

VANISHING IDEALS OVER GRAPHS AND EVEN CYCLES

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ABSTRACT. Let X be an algebraic toric set in a projective space over a finite field. We study the vanishing ideal, $I(X)$, of X and show some useful degree bounds for a minimal set of generators of $I(X)$. We give an explicit combinatorial description of a set of generators of $I(X)$, when X is the algebraic toric set associated to an even cycle or to a connected bipartite graph with pairwise vertex disjoint even cycles. In this case, a formula for the regularity of $I(X)$ is given. We show an upper bound for this invariant, when X is associated to a (not necessarily connected) bipartite graph. The upper bound is sharp if the graph is connected. We are able to show a formula for the length of the parameterized linear code associated with any graph, in terms of the number of bipartite and non-bipartite components.

1. INTRODUCTION

Let \mathbb{P}^{s-1} be a projective space over a finite field \mathbb{F}_q . An *evaluation code*, also known as a *generalized Reed-Muller code*, is a linear code obtained by evaluating the linear space of homogeneous d -forms on a set of points $X \subset \mathbb{P}^{s-1}$ (see Definition 2.1). A linear code obtained in this way, denoted by $C_X(d)$, has length $|X|$. Evaluation codes have been the object of much attention in recent years. To describe their basic parameters (length, dimension and minimum distance), many authors have been using tools coming from Algebraic Geometry and Commutative Algebra, see [2, 3, 7, 11, 17, 19, 22]. Let \mathbb{T}^{s-1} be a projective torus in \mathbb{P}^{s-1} . A *parameterized linear code* is a special type of generalized Reed-Muller code obtained when $X \subset \mathbb{T}^{s-1} \subset \mathbb{P}^{s-1}$ is parameterized by a set of monomials (see Definition 2.5), in this case X is called an *algebraic toric set* because it generalizes the notion of a projective torus. Parameterized linear codes were introduced and studied in [15]. The extra structure on X yields alternative methods to compute the basic parameters of $C_X(d)$.

In this article we focus on linear codes parameterized by the edges of a graph \mathcal{G} (see Definition 2.6). For the study of algebraic toric sets parameterized by the edges of a *clutter*, which is a natural generalization of the concept of graph, we refer the reader to [17, 18]. Not much is known about the parameterized linear codes associated to a general graph. The first results in this direction appear in [10], where the length, dimension and minimum distance of the codes associated to complete bipartite graphs are computed. In [15], one can find a formula for the length of the code associated to a connected graph (see this formula in Proposition 2.7) and also a bound for the minimum distance of the code associated to a connected non-bipartite graph.

An important algebraic invariant associated to a parameterized linear code is the regularity of the ring $S/I(X)$, where S is the coordinate ring of \mathbb{P}^{s-1} , *i.e.*, a polynomial ring in s variables, and $I(X)$ is the vanishing ideal of X (see Definition 2.2). The knowledge of the regularity of $S/I(X)$ is important for applications to coding theory: for $d \geq \text{reg } S/I(X)$ the code $C_X(d)$

2010 *Mathematics Subject Classification*. Primary 13P25; Secondary 14G50, 14G15, 11T71, 94B27, 94B05.

The first author was partially supported by CMUC and FCT (Portugal), through European program COMPETE/FEDER. The second author is a member of the Center for Mathematical Analysis, Geometry and Dynamical Systems. The third author was partially supported by SNI.

coincides with the underlying vector space $\mathbb{F}_q^{|X|}$ and has, accordingly, minimum distance equal to 1. In [24, Corollary 2.31] the authors give bounds for the regularity of $S/I(X)$, when X is the algebraic toric set associated to a connected bipartite graph. In [8] a bound is given for the minimum distance of the codes associated to a graph isomorphic to a cycle of even length, as well as another bound for $\text{reg } S/I(X)$ in this case.

The contents of this paper are as follows. In Section 2, we recall the necessary background. To the best of our knowledge, there is no information available on the parameterized codes arising from disconnected graphs. If \mathcal{G} is an arbitrary graph, in Section 3, Theorem 3.2, we show our first main result, an explicit formula for the length of $C_X(d)$ in terms of the number of bipartite and non-bipartite connected components of the graph.

An earlier result of [15] shows that the vanishing ideal $I(X)$ is minimally generated by a finite set of homogeneous binomials. In Section 4, we study $I(X)$ for an arbitrary algebraic toric set X and show some useful degree bounds for a minimal set of generators of $I(X)$ (see Theorem 4.5 and Proposition 4.6). If the graph \mathcal{G} is an even cycle, another main result of this article is an explicit combinatorial description of a generating set for $I(X)$ consisting of binomials (see Theorem 5.9). This result is generalized to any connected bipartite graph whose cycles are vertex disjoint (see Theorem 5.13). We give examples of bipartite graphs not satisfying this assumption for which $I(X)$ is not generated by the set prescribed in Theorem 5.13 (see Example 5.14).

If the graph \mathcal{G} is an even cycle of length $2k$, using our description of a generating set for $I(X)$, we derive the following formula for the regularity:

$$\text{reg } S/I(X) = (q - 2)(k - 1)$$

(see Theorem 6.2). Then, we give the following upper bound for the regularity of $S/I(X)$ for a general (not necessarily connected) bipartite graph with s edges and m cycles, with disjoint edge sets, of orders $2k_1, \dots, 2k_m$:

$$\text{reg } S/I(X) \leq (q - 2)(s - \sum_{i=1}^m k_i - 1)$$

(see Theorem 6.3). In Corollary 6.5, we show that this estimate is the actual value of $\text{reg } S/I(X)$ if \mathcal{G} is a connected bipartite graph with s edges and with exactly m even cycles, with disjoint vertex sets, of orders $2k_1, \dots, 2k_m$.

The computational algebra techniques of [15] played an important role in discovering some of the results, conjectures, and examples of this paper. Using the computer algebra system *Macaulay2* [12] and the results of [15], one can compute the reduced Gröbner basis, the degree and the regularity of a vanishing ideal $I(X)$ of an algebraic toric set X over a finite field \mathbb{F}_q . This allows us to study and to gain insight on the algebraic invariants of a vanishing ideal that are useful in algebraic coding theory.

For all unexplained terminology and additional information, we refer to [1] (for graph theory), [5] (for the theory of binomial ideals), [4, 13, 20] (for commutative algebra and the theory of Hilbert functions), and [21, 23] (for the theory of linear codes and evaluation codes).

2. PRELIMINARIES

Let $K = \mathbb{F}_q$ be a finite field of order q and fix s a positive integer. Recall that the *projective space* of dimension $s - 1$ over K , denoted by \mathbb{P}^{s-1} , is the quotient space $(K^s \setminus \{0\}) / \sim$ where two vectors $\underline{x}_1, \underline{x}_2$ in $K^s \setminus \{0\}$ are equivalent if $\underline{x}_1 = \lambda \underline{x}_2$ for some $\lambda \in K^* = K \setminus \{0\}$. Denote by \mathbb{T}^{s-1} the subset of \mathbb{P}^{s-1} given by

$$\mathbb{T}^{s-1} = \{[\underline{x}] = [(x_1, \dots, x_s)] \in \mathbb{P}^{s-1} : x_1 \cdots x_s \neq 0\},$$

where $[x]$ is the equivalence class of x . The *projective torus* \mathbb{T}^{s-1} is an Abelian group under componentwise multiplication and is isomorphic to the standard $(s-1)$ -dimensional torus, $(K^*)^{s-1}$, over K .

Consider $S = K[t_1, \dots, t_s] = \bigoplus_{d=0}^{\infty} S_d$, a polynomial ring over the field K with the standard grading. Given a nonempty set of points $X = \{[x_1], \dots, [x_m]\} \subset \mathbb{T}^{s-1} \subset \mathbb{P}^{s-1}$ and letting $f_0 = t_1$, consider, for each d , the map: $\text{ev}_d: S_d \rightarrow K^{|X|}$ given by

$$(2.1) \quad f \mapsto \left(\frac{f(x_1)}{f_0^d(x_1)}, \dots, \frac{f(x_m)}{f_0^d(x_m)} \right), \quad \forall f \in S_d.$$

For each $d \geq 0$, ev_d is a linear map of K -vector spaces. Its image is denoted by $C_X(d)$.

Definition 2.1. The *evaluation code of order d* associated to X is the linear subspace of $K^{|X|}$ given by $C_X(d)$, for $d \geq 0$.

Notice that if $q = 2$ then \mathbb{T}^{s-1} is a point and, accordingly, $C_X(d) = K$, for all d . For this reason, throughout this article we assume that $q > 2$.

Clearly an evaluation code is a linear code, *i.e.*, it is a linear subspace of $K^{|X|}$. Accordingly, one defines the *dimension* of the code as its dimension as a vector space, *i.e.*, as $\dim_K C_X(d)$, its *length* as the dimension of the ambient vector space, which, for evaluation codes, coincides with $|X|$ and, finally, its *minimum distance*, is defined as:

$$\delta_X(d) = \min\{\|\mathbf{w}\| : 0 \neq \mathbf{w} \in C_X(d)\},$$

where $\|\mathbf{w}\|$ is the number of nonzero coordinates of \mathbf{w} . The basic parameters of $C_X(d)$ are related by the *Singleton bound* for the minimum distance:

$$\delta_X(d) \leq |X| - \dim_K C_X(d) + 1.$$

Two of the basic parameters of $C_X(d)$, the dimension and the length, can be expressed using the Hilbert function of the quotient of S by a particular homogeneous ideal. This ideal is the *vanishing ideal* of X , *i.e.*, the ideal of S generated by the homogeneous polynomials of S that vanish on X . Denote it by $I(X)$. Recall that the *Hilbert function* of $S/I(X)$ is given by

$$H_X(d) := \dim_K (S/I(X))_d = \dim_K S_d/I(X)_d = \dim_K C_X(d),$$

see [20]. The unique polynomial $h_X(t) = \sum_{i=0}^{k-1} c_i t^i \in \mathbb{Q}[t]$ of degree $k-1 = \dim S/I(X) - 1$ such that $h_X(d) = H_X(d)$ for $d \gg 0$ is called the *Hilbert polynomial* of $S/I(X)$. The integer $c_{k-1}(k-1)!$, denoted by $\deg S/I(X)$, is called the *degree* or *multiplicity* of $S/I(X)$. In our situation $h_X(t)$ is a nonzero constant because $S/I(X)$ has dimension 1. Furthermore $h_X(d) = |X|$ for $d \geq |X| - 1$, see [13, Lecture 13] and [6]. This means that $|X|$ is equal to the *degree* of $S/I(X)$.

A good parameterized code should have large $|X|$ together with $\dim_K C_X(d)/|X|$ and $\delta_X(d)/|X|$ as large as possible. Here, another algebraic invariant gives an indication of where to look for nontrivial evaluation codes.

Definition 2.2. The *index of regularity* of $S/I(X)$, denoted by $\text{reg } S/I(X)$, is the least integer $\ell \geq 0$ such that $h_X(d) = H_X(d)$ for $d \geq \ell$.

As $S/I(X)$ is a 1-dimensional Cohen-Macaulay graded algebra [6], the index of regularity of $S/I(X)$ is the Castelnuovo-Mumford regularity of $S/I(X)$ [4]. We will refer to $\text{reg}(S/I(X))$ simply as the *regularity* of $S/I(X)$. The regularity is related to the degrees of a minimal generating set of $I(X)$.

Definition 2.3. Let f_1, \dots, f_r be a minimal homogeneous generating set of $I(X)$. The *big degree* of $I(X)$ is defined as $\text{bigdeg } I(X) = \max_i \{\deg(f_i)\}$.

From the definition of the Castelnuovo-Mumford regularity of $S/I(X)$ [4], one has:

Proposition 2.4. $\text{bigdeg } I(X) - 1 \leq \text{reg}(S/I(X))$.

Since $\dim_K C_X(d) = H_X(d)$ and the Hilbert polynomial of $S/I(X)$ is a constant polynomial with constant term equal to the dimension of the ambient vector space, $K^{|X|}$, we deduce that for $d \geq \text{reg } S/I(X)$ the linear code $C_X(d)$ coincides with $K^{|X|}$. This can also be expressed by $\delta_X(d) = 1$ for all $d \geq \text{reg } S/I(X)$. We conclude that the potentially good codes $C_X(d)$ can occur only if $1 \leq d < \text{reg}(S/I(X))$.

For a particular class of evaluation codes, called *parameterized linear codes*, the ideal $I(X)$ has been studied to an extent that it is possible to use algebraic methods, based on elimination theory and Gröbner bases, to compute the dimension and the length of $C_X(d)$, see [15]. Let us briefly describe the notion of a parameterized linear code.

Given an n -tuple of integers, $\nu = (r_1, \dots, r_n) \in \mathbb{Z}^n$, and a vector $\underline{x} = (x_1, \dots, x_n) \in (K^*)^n$, we set $\underline{x}^\nu = x_1^{r_1} \cdots x_n^{r_n} \in K^*$. Let $\nu_1, \dots, \nu_s \in \mathbb{Z}^n$ and let $X^* \subset (K^*)^n$ be the set given by:

$$X^* = \{(\underline{x}^{\nu_1}, \dots, \underline{x}^{\nu_s}) : \underline{x} \in (K^*)^n\}.$$

Consider the multiplicative group structure of $(K^*)^s$ and let $\pi: (K^*)^s \rightarrow \mathbb{T}^{s-1}$ be the quotient map by the *diagonal subgroup* $\Lambda = \{(\lambda, \dots, \lambda) \in (K^*)^s : \lambda \in K^*\}$. Notice that $\mathbb{T}^{s-1} = (K^*)^s / \Lambda$ is the projective torus in \mathbb{P}^{s-1} .

Definition 2.5 ([16],[15]). Let $\nu_1, \dots, \nu_s \in \mathbb{N}^n$. The set of points given by $X = \pi(X^*)$ is called an *algebraic toric set* parameterized by $\nu_1, \dots, \nu_s \in \mathbb{N}^n$. The evaluation codes $C_X(d)$ obtained from an algebraic toric set X are called *parameterized linear codes*.

It is clear that X^* is a subgroup of $(K^*)^s$, since it is the image of the group homomorphism $(K^*)^n \rightarrow (K^*)^s$ given by $\underline{x} \mapsto (\underline{x}^{\nu_1}, \dots, \underline{x}^{\nu_s})$. Denote by $\theta: (K^*)^n \rightarrow X^*$ and by $\tilde{\pi}: X^* \rightarrow X$ the restrictions of the corresponding homomorphisms. Thus, we have the following sequence:

$$(2.2) \quad (K^*)^n \xrightarrow{\theta} X^* \xrightarrow{\tilde{\pi}} X \longrightarrow 1.$$

For a parameterized algebraic toric set X , the vanishing ideal $I(X)$ carries extra structure. We know that, in this situation, $I(X)$ is 1-dimensional Cohen-Macaulay lattice ideal [15]. In particular $I(X)$ is a binomial ideal, *i.e.*, it is generated by binomials. Recall that a binomial in S is a polynomial of the form $t^a - t^b$, where $a, b \in \mathbb{N}^s$ and where, if $a = (a_1, \dots, a_s) \in \mathbb{N}^s$, we set

$$t^a = t_1^{a_1} \cdots t_s^{a_s} \in S.$$

A binomial of the form $t^a - t^b$ is usually referred to as a *pure binomial* [5], although here we are dropping the adjective “pure”.

Let \mathcal{G} be a simple graph with vertex set $V_{\mathcal{G}} = \{v_1, \dots, v_n\}$ and edge set $E_{\mathcal{G}} = \{e_1, \dots, e_s\}$. Throughout the remainder of this article, when dealing with a graph, we shall reserve the use of n and s for the number of vertices and the number of edges of the graph in question. For an edge $e_i = \{v_j, v_k\}$, where $v_j, v_k \in V_{\mathcal{G}}$, let $\nu_i = \underline{e}_j + \underline{e}_k \in \mathbb{N}^n$, where, for $1 \leq j \leq n$, \underline{e}_j is the j -th element of the canonical basis of \mathbb{Q}^n .

Definition 2.6 ([10]). The *algebraic toric set associated to \mathcal{G}* is the toric set parameterized by the n -tuples $\nu_1, \dots, \nu_s \in \mathbb{N}^n$, obtained from the edges of \mathcal{G} . If X is the parameterized toric set associated to \mathcal{G} we call its associated linear code $C_X(d)$ *the parameterized code associated to \mathcal{G}* and we refer to the vanishing ideal of X as the *vanishing ideal over \mathcal{G}* .

If $\underline{x} = (x_1, \dots, x_n) \in (K^*)^n$ and $e_i = \{v_j, v_k\}$ is an edge of \mathcal{G} , we set $\underline{x}^{e_i} = \underline{x}^{v_i} = x_j x_k$, so that the structural map $\theta: (K^*)^n \rightarrow X^*$ is given by $\underline{x} \mapsto (\underline{x}^{e_1}, \dots, \underline{x}^{e_s})$. It is clear that if \mathcal{G} contains isolated vertices, then the associated algebraic toric set X coincides with the algebraic toric set associated to the subgraph of \mathcal{G} obtained by removing these vertices. If \mathcal{G} has a second edge through two vertices, then X is isomorphic to its projection away from the coordinate point of \mathbb{P}^{s-1} corresponding to that edge; which, in turn, coincides with the algebraic toric set defined by the graph obtained from \mathcal{G} by removing the multiple edge. Hence, from the point of view of the algebraic toric set X , the existence of multiple edges in \mathcal{G} is not interesting. If \mathcal{G} has only one edge then it is easy to see that $X = \mathbb{P}^{s-1}$ is a point, $I(X) = 0$ and $C_X(d) = K^*$. Thus throughout the remainder of this article we shall assume that \mathcal{G} is a simple graph with no isolated vertices and with $s \geq 2$.

If \mathcal{G} is a connected graph, the length of $C_X(d)$ has been determined.

Proposition 2.7 ([15, Corollary 3.8]). *Let \mathcal{G} be a connected graph and X its associated algebraic toric set. Then $|X| = (q-1)^{n-1}$ if \mathcal{G} is non-bipartite and $|X| = (q-1)^{n-2}$ if \mathcal{G} is bipartite.*

In particular, since $X \subset \mathbb{T}^{s-1} \subset \mathbb{P}^{s-1}$ and $|\mathbb{T}^{s-1}| = (q-1)^{s-1}$ we see that if \mathcal{G} is a connected non-bipartite graph with $n = s$, then the algebraic toric set parameterized by the edges of \mathcal{G} coincides with \mathbb{T}^{s-1} . In this situation, the vanishing ideal of \mathbb{T}^{s-1} , its invariants and all of the parameters of $C_X(d)$ are known, and are summarized in the following proposition.

Proposition 2.8. ([9, Theorem 1, Lemma 1], [17, Corollary 2.2, Theorem 3.5]) *If \mathbb{T}^{s-1} is the projective torus in \mathbb{P}^{s-1} , then*

- (i) $I(\mathbb{T}^{s-1}) = (\{t_i^{q-1} - t_1^{q-1}\}_{i=2}^s)$;
- (ii) $F_{\mathbb{T}^{s-1}}(t) = (1 - t^{q-1})^{s-1} / (1 - t)^s$;
- (iii) $\text{reg}(S/I(\mathbb{T}^{s-1})) = (s-1)(q-2)$ and $\text{deg}(S/I(\mathbb{T}^{s-1})) = |\mathbb{T}^{s-1}| = (q-1)^{s-1}$;
- (iv) $\dim_K C_{\mathbb{T}^{s-1}}(d) = \sum_{j=0}^{\lfloor d/(q-1) \rfloor} (-1)^j \binom{s-1}{j} \binom{s-1+d-j(q-1)}{s-1}$;
- (v) $\delta_{\mathbb{T}^{s-1}}(d) = (q-1)^{s-(k+2)}(q-1-\ell)$ for all $d < \text{reg}(S/I(\mathbb{T}^{s-1}))$, where $k \geq 0$ and $1 \leq \ell \leq q-2$ are the unique integers such that $d = k(q-2) + \ell$.

In the statement of the result, $F_{\mathbb{T}^{s-1}}(t) = \sum_{i=0}^{\infty} H_{\mathbb{T}^{s-1}}(i)t^i$ is the Hilbert series of $S/I(\mathbb{T}^{s-1})$. The fact that the vanishing ideal in the case of the torus is a complete intersection plays a crucial part in the proof of these results. We know that in practice the vanishing ideal associated to a general graph is far from being a complete intersection. Indeed, by [17, Corollary 4.5] for an algebraic toric set X associated to a graph (or more generally a *clutter*—see [17] for a definition), $I(X)$ is a complete intersection if and only if $X = \mathbb{T}^{s-1}$.

3. THE LENGTH OF PARAMETERIZED CODES OF GRAPHS

We continue to use the notation and definitions used in Section 2. In this section, we show an explicit formula for the length of any parameterized code associated to an arbitrary graph.

Let \mathcal{G} be a simple graph with vertex set $V_{\mathcal{G}} = \{v_1, \dots, v_n\}$ and edge set $E_{\mathcal{G}} = \{e_1, \dots, e_s\}$. Denote by $\mathcal{G}_1, \dots, \mathcal{G}_m$ the connected components of \mathcal{G} . For each $1 \leq j \leq m$, let n_j and s_j denote the number of vertices and edges of \mathcal{G}_j , respectively; so that $n = n_1 + \dots + n_m$ and $s = s_1 + \dots + s_m$. Denote the edges of \mathcal{G}_j by $\{e_{j1}, \dots, e_{js_j}\}$, let $X_j \subset \mathbb{P}^{s_j-1}$ be the algebraic toric set parameterized by \mathcal{G}_j and let

$$(K^*)^{n_j} \xrightarrow{\theta_j} X_j^* \xrightarrow{\tilde{\pi}_j} X_j \longrightarrow 1$$

be the corresponding structural sequences. Since for fixed distinct $j_1 \neq j_2$ the edges $e_{j_1 k_1}$ and $e_{j_2 k_2}$ have no vertex in common and thus $\underline{x}^{e_{j_1 k_1}}$ and $\underline{x}^{e_{j_2 k_2}}$ involve disjoint sets of coordinates of the vector \underline{x} , we deduce that $\theta: (K^*)^n \rightarrow X^*$ is isomorphic to

$$\theta_1 \times \cdots \times \theta_m: (K^*)^{n_1} \times \cdots \times (K^*)^{n_m} \rightarrow X_1^* \times \cdots \times X_m^*.$$

In particular $|X^*| = \prod_{j=1}^m |X_j^*|$. We need to find the order of the kernel of the maps $\tilde{\pi}_j$.

Lemma 3.1. *Let \mathcal{G} be a connected graph. If \mathcal{G} is non-bipartite, then $|\text{Ker } \tilde{\pi}| = \frac{q-1}{2}$ if q is odd and $|\text{Ker } \tilde{\pi}| = q-1$ if q is even. If \mathcal{G} is bipartite, then $|\text{Ker } \tilde{\pi}| = q-1$.*

Proof. Let $\underline{x} \in (K^*)^n$. Then $\theta(\underline{x}) = (1, \dots, 1)$ implies that $\underline{x}^e = 1$ for all $e \in E_{\mathcal{G}}$. Suppose \mathcal{G} is non-bipartite. Then \mathcal{G} contains an odd cycle. We assume, without loss of generality, that the edges in this cycle are

$$e_1 = \{v_1, v_2\}, \dots, e_{2k-1} = \{v_{2k-1}, v_1\},$$

where $v_1, \dots, v_{2k-1} \in V_{\mathcal{G}}$. We deduce that $x_1 x_2 = \cdots = x_{2k-1} x_1 = 1$, which, in turn, implies that $x_1 = \cdots = x_{2k-1} = u \in K^*$ with $u^2 = 1$.

Now, let $v_r \in V_{\mathcal{G}}$ be any vertex of \mathcal{G} . Then, there exists a path

$$\{v_1, v_{j_1}\}, \{v_{j_1}, v_{j_2}\}, \dots, \{v_{j_k}, v_r\}$$

connecting v_1 with v_r . Since $x_1 x_{j_1} = x_{j_1} x_{j_2} = \cdots = x_{j_k} x_r = 1$, we deduce that $x_r = u$. Hence, either $\underline{x} = (1, \dots, 1)$ or $\underline{x} = (-1, \dots, -1)$, from which we conclude that $|\text{Ker } \theta| = 2$ if q is odd and $|\text{Ker } \theta| = 1$ if q is even.

Suppose now that \mathcal{G} is bipartite, and, without loss of generality, denote the bipartition of $V_{\mathcal{G}}$ by $\{v_1, \dots, v_{\ell}\} \cup \{v_{\ell+1}, \dots, v_n\}$. Let v_r be any vertex and let

$$\{v_1, v_{j_1}\}, \{v_{j_1}, v_{j_2}\}, \dots, \{v_{j_k}, v_r\}$$

be a path connecting v_1 with v_r . Notice that $\{v_{j_1}, v_{j_3}, \dots\}$ is a subset of $\{v_{\ell+1}, \dots, v_n\}$ and $\{v_{j_2}, v_{j_4}, \dots\}$ is a subset of $\{v_1, \dots, v_{\ell}\}$. From $x_1 x_{j_1} = x_{j_1} x_{j_2} = \cdots = x_{j_k} x_r = 1$ we deduce that $x_r = x_1$ if $v_r \in \{v_1, \dots, v_{\ell}\}$ or $x_r = x_1^{-1}$ otherwise. Hence $\underline{x} = (x_1, \dots, x_1, x_1^{-1}, \dots, x_1^{-1})$, i.e., the ℓ first coordinates of \underline{x} are equal to x_1 and the remaining ones are equal to x_1^{-1} . Conversely, it is easy to see that any element of $(K^*)^n$ of the form $(u, \dots, u, u^{-1}, \dots, u^{-1})$ belongs to $\text{Ker } \theta$. We deduce that in this case $|\text{Ker } \theta| = q-1$. The proof now follows easily from Proposition 2.7. Indeed, we know that the order of X is $(q-1)^{n-1}$, if \mathcal{G} is non-bipartite and $(q-1)^{n-2}$ otherwise. Hence, $|\text{Ker } \tilde{\pi}| = \frac{q-1}{2}$ if \mathcal{G} is non-bipartite and q is odd, $|\text{Ker } \tilde{\pi}| = q-1$ if \mathcal{G} is non-bipartite and q is even, and $|\text{Ker } \tilde{\pi}| = q-1$ if \mathcal{G} is bipartite. \square

We come to the main result of this section.

Theorem 3.2. *Suppose \mathcal{G} has m connected components, of which γ are non-bipartite. Then,*

$$|X| = \begin{cases} \left(\frac{1}{2}\right)^{\gamma-1} (q-1)^{n-m+\gamma-1}, & \text{if } \gamma \geq 1 \text{ and } q \text{ is odd,} \\ (q-1)^{n-m+\gamma-1}, & \text{if } \gamma \geq 1 \text{ and } q \text{ is even,} \\ (q-1)^{n-m-1}, & \text{if } \gamma = 0. \end{cases}$$

Proof. As in the discussion above, let X_1, \dots, X_m be the parameterized toric sets associated to the connected components of \mathcal{G} . Then $|X^*| = \prod_{j=1}^m |X_j^*|$, which, by Lemma 3.1, is given by

$$|X^*| = \begin{cases} \left(\frac{1}{2}\right)^{\gamma} (q-1)^{n-m+\gamma}, & \text{if } q \text{ is odd,} \\ (q-1)^{n-m+\gamma}, & \text{if } q \text{ is even.} \end{cases}$$

From the proof of Lemma 3.1, it is seen that the kernel of the map $\tilde{\pi}: X^* \rightarrow X$ is equal to Λ , the diagonal subgroup of $(K^*)^s$, if $\gamma = 0$, and it is equal to $\Lambda^2 = \{(\lambda^2, \dots, \lambda^2) \mid \lambda \in K^*\}$ if $\gamma \geq 1$. The subgroup Λ has order $q - 1$. The subgroup Λ^2 has order $q - 1$ if q is even and has order $(q - 1)/2$ if q is odd (this follows readily using the map $\lambda \mapsto (\lambda^2, \dots, \lambda^2)$). As $|X| = |X^*|/|\text{Ker } \tilde{\pi}|$, the result follows. \square

Example 3.3. Let G be the graph whose connected components are a triangle and a square. Thus, $n = 7$, $m = 2$, $\gamma = 1$. Using the formula of Theorem 3.2, we get: (a) $|X| = 1024$ if $q = 5$, and (b) $|X| = 243$ if $q = 2^2$.

4. DEGREE BOUNDS FOR THE GENERATORS OF $I(X)$

We continue to use the notation and definitions used in Section 2. In this section $X \subset \mathbb{P}^{s-1}$ is the algebraic toric set parameterized by $\nu_1, \dots, \nu_s \in \mathbb{N}^n$ and $I(X) \subset S = K[t_1, \dots, t_s]$ is the vanishing ideal of X . We show some degree bounds for a minimal set of generators of $I(X)$ consisting of binomials.

Recall that by [15] we know that $I(X)$ is generated by homogeneous binomials $t^a - t^b$, with $a, b \in \mathbb{N}^s$. There are a number of elementary observations to be made. Let $f = t^a - t^b$ be a nonzero binomial of S . Firstly, since $X \subset \mathbb{T}^{s-1}$, evidently $I(\mathbb{T}^{s-1}) \subset I(X)$, hence $t_i^{q-1} - t_j^{q-1} \in I(X)$, for all $1 \leq i, j \leq s$. Secondly, if $\gcd(t^a, t^b) \neq 1$, then we can factor the greatest common divisor t^c from both t^a and t^b to obtain $t^a - t^b = t^c(t^{a'} - t^{b'})$, for some $a', b' \in \mathbb{N}^s$. Since t^c is never zero on \mathbb{T}^{s-1} , for any $c \in \mathbb{N}^s$, we deduce that $t^a - t^b \in I(X)$ if and only if $t^{a'} - t^{b'} \in I(X)$. Therefore, when looking for “binomial generators” of $I(X)$ we may restrict ourselves to those binomials $t^a - t^b$ such that t^a and t^b have no common divisors. Given $a = (a_1, \dots, a_s) \in \mathbb{N}^s$, we set $|a| = a_1 + \dots + a_s$ and $\text{supp}(a) = \{i : a_i \neq 0\}$. Then, clearly, t^a and t^b have no common divisors if and only if $\text{supp}(a) \cap \text{supp}(b) = \emptyset$.

Definition 4.1. A subgroup of \mathbb{Z}^s is called a *lattice*. A *lattice ideal* is an ideal of the form

$$I(\mathcal{L}) = (\{t^a - t^b : a - b \in \mathcal{L} \text{ and } \text{supp}(a) \cap \text{supp}(b) = \emptyset\}) \subset S$$

for some lattice \mathcal{L} of \mathbb{Z}^s .

Lemma 4.2. Let $L \subset S$ be a lattice ideal generated by $\mathcal{B} = \{t^{a_i} - t^{b_i}\}_{i=1}^r$. Then, (a) $L = I(\mathcal{L})$, where \mathcal{L} is the subgroup of \mathbb{Z}^s generated by $\{a_i - b_i\}_{i=1}^r$, and (b) if $t^{a_i} - t^{b_i}$ is homogeneous for all i and $f = t^a - t^b \in L$, then f is homogeneous.

Proof. Part (a) follows from [14, Lemma 7.6]. To show (b) notice that, from part (a), $f \in I(\mathcal{L})$. Then, $a - b$ is a linear combination of $\{a_i - b_i\}_{i=1}^r$. Thus, if $\mathbf{1} = (1, \dots, 1)$, we get that $|a| - |b|$ is equal to $\langle \mathbf{1}, a - b \rangle = 0$ because $\langle \mathbf{1}, a_i - b_i \rangle = 0$ for all i . Thus, $|a| = \deg(t^a) = \deg(t^b) = |b|$. \square

Lemma 4.3. If $f = t^a - t^b \in I(X)$, then f is homogeneous.

Proof. According to [15, Theorem 2.1], $I(X)$ is lattice ideal generated by homogeneous binomials. Thus, the lemma follows from Lemma 4.2. \square

Lemma 4.4. Let $f = t^a - t^b \in I(X)$, where $a, b \in \mathbb{N}^s$ and $\text{supp}(a) \cap \text{supp}(b) = \emptyset$. Suppose that there exists i such that t_i^{q-1} divides t^a and $\text{supp}(b) \neq \emptyset$. Then, there exists a binomial $g \in I(X)$, with $\deg(g) < \deg(f)$, and there exists j , such that $f - t_j g \in I(\mathbb{T}^{s-1})$.

Proof. Write $t^a = t_i^{q-1}t^{a'}$, with $a' \in \mathbb{N}^s$. Since $\text{supp}(b) \neq \emptyset$, there exists j such that t_j divides t^b . Write $t^b = t_j t^{b'}$, for some $b' \in \mathbb{N}^s$. Then,

$$t^a - t^b = t_i^{q-1}t^{a'} - t_j t^{b'} = (t_i^{q-1} - t_j^{q-1})t^{a'} + t_j(t_j^{q-2}t^{a'} - t^{b'}).$$

Set $g = t_j^{q-2}t^{a'} - t^{b'}$. Then, since $t_i^{q-1} - t_j^{q-1} \in I(X)$, we see that $g \in I(X)$ and, moreover, it is clear that if $g \neq 0$ then $\deg(g) = \deg(f) - 1$. \square

Theorem 4.5. *There exists a set of generators of $I(X)$ which consists of the toric relations $t_i^{q-1} - t_j^{q-1}$ plus a finite set of homogeneous binomials $t^a - t^b$ with $\text{supp}(a) \cap \text{supp}(b) = \emptyset$ and such that the degree of $t^a - t^b$ in each of the variables t_i is $\leq q - 2$.*

Proof. We know that $I(X)$ is generated by binomials [15]. If $\{f_1, \dots, f_r\}$ is a set of binomials generating $I(X)$, then so is the set

$$\mathcal{B} = \{f_1, \dots, f_r\} \cup \{t_i^{q-1} - t_j^{q-1} : 1 \leq i, j \leq s\}.$$

If $f_i \in I(\mathbb{T}^{s-1})$, we have $(\mathcal{B}) = (\{\{f_1, \dots, f_r\} \setminus \{f_i\}\} \cup \{t_i^{q-1} - t_j^{q-1} : 1 \leq i, j \leq s\})$. Thus, we may assume that \mathcal{B} is a generating set of $I(X)$ with $f_i \notin I(\mathbb{T}^{s-1})$ for all i . By the discussion above we may also assume that each f_i is of the form $t^a - t^b$ with $\text{supp}(a) \cap \text{supp}(b) = \emptyset$. We can write $f_1 = t^a - t^b$, with $a, b \in \mathbb{N}^s$. Suppose that there exists i such that t_i^{q-1} divides t^a or t^b . Hence, since f_1 is homogeneous by Lemma 4.3, we deduce that the sets $\text{supp}(a)$ and $\text{supp}(b)$ are both nonempty. Then, from Lemma 4.4, there exists j and a homogeneous binomial $g'_1 \in I(X)$ such that $\deg(g'_1) < \deg(f_1)$ and $f_1 - t_j g'_1 \in I(\mathbb{T}^{s-1})$. We can write $g'_1 = t^c g_1$ for some $c \in \mathbb{N}^s$, where g_1 is a binomial in $I(X)$ whose terms have disjoint support. Clearly,

$$I(X) = (\mathcal{B}) = (\{g_1, f_2, \dots, f_r\} \cup \{t_i^{q-1} - t_j^{q-1} : 1 \leq i, j \leq s\})$$

and $g_1 \notin I(\mathbb{T}^{s-1})$. If there exists i such that t_i^{q-1} divides one of the terms of g_1 , we repeat the previous procedure with g_1 playing the role of f_1 and obtain a binomial g_2 , and so on. Thus, by iterating the previous procedure, we obtain a sequence of homogeneous binomials f_1, g_1, \dots, g_m , with decreasing degrees, such that

$$(4.1) \quad I(X) = (\mathcal{B}) = (\{g_m, f_2, \dots, f_r\} \cup \{t_i^{q-1} - t_j^{q-1} : 1 \leq i, j \leq s\})$$

and $g_m \notin I(\mathbb{T}^{s-1})$. Thus, using the previous procedure enough times, we obtain a binomial $g_m = t^{a'} - t^{b'}$ none of whose terms $t^{a'}$ or $t^{b'}$ is divisible by any t_i^{q-1} , for $1 \leq i \leq s$. If we proceed in this manner, with each of the remaining f_2, \dots, f_r , we reach a generating set satisfying the condition in the statement. \square

The next proposition is intended mainly for practical applications. It gives a bound on the degrees of a minimal set of generators of $I(X)$. It is a valuable tool to use when implementing the calculation of $I(X)$ in a computer algebra software.

Proposition 4.6. *Set $k = \lfloor \frac{s}{2} \rfloor$. If $k \geq 2$, then the vanishing ideal of X has a generating set whose elements have degree $\leq k(q - 2)$.*

Proof. Let $t^a - t^b \in I(X)$ be a homogeneous binomial. Write $a = (a_1, \dots, a_s) \in \mathbb{N}^s$ and $b = (b_1, \dots, b_s) \in \mathbb{N}^s$. By Theorem 4.5, we may assume that $\text{supp}(a) \cap \text{supp}(b) = \emptyset$ and that $0 \leq a_i, b_j \leq q - 2$. Let $r = |\text{supp}(a)|$ and $\ell = |\text{supp}(b)|$. Then, either r or ℓ is $\leq k$, for otherwise:

$$r + \ell \geq 2k + 2 = 2 \lfloor s/2 \rfloor + 2 \geq s + 1,$$

which is impossible. Assume $r \leq k$. Then, $\deg(t^a - t^b) = a_1 + \dots + a_s \leq r(q - 2) \leq k(q - 2)$. \square

As seen in Example 4.7 below, this estimate is sharp. For the graph given there, working over the field \mathbb{F}_3 , the ideal $I(X)$ needs a generator of degree k . If, however, X is the algebraic toric set associated to a cycle \mathcal{G} of order $s = 2k$, then, we will see that by Corollary 5.12, $I(X)$ is generated in degrees $\leq (k-1)(q-2) + 1$.

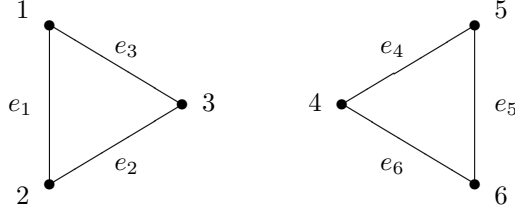


FIGURE 1

Example 4.7. Let \mathcal{G} be the graph in Figure 1 and assume that $q = 3$. Then, using *Macaulay2* [12], we found that $I(X)$ is generated by the (minimal) set of binomials:

$$\begin{aligned} & t_5^2 - t_6^2, \quad t_4^2 - t_6^2, \quad t_3^2 - t_6^2, \quad t_2^2 - t_6^2, \quad t_1^2 - t_6^2, \\ & t_3t_4t_5 - t_1t_2t_6, \quad t_2t_4t_5 - t_1t_3t_6, \quad t_1t_4t_5 - t_2t_3t_6, \quad t_2t_3t_5 - t_1t_4t_6, \quad t_1t_3t_5 - t_2t_4t_6, \\ & t_1t_2t_5 - t_3t_4t_6, \quad t_2t_3t_4 - t_1t_5t_6, \quad t_1t_3t_4 - t_2t_5t_6, \quad t_1t_2t_4 - t_3t_5t_6, \quad t_1t_2t_3 - t_4t_5t_6. \end{aligned}$$

5. GENERATORS OF $I(X)$ FOR EVEN CYCLES AND CERTAIN BIPARTITE GRAPHS

We keep the notation of Section 3: $X \subset \mathbb{P}^{s-1}$ is the algebraic toric set parameterized by a graph \mathcal{G} and $I(X) \subset S = K[t_1, \dots, t_s]$ is the vanishing ideal of X . This section is devoted to giving an explicit description of a binomial generating set for $I(X)$, when $\mathcal{G} = \mathcal{C}_{2k}$ is a cycle of even order, or when \mathcal{G} is a bipartite graph whose cycles are pairwise vertex disjoint.

Proposition 5.1. *Let $f = t^a - t^b \in I(X)$, with $a = (a_1, \dots, a_s)$ and $b = (b_1, \dots, b_s)$, such that $\text{supp}(a) \cap \text{supp}(b) = \emptyset$ and $a_j, b_j \leq q-2$ for all j . (a) If \mathcal{G} is a connected bipartite graph and e_i is an edge of \mathcal{G} which does not belong to any cycle of \mathcal{G} , then $a_i = b_i = 0$. (b) If \mathcal{G} is any graph and \mathcal{G} has an edge e_i with a degree 1 incident vertex, then $a_i = b_i = 0$.*

Proof. (a) Assume, without loss of generality that, $e_i = \{v_1, v_2\}$. In what follows we use the symbol \sqcup to denote a disjoint union of objects. Since \mathcal{G} is bipartite there exist a bipartition $V_{\mathcal{G}} = A \sqcup B$ with, say, $v_1 \in A$ and $v_2 \in B$. Since e_i does not belong to a cycle of \mathcal{G} , the removal of edge e_i produces a disconnected graph $\mathcal{G}_1 \sqcup \mathcal{G}_2$, with $v_1 \in V_{\mathcal{G}_1}$ and $v_2 \in V_{\mathcal{G}_2}$. Let $u \in K^*$ be a generator of the multiplicative group of K . Let us label the vertices of \mathcal{G} with one of the elements u, u^{-1} or 1, according to the rule that we now explain. Let v_r be any vertex. If $v_r \in V_{\mathcal{G}_1}$ label v_r with 1, if $v_r \in V_{\mathcal{G}_2} \cap A$ label v_r with u^{-1} , and if $v_r \in V_{\mathcal{G}_2} \cap B$ label v_r with u . Consider $\underline{x} = (x_1, \dots, x_n) \in (K^*)^n$ where, for $1 \leq r \leq n$, the coordinate x_r takes on the value of the label of v_r . Then $\underline{x}^{e_j} = 1$ if $j \neq i$ and $\underline{x}^{e_i} = u$. Assume that $a_i > 0$, then $b_i = 0$ because a and b have disjoint support. Thus $f(\underline{x}^{e_1}, \dots, \underline{x}^{e_s}) = 0$, implies that $u^{a_i} - 1 = 0$, a contradiction because $1 \leq a_i \leq q-2$. Similarly if $b_i > 0$ we derive a contradiction. Hence, we deduce that $a_i = b_i = 0$. (b) This part follows using a similar argument. \square

Example 5.2. For non-bipartite graphs Proposition 5.1(a) does not hold. Let \mathcal{G} be the graph in Figure 2 and assume that $q = 5$. Then, using *Macaulay2* [12], we found that the binomial $t_1t_2t_4^2t_7 - t_3t_5^2t_6t_8$ is in a minimal generating set of $I(X)$. In this monomial the variables t_4 and t_5 , which are not in any cycle of \mathcal{G} , occur.

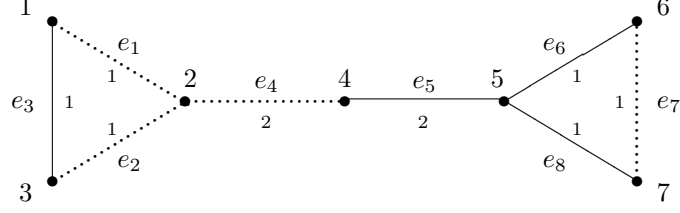


FIGURE 2

Corollary 5.3. *Suppose that $\mathcal{G} = \mathcal{C}_{2k}$ is a cycle of even order. Let $f = t^a - t^b$ be a nonzero homogeneous binomial in $I(X)$, with $a = (a_1, \dots, a_s) \in \mathbb{N}^s$ and $b = (b_1, \dots, b_s) \in \mathbb{N}^s$ such that $\text{supp}(a) \cap \text{supp}(b) = \emptyset$ and $0 \leq a_i, b_j \leq q - 2$. Then $\text{supp}(a) \cup \text{supp}(b) = \{1, \dots, s\}$.*

Proof. Assume, without loss of generality that $s \notin \text{supp}(a) \cup \text{supp}(b)$. Then, f is a polynomial in the variables t_1, \dots, t_{s-1} which vanishes along the projection of X onto the first $s-1$ coordinates. The algebraic toric set obtained after projecting is none other than the algebraic toric set associated with the graph obtained from $\mathcal{G} = \mathcal{C}_{2k}$ by removing the edge e_s , which is a tree. Hence, by Proposition 5.1, none of the remaining variables t_1, \dots, t_{s-1} occurs in f , in other words, $f = 0$, which is a contradiction. \square

From now on, until otherwise stated, we will restrict to the case of $\mathcal{G} = \mathcal{C}_{2k}$, a cycle of order $2k$ with $k \geq 2$. Let $V_{\mathcal{C}_{2k}} = \{v_1, \dots, v_{2k}\}$ and $e_i = \{v_i, v_{i+1}\}$ for $1 \leq i \leq 2k - 1$ and $e_s = e_{2k} = \{v_{2k}, v_1\}$. We are now ready to give a combinatorial description of the generators of $I(X)$ other than those coming from the toric relations. From Theorem 4.5 and Corollary 5.3 we know that there is a set of generators of $I(X)$ consisting of the toric generators $t_i^{q-1} - t_j^{q-1}$ plus a set of binomials of the type $t^a - t^b$ where $a = (a_1, \dots, a_s) \in \mathbb{N}^s$, $b = (b_1, \dots, b_s) \in \mathbb{N}^s$ are such that $\text{supp}(a) \sqcup \text{supp}(b) = \{1, \dots, s\}$ and $0 \leq a_i, b_j \leq q - 2$. Hence to any such binomial one can associate a partition of $\{1, \dots, s\}$. For the remainder of this article, given $r \in \{1, \dots, q - 2\}$ we will fix the following notation:

$$\hat{r} = q - 1 - r.$$

Definition 5.4. Let $\sigma = A \sqcup B$ be a partition of $\{1, \dots, s\}$ and fix $r \in \{1, \dots, q - 2\}$. Define a function $\rho_\sigma^r: \{1, \dots, s\} \rightarrow \{r, \hat{r}\}$, recursively, by setting $\rho_\sigma^r(1) = r$ and,

$$(5.1) \quad \begin{cases} \rho_\sigma^r(i+1) = \widehat{\rho_\sigma^r(i)}, & \text{if } \{i, i+1\} \subset A \text{ or } \{i, i+1\} \subset B \\ \rho_\sigma^r(i+1) = \rho_\sigma^r(i), & \text{otherwise,} \end{cases}$$

for every $1 \leq i \leq s - 1$.

Notice that, for every $i \in \{1, \dots, s - 2\}$, $\rho_\sigma^r(i) = \rho_\sigma^r(i+2)$ if and only if i and $i+2$ are in the same partition. Since s is even, we deduce that $\rho_\sigma^r(1) = \rho_\sigma^r(s-1)$ if and only if 1 and $s-1$ are in the same partition. This implies that $\rho_\sigma^r(1)$ can be defined from $\rho_\sigma^r(s)$ using the same recursive formula. Indeed, working in $\{1, \dots, s\}$ modulo s , the function ρ_σ^r can be recovered recursively, using the above rule, from $\rho_\sigma^r(k)$, for any $k \in \{1, \dots, s\}$. The following lemma will be used in the proofs of some of the results below.

Lemma 5.5. *Let $\sigma = A \sqcup B$ be a partition of $\{1, \dots, s\}$ and $r \in \{1, \dots, q - 2\}$. Consider $i \in A$ and $\sigma' = A' \sqcup B'$ where $A' = A \setminus \{i\}$ and $B' = B \cup \{i\}$. Let $\rho: \{1, \dots, s\} \rightarrow \{r, \hat{r}\}$ be given by $\rho(j) = \rho_\sigma^r(j)$ for every $j \neq i$ and $\rho(i) = \widehat{\rho_\sigma^r(i)}$. Then $\rho = \rho_{\sigma'}^r$, if $i > 1$ or $\rho = \rho_{\sigma'}^{\hat{r}}$, if $i = 1$.*

Proof. We will look first at the case $i = 1$. In this case, $\rho(1) = \widehat{\rho_\sigma^r(1)} = \hat{r} = \rho_{\sigma'}^{\hat{r}}(1)$. If $2 \in A$, then $\rho(2) = \rho_\sigma^r(2) = \hat{r}$, according to the definition of the function ρ , to Definition 5.4 and to the fact that $1 \in A$. But if $2 \in A$, then $2 \in A'$, and $\rho_{\sigma'}^{\hat{r}}(2) = \hat{r}$ since $1 \in B'$. If $2 \in B$, then $\rho(2) = \rho_\sigma^r(2) = r$; but if $2 \in B$, then $2 \in B'$, and $\rho_{\sigma'}^{\hat{r}}(2) = r$. In any case, $\rho(2) = \rho_{\sigma'}^{\hat{r}}(2)$. Let $j \in \{3, \dots, s\}$. By definition, $\rho(j) = \rho_\sigma^r(j)$; and $\rho_\sigma^r(j)$ is determined by σ , by $\rho_\sigma^r(2)$ and by Eq. (5.1). Since $\rho_{\sigma'}^{\hat{r}}(j)$ is determined by σ' , by $\rho_{\sigma'}^{\hat{r}}(2)$ and by Eq. (5.1), since $\rho_\sigma^r(2) = \rho_{\sigma'}^{\hat{r}}(2)$, and since the partitions σ and σ' agree in $\{2, \dots, s\}$, we conclude that $\rho(j) = \rho_\sigma^r(j) = \rho_{\sigma'}^{\hat{r}}(j)$. Therefore, $\rho = \rho_{\sigma'}^{\hat{r}}$. For the case $i > 1$, we use a similar argument to show that $\rho = \rho_\sigma^r$. \square

Given any $\sigma = A \sqcup B$, a partition of $\{1, \dots, s\}$, if, without loss in generality, we choose $1 \in A$, it is clear that given any $r \in \{1, \dots, q-2\}$, there exist unique a and b in \mathbb{N}^s such that $\text{supp}(a) = A$, $\text{supp}(b) = B$, $a_i = \rho_\sigma^r(i)$ if $i \in \text{supp}(a)$ and $b_j = \rho_\sigma^r(j)$ if $j \in \text{supp}(b)$.

Definition 5.6. Let $\sigma = A \sqcup B$ be a partition of $\{1, \dots, s\}$ with $1 \in A$ and let $r \in \{1, \dots, q-2\}$. We denote by f_σ^r the unique binomial $t^a - t^b$ such that $\text{supp}(a) = A$, $\text{supp}(b) = B$, $a_i = \rho_\sigma^r(i)$ if $i \in \text{supp}(a)$ and $b_j = \rho_\sigma^r(j)$ if $j \in \text{supp}(b)$.

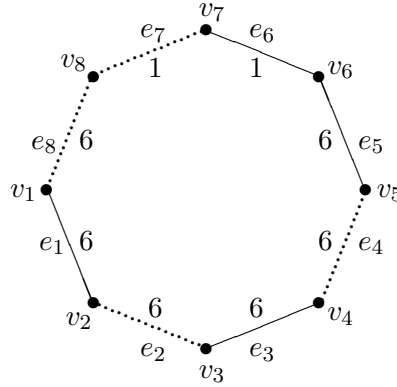


FIGURE 3

The combinatorial data that give rise to a binomial $f_\sigma^r = t^a - t^b$ is clarified by representing it in the graph \mathcal{G} , by putting a label r or \hat{r} to each edge. Figure 3 illustrates the map ρ_σ^6 when $q = 8$, $r = 6$, $s = 8$ and $\sigma = \{1, 3, 5, 6\} \sqcup \{2, 4, 7, 8\}$. The labels of the edges correspond to the exponents of the variables in the corresponding binomial of $I(X)$. Thus, $f_\sigma^6 = t_1^6 t_3^6 t_5^6 t_6 - t_2^6 t_4^6 t_7 t_8$.

Lemma 5.7. Let $\sigma = A \sqcup B$ be a partition of $\{1, \dots, s\}$ and let $r \in \{1, \dots, q-2\}$. Suppose that $1 \in A$ and that there exists $i \in A$ such that $i > 2$ and $i-1 \notin A$. Let σ' be the partition given by $A' \sqcup B'$ where $A' = (A \setminus \{i\}) \cup \{i-1\}$ and $B' = (B \setminus \{i-1\}) \cup \{i\}$. Then $f_\sigma^r \in I(X)$ if and only if $f_{\sigma'}^r \in I(X)$.

Proof. Let $f_\sigma^r = t^a - t^b$. Using the assumption, we can write $t^a = t_i^c t^{a'}$ and $t^b = t_{i-1}^c t^{b'}$, where $c = a_i = b_{i-1}$ and $a', b' \in \mathbb{N}^s$. Then:

$$\begin{aligned} (t_{i-1} t_i)^{\hat{c}} f_\sigma^r &= t_{i-1}^{\hat{c}} t_i^{q-1} t^{a'} - t_{i-1}^{q-1} t_i^{\hat{c}} t^{b'} \\ &= t_{i-1}^{\hat{c}} t_i^{q-1} t^{a'} - t_{i-1}^{\hat{c}} t_{i-1}^{q-1} t^{a'} + t_{i-1}^{\hat{c}} t_{i-1}^{q-1} t^{a'} - t_{i-1}^{q-1} t_i^{\hat{c}} t^{b'} \\ &= t_{i-1}^{\hat{c}} t^{a'} (t_i^{q-1} - t_{i-1}^{q-1}) + (t_{i-1}^{\hat{c}} t^{a'} - t_i^{\hat{c}} t^{b'}) t_{i-1}^{q-1}. \end{aligned}$$

Since t_j is never zero on X we get:

$$f_\sigma^r \in I(X) \Leftrightarrow (t_{i-1} t_i)^{\hat{c}} f_\sigma^r \in I(X) \Leftrightarrow (t_{i-1}^{\hat{c}} t^{a'} - t_i^{\hat{c}} t^{b'}) t_{i-1}^{q-1} \in I(X) \Leftrightarrow t_{i-1}^{\hat{c}} t^{a'} - t_i^{\hat{c}} t^{b'} \in I(X).$$

Now let $a^\sharp, b^\sharp \in \mathbb{N}^s$ be such that $t^{a^\sharp} = t_{i-1}^{\hat{c}} t^{a'}$ and $t^{b^\sharp} = t_i^{\hat{c}} t^{b'}$. Then, $\sigma' = \text{supp}(a^\sharp) \sqcup \text{supp}(b^\sharp)$ is the partition of $\{1, \dots, s\}$ obtained from interchanging $i-1$ and i in $A \sqcup B$. Applying Lemma 5.5 twice, we deduce that $f_{\sigma'}^r = t_{i-1}^{\hat{c}} t^{a'} - t_i^{\hat{c}} t^{b'}$. \square

Lemma 5.8. *Let $\sigma = A \sqcup B$ be a partition of $\{1, \dots, s\}$ with $1 \in A$ and let $r \in \{1, \dots, q-2\}$. If $f_\sigma^r \in I(X)$ then $|A| = |B|$.*

Proof. Let $\ell = |A|$. Using sufficiently many times Lemma 5.7, we may assume that σ is the partition $\{1, \dots, \ell\} \sqcup \{\ell+1, \dots, s\}$. Accordingly,

$$f_\sigma^r = t_1^r t_2^{\hat{r}} \cdots t_\ell^{r'} - t_{\ell+1}^{r'} \cdots t_{s-1}^{\hat{r}} t_s^r,$$

where $r' \in \{r, \hat{r}\}$. Now $\deg(t_j^r \cdots)$, for a monomial consisting of a product of variables with consecutive exponents alternating in $\{r, \hat{r}\}$, is a strictly increasing function with respect to the number of variables involved. Since f_σ^r is homogeneous (by Lemma 4.3) we deduce that $|A| = \ell = s - \ell = |B|$. \square

We come to one of the main results of this section, a combinatorial description of a generating set for vanishing ideals of an even cycle.

Theorem 5.9. *Let $I(X)$ be the vanishing ideal of the algebraic toric set X associated to an even cycle $\mathcal{G} = \mathcal{C}_{2k}$. Then, $I(X)$ is generated by the binomials $t_i^{q-1} - t_j^{q-1}$, $1 \leq i, j \leq s = 2k$, and the binomials f_σ^r obtained from all $r \in \{1, \dots, q-2\}$ and all partitions $\sigma = A \sqcup B$ of $\{1, \dots, s\}$ with $|A| = |B|$.*

Proof. By Theorem 4.5 and Corollary 5.3, we know that $I(X)$ is generated by the binomials of the form $t_i^{q-1} - t_j^{q-1}$, $1 \leq i, j \leq s = 2k$, and the homogeneous binomials $f = t^a - t^b$ with $a = (a_1, \dots, a_s) \in \mathbb{N}^s$ and $b = (b_1, \dots, b_s) \in \mathbb{N}^s$ such that $\text{supp}(a) \sqcup \text{supp}(b) = \{1, \dots, s\}$ and $0 \leq a_i, b_j \leq q-2$. Let f be a binomial of the latter type. We may assume that $1 \in \text{supp}(a)$, for we can always replace f by $-f$ in a generating set of $I(X)$. Set $\sigma = \text{supp}(a) \sqcup \text{supp}(b)$ and let $r = a_1$. Let us show that $f = f_\sigma^r$, i.e., let us show that $a_i = \rho_\sigma^r(i)$, for every $i \in \text{supp}(a) \setminus \{1\}$ and $b_j = \rho_\sigma^r(j)$ for every $j \in \text{supp}(b)$. Let $i \in \text{supp}(a) \setminus \{1\}$ and let $u \in K^*$ be a generator of the multiplicative group of K . Consider $\underline{x} \in (K^*)^n$ given by setting $x_i = u$ and $x_j = 1$ for all $j \neq i$. Then, $f(\underline{x}^{\nu_1}, \dots, \underline{x}^{\nu_s}) = 0$ implies that $u^{a_i-1} u^{a_i} = 1$, if $i-1 \in \text{supp}(a)$ or $u^{a_i} = u^{b_{i-1}}$ if $i-1 \in \text{supp}(b)$. We get, in the first case, $a_i = q-1 - a_{i-1} = \rho_\sigma^r(i)$, and, in the second case, $a_i = b_{i-1} = \rho_\sigma^r(i)$. Similarly, if $j \in \text{supp}(b)$, then $b_j = \rho_\sigma^r(j)$. Since $f_\sigma^r \in I(X)$, by Lemma 5.8, $|A| = |B|$.

To complete the proof let $\sigma = A \sqcup B$ be a partition of $\{1, \dots, s\}$ with $|A| = |B|$, $r \in \{1, \dots, q-2\}$ and let us show that $f_\sigma^r \in I(X)$. By Lemma 5.7, we may assume that σ is the partition $\{1, \dots, k\} \sqcup \{k+1, \dots, s\}$ and $f_\sigma^r = t_1^r t_2^{\hat{r}} \cdots t_k^{r'} - t_{k+1}^{r'} \cdots t_{s-1}^{\hat{r}} t_s^r$, where $r' \in \{r, \hat{r}\}$. Now, let $\underline{x} \in (K^*)^n$. Then $f_\sigma^r(\underline{x}^{\nu_1}, \dots, \underline{x}^{\nu_s}) = x_1^r x_{k+1}^{r'} - x_{k+1}^{r'} x_1^r = 0$, i.e., $f_\sigma^r \in I(X)$. \square

The following conjecture has been verified in a number of examples using *Macaulay2* [12].

Conjecture 5.10. Let X be the algebraic toric set associated to an even cycle $\mathcal{G} = \mathcal{C}_{2k}$ and let λ be the partition $\{1, 3, \dots, 2k-1\} \sqcup \{2, 4, \dots, 2k\}$. If $k \geq 2$, then the set of binomials

$$\mathcal{B} = \{ \{ f_\sigma^r : \sigma = A \sqcup B \text{ is a partition of } \{1, \dots, s\} \text{ with } |A| = |B|, 1 \in A \text{ and } 1 \leq r \leq q-2 \} \\ \cup \{ t_i^{q-1} - t_s^{q-1} : 1 \leq i \leq s-1 \} \setminus \{ f_\lambda^r : 2 \leq r \leq q-2 \} \}$$

is a minimal set of generators and a Gröbner basis of $I(X)$ with respect to the reverse lexicographic order.

Remark 5.11. By Theorem 5.9 and since, for each $2 \leq r \leq q-2$, there exists $g_r \in S$ such that $f_\lambda^r = g_r f_\lambda^1$, we get that \mathcal{B} is a generating set for $I(X)$.

Corollary 5.12. *Let $\mathcal{G} = \mathcal{C}_{2k}$ be an even cycle.*

- (a) *If $f = t^a - t^b$ is an element of \mathcal{B} , then $\deg(f)$ is at most $(q-2)(k-1) + 1$.*
- (b) *Any subset of \mathcal{B} that is also a generating set of $I(X)$ contains an element of the form f_σ^{q-2} , with $\deg(f_\sigma^{q-2}) = (q-2)(k-1) + 1$.*

Proof. We set $M = (q-2)(k-1) + 1$. (a) If $f = t_i^{q-1} - t_j^{q-1}$ for some i, j , then $\deg(f) \leq M$ because $k \geq 2$. Assume that f is not of this form. Then, $f = f_\sigma^r$ for some $1 \leq r \leq q-2$, where σ is the partition $\sigma = A \sqcup B$ and A, B are the supports of a, b respectively, $|A| = |B| = k$ and $1 \in A$. Let i be the cardinality of the set $\{j \in A : f_\sigma^r(j) = r\}$. Then, $\deg(f) = ir + (k-i)\widehat{r}$. If $r = \widehat{r}$, then $r = (q-1)/2$ and $\deg(f) = k(q-1)/2 \leq M$. We may now assume $r \neq \widehat{r}$. If $i = k$, then

$$f_\sigma^r = f_\lambda^1 = t_1 t_3 \cdots t_{2k-1} - t_2 t_4 \cdots t_{2k}.$$

Hence, $\deg(f_\sigma^r) = k \leq M$. To complete the proof we may now assume that $1 \leq i \leq k-1$. In this case, we have

$$\begin{aligned} \deg(f_\sigma^r) &= ir + (k-i)\widehat{r} = i(r - \widehat{r}) + k\widehat{r} \\ &\leq (k-1)(r - \widehat{r}) + k\widehat{r} = (k-1)r + \widehat{r} = r(k-2) + (q-1) \\ &\leq (q-2)(k-2) + (q-1) = (k-1)(q-2) + 1. \end{aligned}$$

Thus, $\deg(f_\sigma^r) \leq M$, as required. To prove (b) let $\mathcal{B}' \subset \mathcal{B}$ be a generating set of $I(X)$. Let $\sigma = \{1, 3, \dots, 2k-3, 2k\} \sqcup \{2, 4, \dots, 2(k-1), 2k-1\}$, then

$$f_\sigma^{q-2} = t_1^{q-2} t_3^{q-2} \cdots t_{2k-3}^{q-2} t_{2k} - t_2^{q-2} t_4^{q-2} \cdots t_{2(k-1)}^{q-2} t_{2k-1}$$

is in \mathcal{B} and has degree M . We will show that $f_\sigma^{q-2} \in \mathcal{B}'$. Since \mathcal{B}' is a generating set of $I(X)$, f_σ^{q-2} is a linear combination, with coefficients in S , of binomials in \mathcal{B}' . These are binomials of the form f_ρ^m , $1 \leq m \leq q-2$, $\deg(f_\rho^m) \leq M$, and of the form $t_i^{q-1} - t_{2k}^{q-1}$ for some $1 \leq i \leq 2k-1$. There is a binomial in \mathcal{B}' , $f_\rho^m = t^{a'} - t^{b'}$, such that

$$t_1^{q-2} t_3^{q-2} \cdots t_{2k-3}^{q-2} t_{2k} = t^c t^{a'}$$

for some monomial t^c . Since $\text{supp}(a') \subset \{1, 3, \dots, 2k-3, 2k\}$ and $t^{a'} - t^{b'}$ cannot be of the form $t_1^{q-1} - t_{2k}^{q-1}$, because of its degree, we deduce that $\sigma = \rho$. From the equality

$$t_1^{q-2} t_3^{q-2} \cdots t_{2k-3}^{q-2} t_{2k} = t^c t^{a'} = t^c (t_1^m t_3^m \cdots t_{2k-3}^m t_{2k}^{\widehat{m}})$$

we conclude that $\widehat{m} = 1$, that is, $m = q-2$. Thus, $f_\sigma^{q-2} = f_\rho^m$. \square

Consider the general case when \mathcal{G} is any graph. Suppose that \mathcal{G} contains a subgraph $\mathcal{H} \cong \mathcal{C}_{2k}$, isomorphic to an even order cycle. Assume without loss of generality that t_1, \dots, t_{2k} are the variables of S corresponding to the edges of \mathcal{H} . Then, given $r \in \{1, \dots, q-2\}$ and a partition $\sigma = A \sqcup B$ of $\{1, \dots, 2k\}$ with $|A| = |B| = k$ and $1 \in A$, the homogeneous binomial $f_\sigma^r \in K[t_1, \dots, t_{2k}] \subset S$ clearly vanishes on the algebraic toric set associated to \mathcal{G} . One could conjecture that together with the binomials $t_i^{q-1} - t_j^{q-1}$, for $1 \leq i, j \leq s$, the binomials obtained in this way, going through all the even cycles of \mathcal{G} , would form a generating set of $I(X)$. This is not true, even for bipartite graphs, as is shown by Example 5.14. This conjecture is true if we restrict to bipartite graphs the cycles of which are vertex disjoint; as we show in Theorem 5.13.

Suppose \mathcal{G} is a bipartite graph the cycles of which have disjoint vertex sets. Let $\mathcal{H}_1, \dots, \mathcal{H}_m$ be the subgraphs of \mathcal{G} isomorphic to some even order cycle, i.e., such that $\mathcal{H}_i \cong \mathcal{C}_{2k_i}$. Let $t_{\epsilon_1^i}, \dots, t_{\epsilon_{2k_i}^i} \in S$ be the variables associated to the edges, $\epsilon_1^i, \dots, \epsilon_{2k_i}^i$ of \mathcal{H}_i . Accordingly, set

$$S_i = K[t_{\epsilon_1^i}, \dots, t_{\epsilon_{2k_i}^i}] \subset S.$$

Finally, denote by $I_i(X)$ the intersection $I(X) \cap S_i$. Then, $I_i(X) \subset S_i$ is equal to $I(X_i)$, the vanishing ideal of the algebraic toric set X_i associated to \mathcal{H}_i . Accordingly, as an ideal over S_i , $I_i(X)$ is generated by the corresponding generating set \mathcal{B}_i , in the variables of S_i , (using the notation of Conjecture 5.10).

Theorem 5.13. *Let \mathcal{G} be a connected bipartite graph, whose (even) cycles $\mathcal{H}_1, \dots, \mathcal{H}_m$ have disjoint vertex sets. Let X be the algebraic toric set associated to \mathcal{G} . Then $I(X)$ is generated by the union of the set $\{t_i^{q-1} - t_j^{q-1} : 1 \leq i, j \leq s\}$ with the set $\mathcal{B}_1(X) \cup \dots \cup \mathcal{B}_m(X)$.*

Proof. By Theorem 4.5, it suffices to show that if $f = t^a - t^b \in I(X)$, with $a = (a_1, \dots, a_s) \in \mathbb{N}^s$, $b = (b_1, \dots, b_s) \in \mathbb{N}^s$, such that $\text{supp}(a) \cap \text{supp}(b) = \emptyset$ and $1 \leq a_i, b_j \leq q-2$, then f belongs to the ideal generated by

$$\mathcal{J} = \{t_i^{q-1} - t_j^{q-1} : 1 \leq i, j \leq s\} \cup \mathcal{B}_1(X) \cup \dots \cup \mathcal{B}_m(X).$$

Recall that f is homogeneous by Lemma 4.3. By Proposition 5.1, we know that $\text{supp}(a) \cup \text{supp}(b)$ is contained in the union of the sets of indices of the variables corresponding to edges of the cycles of \mathcal{G} . In other words, if e_i is an edge not in any edge set of $\mathcal{H}_1, \dots, \mathcal{H}_m$ then $i \notin \text{supp}(a) \cup \text{supp}(b)$. As above, denote by $t_{\epsilon_1^i}, \dots, t_{\epsilon_{2k_i}^i}$ the variables associated to \mathcal{H}_i . We proceed by induction on

$$\mu_f = \{i \in \{1, \dots, m\} : (\text{supp}(a) \cup \text{supp}(b)) \cap \{\epsilon_1^i, \dots, \epsilon_{2k_i}^i\} \neq \emptyset\}.$$

Let $i \in \{1, \dots, m\}$ be such that $(\text{supp}(a) \cup \text{supp}(b)) \cap \{\epsilon_1^i, \dots, \epsilon_{2k_i}^i\} \neq \emptyset$. Consider $a^\sharp, a^b, b^\sharp, b^b \in \mathbb{N}^s$ such that $\text{supp}(a^\sharp) \cup \text{supp}(b^\sharp) \subset \{\epsilon_1^i, \dots, \epsilon_{2k_i}^i\}$, $(\text{supp}(a^b) \cup \text{supp}(b^b)) \cap \{\epsilon_1^i, \dots, \epsilon_{2k_i}^i\} = \emptyset$,

$$t^a = t^{a^\sharp} t^{a^b} \quad \text{and} \quad t^b = t^{b^\sharp} t^{b^b}.$$

By Corollary 5.3, $\text{supp}(a^\sharp) \cup \text{supp}(b^\sharp) = \{\epsilon_1^i, \dots, \epsilon_{2k_i}^i\}$. Since we are assuming $\mathcal{H}_1, \dots, \mathcal{H}_m$ have disjoint vertex sets, setting $t_\ell = 1$ for all $\ell \notin \{\epsilon_1^i, \dots, \epsilon_{2k_i}^i\}$ is equivalent to setting in $\underline{x} \in (K^*)^n$, $x_\ell = 1$ for all $\ell \notin V_{\mathcal{H}_i}$. Hence, making these substitutions and running the argument of the proof of Theorem 5.13, we see that $t^{a^\sharp} - t^{b^\sharp} = f_\sigma^r$, where $r = (a^\sharp)_{\epsilon_1^i} \in \{1, \dots, q-2\}$, (assuming that $\epsilon_1^i \in \text{supp}(a^\sharp)$), and where σ is the partition $\text{supp}(a^\sharp) \sqcup \text{supp}(b^\sharp) = \{\epsilon_1^i, \dots, \epsilon_{2k_i}^i\}$.

Suppose that $\mu_f = 1$. Then $a^b = b^b = 0 \in \mathbb{N}^s$, f_σ^r is homogeneous and we are done.

Suppose that every binomial $g = t^a - t^b \in I(X)$ with $\mu_g \leq m' < m$ is in the ideal generated by \mathcal{J} . Let $f = t^a - t^b \in I(X)$ be a binomial with $\mu_f = m' + 1$. Let $i \in \{1, \dots, m\}$ be such that $(\text{supp}(a) \cup \text{supp}(b)) \cap \{\epsilon_1^i, \dots, \epsilon_{2k_i}^i\} \neq \emptyset$. Consider, as above, $a^\sharp, a^b, b^\sharp, b^b \in \mathbb{N}^s$ such that $t^a = t^{a^\sharp} t^{a^b}$ and $t^b = t^{b^\sharp} t^{b^b}$. Repeating the previous argument we deduce that $t^{a^\sharp} - t^{b^\sharp} = f_\sigma^r$ where, $r = (a^\sharp)_{\epsilon_1^i}$ and $\sigma = \text{supp}(a^\sharp) \sqcup \text{supp}(b^\sharp)$. However, notice that in this case f_σ^r is not necessarily homogeneous. Assume that $|\text{supp}(a^\sharp)| \geq |\text{supp}(b^\sharp)|$. Let $\delta \in \mathbb{N}^s$ be such that $\epsilon_1^i \notin \text{supp}(\delta) \subset \text{supp}(a^\sharp)$, $\delta_\ell = a_\ell^\sharp$ for all $\ell \in \text{supp}(\delta)$ and $|\text{supp}(a^\sharp - \delta)| = k_i$ (where $2k_i$ is the order of \mathcal{H}_i). Set $h = |\text{supp}(\delta)|$, $a' = a^\sharp - \delta$ and let $b' \in \mathbb{N}^s$ be obtained by applying h times Lemma 5.7 to $\sigma = \text{supp}(a^\sharp) \sqcup \text{supp}(b^\sharp)$.

Then $b' = b^\sharp + \hat{\delta}$, where $\hat{\delta}$ has the same support as δ and $(\hat{\delta})_\ell = q - 1 - \delta_\ell$, for every $\ell \in \text{supp}(\hat{\delta})$. Set $\sigma' = \text{supp}(a') \sqcup \text{supp}(b')$. Then $f_{\sigma'}^r = t^{a'} - t^{b'}$ is homogeneous and belongs to $I_i(X)$. Moreover,

$$(5.2) \quad \begin{aligned} f = t^a - t^b &= t^{a'} t^{\delta} t^{a^b} - t^{b^\sharp} t^{b^b} = t^{a'} t^{\delta} t^{a^b} - t^{b'} t^{\delta} t^{a^b} + t^{b'} t^{\delta} t^{a^b} - t^{b^\sharp} t^{b^b} \\ &= f_{\sigma'}^r t^{\delta} t^{a^b} + t^{b^\sharp} (t^{\delta} t^{\delta} t^{a^b} - t^{b^b}). \end{aligned}$$

Now $(\hat{\delta})_\ell + \delta_\ell = q - 1$, for all $\ell \in \text{supp}(\delta)$ and since f is homogeneous, $h = |\text{supp}(\delta)| > |\text{supp}(b^b)|$. Choose $\ell_1, \dots, \ell_h \in \text{supp}(b^b)$, h distinct indices. Let $\gamma \in \mathbb{N}^s$ to be such that $\text{supp}(\gamma) = \{\ell_1, \dots, \ell_h\}$ and $(\gamma)_{\ell_j} = q - 1$, for $j = 1, \dots, h$. Then $t^{\delta} t^{\hat{\delta}} - t^\gamma$ is in the ideal of S generated by \mathcal{J} , since it is in the ideal of the torus. We have

$$(5.3) \quad f = f_{\sigma'}^r t^{\delta} t^{a^b} + t^{b^\sharp} (t^{\delta} t^{\hat{\delta}} t^{a^b} - t^{b^b}) = f_{\sigma'}^r t^{\delta} t^{a^b} + t^{b^\sharp} t^{a^b} (t^{\delta} t^{\hat{\delta}} - t^\gamma) + t^{b^\sharp} (t^\gamma t^{a^b} - t^{b^b}).$$

Let $\gamma^\sharp \in \mathbb{N}^s$ be such that $\text{supp}(\gamma^\sharp) = \{\ell_1, \dots, \ell_h\}$ and $(\gamma^\sharp)_{\ell_j} = (b^b)_{\ell_j}$, for $j = 1, \dots, h$ and set $\gamma^b = \gamma - \gamma^\sharp$ and $b^\natural = b^b - \gamma^\sharp$. Then,

$$(5.4) \quad f = f_{\sigma'}^r t^{\delta} t^{a^b} + t^{b^\sharp} t^{a^b} (t^{\delta^*} - t^\gamma) + t^{b^\sharp} t^{\gamma^\sharp} (t^{\gamma^b} t^{a^b} - t^{b^\natural}),$$

where $g = t^{\gamma^b} t^{a^b} - t^{b^\natural}$ is a homogeneous binomial with $\mu_g \leq m'$. Hence, by induction, g , and therefore f , are in the ideal generated by \mathcal{J} . \square

In Example 5.14, we show that Theorem 5.13 does not hold for general connected bipartite graphs.

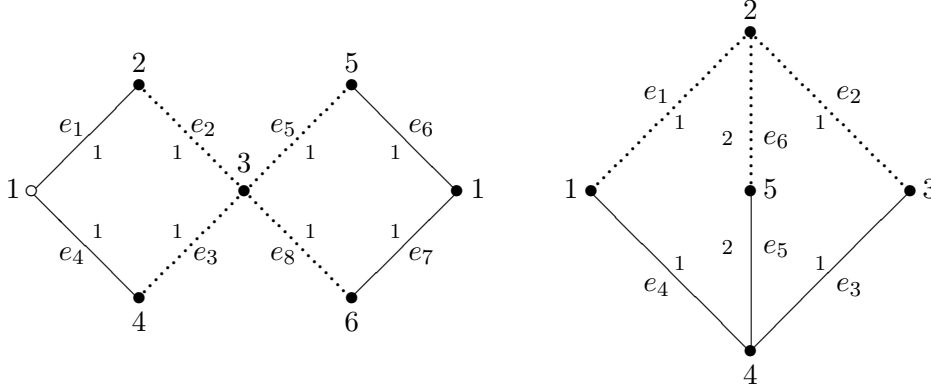


FIGURE 4

Example 5.14. Let \mathcal{G}_1 and \mathcal{G}_2 be the two graphs in Figure 4 (from left to right) and assume that $q = 5$. Notice that we are identifying the two vertices, labeled by 1, in the representation of \mathcal{G}_1 . Thus, \mathcal{G}_1 is a bipartite graph with six vertices and eight edges. Denote by X_1 and X_2 , respectively, the corresponding algebraic toric sets. Then, using *Macaulay2* [12], we found that the binomial $t_1 t_4 t_6 t_7 - t_2 t_3 t_5 t_8$ is in a minimal generating set of $I(X_1)$. In this case, the argument of the proof of Theorem 5.13 does not work, to the extent that if we set t_1, t_2, t_3, t_4 equal to 1, the resulting binomial, $t_6 t_7 - t_5 t_8$, albeit homogeneous, is not of the type f_σ^r for any partition σ of $\{5, 6, 7, 8\}$. The same can be said for the binomial resulting from substituting to 1 the variables t_5, t_6, t_7, t_8 . As to the vanishing ideal of X_2 , we found that there exists a minimal generating set containing $t_1 t_2 t_5^2 - t_3 t_4 t_5^2$, which, when restricted to any of the 3 cycles in \mathcal{G}_2 is not of the type f_σ^r for any partition of the corresponding index set.

6. THE REGULARITY OF $S/I(X)$

In this section we address the question of computing the regularity of $S/I(X)$ for an algebraic toric set X parameterized by a bipartite graph. Theorem 6.3 gives an upper bound for the regularity of $S/I(X)$ for a general bipartite graph. If X is the algebraic toric set parameterized by an even cycle of length $2k$, by Proposition 2.4 and Corollary 5.12, we get

$$\text{bigdeg } I(X) - 1 = (q - 2)(k - 1) \leq \text{reg } S/I(X).$$

This inequality is already known in the literature, see [8, Corollary 3.1] and [24, Corollary 2.19]. We will show that the regularity of $S/I(X)$ is in fact equal to $(q - 2)(k - 1)$, and generalize this result by giving a formula for the regularity of any connected bipartite graph whose cycles have disjoint vertex sets. In the proof of Theorem 6.2, we show the inequality above as an easy consequence of the description of the generators of the ideal $I(X)$.

Lemma 6.1. *Let $1 \leq i \leq s - 2$. Consider the K -automorphism $\sigma_i: S \rightarrow S$ defined by exchanging t_i with t_{i+2} and leaving all other variables fixed. Then, σ_i permutes the elements of the set of all $f_\sigma^r \in I(X)$, for $r \in \{1, \dots, q - 2\}$ and $\sigma = A \sqcup B$ a partition of $\{1, \dots, s\}$ with $|A| = |B|$.*

Proof. Let f_σ^r be a binomial associated to $r \in \{1, \dots, q - 2\}$ and $\sigma = A \sqcup B$ a partition of $\{1, \dots, s\}$. Thus, $f_\sigma^r = t^a - t^b$ where $A = \text{supp}(a)$, $B = \text{supp}(b)$, $a_\ell = \rho_\sigma^r(\ell)$, for all $\ell \in \text{supp}(a)$ and $b_\ell = \rho_\sigma^r(\ell)$, for all $\ell \in \text{supp}(b)$. As $\rho_\sigma^r(\ell) = \rho_\sigma^r(\ell + 2)$ if and only if ℓ and $\ell + 2$ are in the same part of the partition, if i and $i + 2$ are in the same part of the partition then $\sigma_i(f_\sigma^r) = f_\sigma^r$. Suppose that i and $i + 2$ are in different parts of the partition and therefore that $\rho_\sigma^r(i + 2) = \widehat{\rho_\sigma^r(i)}$. Without loss in generality we may write $f_\sigma^r = t_i^{a_i} t^{a'} - t_{i+1}^{\widehat{a_i}} t^{b'}$, where $\text{supp}(a') = \text{supp}(a) \cup \{i\}$ and $\text{supp}(b') = \text{supp}(b) \cup \{i + 2\}$. In this situation, we apply Lemma 5.7 twice, transferring i to the part it does not belong to, and proceeding similarly with $i + 2$. Let σ' be the partition of $\{1, \dots, s\}$ obtained in this way and consider the resulting binomial $f_{\sigma'}^r$. By Lemma 5.7 we see that $f_{\sigma'}^r = t_{i+2}^{a_i} t^{a'} - t_i^{\widehat{a_i}} t^{b'} = \sigma_i(f_\sigma^r)$. \square

Theorem 6.2. *Let X be the algebraic toric set associated to an even order cycle $\mathcal{G} = \mathcal{C}_{2k}$. Then $\text{reg } S/I(X) = (q - 2)(k - 1)$.*

Proof. Recall that $k \geq 2$. Denote by R the graded ring $S/I(X)$. Consider $t_1 \in S$. Since t_1 is regular on R , we have the following exact sequence of graded S -modules:

$$(6.1) \quad 0 \longrightarrow R[-1] \xrightarrow{t_1} R \longrightarrow R/(t_1) \longrightarrow 0,$$

where $R[-1]$ is the graded S -module obtained by a shift in the graduation, *i.e.*, $R[-1]_i = R_{i-1}$. Recall that $H_X(d)$ is, by definition, $\dim_K(S/I(X))_d$, and since $S/I(X)$ is a 1-dimensional ring, the regularity of $S/I(X)$ is the least integer l for which $H_X(d)$ is equal to some constant (indeed equal to $|X|$) for all $d \geq l$. Now, from (6.1) we get $H_X(d) - H_X(d - 1) = \dim_K(R/(t_1))_d$. Hence $\text{reg } S/I(X) = \text{reg } R/(t_1) - 1$. For $d \geq 0$, we define

$$h_d := \dim_K(R/(t_1))_d = H_X(d) - H_X(d - 1).$$

We start by showing that $\text{reg } S/I(X) \leq (q - 2)(k - 1)$. If we show that $h_d = 0$, for $d \geq (q - 2)(k - 1) + 1$, then $H_X(d - 1) = H_X(d)$, for $d - 1 \geq (q - 2)(k - 1)$, and our result follows. Set $S' = K[t_2, \dots, t_s]$. There is a surjection of graded S' -modules

$$\varphi: S' \longrightarrow S/(I(X), t_1) \cong R/(t_1)$$

defined by $\varphi(f) = f + (I(X), t_1)$, for every $f \in S'$. Set $I'(X) = \text{Ker}(\varphi)$, so that

$$S'/I'(X) \cong S/(I(X), t_1).$$

Then, $I'(X)$ is a monomial ideal generated by the monomials obtained by setting $t_1 = 0$ in the generators of $I(X)$; in particular it is generated by t_j^{q-1} , for $2 \leq j \leq s$ and by the monomials t^b in some $f_\sigma^r = t^a - t^b$, for $r \in \{1, \dots, q-2\}$ and σ a partition of $\{1, \dots, s\}$ into 2 parts of equal cardinality. To show that $h_d = 0$, for $d \geq (q-2)(k-1) + 1$, it is enough to show that every monomial M in S' of degree $\geq (q-2)(k-1) + 1$ belongs to $I'(X)$. Since $t_j^{q-1} \in I'(X)$ for all $2 \leq j \leq s$, we may assume that there is no j for which t_j^{q-1} divides the monomial M in question. Let us write it in the following way:

$$M = t_2^{b_1} t_4^{b_2} \dots t_{2k}^{b_k} t_3^{c_1} t_5^{c_2} \dots t_{2k-1}^{c_{k-1}},$$

with $0 \leq b_i, c_j \leq q-2$. We want to show that there exists $f_\sigma^r = t^a - t^b \in I(X)$ such that t^b divides M . By Lemma 6.1, if t^b divides M and there exists r, σ such that $f_\sigma^r = t^a - t^b$, then, for all $i \in \{2, \dots, s-2\}$, $\sigma_i(t^b)$ divides $\sigma_i(M)$ and there exists σ' such that $f_{\sigma'}^r = t^{a'} - \sigma_i(t^b)$. Hence, we may assume that $c_1 \leq c_2 \leq \dots \leq c_{k-1}$ and that $b_1 \geq b_2 \geq \dots \geq b_k$. There are two cases. If $b_k > 0$, then M is divisible by $t_2 t_4 \dots t_{2k}$, which belongs to $I'(X)$, since for $\sigma = \{1, 3, \dots, 2k-1\} \sqcup \{2, 4, \dots, 2k\}$, we have $f_\sigma^1 = t_1 t_3 \dots t_{2k-1} - t_2 t_4 \dots t_{2k}$. The second case is for $b_k = 0$. In this case, from

$$\deg M = \sum_{i=1}^{k-1} (b_i + c_i) \geq (q-2)(k-1) + 1$$

we deduce that there exists $j \in \{1, \dots, k-1\}$ such that $b_j + c_j \geq q-1$. Since $c_j \leq q-2$ we get $b_j \geq 1$. Set $r = b_j$. Notice that then $c_j \geq q-1 - b_j = q-1 - r = \widehat{r}$. Consider the set given by $B = \{2, 4, \dots, 2j, 2j+1, 2j+3, \dots, 2k-1\}$ and let $\sigma = A \sqcup B$ be the partition of $\{1, \dots, s\}$ that it determines. Then:

$$f_\sigma^r = (t_1 t_3 \dots t_{2j-1})^r (t_{2j+2} \dots t_{2k-2} t_{2k})^{\widehat{r}} - (t_2 t_4 \dots t_{2j})^r (t_{2j+1} t_{2j+3} \dots t_{2k-1})^{\widehat{r}} \in I(X).$$

Accordingly, $(t_2 t_4 \dots t_{2j})^r (t_{2j+1} t_{2j+3} \dots t_{2k-1})^{\widehat{r}} \in I'(X)$. Since $b_l \geq b_j = r$, for all $1 \leq l \leq j$, we deduce that $t_{2l}^{b_l}$ divides M , for all $1 \leq l \leq j$. Since $\widehat{r} \leq c_j \leq c_l$, for all $j \leq l \leq k-1$, we deduce that $t_{2l+1}^{\widehat{r}}$ divides M , for all $j \leq l \leq k-1$. In conclusion, $(t_2 t_4 \dots t_{2j})^r (t_{2j+1} t_{2j+3} \dots t_{2k-1})^{\widehat{r}}$ divides M and hence $M \in I'(X)$.

Let us now show that $\text{reg } S/I(X) \geq (q-2)(k-1)$. If we show that $h_d \neq 0$, i.e., $h_d > 0$, for $d = (q-2)(k-1)$, then $H_X(d-1) < H_X(d)$, for $d = (q-2)(k-1)$, and our result follows. It suffices to produce a monomial M of degree $d = (q-2)(k-1)$, such that $M \in (S')_d$ but $M \notin (I'(X))_d$. Consider

$$M = (t_2 \dots t_k)^{q-2} \in (S')_d.$$

Suppose $M \in I'(X)$. Then, as we have seen above,

$$M = \sum_{j=2}^s g_j t_j^{q-1} + \sum_{\sigma, r} h_{\sigma, r} t^b$$

where $g_j, h_{\sigma, r} \in S'$, $f_\sigma^r = t^a - t^b$ and the second summation runs over all partitions σ of $\{1, \dots, s\}$ into 2 parts of equal cardinality and $r \in \{1, \dots, q-2\}$. Since M is a monomial and its degree in each one of the variables is $q-2$, we deduce that M must be a monomial of the form

$$M = h_{\sigma, r} t^b$$

for $h_{\sigma, r} \in S'$, one partition σ of $\{1, \dots, s\}$ into 2 parts of equal cardinality, one $r \in \{1, \dots, q-2\}$ and $f_\sigma^r = t^a - t^b$. But this is not possible because the monomial M has $k-1$ variables, while $h_{\sigma, r} t^b$ has at least k variables. We conclude that $M \notin I'(X)$. \square

Theorem 6.3. *Let \mathcal{G} be a bipartite graph. Let $\mathcal{H}_1, \dots, \mathcal{H}_m$ be subgraphs of \mathcal{G} isomorphic to (even) cycles $\mathcal{H}_i \cong \mathcal{C}_{2k_i}$ that have disjoint edge sets. Then*

$$\operatorname{reg} S/I(X) \leq (q-2)\left(s - \sum_{i=1}^m k_i - 1\right).$$

Proof. For all $1 \leq i \leq m$, let $t_{\epsilon_1^i}, \dots, t_{\epsilon_{2k_i}^i}$ be the variables associated to the edges of \mathcal{H}_i .

Without loss of generality, assume that $t_1 = t_{\epsilon_1^1}$, $t_2 = t_{\epsilon_1^2}$, \dots , $t_m = t_{\epsilon_1^m}$.

Denote by R the quotient $S/I(X)$ and, for $1 \leq i \leq m$, let

$$R_i = R/(t_1, \dots, t_i).$$

Since t_1 is a regular element of R , we have the following short exact sequence of graded S -modules:

$$(6.2) \quad 0 \longrightarrow R[-1] \xrightarrow{t_1} R \longrightarrow R_1 \longrightarrow 0.$$

Furthermore, for all $1 \leq i \leq m-1$, we have exact sequences of graded S -modules:

$$(6.3) \quad R_i[-1] \xrightarrow{t_{i+1}} R_i \longrightarrow R_{i+1} \longrightarrow 0.$$

Claim 1. *For all $1 \leq i \leq m$, $t_j^{q-1} = 0$ in R_i , for all $1 \leq j \leq s$.*

Proof of Claim 1. Since $t_j^{q-1} - t_i^{q-1} \in I(X)$ and $t_i^{q-1} = 0$ in R_i , we deduce that $t_j^{q-1} = 0$ in R_i , for all $1 \leq j \leq s$. \square

Claim 2. *If there exists a nonnegative integer ℓ such that $(R_{i+1})_d = 0$, for all $d \geq \ell$, then $(R_i)_d = 0$ for all $d \geq \ell + q - 2$, where $1 \leq i \leq m - 1$.*

Proof of Claim 2. If $(R_{i+1})_d = 0$, for $d \geq \ell$ then from (6.3) we deduce that for all $d \geq \ell$ the maps $(R_i)_{d-1} \xrightarrow{t_{i+1}} (R_i)_d$ are surjective, i.e., $(R_i)_d = t_{i+1}(R_i)_{d-1}$, for all $d \geq \ell$. Iterating and using Claim 1, we get: $(R_i)_{d+q-2} = t_{i+1}^{q-1}(R_i)_{d-1} = 0$, i.e., $(R_i)_d = 0$ for all $d \geq \ell + q - 2$. \square

Claim 3. *Let t^a be a monomial in S . Suppose that the degree of t^a in the variables associated to \mathcal{H}_i is $\geq (q-2)(k_i - 1) + 1$. Then $t^a = 0$ in R_i .*

Proof of Claim 3. We may assume that t_i does not divide t^a . Defining

$$S_i := K[t_{\epsilon_1^i}, \dots, t_{\epsilon_{2k_i}^i}],$$

we have $I(X_i) \subset S_i$, where $X_i \subseteq \mathbb{P}^{2k_i-1}$ is the set of points parameterized by the edges of the cycle \mathcal{H}_i . It is straightforward to check that $I(X_i) \subset I(X) \subset S$. Let $t^a = t^b t^c$, where t^b is a monomial in $t_{\epsilon_1^i}, \dots, t_{\epsilon_{2k_i}^i}$. It suffices to show $t^b = 0$ in $S_i/(I(X_i) + t_i)$, but since t^b has degree $\geq (q-2)(k_i - 1) + 1$, we can run the same argument as in the proof of Theorem 6.2. \square

Claim 4. *Let $\ell_0 = (q-2)\left(\sum_{i=1}^m (k_i - 1)\right) + (q-2)\left(s - \sum_{i=1}^m 2k_i\right) + 1$. Then $(R_m)_d = 0$, $\forall d \geq \ell_0$.*

Proof of Claim 4. Let t^a be a monomial of degree $d \geq \ell_0$. In view of Claim 3, we may assume that the degree of t^a in the variables associated to \mathcal{H}_i is $\leq (q-2)(k_i - 1)$. Then, the degree of t^a in the remaining $s - \sum_{i=1}^m 2k_i$ variables is $\geq (q-2)\left(s - \sum_{i=0}^m 2k_i\right) + 1$ which implies that one of them is raised to a power $\geq q-1$ and therefore, by Claim 1, $t^a = 0$ in R_m . \square

We now finish the proof of the theorem. Notice that $\ell_0 = (q-2)(s - \sum_{i=1}^m (k_i + 1)) + 1$. Combining Claim 2 with Claim 4 we deduce that $(R_1)_d = 0$, for all $d \geq \ell_0 + (m-1)(q-2)$. Now $\ell_0 + (m-1)(q-2) = (q-2)(s - \sum_{i=1}^m k_i - 1) + 1$ and using (6.2) we see that $(R)_{d-1} \xrightarrow{t_1} (R)_d$ is an isomorphism for all $d \geq (q-2)(s - \sum_{i=1}^m k_i - 1) + 1$. This means that the Hilbert function of R satisfies: $H_X(d-1) = H_X(d)$, for $d-1 \geq (q-2)(s - \sum_{i=1}^m k_i - 1)$. Hence, $\text{reg } R \leq (q-2)(s - \sum_{i=1}^m k_i - 1)$. \square

Remark 6.4. Notice we do not assume that \mathcal{G} is connected nor do we assume that any 2 cycles, \mathcal{H}_1 and \mathcal{H}_2 , in \mathcal{G} have disjoint edge or vertex sets. In fact, we can apply the bound of Theorem 6.3 to both graphs in Figure 4. For \mathcal{G}_1 , on the left, we should use both cycles of order 4. We obtain $\text{reg } S/I(X_1) \leq 3(8-4-1) = 9$. Using *Macaulay2* [12], for $q=5$, we checked that this is the actual value of the regularity. For \mathcal{G}_2 , on the right, we may only use one of the cycles. Then, Theorem 6.3 yields $\text{reg } S/I(X_2) \leq (q-2)(6-2-1) = 3(q-2)$, which, for $q=5$, is not sharp, as the value of $\text{reg } S/I(X_2)$ is 6. The inequality of Theorem 6.3 is an improvement of the inequality given in [24, Corollary 2.31].

Corollary 6.5. *Let \mathcal{G} be a connected bipartite graph, the (even) cycles of which, $\mathcal{H}_1, \dots, \mathcal{H}_m$, with $\mathcal{H}_i \cong C_{2k_i}$, have disjoint vertex sets. Then*

$$\text{reg } S/I(X) = (q-2)(s - \sum_{i=1}^m k_i - 1).$$

Proof. Let $t_{e_1^i}, \dots, t_{e_{2k_i}^i} \in S$ be the set of variables associated to the edges, $e_1^i, \dots, e_{2k_i}^i$ of the even cycle \mathcal{H}_i . We set

$$S_i = K[t_{e_1^i}, \dots, t_{e_{2k_i}^i}] \subset S,$$

and denote by $I_i(X)$ the intersection $I(X) \cap S_i$. Then, $I_i(X) \subset S_i$ is the vanishing ideal of the algebraic toric set associated to \mathcal{H}_i . By Theorem 5.13, $I(X)$ is generated by the set

$$\mathcal{J} = \{t_i^{q-1} - t_j^{q-1} : 1 \leq i, j \leq s\} \cup I_1(X) \cup \dots \cup I_m(X).$$

We proceed by induction on the number of edges of \mathcal{G} . If \mathcal{G} is an even cycle, the result follows from Theorem 6.2. We may assume that e_s is an edge of \mathcal{G} that does not lie on any cycle of \mathcal{G} and that t_s is the variable that corresponds to e_s . For simplicity of notation, we identify the edge e_i with the variable t_i for $i = 1, \dots, s$ and refer to t_i as an edge of the graph \mathcal{G} . Consider the graph \mathcal{G}_1 whose edge set is $\{e_1, \dots, e_{s-1}\}$ (the edge set of \mathcal{G} minus the edge e_s), and whose vertex set is the set of endpoints of the edges e_1, \dots, e_{s-1} . Let X_1 be the algebraic toric set parameterized by the edges of \mathcal{G}_1 . Clearly \mathcal{G}_1 is a bipartite graph whose (even) cycles are again $\mathcal{H}_1, \dots, \mathcal{H}_m$.

Case (I): The graph \mathcal{G}_1 is connected. Let $A(X_1) = K[t_1, \dots, t_{s-1}]/I(X_1)$ be the coordinate ring of X_1 and let $F_{X_1}(t)$ be the Hilbert series of $A(X_1)$. The Hilbert series can be uniquely written as $F_{X_1}(t) = g_1(t)/(1-t)$, where $g_1(t)$ is a polynomial of degree equal to the regularity of $A(X_1)$. Because \mathcal{G}_1 is a connected bipartite graph and has the same cycles as \mathcal{G} , by Theorem 5.13, the vanishing ideal $I(X_1)$ is generated by the set

$$\mathcal{J}_1 = \{t_i^{q-1} - t_j^{q-1} : 1 \leq i, j \leq s-1\} \cup I_1(X) \cup \dots \cup I_m(X)$$

(notice that $I_j(X) = I_j(X_1)$, for $j = 1, \dots, m$). Hence, there is an exact sequence

$$0 \rightarrow A(X_1)[-(q-1)] \xrightarrow{t_1^{q-1}} A(X_1) \rightarrow C = K[t_1, \dots, t_{s-1}]/(I_1(X), \dots, I_m(X), t_1^{q-1}, \dots, t_{s-1}^{q-1}) \rightarrow 0.$$

As a consequence, we get that the Hilbert series $F(C, t)$ of C is given by

$$F(C, t) = F_{X_1}(t)(1 - t^{q-1}) = g_1(t)(1 + t + \dots + t^{q-2}),$$

and $\deg F(C, t) = (q - 2) + \text{reg } A(X_1)$. Since \mathcal{G}_1 is a connected bipartite graph and its even cycles have disjoint vertex sets, by induction we get $\text{reg } A(X_1) = (q - 2)(s - 1 - \sum_{i=1}^m k_i - 1)$, and therefore,

$$(6.4) \quad \deg F(C, t) = (q - 2)(s - \sum_{i=1}^m k_i - 1).$$

From the exact sequence

$$0 \rightarrow (S/I(X))[-1] \xrightarrow{t_s} S/I(X) \rightarrow S/(t_s, I(X)) \rightarrow 0,$$

we get that $F_X(t) = F(S/(t_s, I(X)), t)/(1 - t)$. Thus $\text{reg}(S/I(X)) = \deg F(S/(t_s, I(X)), t)$. Using the isomorphism

$$S/(t_s, I(X)) \simeq K[t_1, \dots, t_{s-1}]/(t_1^{q-1}, \dots, t_{s-1}^{q-1}, I_1(X), \dots, I_m(X)),$$

we obtain that $C \simeq S/(t_s, I(X))$. Hence, by Eq. (6.4), the desired formula follows.

Case (II): The graph \mathcal{G}_1 is disconnected. It is not hard to show that \mathcal{G}_1 has exactly two connected components $\mathcal{G}'_1, \mathcal{G}''_1$. Let E'_1, E''_1 be the edge sets of $\mathcal{G}'_1, \mathcal{G}''_1$ respectively and let X'_1, X''_1 be the algebraic toric sets parameterized by the edges of $\mathcal{G}'_1, \mathcal{G}''_1$ respectively. We may assume that $\mathcal{H}_1, \dots, \mathcal{H}_r$ are the cycles of \mathcal{G}'_1 and $\mathcal{H}_{r+1}, \dots, \mathcal{H}_m$ are the cycles of \mathcal{G}''_1 . By Theorem 5.13, we have that $I(X'_1)$ and $I(X''_1)$ are generated by

$$\begin{aligned} \mathcal{J}'_1 &= \{t_i^{q-1} - t_j^{q-1} : t_i, t_j \in E'_1\} \cup I_1(X) \cup \dots \cup I_r(X) \quad \text{and} \\ \mathcal{J}''_1 &= \{t_i^{q-1} - t_j^{q-1} : t_i, t_j \in E''_1\} \cup I_{r+1}(X) \cup \dots \cup I_m(X), \end{aligned}$$

respectively. We set

$$C'_1 = K[E'_1]/(\{t_i^{q-1}\}_{t_i \in E'_1}, I_1(X), \dots, I_r(X)), \quad C''_1 = K[E''_1]/(\{t_i^{q-1}\}_{t_i \in E''_1}, I_{r+1}(X), \dots, I_m(X)).$$

By the arguments that we used to prove Case (I), and using the induction hypothesis, we get

$$\deg F(C'_1, t) = (q - 2)(|E'_1| - \sum_{i=1}^r k_i), \quad \deg F(C''_1, t) = (q - 2)(|E''_1| - \sum_{i=r+1}^m k_i).$$

Since $K[E'_1]$ and $K[E''_1]$ are polynomial rings in disjoint sets of variables E'_1 and E''_1 , according to [25, Proposition 2.2.20, p. 42], we have an isomorphism

$$C'_1 \otimes_K C''_1 \simeq K[t_1, \dots, t_{s-1}]/(t_1^{q-1}, \dots, t_{s-1}^{q-1}, I_1(X), \dots, I_m(X)) = S/(t_s, I(X)).$$

Altogether, as $F(C'_1 \otimes_K C''_1, t) = F(C'_1, t)F(C''_1, t)$ (see [25, p. 102]), we obtain

$$\begin{aligned} \text{reg } S/I(X) &= \deg F(S/(t_s, I(X)), t) = \deg F(C'_1 \otimes_K C''_1, t) = \deg F(C'_1, t) + \deg F(C''_1, t) \\ &= (q - 2)(|E'_1| + |E''_1| - \sum_{i=1}^m k_i) = (q - 2)(s - \sum_{i=1}^m k_i - 1), \end{aligned}$$

as required. This completes the proof of case (II). \square

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