} i_a)(I_a) \subset I_b, \tag{2.1}$$

and that for all distinct $a, b \in \mathcal{L}$

$$\operatorname{codim}_R(I_a + I_b) \geq 2. \tag{2.2}$$

For simplicity of notation, for 2 distinct indices $a, b \in \mathcal{L}$ we set

$$\varphi_{ab} = \varphi_a + C_{ab} i_a.$$

We will use that for 3 distinct indices $a, b, c \in \mathcal{L}$ we have

$$\varphi_{ac} = \varphi_{ab} + (C_{ac} - C_{ab}) i_a.$$

The proof of the following Proposition will be given in Subsection 2.2.

Proposition 2.1. *Fix distinct $a, b \in \mathcal{L}$. There exists unique $A_{ba} \in R$ such that*

$$\varphi_{ba}(\varphi_{ab}(r)) = A_{ba} r \tag{2.3}$$

for all $r \in I_a$. A_{ba} is a homogeneous element of R with

$$\deg A_{ba} = \deg \varphi_a + \deg \varphi_b,$$

$A_{ba} = A_{ab}$ and

$$(C_{ba} - C_{bc})(C_{ab} - C_{ac}) - A_{ab} \in I_c \tag{2.4}$$

for all $c \in \mathcal{L} \setminus \{a, b\}$.

Definition 2.2. For a nonempty subset $\mathcal{M} \subset \mathcal{L}$ we denote $R_{\mathcal{M}}$ the ring given by the quotient of the polynomial ring $R[y_u \mid u \in \mathcal{M}]$, where $\{y_u \mid u \in \mathcal{M}\}$ is a set of new variables indexed by $\mathcal{M}$, by the ideal generated by the set

$$\{y_u r - \varphi_u(r) \mid u \in \mathcal{M}, r \in I_u\} \cup \{(y_v + C_{vu})(y_u + C_{uv}) - A_{vu} \mid u, v \in \mathcal{M}, u \neq v\}$$

while for $\mathcal{M} = \emptyset$ we set $R_\emptyset = R$. We extend the grading of R to a grading of $R[y_u \mid u \in \mathcal{M}]$ by setting $\deg y_u = \deg \varphi_u$. Since the ideal defining $R_{\mathcal{M}}$ is homogeneous, $R_{\mathcal{M}}$ becomes a graded ring. Given $w \in \mathcal{L} \setminus \mathcal{M}$, we denote $J_{\mathcal{M},w} \subset R_{\mathcal{M}}$ the ideal of $R_{\mathcal{M}}$ generated by the image of the subset

$$I_w \cup \{y_u + C_{uw} \mid u \in \mathcal{M}\}$$

of the polynomial ring $R[y_u \mid u \in \mathcal{M}]$ under the natural ring homomorphism $R[y_u \mid u \in \mathcal{M}] \to R_{\mathcal{M}}$.

Remark 2.3. The geometric meaning of the ideal $J_{\mathcal{M},w} \subset R_{\mathcal{M}}$ is the following. Denote by $I \subset R$ the ideal of R generated by the subset $\cup_{u\in\mathcal{M}} I_u$ (in other words I is the sum of the ideals $I_u, u \in \mathcal{M}$), and by $I^e \subset R_{\mathcal{M}}$ the ideal of $R_{\mathcal{M}}$ generated by the image of I under the natural ring homomorphism $R \to R_{\mathcal{M}}$. The homomorphism $R \to R_{\mathcal{M}}$ induces a scheme morphism $\operatorname{Spec} R_{\mathcal{M}} \to \operatorname{Spec} R$, which restricts to an isomorphism of schemes

$$\operatorname{Spec} R_{\mathcal{M}} \setminus V(I^e) \to \operatorname{Spec} R \setminus V(I),$$

cf. Corollary 2.8 below. The ideal $I_w \subset R$ defines a closed subscheme of $\operatorname{Spec} R$, hence a closed subscheme of $\operatorname{Spec} R \setminus V(I)$, and using the above scheme isomorphism a closed subscheme, say F, of $\operatorname{Spec} R_{\mathcal{M}} \setminus V(I^e)$. Denote by $i\colon \operatorname{Spec} R_{\mathcal{M}} \setminus V(I^e) \to \operatorname{Spec} R_{\mathcal{M}}$ the inclusion morphism. One can show that $J_{\mathcal{M},w}$ is the ideal of $R_{\mathcal{M}}$ corresponding to the scheme-theoretic image (in the sense of [11, Section V.1.1]) $\bar{i}(F)$, which, by definition, is a closed subscheme of $\operatorname{Spec} R_{\mathcal{M}}$.

For simplicity of notation, for distinct $a, b \in \mathcal{L}$ with $a \in \mathcal{M}$ we set

$$y_{ab} = y_a + C_{ab} \in R_{\mathcal{M}}.$$

We will use that for 3 distinct indices $a, b, c \in \mathcal{L}$ with $a \in \mathcal{M}$

$$y_{ac} = y_{ab} + (C_{ac} - C_{ab}).$$

In addition, for distinct $a, b \in \mathcal{L}$ with $a \in \mathcal{M}$ we define the element

$$D_{ab} = A_{ab} - y_{ab} C_{ba}$$

of $R_{\mathcal{M}}$. The meaning of D_{ab} will be clarified in the following theorem, the proof of which will be given in Subsection 2.3.

Theorem 2.4. *Let $\mathcal{M} \subset \mathcal{L}$ be a subset.*

(1) The ring $R_{\mathcal{M}}$ is Gorenstein with $\dim R_{\mathcal{M}} = \dim R$, and the natural map $R \to R_{\mathcal{M}}$ is injective.

(2) Assume there exists $w \in \mathcal{L} \setminus \mathcal{M}$. Then the map $\varphi_w\colon I_w \to R$ has an extension to an $R_{\mathcal{M}}$-homomorphism $\Phi_{\mathcal{M},w}\colon J_{\mathcal{M},w} \to R_{\mathcal{M}}$ uniquely specified by the property $\Phi_{\mathcal{M},w}(y_{uw}) = D_{uw}$, for all $u \in \mathcal{M}$.

(3) With assumptions as in (2), the ideal $J_{\mathcal{M},w}$ of $R_{\mathcal{M}}$ has codimension 1 and the quotient ring $R_{\mathcal{M}}/J_{\mathcal{M},w}$ is isomorphic to R/I_w, hence it is Gorenstein. Moreover, the $R_{\mathcal{M}}$-module $\operatorname{Hom}_{R_{\mathcal{M}}}(J_{\mathcal{M},w}, R_{\mathcal{M}})$ is generated by $\{i_{\mathcal{M},w}, \Phi_{\mathcal{M},w}\}$, where $i_{\mathcal{M},w}\colon J_{\mathcal{M},w} \to R_{\mathcal{M}}$ is the natural inclusion map, and the ring $R_{\mathcal{M}\cup\{w\}}$ is the Kustin–Miller unprojection ring of the pair $J_{\mathcal{M},w} \subset R_{\mathcal{M}}$.

2.1. **Some useful general properties of unprojection.** The main aim of this subsection is to prove Corollary 2.8 which gives general properties of Kustin–Miller unprojection needed in the proof of Theorem 2.4. Unless otherwise mentioned, in this subsection R denotes a commutative ring with identity (not necessarily graded or Noetherian), $I = (f_1, \dots, f_n) \subset R$ is a finitely generated ideal, and $\varphi\colon I \to R$ is an R-homomorphism. Considering $R[S]$, the polynomial ring over R with variable S and setting $g_i = \varphi(f_i)$, for $1 \le i \le n$, we define an ideal $J \subset R[S]$ by:

$$J = (Sf_1 - g_1, \dots, Sf_n - g_n) \subset R[S].$$

Lemma 2.5. *Let* $f = \sum_{j=0}^{m} b_j S^j \in R[S]$*, with possibly* $b_m = 0$*. Then* $f \in J$ *if and only if* $u_0 := b_m \in I$*,* $u_k := b_{m-k} + \varphi(u_{k-1}) \in I$*, for every* $k = 1, \ldots, m$*, and* $u_m = 0$*.*

Proof. Suppose $f \in J$. Then there exist $m' > m$ and $q_i = \sum_{j=0}^{m'} a_{ij} S^j \in R[S]$ with

$$\sum_{j=0}^{m} b_j S^j = \sum_{i=1}^{n} q_i(Sf_i - g_i) = \sum_{j=0}^{m'} \sum_{i=1}^{n} (a_{ij} f_i S^{j+1} - a_{ij} g_i S^j). \tag{2.5}$$

Setting $b_j = 0$, for $m < j \le m' + 1$ and taking coefficients in (2.5) we deduce that:

$$b_0 = -\sum_{i=1}^{n} a_{i0} g_i, \quad b_j = \sum_{i=1}^{n} a_{i(j-1)} f_i - \sum_{i=1}^{n} a_{ij} g_i \quad \text{and} \quad b_{m'+1} = \sum_{i=1}^{n} a_{im'} f_i, \tag{2.6}$$

for all $1 \le j \le m'$. We proceed by induction on m. We first do the case $m = 0$, so assume $f = b_0 \in J$. Since $g_i = \varphi(f_i)$, using (2.6) we get:

$$\begin{aligned} b_0 &= -\textstyle\sum_{i=1}^{n} a_{i0} g_i = -\varphi\big(\sum_{i=1}^{n} a_{i0} f_i\big) = -\varphi\big(\sum_{i=1}^{n} a_{i1} g_i\big) \\ &= -\varphi^2\big(\textstyle\sum_{i=1}^{n} a_{i1} f_i\big) = \cdots = -\varphi^{m'+1}\big(\sum_{i=1}^{n} a_{im'} f_i\big) = 0. \end{aligned}$$

Suppose now that $m > 0$ and that the result holds for any polynomial h of degree in S less than m. Using (2.6) we get:

$$\begin{gathered} 0 = b_{m'+1} = \textstyle\sum_{i=1}^{n} a_{im'} f_i \implies \sum_{i=1}^{n} a_{im'} g_i = 0 \implies \\ 0 = b_{m'} = \textstyle\sum_{i=1}^{n} a_{i(m'-1)} f_i \implies \cdots \implies b_m = \sum_{i=1}^{n} a_{i(m-1)} f_i \in I. \end{gathered}$$

Consider $h = \sum_{j=0}^{m-1} b_j S^j + \varphi(b_m) S^{m-1}$. Since $S b_m - \varphi(b_m) \in J$ and

$$f = h + S^{m-1}(S b_m - \varphi(b_m)) \tag{2.7}$$

we deduce that $f \in J \iff h \in J$. Then, by induction, writing u'_k for the sequence associated to the expression of h, we deduce that $u_1 = b_{m-1} + \varphi(b_m) = u'_0 \in I$,

$$u_{k+1} = b_{m-1-k} + \varphi(u_k) = b_{m-1-k} + \varphi(u'_{k-1}) = u'_k \in I$$

for every $k = 1, \ldots, m-1$, and $u_m = u'_{m-1} = 0$.

Conversely, assume $u_0 = b_m \in I$, $u_k = b_{m-k} + \varphi(u_{k-1}) \in I$, for all $k = 1, \ldots, m$, and $u_m = 0$. Since

$$f = S^{m-1}(S u_0 - \varphi(u_0)) + S^{m-2}(S u_1 - \varphi(u_1)) + \cdots + (S u_{m-1} - \varphi(u_{m-1})) + u_m,$$

we get $f \in J$. ∎

Proposition 2.6. *Let* R *and* $J \subset R[S]$ *be as above. Then:*

1. *the natural homomorphism* $R \to R[S]/J$ *induced by* $R \subset R[S]$ *is injective;*
2. *if* $w \in R[S]/J$ *is such that* $wS \in R \subset R[S]/J$ *then* $w \in I \subset R \subset R[S]/J$*.*

Proof. To prove (1) it is enough to show that if $r \in R$ and $r \in J$ then $r = 0$. This is straightforward from Lemma 2.5. We now prove (2). Write $w = f + J$ for some $f = \sum_{j=0}^{m} b_j S^j$. Let $r \in R$ be such that $fS - r \in J$. By Lemma 2.5, let us consider the sequence u'_k associated to $fS - r$. Then $u'_0 = b_m \in I$ and $u'_k = b_{m-k} + \varphi(u'_{k-1}) \in I$, for $k = 1, \ldots, m$. It suffices to show that $f - u'_m \in J$, which by Lemma 2.5 is true, since its associated sequence u_k is such that $u_k = u'_k$, for $k = 1, \dots, m-1$ and $u_m = b_0 - u'_m + \varphi(u_{m-1}) = b_0 - u'_m + \varphi(u'_{m-1}) = b_0 - u'_m + u'_m - b_0 = 0$. □

Remark 2.7. It is easy to see that the morphism $\operatorname{Spec} R[S]/J \to \operatorname{Spec} R$ induced by the homomorphism $R \to R[S]/J$ restricts to an isomorphism of schemes

$$(\operatorname{Spec} R[S]/J) \setminus V(I^e) \cong \operatorname{Spec} R \setminus V(I),$$

where I^e denotes the ideal of $R[S]/J$ generated by the image of I under the map $R \to R[S]/J$.

Corollary 2.8. *Assume R is a Gorenstein local ring, $I \subset R$ is a codimension 1 ideal with R/I Gorenstein. Denote by R_{un}, as in (1.1), the unprojection ring of the pair $I \subset R$, by $s \in R_{\text{un}}$ the image of S under the quotient map $R[S] \to R_{\text{un}}$, and by I^e the ideal of R_{un} generated by the image of I under the natural map $R \to R_{\text{un}}$. Then the natural map $R \to R_{\text{un}}$ is injective and induces an isomorphism of schemes*

$$\operatorname{Spec} R_{\text{un}} \setminus V(I^e) \cong \operatorname{Spec} R \setminus V(I).$$

Moreover, assume $a \in R_{\text{un}}$. If $as \in R$, then $a \in I \subset R$.

Proof. The injectivity and the last statement follows from Proposition 2.6, while the isomorphism of schemes from Remark 2.7. □

2.2. **Proof of Proposition 2.1.** Assume M is an R-module, and $r \in R$ an element. We say that r is M-regular if the multiplication by r homomorphism $M \to M$, $m \mapsto rm$ is injective. For the proof of Proposition 2.1 we will need the following general lemma.

Lemma 2.9. *Assume R is a Noetherian commutative ring with identity, $I \subset R$ an ideal of R and $M_1, \dots, M_n$ a finite number of finitely generated R-modules. If for every i, with $1 \le i \le n$, exists $a_i \in I$ which is M_i-regular, then there exists $a \in I$ which is M_i-regular for all $1 \le i \le n$.*

Proof. It is well-known ([10, Theorem 3.1]) that for fixed i the subset $\mathcal{U}_i \subset R$ consisting of the elements of R which are not M_i-regular is a finite union of prime ideals. By the assumptions of the Lemma, for fixed i the ideal I is not a subset of $\mathcal{U}_i$. Therefore, by prime avoidance ([10, Lemma 3.3]) I is not a subset of the union $\cup_{i=1}^n \mathcal{U}_i$, which finishes the proof of Lemma 2.9. □

Lemma 2.10. *Fix $a \in \mathcal{L}$. There exists $r_a \in I_a$ which is R-regular, and R/I_c-regular for all $c \in \mathcal{L} \setminus \{a\}$.*

Proof. Since R is Gorenstein, hence Cohen–Macaulay, and $I_a \subset R$ has codimension 1, there exist an R-regular element contained in I_a. Assume $c \in \mathcal{L} \setminus \{a\}$. Using Assumption (2.1) the ideal $I_a + I_c$ of R/I_c has codimension at least 1. Since by our assumptions R/I_c is Gorenstein, hence Cohen–Macaulay, we have that $I_a + I_c$ contains an R/I_c-regular element. Consequently, I_a contains an R/I_c-regular element. The result follows from Lemma 2.9. □

Lemma 2.11. *Fix distinct $a, b \in \mathcal{L}$, and $r_a \in I_a$ which is R-regular, and R/I_c-regular for all $c \in \mathcal{L} \setminus \{a\}$, such an element exists by Lemma 2.10. There exists $r_b \in I_b$, which is R-regular, $R/(r_a)$-regular and R/I_c-regular for all $c \in \mathcal{L} \setminus \{b\}$. In particular, both r_a, r_b and r_b, r_a are R-regular sequences.*

Proof. Arguing as in the proof of Lemma 2.10 and using [10, Exercise 17.4], it is enough to show that I_b contains an $R/(r_a)$-regular element. Since r_a is R-regular, the ideal $(r_a) \subset R$ has codimension 1 in R and the quotient $R/(r_a)$ is Gorenstein, hence Cohen–Macaulay. Since R/I_b is Gorenstein and r_a is an R/I_b-regular element, the ideal $(r_a) + I_b$ has codimension in R exactly 2, so the ideal $I_b + (r_a)$ has codimension in $R/(r_a)$ equal to 1, hence, since $R/(r_a)$ is Cohen–Macaulay, it contains an $R/(r_a)$-regular element. Consequently, I_b contains an $R/(r_a)$-regular element. □

We now start the proof of Proposition 2.1. Fix $r_a \in I_a, r_b \in I_b$ with the properties stated in Lemma 2.11. Since by Assumption (2.1) $\varphi_{ab}(I_a) \subset I_b$, we have

$$r_b\varphi_{ba}(\varphi_{ab}(r_a)) = \varphi_{ba}(r_b\varphi_{ab}(r_a)) = \varphi_{ab}(r_a)\varphi_{ba}(r_b) = r_a\varphi_{ab}(\varphi_{ba}(r_b)). \tag{2.8}$$

Since r_a, r_b and r_b, r_a are R-regular sequences, there exist $A_{ba}, A_{ab} \in R$ such that

$$\varphi_{ba}(\varphi_{ab}(r_a)) = r_a A_{ba} \quad \text{and} \quad \varphi_{ab}(\varphi_{ba}(r_b)) = r_b A_{ab}.$$

The elements A_{ba}, A_{ab} are unique since both r_a and r_b are R-regular. Substituting in (2.8) we get $r_b r_a A_{ba} = r_a r_b A_{ab}$, and using that the product $r_a r_b$ is R-regular, we get $A_{ba} = A_{ab}$. Assume $r \in I_a$. We have

$$r_a r A_{ba} = r(r_a A_{ba}) = r\varphi_{ba}(\varphi_{ab}(r_a)) = \varphi_{ba}(\varphi_{ab}(r r_a)) = r_a\varphi_{ba}(\varphi_{ab}(r)),$$

since r_a is R-regular we get $\varphi_{ba}(\varphi_{ab}(r)) = rA_{ba}$, which proves (2.3).

We now prove that A_{ba} is homogeneous of the stated degree. Denote by A'_{ba} the homogeneous component of A_{ba} of degree equal to $\deg \varphi_a + \deg \varphi_b$. For $r \in I_b$ homogeneous, (2.3) implies by comparing homogeneous components that $rA'_{ba} = rA_{ba}$. Taking into account that I_b is a homogeneous ideal of R, we get that $rA'_{ba} = rA_{ba}$ for all $r \in I_b$. Combining it with (2.3) and the already proven uniqueness of A_{ba} it follows that $A_{ba} = A'_{ba}$.

We now prove (2.4). Fix $c \in \mathcal{L} \setminus \{a, b\}$. We have

$$\begin{aligned}
A_{ab}r_a = \varphi_{ba}(\varphi_{ab}(r_a)) &= (\varphi_b + C_{ba}i_b)\left[(\varphi_a + C_{ab}i_a)(r_a)\right] \\
&= (\varphi_b + C_{bc}i_b)\left[(\varphi_a + C_{ab}i_a)(r_a)\right] + (C_{ba} - C_{bc})\left[(\varphi_a + C_{ab}i_a)(r_a)\right] \\
&= (\varphi_b + C_{bc}i_b)\left[(\varphi_a + C_{ab}i_a)(r_a)\right] + (C_{ba} - C_{bc})\left[(\varphi_a + C_{ac}i_a)(r_a)\right] \\
&\qquad + (C_{ba} - C_{bc})(C_{ab} - C_{ac})r_a.
\end{aligned}$$

Consequently, using that by Assumption (2.1) $\varphi_{ab}(I_a) \subset I_b$, $\varphi_{bc}(I_b) \subset I_c$ and $\varphi_{ac}(I_a) \subset I_c$ we get that $A_{ab}r_a - (C_{ba} - C_{bc})(C_{ab} - C_{ac})r_a \in I_c$. Since r_a is R/I_c-regular, we deduce (2.4), which finishes the proof of Proposition 2.1.

2.3. **Proof of Theorem 2.4.** In this subsection we use the notations introduced in the beginning of this section. We will need the following 3 lemmas for the proof of Theorem 2.4.

Lemma 2.12. *Assume $\mathcal{M} \subset \mathcal{L}$ is a nonempty subset, and $w \in \mathcal{L} \setminus \mathcal{M}$. The natural map $R \to R_{\mathcal{M}}$ induces an isomorphism*

$$R/I_w \cong R_{\mathcal{M}}/J_{\mathcal{M},w}.$$

In particular, the quotient ring $R_{\mathcal{M}}/J_{\mathcal{M},w}$ is Gorenstein, and if $\dim R_{\mathcal{M}} = \dim R$ then $J_{\mathcal{M},w}$ is a codimension 1 ideal of $R_{\mathcal{M}}$.

Proof. Since for distinct $u, v \in \mathcal{M}$

$$y_{uv}y_{vu} - A_{uv} = (y_{uw} + C_{uv} - C_{uw})(y_{vw} + C_{vu} - C_{vw}) - A_{uv},$$

using the definition of $J_{\mathcal{M},w}$ it follows that $R_{\mathcal{M}}/J_{\mathcal{M},w} \cong R/(I_1 + I_w)$, where $I_1 \subset R$ is the ideal generated by the set

$$\{\varphi_{uw}(r) \mid u \in \mathcal{M}, r \in I_u\} \cup \{(C_{uv} - C_{uw})(C_{vu} - C_{vw}) - A_{uv} \mid u, v \in \mathcal{M}, u \neq v\}.$$

To prove Lemma 2.12 it is enough to show that $I_1 \subset I_w$, and this follows by combining Assumption (2.1) with Proposition 2.1. □

Lemma 2.13. *Assume $\mathcal{M} \subset \mathcal{L}$ is a nonempty subset, $u \in \mathcal{M}$ and $w \in \mathcal{L} \setminus \mathcal{M}$. Then, for all $r \in I_w$, the following equality holds in $R_{\mathcal{M}}$:*

$$rA_{uw} = \varphi_{wu}(r)y_{uw}. \tag{2.9}$$

Proof. Set $x = \varphi_{wu}(r)$. By Assumption (2.1) $x \in I_u$. Using Proposition 2.1 and that the equality $\varphi_{uw}(x) = xy_{uw}$ holds in $R_{\mathcal{M}}$ (since it is equivalent to $\varphi_u(x) = xy_u$), we get $rA_{uw} = \varphi_{uw}(\varphi_{wu}(r)) = \varphi_{uw}(x) = xy_{uw}$ and (2.9) follows. □

Lemma 2.14. *Assume that for a nonempty subset $\mathcal{M} \subset \mathcal{L}$ the natural map $R \to R_{\mathcal{M}}$ is injective. Consider distinct $u, v \in \mathcal{M}$ and $w \in \mathcal{L} \setminus \mathcal{M}$. Then the following equality holds in $R_{\mathcal{M}}$:*

$$y_{uw}D_{vw} = y_{vw}D_{uw}. \tag{2.10}$$

Proof. We start by showing that

$$y_{uw}D_{vw} - y_{vw}D_{uw} \tag{2.11}$$

is in the image of the natural map $R \to R_{\mathcal{M}}$. Substituting in (2.11) y_{uw} with $y_{uv} + (C_{uw} - C_{uv})$, y_{vw} with $y_{vu} + (C_{vw} - C_{vu})$ and expanding D_{vw} and D_{uw}:

$$\begin{aligned} y_{uw}D_{vw} - y_{vw}D_{uw} = \\ &[y_{uv} + (C_{uw} - C_{uv})]\left(A_{vw} - [y_{vu} + (C_{vw} - C_{vu})]\,C_{wv}\right) \\ &- [y_{vu} + (C_{vw} - C_{vu})]\left(A_{uw} - [y_{uv} + (C_{uw} - C_{uv})]\,C_{wu}\right). \end{aligned}$$

It is enough to show that, after expanding, the sum of the terms involving the variables y_{uv} and y_{vu} is in the image of $R \to R_{\mathcal{M}}$. Since $y_{uv}y_{vu} = A_{uv}$ in $R_{\mathcal{M}}$, it suffices to show that

$$y_{uv}A_{vw} - y_{uv}(C_{vw} - C_{vu})C_{wv} - y_{vu}(C_{uw} - C_{uv})C_{wv}$$
$$-y_{vu}A_{uw} + y_{vu}(C_{uw} - C_{uv})C_{wu} + y_{uv}(C_{vw} - C_{vu})C_{wu}$$

which is equal to

$$y_{uv}\left[A_{vw} - (C_{wv} - C_{wu})(C_{vw} - C_{vu})\right] - y_{vu}\left[A_{uw} - (C_{wu} - C_{wv})(C_{uw} - C_{uv})\right]$$

is in the image of $R \to R_{\mathcal{M}}$. This follows using Proposition 2.1 and the definition of $R_{\mathcal{M}}$. We have shown that $y_{uw}D_{vw} - y_{vw}D_{uw}$ is in the image of $R \to R_{\mathcal{M}}$.

We claim that

$$r(y_{uw}D_{vw} - y_{vw}D_{uw}) = 0 \tag{2.12}$$

holds in $R_{\mathcal{M}}$ for any $r \in I_w$. Using Lemma 2.13, we have

$$ry_{uw}D_{vw} = y_{uw}(rA_{vw} - ry_{vw}C_{wv}) = y_{uw}(\varphi_{wv}(r)y_{vw} - ry_{vw}C_{wv}) = y_{uw}y_{vw}\varphi_w(r)$$

and similarly $ry_{vw}D_{uw} = y_{vw}y_{uw}\varphi_w(r)$. Thus (2.12) follows. Now, since $I_w \subset R$ has codimension 1 and R is Cohen-Macaulay, I_w contains an R-regular element. Hence combining (2.12) with the previously shown fact that $y_{uw}D_{vw} - y_{vw}D_{uw}$ is in the image of $R \to R_{\mathcal{M}}$ and that, by assumption, this map is injective, we deduce Equality (2.10). □

We start the proof of Theorem 2.4 by using complete induction on the cardinality of $\mathcal{M}$. Assume first that $\mathcal{M}$ is the empty set. Then $R_{\mathcal{M}} = R$ so statement (1) of Theorem 2.4 is trivially true. Assume there exists $w \in \mathcal{L} \setminus \mathcal{M}$. Then $J_{\mathcal{M},w} = I_w$ so (2) and (3) are trivially true, since there are no y_u. This proves Theorem 2.4 for the case that $\mathcal{M}$ has cardinality 0.

Suppose Theorem 2.4 is true for all $\mathcal{M} \subset \mathcal{L}$ of cardinality strictly less than n, where $n \geq 1$. Consider $\mathcal{M} \subset \mathcal{L}$ of cardinality n, and fix an element $v \in \mathcal{M}$. We denote by $\mathcal{N}$ the set $\mathcal{M} \setminus \{v\}$. Since the cardinality of $\mathcal{N}$ is strictly less than n, by the inductive hypothesis applied to $\mathcal{N}$ and to $v \in \mathcal{M} \setminus \mathcal{N}$ we have that $R_{\mathcal{N}}$ is Gorenstein with $\dim R_{\mathcal{N}} = \dim R$, the map $R \to R_{\mathcal{N}}$ is injective and that $R_{\mathcal{M}}$ is the unprojection of $J_{\mathcal{N},v} \subset R_{\mathcal{N}}$. Consequently, using [21, Theorem 1.5] $R_{\mathcal{M}}$ is Gorenstein with $\dim R_{\mathcal{M}} = \dim R_{\mathcal{N}} = \dim R$, and by Corollary 2.8 the natural map $R_{\mathcal{N}} \to R_{\mathcal{M}}$ is injective, hence the natural map $R \to R_{\mathcal{M}}$ is also injective. If $\mathcal{L} \setminus \mathcal{M} = \emptyset$ then there is nothing left to show. Assume this is not the case and let $w \in \mathcal{L} \setminus \mathcal{M}$. We will make use in our argument of the chain of strict inclusions

$$\mathcal{N} \ \subset \ \mathcal{M} = \mathcal{N} \cup \{v\} \ \subset \ \mathcal{M} \cup \{w\}.$$

Using Lemma 2.12 we have that $J_{\mathcal{M},w}$ is a codimension 1 ideal of $R_{\mathcal{M}}$ with the quotient ring $R_{\mathcal{M}}/J_{\mathcal{M},w}$ being isomorphic to R/I_w, hence Gorenstein. We use the identification of $R_{\mathcal{N}}$ with a subring of $R_{\mathcal{M}}$ to consider the ideal $J_{\mathcal{N},w} \subset R_{\mathcal{N}}$. We fix an $R_{\mathcal{M}}$-regular element of $J_{\mathcal{M},w}$ (such an element exists since $R_{\mathcal{M}}$ is Gorenstein, hence Cohen–Macaulay, and $J_{\mathcal{M},w}$ is a codimension 1 ideal), say

$$br_1 + \sum_{u \in \mathcal{M}} b_u(y_u + C_{uw}) \in R_{\mathcal{M}}$$

with $r_1 \in I_w$ and $b, b_u \in R_{\mathcal{M}}$, and define the element

$$s = \frac{b\varphi_w(r_1) + \sum_{u\in\mathcal{M}} b_u D_{uw}}{br_1 + \sum_{u\in\mathcal{M}} b_u(y_u + C_{uw})} \quad \in K(R_{\mathcal{M}}),$$

where $K(R_{\mathcal{M}})$ is the total quotient ring of $R_{\mathcal{M}}$, that is the localisation of $R_{\mathcal{M}}$ with respect to the multiplicatively closed subset of nonzero divisors of $R_{\mathcal{M}}$, cf. [10, p. 60].

Lemma 2.15. *We have the following equalities inside $K(R_{\mathcal{M}})$:*

$$sr = \varphi_w(r) \tag{2.13}$$

for all $r \in I_w$, and

$$sy_{uw} = D_{uw} \tag{2.14}$$

for all $u \in \mathcal{M}$.

Proof. To prove (2.13) it is enough to show that for all $r \in I_w$

$$rD_{uw} = \varphi_w(r)(y_u + C_{uw}),$$

which is true being a restatement of (2.9), and that $r\varphi_w(r_1) = r_1\varphi_w(r)$ which is true, since $r\varphi_w(r_1) = \varphi_w(rr_1) = r_1\varphi(r)$. To prove (2.14) is is enough to prove that for all $v \in \mathcal{M}$,

$$\varphi_w(r_1)(y_v + C_{vw}) = D_{vw}r_1,$$

which is true being a restatement of (2.9), and that for all $v \in \mathcal{M}$ with $v \neq u$ we have $(y_v + C_{vw})D_{uw} = (y_u + C_{uw})D_{vw}$, which is true, being a reformulation of (2.10). This finishes the proof of Lemma 2.15. □

Using Lemma 2.15, the multiplication by s which, a priori, is only an $R_{\mathcal{M}}$-homomorphism $R_{\mathcal{M}} \to K(R_{\mathcal{M}})$ extends $\varphi_w \colon I_w \to R$ and has its image contained inside $R_{\mathcal{M}} \subset K(R_{\mathcal{M}})$. Consequently, it defines an $R_{\mathcal{M}}$-homomorphism

$$\Phi_{\mathcal{M},w} \colon J_{\mathcal{M},w} \to R_{\mathcal{M}}.$$

This shows (2) of the induction step.

We shall now prove that

$$\operatorname{Hom}_{R_{\mathcal{M}}}(J_{\mathcal{M},w}, R_{\mathcal{M}}) = (i_{\mathcal{M},w}, \Phi_{\mathcal{M},w}). \tag{2.15}$$

Let $\psi \in \operatorname{Hom}_{R_{\mathcal{M}}}(J_{\mathcal{M},w}, R_{\mathcal{M}})$. Write $\psi(y_{vw}) = a + by_{vw}$, with $a \in R_{\mathcal{N}}$ and $b \in R_{\mathcal{M}}$, and set

$$\psi_1 = \psi - bi_{\mathcal{M},w}.$$

By construction $\psi_1(y_{vw}) = a \in R_{\mathcal{N}}$. We claim that $\psi_1(r) \in R_{\mathcal{N}}$ for all $r \in J_{\mathcal{N},w} \subset R_{\mathcal{N}}$. Indeed, for $r \in J_{\mathcal{N},w} \subset J_{\mathcal{M},w}$ we have

$$y_{vw}\psi_1(r) = r\psi_1(y_{vw}) \in R_{\mathcal{N}}$$

and the claim follows from Corollary 2.8. We deduce that the restriction of ψ_1 to $J_{\mathcal{N},w}$ is a well-defined $R_{\mathcal{N}}$-homomorphism $J_{\mathcal{N},w} \to R_{\mathcal{N}}$. Let us denote this homomorphism by ψ_2. Since the cardinality of $\mathcal{N}$ is strictly less than n, by the induction hypothesis

$$\operatorname{Hom}_{R_{\mathcal{N}}}(J_{\mathcal{N},w}, R_{\mathcal{N}}) = (i_{\mathcal{N},w}, \Phi_{\mathcal{N},w}) \tag{2.16}$$

as $R_{\mathcal{N}}$-modules. Hence there exist $a_1, a_2 \in R_{\mathcal{N}}$ such that

$$\psi_2 = a_1 i_{\mathcal{N},w} + a_2 \Phi_{\mathcal{N},w}. \tag{2.17}$$

Consider the $R_{\mathcal{M}}$-homomorphism $\psi_3 \colon J_{\mathcal{M},w} \to R_{\mathcal{M}}$ given by

$$\psi_3 = \psi_1 - a_1 i_{\mathcal{M},w} - a_2 \Phi_{\mathcal{M},w}.$$

Lemma 2.16. *The $R_{\mathcal{M}}$-homomorphism $\psi_3 \colon J_{\mathcal{M},w} \to R_{\mathcal{M}}$ is the zero homomorphism.*

Proof. Since $\Phi_{\mathcal{N},w}$ coincides with the restriction of $\Phi_{\mathcal{M},w}$ to $J_{\mathcal{N},w}$, using (2.17) we deduce that $\psi_3(r) = 0$ for all $r \in J_{\mathcal{N},w}$, in particular $\psi_3(r) = 0$ for all $r \in I_w$ and $\psi_3(y_{uw}) = 0$ for all $u \in \mathcal{N}$. Taking into account that $J_{\mathcal{M},w}$ is the ideal of $R_{\mathcal{M}}$ generated by $J_{\mathcal{N},w} \cup \{y_{vw}\}$, to prove Lemma 2.16 it is enough to show that $\psi_3(y_{vw}) = 0$. Denote by $J_{\mathcal{N},v,w} \subset R_{\mathcal{N}}$ the ideal of $R_{\mathcal{N}}$ generated by

$$I_v \cup I_w \cup \{y_{uw} \mid u \in \mathcal{N}\}.$$

We will first show that

$$r\psi_3(y_{vw}) = 0 \tag{2.18}$$

for all $r \in J_{\mathcal{N},v,w}$. Indeed, for $u \in \mathcal{N}$

$$y_{uw}\psi_3(y_{vw}) = \psi_3(y_{uw}y_{vw}) = y_{vw}\psi_3(y_{uw}) = 0,$$

for $r \in I_w$

$$r\psi_3(y_{vw}) = \psi_3(ry_{vw}) = y_{vw}\psi_3(r) = 0,$$

while for $r \in I_v$, $ry_{vw} = \varphi_{vw}(r)$, which by Assumption (2.1) is in I_w, hence

$$r\psi_3(y_{vw}) = \psi_3(ry_{vw}) = \psi_3(\varphi_{vw}(r)) = 0.$$

It remains to prove that (2.18) implies that $\psi_3(y_{vw}) = 0$, and for that it is enough to show that $J_{\mathcal{N},v,w}$ contains an $R_{\mathcal{N}}$-regular element. Since $R_{\mathcal{N}}$ is Gorenstein, hence Cohen–Macaulay, it is enough to show that the ideal $J_{\mathcal{N},v,w}$ has codimension in $R_{\mathcal{N}}$ at least 1. Consider the natural surjection

$$\frac{R}{I_v + I_w}[y_v] \to \frac{R_{\mathcal{N}}}{J_{\mathcal{N},v,w}}.$$

Since, by Assumption (2.2), $\operatorname{codim}_R(I_v + I_w) \geq 2$ we deduce that $J_{\mathcal{N},v,w}$ has codimension in $R_{\mathcal{N}}$ at least 1, which finishes the proof of Lemma 2.16. □

Using Lemma 2.16, we get $\psi \in (i_{\mathcal{M},w}, \Phi_{\mathcal{M},w})$, which proves Equation (2.15). It follows immediately from the definitions that $R_{\mathcal{M}\cup\{w\}}$ is the unprojection of type Kustin–Miller of the pair $J_{\mathcal{M},w} \subset R_{\mathcal{M}}$, which finishes the proof of Theorem 2.4.

3. The case of a complete intersection

In this section we describe an application of Theorem 2.4 to a case where R is the quotient of a polynomial ring by an ideal generated by a regular sequence. After introducing the initial data, we calculate explicitly the unprojection ring $R_{\mathcal{L}}$ in Proposition 3.4 and some of its numerical invariants in Proposition 3.6. These results will play an important role in the construction of the 7 families of Calabi-Yau 3-folds in Section 4.

Let S denote the ambient polynomial ring over a field $\mathbb{K}$ given by

$$S = \mathbb{K}[x_{ij} \mid 1 \leq i \leq n,\ 1 \leq j \leq k_i],$$

where $n \geq 2$ and, for every $1 \leq i \leq n$ we have $k_i \geq 1$. We set $N = k_1 + \cdots + k_n$. We fix an integer d and, for each variable x_{ij}, we set the degree of x_{ij} to be a positive integer, subject to the condition

$$\sum_{j=1}^{k_i} \deg(x_{ij}) = d$$

for every $1 \leq i \leq n$. For $i \in \{1, \ldots, n\}$ we denote the product $x_{i1}x_{i2}\cdots x_{ik_i}$ by X_i. Then, for $m \in \{1, \ldots, n-1\}$, we consider the degree d homogeneous polynomial given by:

$$f_m = a_{m1}X_1 + a_{m2}X_2 + \cdots + a_{mn}X_n,$$

where $a_{ml} \in \mathbb{K}$ are general. Finally, setting $I_X = (f_1, \ldots, f_{n-1})$, we define $R = S/I_X$. It is easy to see that $f_1, \ldots, f_{n-1}$ is a regular sequence in S. Indeed, using linear algebra and the generality of a_{ml} we see that $(f_1, \ldots, f_{n-1}, x_{n1}) = (X_1, X_2, \ldots, X_{n-1}, x_{n1})$ which is an ideal of S of codimension n, hence the ideal $(f_1, \ldots, f_{n-1})$ has codimension $n-1$ and the claim follows from the fact that S is Cohen–Macaulay. Since S is Gorenstein and $f_1, \ldots, f_{n-1}$ is a regular sequence of S, the ring R is Gorenstein.

For $i \in \{1, \ldots, n\}$ set $\Theta_i = \{x_{i1}, x_{i2}, \ldots, x_{ik_i}\}$ and consider $\Theta_1 \times \cdots \times \Theta_n$. For each $u \in \Theta_1 \times \cdots \times \Theta_n$, write $u = (u_1, \ldots, u_n)$ with $u_i \in \{x_{i1}, x_{i2}, \ldots, x_{ik_i}\}$ and consider $I'_u \subset S$ the ideal generated by $\{u_1, \ldots, u_n\}$. It is clear that $I_X \subset I'_u$. Denote by I_u the ideal of R given by I'_u/I_X. Since both I_X and I'_u are generated by S-regular sequences and $\dim S/I'_u = \dim S/I_X - 1$, the ideal I_u is a codimension 1 homogeneous ideal of R and R/I_u is Gorenstein.

Let B denote the matrix of coefficients:

$$B = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{(n-1)1} & \cdots & a_{(n-1)n} \end{bmatrix}. \tag{3.1}$$

Denote by Δ_i the ith entry of the $n \times 1$ matrix $\wedge^{n-1}B$, in other words, Δ_i equals $(-1)^i$ times the determinant of the submatrix of B obtained by deleting the ith column. By the generality assumption on a_{ml}, we know that $\Delta_i \neq 0$, for all i. For

$1 \leq i \leq n$ and $1 \leq j \leq k_i$, set

$$\widehat{x_{ij}} = \prod_{\substack{1 \leq a \leq k_i \\ a \neq j}} x_{ia}.$$

Notice that $\widehat{x_{ij}} = \frac{X_i}{x_{ij}}$, in $K(S)$, the ring of fractions of S. Given $u = (u_1, \dots, u_n) \in \Theta_1 \times \cdots \times \Theta_n$, we can write the generators of I_X in matrix format as

$$\begin{bmatrix} f_1 \\ \vdots \\ f_{n-1} \end{bmatrix} = \begin{bmatrix} a_{11}\widehat{u_1} & \dots & a_{1n}\widehat{u_n} \\ \vdots & \ddots & \vdots \\ a_{(n-1)1}\widehat{u_1} & \dots & a_{(n-1)n}\widehat{u_n} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}. \tag{3.2}$$

Denote by Q the $(n-1) \times n$ matrix of (3.2) and by $(\wedge^{n-1} Q)_i$ the determinant of the submatrix of Q obtained by deleting the ith column multiplied by $(-1)^i$. Following [20, Theorem 4.3] the map $\varphi_u \in \operatorname{Hom}_R(I_u, R)$ given by

$$u_i + I_X \mapsto (\wedge^{n-1} Q)_i + I_X,$$

together with the inclusion $i_u \colon I_u \to R$ generate $\operatorname{Hom}_R(I_u, R)$. Notice that

$$(\wedge^{n-1} Q)_i = \Delta_i \prod_{\substack{1 \leq a \leq n \\ a \neq i}} \widehat{u_a}. \tag{3.3}$$

Remark 3.1. It is easy to check that $\operatorname{codim}_R(I_u + I_v) \geq 2$ if and only if $u \neq v$. However, if we do not impose any extra assumption on u, v the existence of $C_{uv} \in R$ such that $(\varphi_u + C_{uv} i_u)(I_u) \subset I_v$, an assumption of Theorem 2.4, can fail. For an example see [18, Remark 3.1].

Lemma 3.2. *Let $\mathcal{L}$ be a subset of $\Theta_1 \times \cdots \times \Theta_n$ such that for distinct $u, v \in \mathcal{L}$ there exist $1 \leq i_1 < i_2 \leq n$ such that $u_{i_1} \neq v_{i_1}$ and $u_{i_2} \neq v_{i_2}$, where $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)$. Then, for distinct $u, v \in \mathcal{L}$, we have $\operatorname{codim}_R(I_u + I_v) \geq 3$ and $\varphi_u(I_u) \subset I_v$.*

Proof. It is clear that under this assumption $\operatorname{codim}_R(I_u + I_v) \geq 3$ for every $u \neq v$ in $\mathcal{L}$. Let i_1, i_2 be such that $u_{i_1} \neq v_{i_1}$ and $u_{i_2} \neq v_{i_2}$, so that v_{i_1} divides $\widehat{u_{i_1}}$ and v_{i_2} divides $\widehat{u_{i_2}}$. By (3.3), we deduce that $\varphi_u(u_i) \in I_v$ for every $1 \leq i \leq n$. □

For the following we fix a subset $\mathcal{L}$ of $\Theta_1 \times \cdots \times \Theta_n$ of cardinality ≥ 2 satisfying the assumptions of Lemma 3.2. This assumption implies that $k_{i_1} \geq 2$ and $k_{i_2} \geq 2$, for at least two indices i_1, i_2; hence the total number N of variables of S is ≥ 4. By Proposition 2.1, for any $u \neq v$ there exists $A_{uv} \in R$ such that $R_{\mathcal{L}}$, the parallel unprojection of $\{I_u \mid u \in \mathcal{L}\}$ in R, is given as the quotient of $R[y_u \mid u \in \mathcal{L}]$ by the ideal generated by

$$\{y_u r - \varphi_u(r) \mid u \in \mathcal{L}, r \in I_u\} \cup \{y_u y_v - A_{uv} \mid u, v \in \mathcal{L}, u \neq v\}.$$

Lemma 3.3. *Fix indices $i_1 \neq i_2$ such that $u_{i_1} \neq v_{i_1}$ and $u_{i_2} \neq v_{i_2}$. We have*

$$A_{uv} = \Delta_{i_1} \Delta_{i_2} \frac{\widehat{u_{i_2}}}{v_{i_2}} \frac{\widehat{v_{i_1}}}{u_{i_1}} \left[\prod_{\substack{1 \leq a \leq n \\ a \neq i_1, i_2}} \widehat{u_a} \right] + I_X.$$

Proof. Following the proof of Proposition 2.1, to calculate A_{uv} we start by identifying $r_u \in I_u$ and $r_v \in I_v$ such that r_u, r_v is a regular sequence. It is clear that u_{i_1}, v_{i_2} satisfy this condition. According to the proof of Proposition 2.1, $A_{uv} \in R$ can be computed by factoring $u_{i_1} v_{i_2}$ in $\varphi_u(u_{i_1} + I_X)\varphi_v(v_{i_2} + I_X)$. Now,

$$\varphi_u(u_{i_1} + I_X)\varphi_v(v_{i_2} + I_X) = \Delta_{i_1}\Delta_{i_2}\left[\prod_{a \neq i_1} \widehat{u_a}\right]\left[\prod_{b \neq i_2} \widehat{v_b}\right] + I_X$$

and since $u_{i_2} \neq v_{i_2}$, we know that v_{i_2} divides $\widehat{u_{i_2}}$ and likewise u_{i_1} divides $\widehat{v_{i_1}}$. Hence we deduce the result. □

Consider the polynomial ring $S_{\mathcal{L}} = \mathbb{K}[x_{ij}, y_u \mid 1 \leq i \leq n,\ 1 \leq j \leq k_i,\ u \in \mathcal{L}]$, where

$$\deg(y_u) = (n-1)d - \sum_{i=1}^{n} \deg(u_i), \tag{3.4}$$

and the ideal of $S_{\mathcal{L}}$ generated by:

$$\begin{aligned} E = &\{f_m \mid 1 \leq m \leq n-1\} \cup \{y_u u_i - \Delta_i \prod_{\substack{1 \leq a \leq n \\ a \neq i}} \widehat{u_a} \mid u \in \mathcal{L}, 1 \leq i \leq n\} \\ &\cup \{y_u y_v - \Delta_{i_1}\Delta_{i_2} \frac{\widehat{u_{i_2}}}{v_{i_2}} \frac{\widehat{v_{i_1}}}{u_{i_1}} \prod_{\substack{1 \leq a \leq n \\ a \neq i_1, i_2}} \widehat{u_a}\widehat{v_a} \mid u, v \in \mathcal{L}, u \neq v\} \end{aligned} \tag{3.5}$$

where, for distinct $u, v \in \mathcal{L}$, i_1, i_2 are two indices, depending of u and v, such that u and v have distinct i_1 and i_2 components. From Definition 2.2, using Lemma 3.3 the proof of the following proposition is immediate.

Proposition 3.4. *We have $R_{\mathcal{L}} = S_{\mathcal{L}}/(E)$.*

Remark 3.5. In the following proposition, we will use the following general facts. Assume A is a Gorenstein graded ring, and denote by A_{un} the Kustin–Miller unprojection of a codimension 1 homogeneous ideal $I \subset A$ with A/I Gorenstein. Then $\dim A_{\text{un}} = \dim A$ (see [21]) and if the canonical module of A is $A(k)$, then the canonical module of A_{un} is $A_{\text{un}}(k)$, see [17, Remark 2.23]. Moreover, using again [17, Remark 2.23], there is the following relation between the Hilbert series of A_{un}, A and A/I_u:

$$H_{A_{\text{un}}}(t) = H_A(t) + H_{A/I}(t)\frac{t^{\deg(y)}}{1 - t^{\deg(y)}}, \tag{3.6}$$

where $y \in A_{\text{un}}$ is the new unprojection variable.

Proposition 3.6. *The ring $R_{\mathcal{L}}$ is a Gorenstein graded ring of dimension $N - (n - 1)$. Its canonical module is given by $R_{\mathcal{L}}(-d)$ and its degree as an $S_{\mathcal{L}}$-module is*

$$\frac{d^{n-1}}{\prod_{i=1}^{n}\prod_{j=1}^{k_i} \deg(x_{ij})} + \sum_{(u_1,\dots,u_n) \in \mathcal{L}} \frac{\prod_{i=1}^{n} \deg(u_i)}{(n-1)d - \sum_{i=1}^{n} \deg(u_i)}. \tag{3.7}$$

Proof. Recall d is the common value of $\sum_{j=1}^{k_i} \deg(x_{ij})$ for $1 \leq i \leq n$. By Theorem 2.4, $R_{\mathcal{L}}$ is Gorenstein. By the same theorem, parallel unprojection can be factored in a sequence of Kustin–Miller unprojections. Since $\dim R = N - (n-1)$ and the canonical module of R is $R(-d)$ it follows by Remark 3.5 that $\dim R_{\mathcal{L}} = N - (n-1)$ and that the canonical module of $R_{\mathcal{L}}$ is $R_{\mathcal{L}}(-d)$. Finally, (3.7) follows iterating (3.6). □

4. Examples and applications

This section contains examples and applications of parallel Kustin–Miller unprojection. In Remarks 4.1 and 4.2 we point out that the $\binom{n}{2}$ Pfaffians format and the algebra of the Castelnuovo blowdown of 6 disjoint lines in a smooth cubic surface are special instances of parallel Kustin–Miller unprojection. In Subsection 4.1 we sketch the construction of the 7 families of Calabi–Yau 3-folds listed in Table I of the Introduction. Subsection 4.2 contains a detailed treatment of one of the families, namely Case 3.2.

Remark 4.1. Assume $X \subset \mathbb{P}^3$ is a smooth cubic complex surface. It is well-known (cf. [13, Section V.4]) that X contains exactly 27 distinct lines, and moreover we can find 6 of them, say $l_1, \dots l_6$, such that $l_i \cap l_j = \emptyset$ when $i \neq j$. We denote R the homogeneous coordinate ring of X. We set $\mathcal{L} = \{1, \dots, 6\}$, and for $j \in \mathcal{L}$ we denote by $I_j \subset R$ the homogeneous ideal of the line $l_j \subset X$. One can show that the set $\{I_j \mid j \in \mathcal{L}\}$ satisfies the assumptions of our main theorem (see [18, Section 4.1]), and that the ring $R_{\mathcal{L}}$ corresponds to the Castelnuovo blowdown ([13, Theorem V.5.7]) of the six (-1)-lines $l_1, \dots, l_6$ of X.

Remark 4.2. In Subsection 4.1 the $\binom{n}{2}$ Pfaffians format introduced in [17, Section 2] will be used for the construction of 3 of the 7 families of Calabi–Yau 3-folds. [18, Section 4.2] contains a proof that this format (when tensored over $\mathbb{Z}$ with a field) is a special case of the theory of parallel unprojection developed in the present paper.

4.1. **Construction of 7 Calabi–Yau families.** In this subsection we sketch the explicit construction, using parallel unprojection, of the 7 families of Calabi–Yau 3-folds in weighted projective space listed in Table I of the Introduction. In Subsection 4.2 we will give a detailed treatment of one of the cases, namely Case 3.2. We checked that for each of the other 6 cases one can argue in a similar way as in Subsection 4.2 in order to calculate the singular locus of the general member of each family. In Remark 4.3 we make some comments about their geometry.

The main idea for the construction is to parallel unproject a complete intersection Fano 4-fold in usual projective space into weighted projective space and take a hypersurface section to produce a Calabi–Yau 3-fold. If $W \subset \mathbb{P}^n$ is a nondegenerate complete intersection Fano 4-fold, then $n \leq 8$. In order to use the results of Section 3 we restrict ourselves to considering only complete intersections by forms of the same degree. A little analysis shows the possibilities contained in Table II below.

For each case, say $W \subset \mathbb{P}^n$, unprojecting a suitable set of b linear subspaces of dimension 3 contained in W produces a subscheme $V \subset \mathbb{P}(1^{n+1}, a^b)$, where a, that can be obtained using formula (3.4), is the case number (i.e., $a = 2$ in the cases W_4

Case 0	$W_2 \subset \mathbb{P}^5$
Case 1	$W_3 \subset \mathbb{P}^5$, $W_{2,2} \subset \mathbb{P}^6$
Case 2	$W_4 \subset \mathbb{P}^5$, $W_{2,2,2} \subset \mathbb{P}^7$
Case 3	$W_5 \subset \mathbb{P}^5$, $W_{3,3} \subset \mathbb{P}^6$, $W_{2,2,2,2} \subset \mathbb{P}^8$

Table II

and $W_{2,2,2}$ and so on). Notice, however, that Case 0 does not lead to a Calabi–Yau 3-fold, since the new unprojection variables turn out to have degree $a = 0$. Hence, we are left with Cases 1,2 and 3. We obtain a Calabi–Yau 3-fold X by taking a hypersurface section of degree $4 - a$.

Our strategy was to perform the above construction within the framework of the $\binom{n}{2}$ Pfaffians format [17] (in the hypersurface case) or the format of Section 3. In order to make the computations simpler and more symmetrical, in some of the cases we increased the dimension of the ambient projective space of W, to be able to find equations for the loci as disjoint as feasible. Say, in the case of $W_4 \subset \mathbb{P}^5$ (Case 2.1), we first look for 4 loci in $\mathbb{P}^5$ contained in W_4. Each loci is given by 2 linear equations x_i, z_i. Ideally the collection x_i, z_i, $1 \leq i \leq 4$, would be linearly independent, and the equation defining W_4 would be a general element of degree 4 of $\cap_{i=1}^4 (x_i, z_i)$. Hence we worked over $\mathbb{P}^7$, with homogeneous coordinates x_i, z_i, for $1 \leq i \leq 4$, and set $\widetilde{W}_4 \subset \mathbb{P}^7$ equal to $V(F)$, where $F \in \cap_{i=1}^4 (x_i, z_i)$ is a general element of degree 4. After unprojecting $\widetilde{W}_4$, we took 2 general linear sections and a general quadratic section producing a 3-fold X. For all 7 cases the steps are similar, and are described briefly in what follows (the degree of V was computed using Proposition 3.6):

Case 1.1: $\widetilde{W}_3 = V(F) \subset \mathbb{P}^5$, variables x_i, z_i, for $1 \leq i \leq 3$, loci $= \{(x_i, z_i) \mid 1 \leq i \leq 3\}$, F general of degree 3 in $\bigcap_{i=1}^3 (x_i, z_i)$. Parallel unprojection gives $V \subset \mathbb{P}^8$, $\deg V = 6$. X is a section of V by a general cubic hypersurface. Then $X \subset \mathbb{P}^8$ is a smooth Calabi–Yau 3-fold with $\deg X = 18$.

Case 1.2: $\widetilde{W}_{2,2} = V(F, G) \subset \mathbb{P}^6$, variables z and x_{ij}, for $1 \leq i \leq 3$, $1 \leq j \leq 2$, the set of loci is given by $\{(x_{1i_1}, x_{2i_2}, x_{3i_3})\}$, where all indices $i_p \in \{1, 2\}$ and exactly 3 or 1 of the i_p are equal to 1, F, G are general elements of the linear system $\langle x_{i1}x_{i2}, 1 \leq i \leq 3\rangle$. Notice that the variable z does not occur in F, G, and hence $\widetilde{W}_{2,2}$ is a cone. Parallel unprojection gives $V \subset \mathbb{P}^{10}$, $\deg V = 8$. X is a section of V by a general cubic hypersurface. Then $X \subset \mathbb{P}^{10}$ is a smooth Calabi–Yau 3-fold with $\deg X = 24$.

Case 2.1: $\widetilde{W}_4 = V(F) \subset \mathbb{P}^7$, variables x_i, z_i, for $1 \leq i \leq 4$, loci $= \{(x_i, z_i) \mid 1 \leq i \leq 4\}$, F general of degree 4 in $\bigcap_{i=1}^4 (x_i, z_i)$. Parallel unprojection gives $V \subset \mathbb{P}(1^8, 2^4)$, $\deg V = 6$. X is a section of V by 2 general linear and 1 general quadratic hypersurfaces. Then $X \subset \mathbb{P}(1^6, 2^3)$ is a smooth Calabi–Yau 3-fold with $\deg X = 12$.

Case 2.2: $\widetilde{W}_{2,2,2} = V(F, G, H) \subset \mathbb{P}^7$, variables x_{ij}, for $1 \leq i \leq 4$, $1 \leq j \leq 2$, the set of loci is given by $\{(x_{1i_1}, x_{2i_2}, x_{3i_3}, x_{4i_4})\}$, where all indices $i_p \in \{1, 2\}$ and exactly 4, 2 or 0 of the i_p are equal to 1, F, G, H general elements of the linear system $\langle x_{i1}x_{i2}, 1 \leq i \leq 4\rangle$. Parallel unprojection gives $V \subset \mathbb{P}(1^8, 2^8)$, $\deg V = 12$. X is a section of V by a general quadratic hypersurface. Then $X \subset \mathbb{P}(1^8, 2^7)$ is a smooth Calabi–Yau 3-fold with $\deg X = 24$.

Case 3.1: $\widetilde{W}_5 = V(F) \subset \mathbb{P}^9$, variables x_i, z_i, for $1 \le i \le 5$, loci $= \{(x_i, z_i) \mid 1 \le i \le 5\}$, F general of degree 5 in $\bigcap_{i=1}^5 (x_i, z_i)$. Parallel unprojection gives $V \subset \mathbb{P}(1^{10}, 3^5)$, $\deg V = \frac{20}{3}$. X is a section of V by 4 general linear hypersurfaces. Then $X \subset \mathbb{P}(1^6, 3^5)$ is a singular Calabi–Yau 3-fold with 5 quotient singularities of type $\frac{1}{3}(1,1,1)$ and $\deg X = \frac{20}{3}$.
Case 3.2: (treated in more detail in Subsection 4.2) $\widetilde{W}_{3,3} = V(F, G) \subset \mathbb{P}^8$, variables x_{ij}, for $1 \le i, j \le 3$, loci $= \{(x_{1i}, x_{2i}, x_{3i}) \mid 1 \le i \le 3\} \cup \{(x_{1i}, x_{2j}, x_{3k}) \mid \{i, j, k\} = \{1, 2, 3\}\}$, F, G general elements of the linear system $\langle x_{i1}x_{i2}x_{i3}, 1 \le i \le 3\rangle$. Parallel unprojection gives $V \subset \mathbb{P}(1^9, 3^9)$, $\deg V = 12$. X is a section of V by 3 general linear hypersurfaces. Then $X \subset \mathbb{P}(1^6, 3^9)$ is a singular Calabi–Yau 3-fold with 9 quotient singularities of type $\frac{1}{3}(1,1,1)$ and $\deg X = 12$.
Case 3.3: $\widetilde{W}_{2,2,2,2} = V(F, G, H, K) \subset \mathbb{P}^9$, variables x_{ij}, for $1 \le i \le 5$, $1 \le j \le 2$, the set of loci is given by $\{(x_{1i_1}, x_{2i_2}, x_{3i_3}, x_{4i_4}, x_{5i_5})\}$, where all indices $i_p \in \{1, 2\}$ and exactly 5 or 3 or 1 of the i_p are equal to 1, F, G, H, K are general elements of the linear system $\langle x_{i1}x_{i2}, 1 \le i \le 5\rangle$. Parallel unprojection gives $V \subset \mathbb{P}(1^{10}, 3^{16})$, $\deg V = \frac{64}{3}$. X is a section of V by 2 general linear hypersurfaces. Then $X \subset \mathbb{P}(1^8, 3^{16})$ is a singular Calabi–Yau 3-fold with 16 quotient singularities of type $\frac{1}{3}(1,1,1)$ and $\deg X = \frac{64}{3}$.

Remark 4.3. We make some brief comments on the geometry of the families constructed. It was pointed out to us by Miles Reid that the Calabi–Yau 3-fold X obtained in Case 1.1 can also be described as a $(3,3)$-divisor inside $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$. This is particularly clear for the special choice of $F = x_1x_2x_3 - z_1z_2z_3$, which falls within the framework of Section 3, and whose unprojection $V \subset \mathbb{P}^8$ according to (3.5) is defined by:

$$\operatorname{rank} \begin{pmatrix} y_1 & z_3 & x_2 \\ x_3 & y_2 & z_1 \\ z_2 & x_1 & y_3 \end{pmatrix} \le 1,$$

where y_1, y_2, y_3 are the unprojection variables. Using this, we got $h^{1,1}(X) = 2$ and $h^{1,2}(X) = 83$. Since in Cases 3.1, 3.2 and 3.3 we only take linear hypersurface sections, it is not hard to see that for each of those cases the Calabi–Yau 3-fold X obtained is birational to a nodal complete intersection Z. This variety is simply the intersection of $\widetilde{W}$ with the linear hypersurfaces used to construct X. The 3-fold Z has a small resolution of singularities, which we denote by $\widehat{Z}$. Then, using the method described in [12, Remark 4.11], we found that $h^{1,1}(\widehat{Z}) = 6$, $h^{1,2}(\widehat{Z}) = 36$ for Case 3.1, $h^{1,1}(\widehat{Z}) = 21$, $h^{1,2}(\widehat{Z}) = 12$ for Case 3.2 and $h^{1,1}(\widehat{Z}) = 27$, $h^{1,2}(\widehat{Z}) = 11$ for Case 3.3.

So far we have not been able to compute the Hodge numbers of the Calabi–Yau 3-folds in the cases 1.2, 2.1 and 2.2. It seems unlikely that they can be treated as the cases 3.1, 3.2 and 3.3 (i.e., using the method described in [12, Remark 4.11]) since in the former cases we cut V by a hypersurface of degree > 1 the equation of which involves the unprojection variables and thus X is not birational to a complete intersection inside the original $\widetilde{W}$. Due to the high codimensions of the embeddings of the Calabi–Yau 3-folds constructed, our efforts to compute Hodge numbers by counting rational points over finite fields were unsuccessful. It will be interesting to investigate alternative descriptions for the 3-folds like the one stated for Case 1.1.

4.2. **Detailed study of Case 3.2.** In Subsection 4.1 we sketched the construction, via parallel Kustin–Miller unprojection, of 7 families of Calabi–Yau 3-folds. In what follows we give the details of the construction for the family corresponding to Case 3.2, which is a family of degree 12 Calabi–Yau 3-folds $X \subset \mathbb{P}(1^6, 3^9)$. The most difficult part of the arguments is controlling the singular locus of the general member of the family. As already mentioned above, we checked that for each of the other 6 families the same way of arguing also allow us to calculate the singular locus of the general member.

We set $\mathbb{K} = \mathbb{C}$, the field of complex numbers, and consider the polynomial ring

$$S = \mathbb{K}[x_{ij} \mid 1 \le i, j \le 3],$$

where we put $\deg x_{ij} = 1$ for all $1 \le i, j \le 3$. We define an S-regular sequence f_1, f_2 as in Section 3. Namely,

$$f_1 = a_{11}X_1 + a_{12}X_2 + a_{13}X_3,$$
$$f_2 = a_{21}X_1 + a_{22}X_2 + a_{23}X_3,$$

where $X_i = x_{i1}x_{i2}x_{i3}$, for $i = 1, 2, 3$ and $a_{ij} \in \mathbb{K}$ are general. We set $R = S/(f_1, f_2)$. We have $\omega_R \cong R(d)$, where $d = 3+3-9 = -3$, and we get a corresponding projective 6-fold, $\operatorname{Proj} R \subset \mathbb{P}^8 = \mathbb{P}(1^9)$. We define the index set

$$\mathcal{L} = \{(i, i, i) \mid 1 \le i \le 3\} \cup \{(i, j, k) \in \{1, 2, 3\}^3 \mid \{i, j, k\} = \{1, 2, 3\}\}.$$

Notice that $\mathcal{L}$ has 9 elements and if $u \ne v \in \mathcal{L}$ the set $\{i \in \{1, 2, 3\} \mid u_i = v_i\}$ has at most 1 element. Moreover, $\mathcal{L}$ is a maximal subset of $\{1, 2, 3\}^3$ with this property. For $u = (a, b, c) \in \mathcal{L}$, we define the ideal I_u of R given by $I_u = (x_{1a}, x_{2b}, x_{3c})$. It is clear that $\mathcal{L}$ satisfies the conditions of Lemma 3.2. The degree of each new unprojection variable is 3, and thus we obtain a projectively Gorenstein 6-fold $\operatorname{Proj} R_{\mathcal{L}} \subset \mathbb{P}(1^9, 3^9)$. In what follows, we show that the intersection of $\operatorname{Proj} R_{\mathcal{L}}$ with 3 general degree 1 hypersurfaces of $\mathbb{P}(1^9, 3^9)$ yields a codimension 11 Calabi–Yau 3-fold $X \subset \mathbb{P}(1^6, 3^9)$ with 9 quotient singularities of type $\frac{1}{3}(1, 1, 1)$. The rest of this subsection will mostly be about controlling the singular loci of the construction.

The following proposition specifies the singular locus of $\operatorname{Spec} R$, which we denote by $\operatorname{Sing}(\operatorname{Spec} R)$.

Proposition 4.4. *We have*

$$\operatorname{Sing}(\operatorname{Spec} R) = \bigcup V(x_{t_1,p_1}, x_{t_1,p_2}, x_{t_2,p_3}, x_{t_2,p_4}, x_{t_3,p_5}),$$

where the union is for t_1, t_2, t_3 with $\{t_1, t_2, t_3\} = \{1, 2, 3\}$ and $p_1, \dots, p_5 \in \{1, 2, 3\}$ with $p_1 \ne p_2$ and $p_3 \ne p_4$. In particular, $\dim \operatorname{Spec} R - \dim \operatorname{Sing}(\operatorname{Spec} R) = 3$.

Proof. Denote, for simplicity, $V(x_{t_1,p_1}, x_{t_1,p_2}, x_{t_2,p_3}, x_{t_2,p_4}, x_{t_3,p_5})$ by $V(t_i, p_j)$. We set, for $1 \le i \le 3$,

$$A_i = \frac{\partial X_1}{\partial x_{1i}}, \quad B_i = \frac{\partial X_2}{\partial x_{2i}}, \quad C_i = \frac{\partial X_3}{\partial x_{3i}}.$$

Since the Jacobian matrix of f_1, f_2 is, in block format, equal to

$$\begin{pmatrix} a_{11}(A_1, A_2, A_3) & a_{12}(B_1, B_2, B_3) & a_{13}(C_1, C_2, C_3) \\ a_{21}(A_1, A_2, A_3) & a_{22}(B_1, B_2, B_3) & a_{23}(C_1, C_2, C_3) \end{pmatrix},$$

from the generality of a_{ij} it follows that the vanishing of all 2×2 minors of Jacobian matrix is equivalent to

$$A_iB_j = A_iC_j = B_iC_j = 0 \tag{4.1}$$

for all $1 \le i, j \le 3$. Assume $P \in \operatorname{Sing}(\operatorname{Spec} R)$. If all A_i, B_i, C_i vanish at P, then it is clear that P is contained in at least one of the loci $V(t_i, p_j)$. Assume this is not the case, by symmetry we can assume that A_1 does not vanish at P. Then we get $B_j = C_j = 0$ at P for all $1 \le j \le 3$, which implies that at least 2 variables x_{2i} and at least 2 variables x_{3j} vanish at P. Using the equation f_1 we get that at least 1 of the x_{1j} also vanishes, so P is again contained in at least one of the loci $V(t_i, p_j)$. Conversely, by (4.1) it is clear that a point contained in a loci $V(t_i, p_j)$ is a singular point of $\operatorname{Spec} R$. □

Proposition 4.5. *We have* $\operatorname{Sing}(\operatorname{Spec} R) \subset \bigcup_{u\in\mathcal{L}} V(I_u)$.

Proof. Using Proposition 4.4, it is enough to show that given t_1, t_2, t_3 with $\{t_1, t_2, t_3\} = \{1, 2, 3\}$ and $p_1, \dots, p_5 \in \{1, 2, 3\}$ with $p_1 \ne p_2$ and $p_3 \ne p_4$ there exists $u \in \mathcal{L}$ with

$$I_u \subset (x_{t_1,p_1}, x_{t_1,p_2}, x_{t_2,p_3}, x_{t_2,p_4}, x_{t_3,p_5}).$$

If $p_5 \in \{p_1, p_2\}$ and $p_5 \in \{p_3, p_4\}$ we set $u = (p_5, p_5, p_5)$ and the Proposition is true. Assume this is not the case, then $\{p_5, p_1, p_2\} = \{1, 2, 3\}$ or $\{p_5, p_3, p_4\} = \{1, 2, 3\}$. Without loss of generality (due to symmetry) assume that $\{p_5, p_1, p_2\} = \{1, 2, 3\}$. At least one of p_3 and p_4 is not equal to p_5, let us assume without loss of generality that $p_3 \ne p_5$. Then, either $\{p_1, p_3, p_5\} = \{1, 2, 3\}$ or $\{p_2, p_3, p_5\} = \{1, 2, 3\}$. By symmetry we can assume without loss of generality that $\{p_1, p_3, p_5\} = \{1, 2, 3\}$. Then we set $u \in \mathcal{L}$ to be the triple with t_1 coordinate equal to p_1, t_2 coordinate equal to p_3 and t_2 coordinate equal to p_5. This finishes the proof of Proposition 4.5. □

Remark 4.6. Assume $\mathcal{M} \subset \mathcal{L}$ is a subset. Combining Proposition 4.5 with the natural isomorphism of schemes (cf. Corollary 2.8)

$$\operatorname{Spec} R \setminus \bigcup_{u\in\mathcal{M}} V(I_u) \cong \operatorname{Spec} R_{\mathcal{M}} \setminus \bigcup_{u\in\mathcal{M}} V(I_u^e),$$

where $I_u^e \subset R_{\mathcal{M}}$ denotes the ideal of $R_{\mathcal{M}}$ generated by the image of I_u under the natural map $R \to R_{\mathcal{M}}$, we get that

$$\operatorname{Sing}(\operatorname{Spec} R_{\mathcal{M}}) \subset \bigcup_{u\in\mathcal{L}} V(I_u^e).$$

Proposition 4.7. *Assume $\mathcal{M} \subset \mathcal{L}$ is a nonempty subset. We have*

$$\operatorname{Sing}(\operatorname{Spec} R_{\mathcal{M}}) \subset V(y_u \mid u \in \mathcal{M}).$$

Proof. Fix $u \in \mathcal{M}$. From the equations defining $R_{\mathcal{M}}$, there exist exactly $2 + \#\mathcal{M}$ involving the variable y_u, namely those specified by the products $y_u u_i$, for $1 \le i \le 3$, and $y_u y_v$, for $v \in \mathcal{M} \setminus \{u\}$, cf. Equation (3.5). Call the first three g_1, g_2, g_3 and the rest g_v, for $v \in \mathcal{M} \setminus \{u\}$. Consider the submatrix N of the Jacobian matrix of the polynomials $\{g_i, g_v \mid 1 \le i \le 3, v \in \mathcal{M} \setminus \{u\}\}$ corresponding to differentiation with respect to the variables, u_i, y_v, where $1 \le i \le 3, v \in \mathcal{M} \setminus \{u\}$. Looking at Equation (3.5) we get that, for $1 \le i \le 3$, g_i does not involve any y_v for $v \ne u$

and also does not involve any u_j for $j \neq i$. Moreover, for $v \in \mathcal{M} \setminus \{u\}$, g_v does not involve any y_w, for $w \in \mathcal{M} \setminus \{u, v\}$. Consequently, N is upper triangular with all diagonal entries equal to y_u, hence its determinant is a power of y_u. Since the codimension of $R_{\mathcal{M}}$ is $2 + \#\mathcal{M}$, Proposition 4.7 follows. □

The next Proposition, which is a key ingredient for Theorem 4.10, tells us that under an unprojection the singular locus improves. Let R_u denote $R_{\mathcal{M}}$ when $\mathcal{M} = \{u\}$.

Proposition 4.8. *Fix $u \in \mathcal{L}$. Denote by F the intersection (inside $\operatorname{Spec} R_u$) of $\operatorname{Sing}(\operatorname{Spec} R_u)$ with $V(I_u^e)$. Then, $\dim \operatorname{Spec} R_u - \dim F \geq 4$.*

Proof. By Proposition 4.7 $F \subset V(y_u)$. Looking at Equation (3.5) the 3 equations of R_u involving y_u imply the vanishing of 2 more variables. A direct observation of the Jacobian matrix of the equations of R_u gives us the vanishing of an additional variable, and Proposition 4.8 follows. □

Proposition 4.9. *Assume $u, v \in \mathcal{L}$ distinct. Denote by $G \subset \operatorname{Spec} R_{\mathcal{L}}$ the closed subscheme defined by $I_u^e + I_v^e + (y_u \mid u \in \mathcal{L})$. Then, $\dim \operatorname{Spec} R_{\mathcal{L}} - \dim G \geq 4$.*

Proof. The set $\{i \in \{1,2,3\} \mid u_i = v_i\}$ is either empty or has 1 element. If it is empty, the $15 = 6 + 9$ variables appearing in $I_u^e + I_v^e + (y_u \mid u \in \mathcal{L})$ already give the desired codimension. Assume it is not empty, then there are 14 variables appearing in $I_u^e + I_v^e + (y_u \mid u \in \mathcal{L})$. By Equation (3.5), the quadratic equation of $R_{\mathcal{L}}$ involving $y_u y_v$ gives us an additional variable vanishing, which finishes the proof of Proposition 4.9. □

Theorem 4.10. *We have $\dim \operatorname{Spec} R_{\mathcal{L}} - \dim \operatorname{Sing}(\operatorname{Spec} R_{\mathcal{L}}) \geq 4$.*

Proof. Assume F is an irreducible component of $\operatorname{Sing}(\operatorname{Spec} R_{\mathcal{L}})$. Using Remark 4.6, there exists $u \in \mathcal{L}$ such that $F \subset V(I_u^e)$. There are now two cases.

Case 1. For all $v \in \mathcal{L} \setminus \{u\}$ we have that F is not a subset of $V(I_v^e)$. Using the birationality of the morphism induced by the natural ring homomorphism $R_u \to R_{\mathcal{L}}$, cf. Remark 4.6, the result follows using Proposition 4.8.

Case 2. Assume there exists $v \in \mathcal{L} \setminus \{u\}$ such that $F \subset V(I_v^e)$. Proposition 4.7 gives $F \subset V(I_u^e + I_v^e + (y_u \mid u \in \mathcal{L}))$, and the result follows from Proposition 4.9. □

Remark 4.11. Arguing similarly to the proof of Theorem 4.10 we get that for any (not necessarily nonempty) subset $\mathcal{M} \subset \mathcal{L}$ we have

$$\dim \operatorname{Spec} R_{\mathcal{M}} - \dim \operatorname{Sing}(\operatorname{Spec} R_{\mathcal{M}}) \geq 3.$$

Combining it with [10, Theorem 18.15] we get that $R_{\mathcal{M}}$ is a direct product of normal domains, and since it is positively graded with degree-0 part a field we get that $R_{\mathcal{M}}$ is a normal domain.

Theorem 4.12. *Assume $h_1, h_2, h_3 \in R_{\mathcal{L}}$ are 3 general degree 1 homogeneous elements. The ring $R_X = R_{\mathcal{L}}/(h_1, h_2, h_3)$ is a normal Gorenstein domain with $\dim R_X = \dim R_{\mathcal{L}} - 3 = 7 - 3 = 4$, and $\operatorname{Spec} R_X \setminus \{P_0\}$ is smooth, where $P_0 = (x_{ij}, y_u \mid 1 \leq i, j \leq 3, u \in \mathcal{L})$.*

Proof. Since for the specific choices $h_1 = x_{11}$, $h_2 = x_{12}$, $h_3 = x_{13}$ one can check that $R_{\mathcal{L}}/(h_1, h_2, h_3)$ is the quotient of a polynomial ideal by a monomial ideal and $\dim R_{\mathcal{L}}/(h_1, h_2, h_3) = \dim R_{\mathcal{L}} - 3$, it follows that for 3 general degree 1 homogeneous elements h_1, h_2, h_3, $\dim R_X = \dim R_{\mathcal{L}} - 3$. Since by Theorem 2.4 $R_{\mathcal{L}}$ is Gorenstein, hence Cohen-Macaulay, h_1, h_2, h_3 is a regular sequence for R and R_X is again Gorenstein.

We will show that $\operatorname{Spec} R_X \setminus \{P_0\}$ is smooth. Consider the base locus of the linear system of degree 1 homogeneous elements of $S_{\mathcal{L}}$, $Z_1 = V(x_{ij}, 1 \le i, j \le 3) \subset \operatorname{Spec} R_X$. Using Equation (3.5)

$$Z_1 = \bigcup_{u \in \mathcal{L}} V(x_{ij}, y_v \mid 1 \le i, j \le 3, v \in \mathcal{L} \setminus \{u\}). \tag{4.2}$$

Arguing as in the proof of Proposition 4.7 each point of $Z_1 \setminus \{P_0\}$ is a smooth point of $\operatorname{Spec} R_X$. Since by Theorem 4.10 $\dim \operatorname{Sing}(\operatorname{Spec} R_{\mathcal{L}}) \le 3$, applying Bertini Theorem (cf. [3, Theorem 1.7.1]) we get that $\operatorname{Spec} R_X \setminus \{P_0\}$ is smooth. Using [10, Theorem 18.15] and arguing as in Remark 4.11 we get that R_X is a normal domain, which finishes the proof of Theorem 4.12. □

Proposition 4.13. *The scheme* $X = \operatorname{Proj} R_X$ *is integral,* $\dim X = 3$*, and* $\omega_X \cong \mathcal{O}_X$*. The singular locus of* X *consists of* 9 *isolated* $\frac{1}{3}(1,1,1)$ *singular points. In addition,* $h^1(\mathcal{O}_X) = 0$*.*

Proof. Regard X as the subvariety of $\mathbb{P}(1^9, 3^9)$ given by the equations (3.5) together with h_1 ,h_2, h_3. Since $\operatorname{Spec} R_X$ can be seen as its affine cone, using Theorem 4.12 we have that X is integral and 3-dimensional. The equality $\omega_X \cong \mathcal{O}_X$ follows from Proposition 3.6. By projective Gorensteiness of R_X we have $h^1(\mathcal{O}_X) = 0$.

For $u \in \mathcal{L}$ we denote by P_u the point of X corresponding to the ideal $(x_{ij}, y_v \mid 1 \le i, j \le 3, v \in \mathcal{L} \setminus \{u\})$ of R_X. Using (4.2), we get from Theorem 4.12 that X is smooth outside the 9 points $\{P_u \mid u \in \mathcal{L}\}$. Fix $u \in \mathcal{L}$. Around P_u we have $y_u = 1$. Looking at equations (3.5) we can eliminate the variables y_v for $v \in \mathcal{L} \setminus \{u\}$ and u_1, u_2, u_3, since these variables appear in the set of equations multiplied by y_u. This means that P_u is a quotient singularity of type $\frac{1}{3}(1,1,1)$. □

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