CODES OVER A WEIGHTED TORUS

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Abstract. We define the notion of weighted projective Reed–Muller codes over a subset $X \subset \mathbb{P}(w_1, \ldots, w_s)$ of a weighted projective space over a finite field. We focus on the case when X is a projective weighted torus. We show that the vanishing ideal of X is a lattice ideal and relate it with the lattice ideal of a minimal presentation of the semigroup algebra of the numerical semigroup $Q = \langle w_1, \ldots, w_s \rangle \subset \mathbb{N}$. We compute the index of regularity of the vanishing ideal of X in terms of the weights of the projective space and the Frobenius number of Q. We compute the basic parameters of weighted projective Reed–Muller codes over a 1-dimensional weighted torus and prove they are maximum distance separable codes.

1. Introduction

A standard projective Reed–Muller code, $C_X(d)$, is the image of the degree d homogeneous component of a standard polynomial ring $K[t_1, \ldots, t_s]$ over a finite field K by a homomorphism defined by evaluation of forms of degree d on the points of and arbitrary subset $X \subset \mathbb{P}^s$. In this work we define the notion of *weighted projective Reed–Muller code* (see Definition 3.1). This notion differs from the standard definition in that the grading of $K[t_1, \ldots, t_s]$, which is given by $\deg(t_i) = w_i \geq 1$, for coprime integers w_i, is not necessarily the standard one. We focus on the family of codes $C_{\mathbb{T}}(d)$ associated to a weighted $(s-1)$-dimensional projective torus $\mathbb{T}(w_1, \ldots, w_s)$ (see Definition 2.4).

Standard projective Reed–Muller codes of order $d \leq q$ were defined and studied by Lachaud in [17, 18] and, for all $d \geq 0$, by Sørensen in [31]. Much of the recent research on projective Reed–Muller codes over an arbitrary subset of $X \subset \mathbb{P}^s$ focuses on the computation of their basic parameters: *length*, *dimension* and *minimum distance* (see Definition 3.3). When $X = \mathbb{P}^s$

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all the basic parameters are known (*cf.* [18, 31]); in particular, projective Reed–Muller codes over $\mathbb{P}^1$ are *maximum distance separable* codes. In general, an approach to this computation using commutative algebra (as in [26]) relies on a good understanding of $I_X \subset K[t_1, \dots, t_s]$, the vanishing ideal of X. Many authors have studied projective Reed–Muller codes over a subset $X \subset \mathbb{P}^s$ for which the ideal I_X is well understood; *e.g.*, when X is the set of rational points of a complete intersection, *cf.* [1, 4, 5, 7, 11, 15, 29, 30], when X is the Segre embedding of the product of two projective spaces, *cf.* [12], when X is the Veronese variety *cf.* [13], when X is an affine Cartesian product *cf.* [3, 19, 10], when X is the projective torus in $\mathbb{P}^s$ and, more generally, when X is an *algebraic toric subset*, *cf.* [21, 22, 23, 25, 28, 29].

The advantage of working with subsets of the torus is that for a certain subclass of these subsets (consisting of algebraic toric subsets, as defined by Villarreal *et al.* in [25, 27]) the ideal I_X is a lattice ideal. Like in the standard case, $I_{\mathbb{T}}$, the vanishing ideal of the weighted torus $\mathbb{T}(w_1, \dots, w_s)$, is also a lattice ideal. Indeed, we show that $I_{\mathbb{T}}$ is Cohen–Macaulay, 1-dimensional and can be obtained from the lattice ideal of a minimal presentation of the semigroup algebra of the numerical semigroup $Q = \langle w_1, \dots, w_s \rangle \subset \mathbb{N}$ (*cf.* Theorem 2.8). The lattice ideal of a minimal presentation of the semigroup algebra was first studied by Herzog in [16]. He gives a sufficient condition for this ideal to be a complete intersection (see Remark 2.12), which, combined with our results, is also a sufficient condition for $I_{\mathbb{T}}$ to be a complete intersection. The relation between $I_{\mathbb{T}}$ and the lattice ideal of a minimal presentation of Q enables the computation of the Hilbert Series and the index of regularity of $K[t_1, \dots, t_s]/I_{\mathbb{T}}$ in terms of $w_1, \dots, w_s$ and the Frobenius number of Q (*cf.* Theorem 3.8 and Corollary 3.9). The importance, from a coding theory point of view, of the knowledge of the index of regularity is clearer in the case of standard projective Reed–Muller codes. Here, the function $\dim_K C_X(d)$ is strictly increasing and the value of d for which $\dim_K C_X(d)$ becomes constant and equal to the dimension of the ambient space (thus, for which $C_X(d)$ becomes the trivial code) is precisely given by the index of regularity. In the weighted case $\dim_K C_X(d)$ is not necessarily an increasing function and we may get some trivial codes $C_X(d)$ before d reaches the index of regularity (*cf.* Example 3.11).

The structure of the article is as follows. In Section 2 we study the vanishing ideal of a weighted projective torus. The basic definitions are recalled. We show that $I_{\mathbb{T}}$ is a 1-dimensional, Cohen-Macaulay lattice ideal and relate it to the lattice ideal of a minimal presentation of the semigroup algebra of $Q = \langle w_1, \dots, w_s \rangle$. In Section 3 we define weighted projective Reed-Muller codes. We compute the length of the weighted projective Reed–Muller codes over the weighted torus (*cf.* Proposition 3.5) and we compute the Hilbert Series and the index of regularity of $K[t_1, \dots, t_s]/I_{\mathbb{T}}$ (*cf.* Theorem 3.8 and Corollary 3.9). In Section 4 we study projective Reed–Muller codes over a

1-dimensional weighted torus $\mathbb{T}(w_1, w_2)$. We compute their dimensions and minimum distances (*cf.* Proposition 4.1 and Theorem 4.4) and we show they are maximum distance separable codes.

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2. Vanishing ideal of a weighted torus

Let K be a finite field with q elements. We denote by K^* the cyclic group of invertible elements of K. Given two s-tuples $\mathbf{v} = (n_1, \ldots, n_s) \in \mathbb{N}^s$ and $u = (u_1, \ldots, u_s) \in R^s$, where R is a commutative ring with identity, $u^{\mathbf{v}}$ denotes $u_1^{n_1} \cdots u_s^{n_s} \in R$.

Definition 2.1. Set $t = (t_1, \ldots, t_s)$. An ideal $I \subset K[t_1, \ldots, t_s]$ generated by *binomials*, *i.e.* polynomials of the form $\alpha t^{\mathbf{a}} - \beta t^{\mathbf{b}}$, for some $\alpha, \beta \in K$ and $\mathbf{a}, \mathbf{b} \in \mathbb{N}^s$, is called a *binomial ideal.*

We will only deal with *pure binomials*, *i.e.*, binomials for which $\alpha = \beta = 1$.

Definition 2.2. Let $\mathcal{L} \subset \mathbb{Z}^s$ be a lattice. The lattice ideal $I_{\mathcal{L}} \subset K[t_1, \ldots, t_s]$ is the ideal generated by $t^{\mathbf{a}} - t^{\mathbf{b}}$ for all $\mathbf{a}, \mathbf{b} \in \mathbb{N}^s$ such that $\mathbf{a} - \mathbf{b} \in \mathcal{L}$.

Definition 2.3. Let $w_1, \ldots, w_s$ be positive integers satisfying $\gcd(w_1, \ldots, w_s) = 1$. The *weighted projective space with weights* $w_1, \ldots, w_s$, that we denote by $\mathbb{P}(w_1, \ldots, w_s)$, is the quotient $(K^s \setminus 0)/K^*$, where $\lambda \in K^*$ acts by $\lambda(x_1, \ldots, x_s) = (\lambda^{w_1} x_1, \ldots, \lambda^{w_s} x_s)$.

Let $K[t_1, \ldots, t_s]$ be the coordinate ring of $\mathbb{P}(w_1, \ldots, w_s)$, endowed with the grading given by $\deg(t_i) = w_i$, for all $1 \leq i \leq s$. Set $\mathbf{w} = (w_1, \ldots, w_s)$. We remark that a binomial $t^{\mathbf{a}} - t^{\mathbf{b}}$, with $\mathbf{a}, \mathbf{b} \in \mathbb{N}^s$ is homogeneous if and only if $\langle \mathbf{a} - \mathbf{b}, \mathbf{w} \rangle = 0$, where $\langle \cdot \rangle$ is the canonical inner product.

Definition 2.4. The *weighted projective torus*, $\mathbb{T}(w_1, \ldots, w_s) \subset \mathbb{P}(w_1, \ldots, w_s)$ is the set:

$$\mathbb{T}(w_1, \ldots, w_s) := \{(x_1, \ldots, x_s) \in \mathbb{P}(w_1, \ldots, w_s) : x_1 \cdots x_s \neq 0\}.$$

Definition 2.5. For a set $X \subset \mathbb{P}(w_1, \ldots, w_s)$ the ideal of $K[t_1, \ldots, t_s]$ generated by all homogeneous polynomials that vanish on X is called the *vanishing ideal* of X and is denoted by I_X. We denote the vanishing ideal of $\mathbb{T}(w_1, \ldots, w_s)$ by $I_{\mathbb{T}}$.

Over an infinite field $I_{\mathbb{T}} = (0)$. its multiplicative group is cyclic of order $q - 1$, we get

$$\left(t_2^{w_1(q-1)} - t_1^{w_2(q-1)}, \ldots, t_s^{w_1(q-1)} - t_1^{w_s(q-1)}\right) \subset I_{\mathbb{T}}.$$

For general $w_1, \ldots, w_s$ this inclusion is strict. The precise structure of a minimal generating set of $I_{\mathbb{T}}$ is closely related with the numerical semigroup $Q = \langle w_1, \ldots, w_s \rangle \subset \mathbb{N}$, *cf.* Remark 2.12.

The proof of the next lemma follows closely that of Theorem 2.1 in [25].

Lemma 2.6. *In the polynomial ring extension* $K[t_1,\dots,t_s]\subset K[t_1,\dots,t_s,y_1,\dots,y_s,z]$,

$$I_{\mathbb{T}}=(\{t_i-y_iz^{w_i}\}_{i=1}^s\cup\{y_i^{q-1}-1\}_{i=1}^s)\cap K[t_1,\dots,t_s]. \tag{2.1}$$

In particular, $I_{\mathbb{T}}$ *is generated by homogeneous pure binomials.*

Proof. Let J denote the ideal on the right hand side of (2.1). We start by showing that $I_{\mathbb{T}}\subset J$. Let $f=\sum_{j=1}^r\alpha_jt^{\mathbf{v}_j}\in I_{\mathbb{T}}$, for some $\alpha_j\in K$ and $\mathbf{v}_j\in\mathbb{N}^s$ be a homogeneous polynomial of degree d. Let $\mathbf{u}=(u_1,\dots,u_s)\in\mathbb{N}^s$, then

$$t^{\mathbf{u}}=t_1^{u_1}\cdots t_s^{u_s}=(t_1-y_1z^{w_1}+y_1z^{w_1})^{u_1}\cdots(t_s-y_sz^{w_s}+y_sz^{w_s})^{u_s}=\sum_{i=1}^s(t_i-y_iz^{w_i})g_{\mathbf{u},i}+z^dy^{\mathbf{u}},$$

where $g_{\mathbf{u},j}\in K[t_1,\dots,t_s,y_1,\dots,y_s,z]$ and $d=\sum_{i=1}^s u_iw_i$. Notice that since f is homogeneous of degree d, this sum with $\mathbf{u}$ replaced by $\mathbf{v}_j$ equals d. Expanding each $t^{\mathbf{v}_j}$ in this way, we get

$$f=\sum_{i=1}^s(t_i-y_iz^{w_i})g_i+z^d\sum_{j=1}^r\alpha_jy^{\mathbf{v}_j},$$

where $g_i=\sum_{j=1}^r g_{\mathbf{v}_j,i}$. Dividing the polynomial $\sum_{j=1}^r\alpha_jy^{\mathbf{v}_j}$ by the set $\{y_i^{q-1}-1\}_{i=1}^s$ in the polynomial ring $K[y_1,\dots,y_s]$, we deduce that there exist $h_i,g\in K[y_1,\dots,y_s]$, with g of degree $<q-1$ in each variable such that

$$f=\sum_{i=1}^s(t_i-y_iz^{w_i})g_i+z^d\sum_{i=1}^s(y_i^{q-1}-1)h_i+z^dg. \tag{2.2}$$

Let us show that $g(x_1,\dots,x_s)=0$ for all $(x_1,\dots,x_s)\in(K^*)^s$, since, by the Combinatorial Nullstellensatz, this implies that $g=0$. Regarding $(x_1,\dots,x_s)\in(K^*)^s$ as a system of homogeneous coordinates of a point in $\mathbb{T}(w_1,\dots,w_s)$ we get, by assumption, $f(x_1,\dots,x_s)=0$. Hence, setting $t_i=y_i=x_i$ and $z=1$ in (2.2):

$$0=\sum_{i=1}^s(x_i^{q-1}-1)h_i(x_1,\dots,x_s)+g(x_1,\dots,x_s)=g(x_1,\dots,x_s).$$

To show the reverse inclusion we start by remarking that J, being an elimination of an ideal generated by pure binomials, is itself generated by pure binomials (*cf.* [8]). It suffices to show that any pure binomial in J is also in $I_{\mathbb{T}}$. In passing we will show that such a binomial is homogeneous. This will imply the assertion that $I_{\mathbb{T}}$ is generated by homogeneous pure binomials.

Let $t^{\mathbf{a}}-t^{\mathbf{b}}\in J$ for some $\mathbf{a},\mathbf{b}\in\mathbb{N}^s$. Then there exist $g_i,h_i\in K[t_1,\ldots,t_s,y_1,\ldots,y_s,z]$ such that

$$t^{\mathbf{a}}-t^{\mathbf{b}}=\sum_{i=1}^{s}(t_i-y_iz^{w_i})g_i+\sum_{i=1}^{s}(y_i^{q-1}-1)h_i. \tag{2.3}$$

In (2.3), substituting 1 for y_i and z^{w_i} for t_i we get $z^{a_1w_1}\cdots z^{a_sw_s}-z^{b_1w_1}\cdots z^{b_sw_s}=0$ and therefore $\langle\mathbf{a},\mathbf{w}\rangle=\langle\mathbf{b},\mathbf{w}\rangle$, *i.e.*, $t^{\mathbf{a}}-t^{\mathbf{b}}$ is homogeneous. Finally, to show that it vanishes on an arbitrary point $(x_1,\ldots,x_s)$ of the weighted torus, we use (2.3) but this time substituting t_i and y_i by x_i and z by 1. □

Remark 2.7. More generally, it can be shown that Lemma 2.6 holds for the vanishing ideal of a *weighted toric subset* $X\subset\mathbb{T}(w_1,\ldots,w_s)$ parameterized by $\mathbf{v}_1,\ldots,\mathbf{v}_s\in\mathbb{N}^n$, for some integer $n>0$. More precisely, by analogy with the standard case (*cf.* [25, §2]), when X is the subset defined by:

$$X=\{(\underline{x}^{\mathbf{v}_1},\ldots,\underline{x}^{\mathbf{v}_s})\in\mathbb{P}(w_1,\ldots,w_s):\underline{x}\in(K^*)^n\},$$

where $w_i=\frac{1}{d}\sum_{j=1}^n(\mathbf{v}_i)_j$, with $d=\gcd\{\sum_{j=1}^n(\mathbf{v}_1)_j,\ldots,\sum_{j=1}^n(\mathbf{v}_s)_j\}$. In the case when $\mathbf{v}_1,\ldots,\mathbf{v}_s$ coincide with the incidence vectors of a uniform clutter (and, in particular, of a graph), this notion coincides with the notion of toric subset parameterized by $\mathbf{v}_1,\ldots,\mathbf{v}_s$, as defined in [25, 28].

We denote by $\mathbf{w}^\perp$ the orthogonal in $\mathbb{R}^s$ of $\langle\mathbf{w}\rangle$, with respect to the canonical inner product.

Theorem 2.8. *Let* $\mathcal{L}=(q-1)\big(\mathbf{w}^\perp\cap\mathbb{Z}^s\big)$. *Then* $I_{\mathbb{T}}=I_{\mathcal{L}}$ *and, in particular,* $I_{\mathbb{T}}$ *is* 1*-dimensional and Cohen–Macaulay.*

Proof. To prove the equality we need to show the inclusion $I_{\mathbb{T}}\subset I_{\mathcal{L}}$. Let $t^{\mathbf{a}}-t^{\mathbf{b}}\in K[t_1,\ldots,t_s]$ be a homogeneous binomial vanishing on $\mathbb{T}(w_1,\ldots,w_s)$. Fix $i\in\{1,\ldots,s\}$ and let α be a generator of K^*. Consider $(1,\ldots,1,\alpha,1,\ldots,1)\in\mathbb{T}(w_1,\ldots,w_s)$, with α in the i-th position. Evaluating $t^{\mathbf{a}}-t^{\mathbf{b}}$ at this point we obtain

$$\alpha^{a_i}-\alpha^{b_i}=0\iff\alpha^{a_i-b_i}=1\iff a_i-b_i\equiv 0\pmod{q-1}.$$

Since this holds for any $i\in\{1,\ldots,s\}$ we obtain $a-b\in(q-1)\mathbb{N}^s$. As $t^{\mathbf{a}}-t^{\mathbf{b}}$ is homogeneous we get $\langle\mathbf{a}-\mathbf{b},\mathbf{w}\rangle=0$ and hence $\mathbf{a}-\mathbf{b}\in\mathcal{L}$. This proves that $I_{\mathbb{T}}=I_{\mathcal{L}}$. Since the rank of $\mathcal{L}$ is $s-1$ we deduce that $I_{\mathbb{T}}$ is 1-dimensional (*cf.* [20, Proposition 7.5]). Furthermore, since

$$\big(t_2^{w_1(q-1)}-t_1^{w_2(q-1)},\ldots,t_s^{w_1(q-1)}-t_1^{w_s(q-1)}\big)\subset I_{\mathbb{T}} \tag{2.4}$$

and consequently $V(I_{\mathbb{T}},t_i)=\{0\}$, by [24, Proposition 5.3], $I_{\mathbb{T}}$ is Cohen–Macaulay. □

Definition 2.9. Let $Q \subset \mathbb{N}$ denote the submonoid of $\mathbb{N}$ generated by $w_1, \dots, w_s \in \mathbb{N}$. Since $\gcd(w_1, \dots, w_s) = 1$, Q is a *numerical semigroup*, *i.e.*, it has finite complement. The semigroup algebra, denoted by $K[Q]$ is the subalgebra of the polynomial ring $K[z]$ given by $K[z^{w_1}, \dots, z^{w_s}]$.

Lemma 2.10. *Let $\mathcal{L}^\flat = \mathbf{w}^\perp \cap \mathbb{Z}^s$ and consider $I_{\mathcal{L}^\flat} \subset K[t_1, \dots, t_s]$ the corresponding lattice ideal. Then $I_{\mathcal{L}^\flat}$ is a homogeneous ideal and $K[t_1, \dots, t_s]/I_{\mathcal{L}^\flat} \simeq K[Q]$.*

Proof. See [20, Theorem 7.3] and also [16, Proposition 1.4]. □

The following lemma yields a relation between $I_{\mathbb{T}}$ and $I_{\mathcal{L}^\flat}$.

Lemma 2.11. *Let $\mathcal{L}^\flat = \mathbf{w}^\perp \cap \mathbb{Z}^s$. Suppose that $I_{\mathcal{L}^\flat} = (t^{\mathbf{a}_1} - t^{\mathbf{b}_1}, \dots, t^{\mathbf{a}_r} - t^{\mathbf{b}_r})$, for some $\mathbf{a}_i, \mathbf{b}_i \in \mathbb{N}^s$. Then $I_{\mathbb{T}} = (t^{(q-1)\mathbf{a}_1} - t^{(q-1)\mathbf{b}_1}, \dots, t^{(q-1)\mathbf{a}_r} - t^{(q-1)\mathbf{b}_r})$.*

Proof. Let $J = (t^{(q-1)\mathbf{a}_1} - t^{(q-1)\mathbf{b}_1}, \dots, t^{(q-1)\mathbf{a}_r} - t^{(q-1)\mathbf{b}_r})$. Since $(q-1)\mathbf{a}_i - (q-1)\mathbf{b}_i \in \mathcal{L}$, the inclusion $J \subset I_{\mathbb{T}}$ is clear. Conversely, let $t^{\mathbf{a}} - t^{\mathbf{b}} \in I_{\mathbb{T}}$. Then there exist $\mathbf{c}^+, \mathbf{c}^- \in \mathbb{N}^s$ such that $\mathbf{c}^+ - \mathbf{c}^- \in \mathcal{L}^\flat$ and $\mathbf{a} - \mathbf{b} = (q-1)(\mathbf{c}^+ - \mathbf{c}^-)$. Since $t^{\mathbf{c}^+} - t^{\mathbf{c}^-} \in I_{\mathcal{L}^\flat}$ there exist $h_i \in K[t_1, \dots, t_s]$ such that $t^{\mathbf{c}^+} - t^{\mathbf{c}^-} = \sum_{j=1}^r (t^{\mathbf{a}_j} - t^{\mathbf{b}_j}) h_j$. Substituting in this equality t_i^{q-1} for t_i for every $i = 1, \dots, s$, we deduce that $t^{\mathbf{a}} - t^{\mathbf{b}} \in J$. Hence $I_{\mathbb{T}} \subset J$. □

Remark 2.12. In [16], Herzog shows that if the condition

$$\operatorname{lcm}(\gcd\{w_1, \dots, w_{i-1}\}, w_i) \in \langle w_1, \dots, w_{i-1} \rangle \tag{2.5}$$

is satisfied, for every $i = 2, \dots, s$, then $I_{\mathcal{L}^\flat}$ is generated by the binomials $t_i^{c_i} - \prod_{j=1}^{i-1} t_j^{r_{ij}}$, where, for $i = 2, \dots, s$, $c_i = \gcd\{w_1, \dots, w_{i-1}\} / \gcd\{w_1, \dots, w_i\}$, and $r_{ij} \in \mathbb{N}$ are nonnegative integers such that $\operatorname{lcm}(\gcd\{w_1, \dots, w_{i-1}\}, w_i) = c_i w_i = \sum_{j=1}^{i-1} r_{ij} w_j$. In particular, in this situation, $I_{\mathcal{L}^\flat}$ is a complete intersection and, by Lemma 2.11, $I_{\mathbb{T}}$ is also a complete intersection. It can be checked that (2.5) is satisfied for $s = 2$ or if $w_1 = 1$ or 2. The case when $w_i = 1$ for all $i = 1, \dots, s$, is worth highlighting, for these w_i (2.4) is an equality. If $s = 3$ then (2.5) is also a necessary condition for $I_{\mathcal{L}^\flat}$ to be a complete intersection (*cf.* [16, Theorem 3.10]). This condition is no longer necessary when $s = 4$; it can be shown (*cf.* [9, Example 3.9]) that if $\mathbf{w} = (20, 30, 33, 44)$ then $I_{\mathcal{L}^\flat}$ is a complete intersection, despite the fact that no ordering of $20, 30, 33, 44$ satisfies (2.5). A recursive method for deciding whether, for a given $\mathbf{w}$, the ideal $I_{\mathcal{L}^\flat}$ is a complete intersection is given by Delorme in [6].

3. Weighted projective Reed–Muller codes

Definition 3.1. Let $X \subset \mathbb{P}(w_1, \dots, w_s)$ and set $m = |X|$. Fix $\underline{x}_1, \dots, \underline{x}_m \in K^s$ systems of homogeneous coordinates for the points of X. Given $d \geq 0$, let $\operatorname{ev}_d \colon K[t_1, \dots, t_s]_d \to K^m$ be the map defined by $f \mapsto (f(\underline{x}_1), \ldots, f(\underline{x}_m))$ for all $f \in K[t_1, \ldots, t_s]_d$. The image of ev_d, denoted by $C_X(d)$, is called *the weighted projective Reed–Muller code over* X (or, if the context is clear, simply the code over X) *of order* d.

Remark 3.2. Let $\underline{x}'_1, \ldots, \underline{x}'_m$ be a different choice of homogeneous coordinates of the points of X. Denote by ev'_d the corresponding evaluation map. Then, there exist $\lambda_1, \ldots, \lambda_m \in K^*$ such that $\mathrm{ev}'_d(K[t_1, \ldots, t_s]_d)$ is the image of $\mathrm{ev}_d(K[t_1, \ldots, t_s]_d)$ by the linear map defined by $(y_1, \ldots, y_m) \mapsto (\lambda_1 y_1, \ldots, \lambda_m y_m)$, for every $(y_1, \ldots, y_m) \in K^m$. *I.e.*, the 2 codes are equivalent.

Definition 3.3. The basic parameters of a linear code $0 \neq C \subset K^m$ are the *length*, the *dimension* and the *minimum distance*. The length is m, the dimension of the ambient vector space; the dimension is its dimension as a vector space and the minimum distance, that is denoted by δ_C, is given by $\min\{\|\underline{x}\| : \underline{x} \in C \setminus 0\}$ where $\|\underline{x}\|$ is the *Hamming weight* of $\underline{x}$, *i.e.*, the number of nonzero components of $\underline{x}$. A code is said *maximum distance separable* if the *singleton bound*:

$$\delta_C \leq \mathrm{length}(C) - \dim_K C + 1,$$

which is always satisfied for a linear code, is an equality.

Remark 3.4. If $C = C_X(d)$ is a weighted projective Reed–Muller code over $X \subset \mathbb{P}(w_1, \ldots, w_s)$ of order d, the length is equal to $|X|$ (thus is independent of d) and the minimum distance can be computed as m minus the maximum number of zeros a homogeneous polynomial of degree d can attain on X without belonging to I_X. Maximum distance separable codes are codes that, for their length and dimension, maximize minimum distance, in other words, have maximum error-correcting capability.

In this work we shall focus on the codes over $X = \mathbb{T}(w_1, \ldots, w_s)$. We abbreviate the notation for the codes to $C_{\mathbb{T}}(d)$ and for their minimum distance to $\delta_{\mathbb{T}}(d)$.

Proposition 3.5. *The length of* $C_{\mathbb{T}}(d)$ *is* $(q-1)^{s-1}$.

Proof. The length of $C_{\mathbb{T}}(d)$ coincides with the cardinality of the set of K-points of X. Since this set can be seen as the quotient $(K^*)^s/K^*$ by the induced action of K^*, all we need to check is that the orbits have cardinality $q-1$. Assume that

$$\lambda(x_1, \ldots, x_s) = \mu(x_1, \ldots, x_s)$$

for some $\lambda, \mu \in K^*$. Then $\lambda^{w_i} x_i = \mu^{w_i} x_i \iff (\lambda/\mu)^{w_i} = 1$, hence $\mathrm{ord}(\lambda/\mu)$ divides w_i, for all i. Since, by assumption, $\gcd\{w_1, \ldots, w_s\} = 1$ we deduce that $\lambda = \mu$. $\square$

Let M be a finitely generated graded $K[t_1,\ldots,t_s]$-module. The Hilbert function of M is the function $\varphi_M\colon \mathbb{Z}\to\mathbb{Z}$ defined by $\varphi_M(d)=\dim_K M_d$. This function is quasi-polynomial, *i.e.*, there exist a positive integer g (the period) and $P_0,\ldots,P_{g-1}$, polynomials, such that, for $d\gg 0$, $\varphi_M(d)=P_i(d)$, where $d\equiv i \pmod g$ (*cf.* Serre's Theorem [2, Theorem 4.4.3]).

Definition 3.6. The *index of regularity of* M is the least $r\geq 0$ such that for $d\geq r$,

$$d\equiv i \pmod g \implies \varphi_M(d)=P_i(d).$$

The *Hilbert series of* M is given by $H_M(t)=\sum_{d=0}^{\infty}\varphi_M(d)t^d$. $H_M(t)$ is a rational function (*cf.* [32, Proposition 4.1.3] or [2, Proposition 4.4.1]) and its degree is called the a-invariant of M. By Serre's theorem the index of regularity of M is equal to $\deg H_M(t)+1$.

When $M=K[t_1,\ldots,t_s]/I_{\mathbb{T}}$, we abbreviate the notation for its Hilbert function and Hilbert series to $\varphi_{\mathbb{T}}$ and $H_{\mathbb{T}}(T)$, respectively. We remark that $\dim_K C_{\mathbb{T}}(d)=\varphi_{\mathbb{T}}(d)$.

Remark 3.7. If one of the weights is equal to 1, say $w_1=1$, then multiplication by t_1 induces a monomorphism $C_{\mathbb{T}}(d)\hookrightarrow C_{\mathbb{T}}(d+1)$. Moreover, one can also check that $\delta_{\mathbb{T}}(d)\geq\delta_{\mathbb{T}}(d+1)$. In the standard case (*i.e.* when all $w_i=1$), for d up to the index of regularity minus 1, this inclusion is proper and the inequality is strict, implying that $\dim_K C_{\mathbb{T}}(d)$ is strictly increasing and $\delta_{\mathbb{T}}(d)$ is strictly decreasing.

Let $G:=\mathbb{N}\setminus Q$ denote the set of gaps of the numerical semigroup Q and denote by $\mathsf{g}_Q:=\max G$ the Frobenius number of Q.

Theorem 3.8. *The Hilbert series of* $K[t_1,\ldots,t_s]/I_{\mathbb{T}}$ *is given by*

$$H_{\mathbb{T}}(t)=\frac{\left(\frac{1}{1-t^{q-1}}-\sum_{a\in G}t^{a(q-1)}\right)\prod_{i=1}^{s}(1-t^{w_i(q-1)})}{\prod_{i=1}^{s}(1-t^{w_i})}. \tag{3.1}$$

Proof. Let $M=K[t_1,\ldots,t_s]/I_{\mathcal{L}^\flat}$. By Lemma 2.10, $M\simeq K[Q]$, hence

$$H_M=\sum_{a\in Q}t^a=\tfrac{1}{1-t}-\sum_{a\in G}t^a=\frac{\left(\frac{1}{1-t}-\sum_{a\in G}t^a\right)\prod_{i=1}^{s}(1-t^{w_i})}{\prod_{i=1}^{s}(1-t^{w_i})}. \tag{3.2}$$

Notice that, since $(1-t^{w_1})/(1-t)$ is a polynomial, the numerator of (3.2) is a polynomial. Let $I_{\mathcal{L}^\flat}=(t^{\mathbf{a}_1}-t^{\mathbf{b}_1},\ldots,t^{\mathbf{a}_r}-t^{\mathbf{b}_r})$, for some $\mathbf{a}_i,\mathbf{b}_i\in\mathbb{N}^s$. Then, by Lemma 2.11,

$$I_{\mathbb{T}}=(t^{(q-1)\mathbf{a}_1}-t^{(q-1)\mathbf{b}_1},\ldots,t^{(q-1)\mathbf{a}_r}-t^{(q-1)\mathbf{b}_r})$$

and by [22, Lemma 3.7] we get (3.1). □

Corollary 3.9. *The index of regularity of* $K[t_1,\ldots,t_s]/I_{\mathbb{T}}$ *is* $(q-2)\big(\sum_{i=1}^{s}w_i+\mathsf{g}_Q\big)+\mathsf{g}_Q+1$.

Remark 3.10. From the formula for Hilbert series of $K[t_1,\ldots,t_n]/I_{\mathbb{T}}$ given in (3.1), one can show directly that $\varphi_{\mathbb{T}}(d)=(q-1)^{s-1}$, for all $d\gg 0$. Indeed, assuming that $\gcd(w_1,q-1)=1$ (such w_i exists since $\gcd(w_1,\ldots,w_s)=1$) from (3.1) we get:

$$H_{\mathbb{T}}=\left(\textstyle\sum_{a\in Q}t^{a(q-1)}\right)(1+t^{w_1}+\cdots+t^{w_1(q-2)})\textstyle\prod_{i=2}^{s}(1+t^{w_i}+\cdots t^{w_i(q-2)}).$$

It will suffice to show that for all $d\gg 0$, the coefficient of t^d in

$$\left(\textstyle\sum_{a\in Q}t^{a(q-1)}\right)(1+t^{w_1}+\cdots+t^{w_1(q-2)}) \tag{3.3}$$

is equal to 1. Since $\gcd(w_1,q-1)=1$, the integers $0,w_1,\ldots,w_1(q-2)$ form a full system of residues modulo $q-1$. Hence no power of t appears repeated in the expansion of (3.3). To see that for all $d\gg 0$ the power t^d occurs in the expansion (3.3), let $0\le k\le q-2$ be such that $d\equiv w_1k \pmod{q-1}$. Since $d\gg 0$, $(d-w_1k)/(q-1)\in Q$, hence t^{d-w_1k} occurs in $\sum_{a\in Q}t^{a(q-1)}$ and this implies that t^d occurs in the expansion of (3.3). We conclude that in our case, the polynomials expressing the Hilbert function of $K[t_1,\ldots,t_n]/I_{\mathbb{T}}$ for large d are constant and equal to $(q-1)^{s-1}$. In particular, for d larger than or equal to the index of regularity of this module, which is given by the preceding corollary, the Hilbert function is equal to $(q-1)^{s-1}$. By consequence, for d in this range the codes $C_{\mathbb{T}}(d)$ are trivial.

Example 3.11. Suppose $K=\mathrm{GF}(4)$, $X=\mathbb{T}(3,4,5)$ and consider the corresponding family of codes $C_{\mathbb{T}}(d)$. By Proposition 3.5, these are codes of length 9. Using [14], we can check that the ideal $I_{\mathbb{T}}$ is minimally generated by the binomials $t_2^6+t_1^3t_3^3$, $t_1^9+t_2^3t_3^3$, $t_1^6t_2^3+t_3^6$ and thus is not a complete intersection. From Theorem 3.8,

$$H_{\mathbb{T}}(t)=\frac{1-t^{24}-t^{27}-t^{30}+t^{39}+t^{42}}{(1-t^5)(1-t^4)(1-t^3)}.$$

Hence the index of regularity of $K[t_1,\ldots,t_s]/I_{\mathbb{T}}$ is $42-12+1=31$. This number can also be computed using Corollary 3.9. Table 1 shows the dimension and minimum distance of $C_{\mathbb{T}}(d)$, for $d=0,\ldots,30$, computed using [14]. One feature to bear in mind is that, unlike standard projective Reed–Muller codes, $\dim_K C_{\mathbb{T}}(d)$ is not strictly increasing and $\delta_{\mathbb{T}}(d)$ is not strictly decreasing. Nevertheless, this family of codes is not necessarily redundant. For example, the two 4-dimensional codes with equal minimum distance ($d=15$ and 16) are not equivalent. Indeed, these codes have generating matrices in standard form $(I_4|A)$ and $(I_4|B)$ where $A,B\in M_{4\times 5}\,\mathrm{GF}(4)$ are given by:

$$\begin{pmatrix}1&1&0&0&0\\0&0&1&1&0\\1&\alpha&\alpha&\alpha+1&\alpha+1\\1&\alpha&\alpha&\alpha+1&\alpha\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}\alpha+1&\alpha+1&\alpha+1&1&1\\0&0&1&0&1\\\alpha&\alpha+1&\alpha+1&1&1\\0&1&0&1&0\end{pmatrix}.$$

Table 1. Parameters of $C_{\mathbb{T}}(d)$, with $\mathbf{w}=(3,4,5)$ and $K=\mathrm{GF}(4)$

d	0	1	2	3	4	5	6	7	8	9	10	11	12	13
dim	1	0	0	1	1	1	1	1	2	2	2	2	3	3
δ	9	–	–	9	9	9	9	9	6	6	6	6	6	6

d	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
dim	3	4	4	4	5	5	6	6	6	7	7	8	8	7	9	9	8
δ	6	3	3	4	3	3	3	3	3	2	2	2	2	2	1	1	2

4. Codes over $\mathbb{T}(w_1,w_2)$

In this section we study the weighted projective Reed–Muller codes over a 1-dimensional torus $\mathbb{T}(w_1,w_2)$. In this case $I_{\mathbb{T}}$ is always a complete intersection (*cf.* Remark 2.12):

$$I_{\mathbb{T}}=\big(t_1^{(q-1)w_2}-t_2^{(q-1)w_1}\big).$$

By a classical result of Sylvester, the Frobenius number of $Q=\langle w_1,w_2\rangle$ is $\mathsf{g}_Q=w_1w_2-w_1-w_2$. According to Corollary 3.9, the index of regularity of $K[t_1,\dots,t_s]/I_{\mathbb{T}}$ is $(q-1)w_1w_2-w_1-w_2+1$. Hence, we restrict to the range $1\leq d\leq (q-1)w_1w_1-w_1-w_2$. We will show below in Corollary 4.5 that all weighted projective Reed–Muller codes over a 1-dimensional (weighted) torus are maximum distance separable codes.

Given a semigroup $Q\subset\mathbb{N}$, let us denote by $\chi_Q\colon\mathbb{N}\to\{0,1\}$ the characteristic function of $Q\subset\mathbb{N}$, *i.e.*, the function given by $\chi_Q(d)=1$ if and only if $d\in Q$ and $\chi_Q(d)=0$ otherwise. We use this function for the semigroup $Q=\langle w_1,w_2\rangle$ only; to ease notation we will write simply χ.

Proposition 4.1. *Let $0\leq d\leq w_1w_2(q-1)-w_1-w_2$. Write $d=kw_1w_2+l$, where $k\geq 0$ and $0\leq l<w_1w_2$. Then,* $\dim_K C_{\mathbb{T}}(d)=k+\chi(l)$.

Proof. The Hilbert series of $I_{\mathbb{T}}$ is

$$H_{\mathbb{T}}(t)=\frac{1-t^{(q-1)w_1w_2}}{(1-t^{w_1})(1-t^{w_2})}=\big(1+t^{w_1}+\cdots+t^{(q-1)w_1w_2-w_1}\big)\big(1+t^{w_2}+t^{2w_2}+\cdots\big).$$

Hence, the dimension of $C_{\mathbb{T}}(d)$ coincides with the coefficient of the monomial in t^d on the right hand side of the above equation. Suppose that $a,b\in\mathbb{N}$ are such that $d=aw_1+bw_2$. Then $aw_1\leq aw_1+bw_2<(q-1)w_1w_2$ implies that $a\leq(q-1)w_2-1$. Hence the coefficient of t^d on the right hand side of the equation, is the number of pairs $(a, b) \in \mathbb{N}^2$ such that $d = aw_1 + bw_2$, or, in the context of numerical semigroups, the number of *factorizations* of d in $Q = \langle w_1, w_2 \rangle$.

Let us compute this number. Since $\mathsf{g}_Q = w_1w_2 - w_1 - w_2$, we see that $l + (1 - \chi(l))w_1w_2 \in Q$. Let $a, b \in \mathbb{N}$ be such that $l + (1 - \chi(l))w_1w_2 = aw_1 + bw_2$. Then

$$d = kw_1w_2 + l = (a + iw_2)w_1 + (b + (k - 1 + \chi(l) - i)w_1)w_2$$

for $i = 0, \ldots, k - 1 + \chi(l)$, yields $k + \chi(l)$ distinct factorizations of d. Consider $\{(a_i, b_i)\}_{i=1}^r$ the set of all factorizations of d. We may assume $a_r > a_{r-1} > \cdots > a_1 \geq 0$. Since the difference $a_i - a_{i-1}$ must be divisible by w_2 we get $a_r \geq (r - 1)w_2$. Therefore

$$d - (r - 1)w_1w_2 = (a_r - (r - 1)w_2)w_1 + b_rw_2 \in Q = \langle w_1, w_2 \rangle \,. \tag{4.1}$$

Additionally, $(r - 1)w_1w_2 \leq a_rw_1 \leq d = kw_1w_2 + l \leq kw_1w_2$, hence $r \leq k + 1$. Now, if $r = k + 1$, then, by (4.1), $l \in Q$. This shows that $r \leq k + \chi(l)$. □

Let us denote by $\alpha \in K^*$ a choice of generator of the cyclic group K^*. Given a homogeneous $f \in K[t_1, \ldots, t_s]$, we denote by $V(f)$ its set of zeros in $\mathbb{P}(w_1, \ldots, w_s)$.

Lemma 4.2. *For each $0 \leq r \leq q - 2$, $V(t_1^{w_2} - \alpha^r t_2^{w_1}) \subset \mathbb{T}(w_1, w_2)$ consists of a single point. Moreover, as r varies in $\{0, \ldots, q - 2\}$, every point of $\mathbb{T}(w_1, w_2)$ is obtained in this way.*

Proof. Fix $a, b \in \mathbb{Z}$ such that $aw_1 + bw_2 = 1$. Clearly $(\alpha^{rb}, \alpha^{-ra}) \in V(t_1^{w_2} - \alpha^r t_2^{w_1})$. Suppose $(x_1, x_2) \in \mathbb{T}(w_1, w_2)$ belongs to $V(t_1^{w_2} - \alpha^r t_2^{w_1})$, *i.e.*, $x_1^{w_2} = \alpha^r x_2^{w_1}$. Then:

$$(x_1, x_2) = (x_1(x_1^{-a}x_2^{-b})^{w_1}, x_2(x_1^{-a}x_2^{-b})^{w_2}) = (x_1^{bw_2}x_2^{-bw_1}, x_1^{-aw_2}x_2^{aw_1}) = (\alpha^{rb}, \alpha^{-ra}).$$

Hence $V(t_1^{w_2} - \alpha^r t_2^{w_1}) = \{(\alpha^{rb}, \alpha^{-ra})\}$. To show that every point in $(x_1, x_2) \in \mathbb{T}(w_1, w_2)$ is the zero of one such polynomial it suffices to notice that $x_1^{w_2}/x_2^{w_1} = \alpha^r$, for some $0 \leq r \leq q - 2$. □

Proposition 4.3. *Let $f \in K[t_1, t_2]$ be nonzero, homogeneous of degree d. Write $d = kw_1w_2 + l$, where $k \geq 0$ and $0 \leq l < w_1w_2$. Then $|V(f) \cap \mathbb{T}(w_1, w_2)| \leq k - 1 + \chi(l)$.*

Proof. We argue by induction on k. Suppose that $k = 0$. Then $d < w_1w_2$ and, by an argument similar to the one used in the proof of Proposition 4.1, we deduce that there is only one factorization of d in Q, hence f is a monomial and thus $|V(f) \cap \mathbb{T}(w_1, w_2)| = 0$. Additionally, if $k = 0$ then $l = d \in Q$ and so $\chi(l) = 1$ and the inequality of the statement holds.

Suppose $k \geq 1$. Let us write $f = gt_1^at_2^b$, for some $g \in K[t_1, t_2]$, such that neither t_1 nor t_2 divides g. Let $d' = d - aw_1 - bw_2$, be the degree of g. If $d' < w_1w_2$ then $g = 1$. In this situation

$V(f) \cap \mathbb{T}(w_1, w_2)$ is empty and there is nothing to show. Suppose $d' \geq w_1 w_2$. Let us write $d' = k' w_1 w_2 + l'$ for $1 \leq k' \leq k$ and $0 \leq l' < w_1 w_2$. If $k' < k$, by induction we get:

$$|V(f) \cap \mathbb{T}(w_1, w_2)| = |V(g) \cap \mathbb{T}(w_1, w_2)| \leq k' - 1 + \chi(l') \leq k - 1 + \chi(l). \tag{4.2}$$

If $k = k'$ then $l = l' + a w_1 + b w_2$ and thus $\chi(l') \leq \chi(l)$. We may assume there exists $(x_1, x_2) \in V(f) \cap \mathbb{T}(w_1, w_2)$. Let us write $g = \sum_{i=0}^{r} \alpha_i t_1^{a_i} t_2^{b_i}$, with $r \geq 1$ and, without loss in generality, $0 = a_0 \leq a_1 \leq \cdots \leq a_r$. Since $w_1 a_i + w_2 b_i = d' = w_2 b_0$, we deduce that there exist $m_i \geq 0$ such that $a_i = m_i w_2$ and $b_i = b_0 - m_i w_1$. Hence, we may write in $\operatorname{Frac} K[t_1, t_2]$:

$$g = t_2^{b_0} \sum_{i=0}^{r} \alpha_i \left(\frac{t_1^{w_2}}{t_2^{w_1}} \right)^{m_i} = t_2^{b_0} G\big(t_1^{w_2}/t_2^{w_1}\big),$$

where $G(z) = \sum_{i=0}^{r} \alpha_i z^{m_i} \in K[z]$ has degree $m_r = a_r/w_2$. We see that $x_1^{w_2}/x_2^{w_1}$ is a zero of G. Let $0 \leq r \leq q-2$ be such that $\alpha^r = x_1^{w_2}/x_1^{w_1}$. Then $G(z) = H(z)(z - \alpha^r)$, for some $H \in K[z]$, of degree $a_r/w_2 - 1$. Accordingly, $g = t_2^{b_0} H\big(t_1^{w_2}/t_2^{w_1}\big)\big(t_1^{w_2}/t_2^{w_1} - \alpha^r\big)$. Since

$$b_0 - w_1 \geq w_1(a_r/w_2 - 1) \iff b_0 w_2 \geq w_1 a_r \iff d' \geq w_1 a_r,$$

clearing denominators, we conclude that there exists $h \in K[t_1, t_2]$, homogeneous, such that $g = (t_1^{w_2} - \alpha^r t_2^{w_1})h$. Since the degree of h is $(k'-1)w_1 w_2 + l'$, by induction and Lemma 4.2

$$|V(f) \cap \mathbb{T}(w_1, w_2)| = |V(h) \cap \mathbb{T}(w_1, w_2)| + 1 \leq (k'-1) - 1 + \chi(l') + 1 \leq k - 1 + \chi(l).$$

□

We now address the computation of the minimum distance of the weighted projective Reed–Muller codes over a weighted torus. Recall that the minimum distance is defined for a nonzero code. Thus, the assumption that $d \in Q$, equivalent to $\dim_K C_{\mathbb{T}}(d) \neq 0$, is necessary in the statement of the theorem.

Theorem 4.4. *If $0 \leq d \leq w_1 w_2 (q-1) - w_1 - w_2$ and $d \in Q$. Write $d = k w_1 w_2 + l$ with $k \geq 0$ and $0 \leq l < w_1 w_2$. Then the minimum distance of the evaluation code $C_{\mathbb{T}}(d)$ is $(q-1) - k + 1 - \chi(l)$.*

Proof. Let $f \in K[t_1, t_2]$ be a homogeneous polynomial of degree d. Then, by Proposition 4.3, f has at most $k - 1 + \chi(l)$ zeros on $T(w_1, w_2)$. Since, by Proposition 3.5, the length of $C_{\mathbb{T}}(d)$ is $q - 1$ we get $\delta_{\mathbb{T}}(d) \geq (q-1) - k + 1 - \chi(l)$. To prove the reverse inequality, we split the proof into 2 cases. If $\chi(l) = 1$, let $a, b \in \mathbb{N}$ be such that $l = a w_1 + b w_2$. Then, the polynomial

$$f = t_1^a t_2^b \textstyle\prod_{i=1}^{k} (t_1^{w_2} - \alpha^i t_2^{w_1})$$

has degree d and, since $d \leq w_1 w_2 (q-1) - w_1 - w_2$ implies that $0 \leq k \leq q-2$, by Lemma 4.2, has exactly $k = k - 1 + \chi(l)$ zeros on $\mathbb{T}(w_1, w_2)$. If $\chi(l) = 0$ then, since $d = k w_1 w_2 + l \in Q$ we must have $k > 0$. Additionally, since $d - w_1 w_2 (k-1) \geq w_1 w_2 > \mathsf{g}_Q$, there exist $a, b \in \mathbb{N}$ such that $aw_1 + bw_2 = d - w_1w_2(k-1)$. Then the polynomial $f = t_1^a t_2^b \prod_{i=1}^{k-1}(t_1^{w_2} - \alpha^i t_2^{w_1})$ has degree d and has exactly $k-1 = k-1+\chi(l)$ zeros on $\mathbb{T}(w_1, w_2)$. $\square$

Corollary 4.5. *The weighted projective Reed–Muller codes $C_{\mathbb{T}}(d)$ over a (weighted) torus are maximum distance separable codes, i.e., for $d \in Q$, $\delta_{\mathbb{T}}(d) = (q-1) - \dim C_{\mathbb{T}}(d) + 1$.*

References

[1] E. Ballico and C. Fontanari, *The Horace method for error-correcting codes*, Appl. Algebra Engrg. Comm. Comput. **17** (2006), no. 2, 135–139.

[2] W. Bruns and J. Herzog, *Cohen–Macaulay rings*, Cambridge Studies in Advanced Mathematics, **39**. Cambridge University Press, Cambridge, (2nd. Edition, 1998).

[3] C. Carvalho, *On the second Hamming weight of some Reed–Muller type codes*, Finite Fields Appl. **24** (2013), 88–94.

[4] A. Couvreur, *The dual minimum distance of arbitrary-dimensional algebraic-geometric codes*, J. Algebra **350** (2012), 84–107.

[5] A. Couvreur and I. Duursma, *Evaluation codes from smooth quadric surfaces and twisted Segre varieties*, Des. Codes Cryptogr. **66** (2013), no. 1-3, 291–303.

[6] C. Delorme, *Sous-monoïdes d'intersection complète de* $\mathbb{N}$, Ann. Sci. École Norm. Sup. (4) **9** (1976), no. 1, 145–154.

[7] I. Duursma, C. Rentería and H. Tapia-Recillas, *Reed–Muller codes on complete intersections*, Appl. Algebra Engrg. Comm. Comput. **11** (2001), no. 6, 455–462.

[8] D. Eisenbud and B. Sturmfels, *Binomial ideals*, Duke Math. J. **84** (1996), no. 1, 1–45.

[9] K. G. Fischer and J. Shapiro, *Mixed matrices and binomial ideals*, J. Pure Appl. Algebra **113** (1996), no. 1, 39–54.

[10] O. Geil and C. Thomsen, *Weighted Reed-Muller codes revisited*, Des. Codes Cryptogr. **66** (2013), no. 1–3, 195–220.

[11] L. Gold, J. Little and H. Schenck, *Cayley–Bacharach and evaluation codes on complete intersections*, J. Pure Appl. Algebra **196** (2005), no. 1, 91–99.

[12] M. González-Sarabia, C. Rentería and H. Tapia-Recillas, *Reed–Muller-type codes over the Segre variety*, Finite Fields Appl. **8** (2002), no. 4, 511–518.

[13] C. Rentería and H. Tapia-Recillas, *Reed–Muller type codes on the Veronese variety over finite fields*, in Coding Theory, Cryptography and Related Areas (J. Buchmann, T. Hoholdt, H. Stichtenoth, and H. Tapia-Recillas, Eds.), pp. 237–243, Springer-Verlag, Berlin, 2000,

[14] D. R. Grayson and M. E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, Available at http://www.math.uiuc.edu/Macaulay2/.

[15] J. P. Hansen, *Linkage and codes on complete intersections*, Appl. Algebra Engrg. Comm. Comput. **14** (2003), no. 3, 175–185.

[16] J. Herzog, *Generators and relations of abelian semigroups and semigroup rings*, Manuscripta Math. **3** (1970), 17–193.

[17] G. Lachaud, *Projective Reed–Muller codes*, Coding theory and applications (Cachan, 1986), 125–129, Lecture Notes in Comput. Sci., **311**, Springer, Berlin, 1988.

[18] G. Lachaud, *The parameters of projective Reed–Muller codes*, Discrete Math. **81** (1990), no. 2, 217–221.

[19] H. H. López, C. Renteria-Márquez and R. H. Villarreal, *Affine cartesian codes*, Des. Codes Cryptogr. **71** (2014), no. 1, 5–19.

[20] E. Miller and B. Sturmfels, *Combinatorial Commutative Algebra*, Graduate Texts in Mathematics **227**, Springer-Verlag, New York, 2005.

[21] J. Neves and M. Vaz Pinto, *Vanishing ideals over complete multipartite graphs*, J. Pure Appl. Algebra **218** (2014), 1084–1094.

[22] J. Neves, M. Vaz Pinto and R. H. Villarreal, *Regularity and algebraic properties of certain lattice ideals*, arXiv:1301.4249, Preprint 2013

[23] J. Neves, M. Vaz Pinto and R. H. Villarreal, *Vanishing ideals over graphs and even cycles*, Comm. Algebra (in press).

[24] L. O'Carroll, F. Planas-Vilanova and R. H. Villarreal, *Degree and algebraic properties of lattice and matrix ideals*, SIAM J. Discrete Math. (in press)

[25] C. Rentería, A. Simis and R. H. Villarreal, *Algebraic methods for parameterized codes and invariants of vanishing ideals over finite fields*, Finite Fields Appl. **17** (2011), no. 1, 81–104.

[26] C. Rentería and H. Tapia-Recillas, *Reed–Muller codes: an ideal theory approach*, Comm. Algebra **25** (1997), no. 2, 401–413.

[27] E. Reyes, R. H. Villarreal and L. Zárate, *A note on affine toric varieties*, Linear Algebra Appl. **318** (2000), no. 1-3, 173–179.

[28] E. Sarmiento, M. Vaz Pinto and R. H. Villarreal, *The minimum distance of parameterized codes on projective tori*, Appl. Algebra Engrg. Comm. Comput. **22** (2011), no. 4, 249–264.

[29] E. Sarmiento, M. Vaz Pinto and R. H. Villarreal, *On the vanishing ideal of an algebraic toric set and its parametrized linear codes*, J. Algebra Appl. **11** (2012), no. 4, 1250072, 16 pp.

[30] I. Soprunov, *Toric complete intersection codes*, J. Symbolic Comput. 50 (2013), 374–385.

[31] A. B. Sørensen, *Projective Reed–Muller codes*, IEEE Trans. Inform. Theory 37 (1991), no. **6**, 1567–1576.

[32] R. H. Villarreal, *Monomial Algebras*, Monographs and Textbooks in Pure and Applied Mathematics **238**, Marcel Dekker, New York, 2001.

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