PRIME KNOT COMPLEMENTS WITH MERIDIONAL ESSENTIAL SURFACES OF ARBITRARILY HIGH GENUS

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ABSTRACT. We show the existence of infinitely many prime knots each of which having in their complements meridional essential surfaces with two boundary components and arbitrarily high genus.

1. Introduction

Since the work of Haken and Waldhausen, it is common to study 3-manifolds, as knot complements, by their decomposition along surfaces into submanifolds. A very important class of surfaces used in these decompositions are the essential surfaces, which has motivated research on the properties and existence of closed essential surfaces or meridional essential surfaces in knot complements in S^3. A particularly interesting phenomena is the existence of knots with the property that their complements have closed essential surfaces of arbitrarily high genus. The first examples of knots with this property were given by Lyon [13], where he proves the existence of fibered knots complements with closed essential surfaces. Later Oertel [16] and recently Li [10] also give examples of knots having closed essential surfaces of arbitrarily high genus in their complements. Oertel uses the planar surfaces from the tangles defining the Montesino knots to construct and characterize the essential surfaces. Lyon and Li use connected sum of knots on their constructions and afterwards sattelite knots to obtain primeness of the desired examples.
In this paper we consider meridional surfaces instead, and prove that there is also no general bound for the genus of meridional essential surfaces in the complements of (prime) knots. In fact, we construct prime knots each of which with meridional essential surfaces in their complements having only two boundary components and arbitrarily high genus. Then, in particular, we prove that some prime knots have the property that they can be decomposed by surfaces of all positive genus as composite knots are decomposed by spheres. The results of this paper are summarized in the following theorem and its corollary.

Theorem 1. *There are infinitely many prime knots each of which having the property that its complement has a meridional essential surface of genus g and two boundary components for all positive g.*

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Corollary 1.1. *There are infinitely many knots each of which having the property that its complement has a meridional essential surface of genus g and two boundary components for all $g \geq 0$.*

From [1], at least one of the swallow-follow surfaces obtained from the meridional essential surfaces in Theorem 1 is also essential and of higher genus. Hence, the knots from the theorem are also examples of knots having closed essential surfaces of arbitrarily high genus in their complements.
Together with the literature already cited, this paper joins several other contributions to unserstand better knots with respect to the existence of closed or meridional essential surfaces in their complements: In [14] Menasco studies essential surfaces in alternating links complements; Finkelstein-Moriah [5] and Lustig-Moriah [12] prove the existence of meridional essential and closed essential surfaces, respectively, for a large class of links characterized by a certain $2n$-plat projection; There is also the work of Finkelstein [4] and Lozano-Przytycki [11] describing closed incompressible surfaces in closed 3-braids; More recently, after Gordon-Reid [8] proved that tunnel number one knots have no meridional planar essential surface in their complements, Eudave-Muñoz [2], [3] proved that some of these knots actually have meridional or closed essential surfaces in their complements.
The proof of Theorem 1 follows a similar philosophy as in Lyon's paper [13], where he uses the connected sum of two knots and essential surfaces in their exteriors. We could follow the same construction if our aim was only to construct a knot exterior with merdidional essential surfaces of arbitrarily high genus. However, as we want the surfaces to have two boundary components we cannot use the sattelite construction to obtain primeness of the knot. As we also want the knots to be prime we cannot use a connected sum as the base for the construction. So, instead of using composite knots we consider a decomposition of prime knots along certain essential tori separating the knot into two arcs. The main techniques for the proof are classical in 3-manifold topology, as innermost curve arguments and branched surface theory. The reference used for standard definitions and notation in knot theory is Rolfsen's book [18]. Throughout this paper all submanifolds are assumed to be in general position and we work in the piecewise linear category.

2. Construction of the knots

In our construction we use 2-*string essential free tangles*, that we define as follows: A n-*string tangle* is a pair (B, σ) where B is a 3-ball and σ is a collection of n properly embedded disjoint arcs in B. We say that (B, σ) is *essential* if for every disk D properly embedded in $B - \sigma$ then ∂D bounds a disk in $\partial B - \partial\sigma$. The tangle is said to be *free* if the fundamental group of $B - \sigma$ is free, or, equivalently, if the closure of $B - N(\sigma)$ is a handlebody.
Let H be a solid torus and γ an embedded graph in H, as in Figure 1. The graph γ is topologically a circle connected to two segments, a_1 and a_2, at a boundary point of each. The other two boundary points of $a_1 \cup a_2$ are in ∂H. There is a separating disk D_H in H intersecting γ transversely at a point of each segment a_1 and a_2, and decomposing H into a solid torus and a 3-ball B_H where $(B_H, B_H \cap \gamma)$ is a 2-string essential free tangle[1] with $B_H \cap \gamma$ two knotted arcs in B_H. (See Figure 1(b).)

[1]See the Appendix, section 4, for an example of a 2-string essential free tangle with both strings knotted.

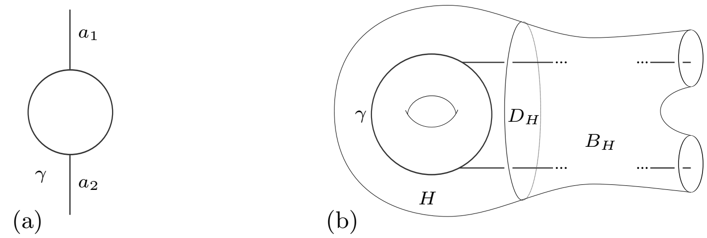


FIGURE 1: The graph γ, in (a), and its embedding into the solid torus H, in (b).

Denote by T a regular neighborhood of γ in H and suppose there is a properly embedded arc s in T, as in Figure 2(a), with the boundary of s in $T \cap \partial H$. Assume

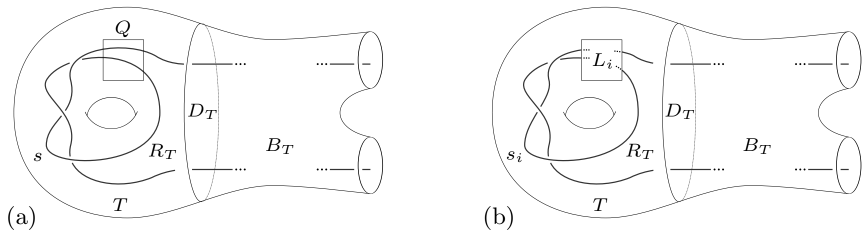


FIGURE 2: The solid torus T with the string s, in (a), and the solid torus T_i with the string s_i, in (b).

there is a separating disk D_T in T intersecting s at two points and decomposing T into a 3-ball B_T and a solid torus R_T. The boundary of s is in ∂B_T and $(B_T, B_T \cap s)$ is a 2-string essential free tangle with the two arcs $B_T \cap s$ in B_T being knotted. The string $R_T \cap s$ in the solid torus R_T is such that when capped off by an arc in D_T we get the $(2,-3)$-torus knot boundary parallel in R_T.
We say that an arc properly embedded in a solid torus is *essential* if it is not *boundary parallel*, that is the arc does not co-bound an embedded disk in the solid torus with a segment in the boundary of the solid torus, and if the boundary of the solid torus is incompressible in the complement of the arc. In Lemma 1 we prove that s is essential in T.
Consider a ball Q in $T - B_T$ intersecting s at two parallel trivial arcs, as in Figure 2(a), and an infinite collection of knots L_i, $i \in \mathbb{N}$. We replace the two parallel trivial arcs by two parallel arcs with the pattern[2] of a knot L_i, as in Figure 2(b). After this tangle replacement, we denote by s_i the string obtained from s, by T_i the solid torus T containing s_i, by γ_i the graph γ whose regular neighborhood is T_i, and by H_i the solid torus H containing T_i. Let $E_H(T)$ be the exterior of T in H, that is the closure of $H - T$, and $E_T(s)$ be the exterior of $N(s)$ in T, that is the closure of $T - N(s)$. The following lemmas are relevant for the next section, and they are also valid if we replace s by s_i, T by T_i and H by H_i in their statements.

Lemma 1.
(a) *The surfaces ∂H and ∂T are incompressible in $E_H(T)$.*

[2]By a properly embedded arc in a ball B having the *pattern* of a knot K we mean that when we cap off the arc with a string in ∂B we get the knot K.

(b) *The arc s is essential in T.*

Proof.
(a) First we prove that ∂H is incompressible in $E_H(T)$. As T is a regular neighborhood of γ this is equivalent to prove that ∂H is incompressible in $H-\gamma$. The graph γ in H is defined by a circle c and two segments a_1 and a_2, each with an end in the circle and the other end in ∂H, and H is a regular neighborhood of c. Hence, the boundary of a properly embedded disk D in H disjoint from c bounds a disk O in ∂H. Furthermore, as D is disjoint from γ and each segment a_1 and a_2 intersects ∂H at a single point, the disk O is disjoint from γ. Then, the boundary of every embedded disk in $E_H(T)$ with boundary in ∂H bounds a disk in $\partial H - \partial H \cap \partial T$, which means ∂H is incompressible in $E_H(T)$.
We prove similarly that ∂T is incompressible in $E_H(T)$. Let D be a properly embedded disk in $E_H(T)$ with boundary in ∂T. We have $T = N(c) \cup N(a_1) \cup N(a_2)$. As a_1 and a_2 have each an end in ∂H and in c, we can isotope the boundary of D to $N(c)$. As H is a regular neighborhood of c we have that ∂D bounds a disk O in $\partial N(c)$. As a_1 and a_2 have each only one end in c, we have that O is a disk in ∂T. Hence, ∂T is incompressible in $E_H(T)$.

(b) To prove that s is essential in T we have to prove that ∂T is incompressible in $E_T(s)$ and that s is not boundary parallel. We start by showing that ∂T is incompressible in $E_T(s)$. As the tangle $(B_T, B_T \cap s)$ is essential, the boundary of a properly embedded disk in $B_T - (s \cap B_T)$ bounds a disk in $\partial B_T - (s \cap \partial B_T)$. Also, the string $R_T \cap s$ in the solid torus R_T when capped off by an arc in D_T is the $(2,-3)$-torus knot boundary parallel in R_T. Hence, every disk in $R_T - R_T \cap s$ with boundary in ∂T has boundary bounding a disk in ∂T. Also, if a disk in $R_T - R_T \cap s$ has boundary in D_T then its boundary bounds a disk in D_T disjoint from $s \cap D_T$. Suppose D is a disk properly embedded in $E_T(s)$ with boundary in ∂T. If D is disjoint from D_T then ∂D bounds a disk in $\partial T - s \cap \partial T$. So, if D is a compressing disk for ∂T in $E_T(s)$ it intersects D_T. Hence, we assume that D intersects D_T transversely in a collection of arcs and simple closed curves, with $|D \cap D_T|$ minimal. If D intersects D_T in simple closed curves then consider an innermost one in D and the respective innermost disk O. From the previous observations, we have that ∂O bounds a disk in D_T. Therefore, by an isotopy of D along the ball bounded by $D \cup D_T$ we can reduce $|D \cap D_T|$, which contradicts its minimality. Hence, $D \cap D_T$ is a collection of arcs. Consider an outermost arc α between the arcs $D \cap D_T$ in D and the respective outermost disk, that we also denote by O. If O is in B_T then ∂O bounds a disk O' in ∂B_T intersecting $D_T - s$ at a disk. Suppose now that O is in R_T. If O is essential in R_T then O intersects at least twice the $(2,-3)$-torus knot obtained from $R_T \cap s$ by capping off the ends of this string in D_T. However, ∂O intersects at most once this knot, whether α separates the components of $D_T \cap s$ in D_T or not. This implies that O intersects $R_T \cap s$, which is contradiction with O being disjoint from s. Therefore, O is inessential in R_T and ∂O bounds a disk O' in ∂R_T intersecting $D_T - s$ at a disk. In both cases, O in B_T or in R_T, ∂O bounds a disk O' intersecting $D_T - s$ at a disk. If we isotope D along the ball bounded by $O \cup O'$ we reduce $|D \cap D_T|$, contradicting its minimality. Hence, ∂T is incompressible in $E_T(s)$.
Now we prove that s is not boundary parallel in T. Suppose that D is now a disk embedded in T co-bounded by s and an arc b in ∂T. Following a similar argument

as before we can prove that D does not intersect D_T at simple closed curves and arcs with both ends in b. Hence, $D \cap D_T$ is a collection of two arcs, each with an end in s and the other end in b. However, the disk components these arcs separate from D imply that the strings of the tangle $(B_T, B_T \cap s)$ are trivial, which contradicts this tangle being essential. Hence, s is not boundary parallel in T and, together with ∂T being incompressible in the exterior of s in T, we have that s is essential in T. □

Lemma 2. *There is no properly embedded disk in $E_H(T)$*

(a) *intersecting one of the disks of $T \cap \partial H$ at a single point; or*
(b) *with boundary the union of an arc in ∂T and an arc in ∂H, and not bounding a disk in $\partial E_H(T)$.*

Proof. Let D be a properly embedded disk in $E_H(T)$. Following an argument as in Lemma 1 we can assume that $|D \cap D_H|$ is minimal and that $D \cap D_H$ is a collection of essential arcs in $D_H - D_H \cap T$ with ends in $D_H \cap T$ and ∂D_H. Consider also $B_H \cap T$, which is a collection of two cylinders C_1 and C_2, and assume that $|D \cap (B_H \cap T)|$ is minimal. If some arcs of $\partial D \cap C_i$ have ends in the same boundary component of the annlus $\partial C_i - C_i \cap \partial B_H$, $i = 1, 2$, then by using an innermost curve argument we can reduce $|D \cap (B_H \cap T)|$ and contradict its minimality. Therefore, $D \cap C_i$ is a collection of essential arcs in the annulus $\partial C_i - C_i \cap \partial B_H$, $i = 1, 2$.

(a) Assume that D intersects $T \cap \partial H$ exactly once at $C_1 \cap \partial H$. As $D \cap D_H$ is a collection of arcs, the components of $D \cap B_H$ are a collection of disks. Consequently, one component of $D \cap B_H$ is a disk in $B_H - B_H \cap T$ intersecting $C_1 \cap \partial H$ once. This means that $C_1 \cap \partial H$ is primitive with respect to the complement of $B_H \cap T$ in B_H. Hence, as the complement of $C_1 \cup C_2$ is a handlebody (because $(B_H, B_H \cap \gamma)$ is a free tangle), the complement of C_2 in B_H is a solid torus. Then the core of C_2 is unknotted, which is a contradiction to $B_H \cap \gamma$ being a collection of two knotted arcs in B_H.

(b) Suppose the disk D is as in the statement with $\partial D = a \cup b$, where a is an arc in ∂T and b an arc in ∂H. The intersection of D with $T \cap \partial H$ is the boundary of a (and b), and notice that $C_1 \cup C_2$ intersects ∂H at $T \cap \partial H$. From the statement (a) of this lemma the boundary of a (and b) is in one disk component of $T \cap \partial H$, that without loss of generality we assume to be $C_1 \cap \partial H$. As $D \cap C_i$ is a collection of essential arcs in $\partial C_i - C_i \cap \partial B_H$ and ∂D intersects $\partial T \cap \partial H$ in two points, the arc a intersects D_H at two points. Furthermore, by an innermost arc argument and the minimality of $|D \cap D_H|$, there are no arcs of $D \cap D_H$ with both end points in $b \cap D_H$.

FIGURE 3: The arcs $D \cap D_H$ in D when a single arc intersects a, in (a), and when two arcs intersect a, in (b).

The points of $a \cap D_H$ cobound one or two arcs of $D \cap D_H$. Suppose $a \cap D_H$ is the boundary of a single arc, α, component of $D \cap D_H$, as in Figure 3(a). Hence, the arc α separates a disk from D in B_H that cuts a ball from $B_H - C_1$ contaning C_2, otherwise ∂D bounds a disk in $\partial E_H(T)$. As there are no local knots in the free tangle $(B_H, B_H \cap s)$, this case implies that either the core of C_2 corresponds to a trivial string or a string parallel to the core of C_1, which contradicts this tangle being a 2-string essential free tangle (see Lemma 2.1 in [15]). If each point of $a \cap D_H$ cobounds an arc of $D \cap D_H$ with the other end in b, as in Figure 3(b), then the core C_1 corresponds to a string of $(B_H, B_H \cap \gamma)$ that is trivial, which also contradicts this tangle being essential. □

To construct the knots to prove the main theorem of this paper we identify the solid tori H and H_i along their boundaries, by identifying a meridian of one boundary to a longitude of the other and defining a Heegaard decomposition $H \cup H_i$ of S^3, such that ∂s is identified with ∂s_i. From construction, $K_i = s \cup s_i$ is a knot in S^3, for $i \in \mathbb{N}$, and in the next proposition we prove these knots are prime.

Proposition 1. *The knots K_i, $i \in \mathbb{N}$, are an infinite collection of distinct prime knots.*

To prove that the knots K_i are prime we use the following technical result. Let K and L be non-trivial knots. Take a ball B intersecting K in two parallel trivial arcs with the tangle $(B^c, B^c \cap K)$ being locally unknotted. We replace the arcs of $B \cap K$ in B by two parallel arcs with the pattern of L, where each new arc has the same boundary component as one of the replaced arcs. We denote the resulting knot from this construction by K_L.

Lemma 3. *The knot K_L is prime.*

Proof. If the knot K_L is trivial then it bounds a disk D in S^3. Then, ∂D intersects ∂B at four points. Suppose that $|D \cap \partial B|$ is minimal. By an innermost curve argument, as used before, we can show that $D \cap \partial B$ is a collection of two arcs. The strings of $B \cap K_L$ are knotted and each can't co-bound an outermost disk of $D - D \cap \partial B$ with an arc in ∂B. Hence, the arcs of $D \cap \partial B$ have an end on each string of $B \cap K_L$ and co-bound together with the strings a disk in B. Each arc of $D \cap \partial B$ also co-bounds a disk with a string of $K_L \cap B^c$. Therefore, if we replace the tangle $(B, B \cap K_L)$ with the tangle $(B, B \cap K)$ we obtain a disk in S^3 bounded by K, which is a contradiction because K is knotted. Hence, the knot K_L is also non-trivial.
Now we prove that K_L is prime. Suppose there is a decomposing sphere S for K_L. As $(B, B \cap K_L)$ is defined by two parallel strings in B, using the disk co-bounded by the two strings $B \cap K_L$ in B we can show that S can be assumed disjoint from B. However, this means that S is in B^c, which contradicts $(B^c, B^c \cap K_L)$ being locally unknotted. □

As for the construction of the knots K_i, we construct a knot K by identifying two copies of H, say H and H', by identifying a meridian of one boundary to a longitude of the other and defining a Heegaard decomposition $H \cup H'$ of S^3, such that the two copies of s, say s and s' resp., are also identified along their boundaries. As s is essential in H we have that ∂H defines a meridional incompressible surface in the exterior of K, which means that K is not trivial. We also denote the copy of the solid torus T of H in H' by T'.

We will use this knot K, the knots L_i and the construction of Lemma 3 to define the knots K_i, but first we need the following lemma. Let Q be the ball as in Figure 2 and Q^c its complement.

Lemma 4. *The tangle $(Q^c, Q^c \cap K)$ is locally unknotted.*

Proof. Suppose $(Q^c, Q^c \cap K)$ is locally knotted. Then there is a sphere S bounding a ball P intersecting $Q^c \cap K$ at a single knotted arc. We have that s and s' have no local knots in $T \cup T'$. Then S intersects T or T'.
Consider the intersection of S with ∂T and $\partial T'$, and suppose it has a minimal number of components. From the construction of the knot K the cores of the solid tori T and T' define a two component link with each component being unknotted. As the tangle $(B_T, B_T \cap s)$ is free and essential we can assume that S is disjoint from B_T (and similarly, that S is disjoint from $B_{T'}$).
The intersections of S with the boudaries of T and T' is a collection of simple closed curves. As S is disjoint from B_T and $B_{T'}$ the curves of intersection are either in $\partial T - B_T$ or in $\partial T' - B_{T'}$. Consider E a disk component of S separated by $\partial T \cup \partial T'$ from S. Suppose E is not in $T \cup T'$ and its boundary is in ∂T (or similarly $\partial T'$). From the minimality of $|S \cap \partial(T \cup T')|$ and as S^3 does not have a $S^2 \times S^1$ or a Lens space summand, we have that ∂E is a longitude of ∂T. Therefore, the core of T bounds a disk disjoint from T', which is a contradiction to the cores of T and T' being linked from construction. Hence, E is in T or T'. If E is in T (or similarly in T') and is disjoint from s then as s is essential in T we have that ∂E bounds a disk in $\partial T - s$. In this case we can reduce the number of components of S intersection with $\partial T \cup \partial T'$, which is a contradiction to its minimality. Then, we can assume that all disks E intersect s or s'. If some disk E intersects either s or s' at two points then some other disk component of S separated by $\partial T \cup \partial T'$ is disjoint from s and s', which is a contradiction to all disks E intersecting s or s'. Then, there is an essential disk E in T (or similarly, in T') that intersects s at a single point. As before, let R_T be the solid torus separated by D_T from T. From the construction of s in T, if we cap off $R_T \cap s$ with an arc in D_T we get a torus knot. Then any essential disk in R_T intersects the knot in more than one point. As E is disjoint from B_T it is a non-separating disk in R_T intersecting the torus knot at a single point, which is a contradiction. Hence, $(Q^c, Q^c \cap K)$ is locally unknotted. □

Proof of Proposition 1. The knots K_i are the knots K_{L_i} obtained from the knots K and L_i with a construction as in Lemma 3. From Lemmas 3 and 4 the knots K_i are prime.
Each knot K_i is also sattelite with companion knot L_i and pattern knot K. Then, from the unicity of JSJ-decomposition of compact 3-manifolds and as the knots L_i are distinct we have that the knots K_i, $i \in \mathbb{N}$, are an infinite collection of distinct prime knots. □

3. Knots with meridional essential surfaces for all genus

In this section we prove Theorem 1, and its corollary, by showing the knots K_i, $i \in \mathbb{N}$, have meridional essential surfaces of all positive genus and two boundary components. We start by constructing these surfaces, denoted by $F_1, \dots F_g, \dots$ where F_g has genus g, in the complement of an arbitrary knot K_i, and afterwards we prove they are essential in $E(K_i)$. In this construction we denote the boundaries of s and s_i by $\partial_1 s$ $(= \partial_1 s_i)$ and $\partial_2 s$ $(= \partial_2 s_i)$. Denote by X (resp., Y) the punctured

torus ∂T (resp., ∂T_i) obtained by cutting the interior of the discs $\partial T \cap \partial H$ (resp., $\partial T_i \cap \partial H$) . We also denote by $\partial_i X$ (resp., $\partial_i Y$) the boundary component of X (resp., Y) related to $\partial_i s$, $i = 1, 2$.
The surface F_1 is defined as X together with the annuli cut by ∂X from $\partial H \cap E(K_i)$, that we denote by O_i, $i = 1, 2$, with respect to $\partial_i X$. The surface F_2 is obtained from X and Y by gluing two copies of O_1 to $\partial_1 X$ and $\partial_1 Y$, pushing them slightly into H and H_i respectively, and identifying the boundary components $\partial_2 X$ and $\partial_2 Y$. In Figure 4 we have a schematic representation of F_1 and F_2.

FIGURE 4: A schematic diagram of surface F_1, in (a), and surface F_2, in (b).

To construct the surfaces F_g, for $g \geq 3$, we follow a general procedure as explained next. In H_i consider a copy of Y and an annulus A, around s_i, defined by $\partial N(s_i) - (\partial N(s_i) \cap \partial H_i)$. We denote by Z the surface obtained by identifying Y and A along the boundaries $\partial_1 Y$ and $\partial_1 A$. Let $n = g - 1$ and $A_1, \ldots, A_{n-2}$ be disjoint copies of A disjoint from Z. Consider also n disjoint copies of X in H, denoted by $X_1, \ldots, X_n$. Denote $\partial_1 X_j$ (resp., $\partial_2 X_j$) the boundary component of X_j around $\partial_1 s$ (resp., $\partial_2 s$). Similarly, we label the boundary components of A_j by $\partial_1 A_j$ and $\partial_2 A_j$. To construct F_g we start by attaching $\partial_2 X_n$ and $\partial_2 X_{n-1}$ to the two boundary components of Z respecting the order from $\partial_2 s$. If $g \geq 4$ we also attach $\partial_2 X_{n-2}, \ldots, \partial_2 X_1$ to $\partial_2 A_{n-2}, \ldots, \partial_2 A_1$, respectively, and $\partial_1 X_n, \ldots, \partial_1 X_3$ to $\partial_1 A_{n-2}, \ldots, \partial_1 A_1$, respectively. The surface F_g has two boundary components ($\partial_1 X_1$ and $\partial_1 X_2$) and Euler characteristic $-2g$, which means the genus of F_g is g. In Figure 5 we have a schematic representation of F_3 and F_4, and in Figure 6 a representation of the general construction of F_g.

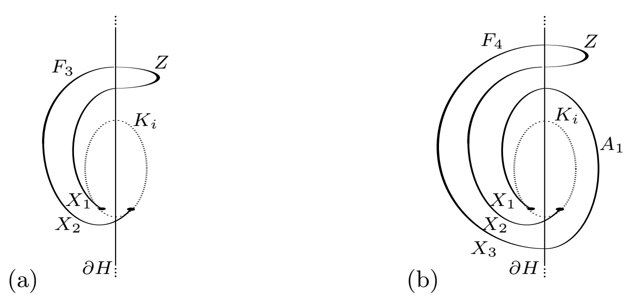


FIGURE 5: A schematic diagram of surface F_3, in (a), and surface F_4, in (b).

Lemma 5. *The surfaces F_1 and F_2 are essential in the exterior of the knot K_i, $i \in \mathbb{N}$.*

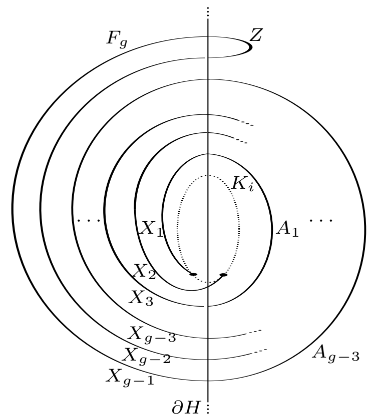


FIGURE 6: A schematic general representation of the surface F_g, for $g \geq 3$.

Proof. For $g = 1, 2$ we assume that F_g is not essential in $E(K_i)$ and prove this leads to a contradiction using innermost curve arguments. We denote generically by D a compressing or a boundary compressing disk for F_g in $E(K_i)$. In case D is a boundary compressing disk then $\partial D = a \cup b$ where a is an arc in $\partial E(K_i) - F_g$, with one end in each component of ∂F_g, and b is an arc in F_g. We also assume $|D \cap \partial H|$ to be minimal. Consequently, using an innermost curve argument, as in Lemma 1, we have that D does not intersect ∂H in simple closed curves.

Suppose $g = 1$. By a small isotopy of a neighborhood of ∂F_1 into H if necessary, we can assume that F_1 is in H. If D is a compressing disk for F_1 in $E(K_i)$ then $D \subset H$, as D cannot intersect ∂H in simple closed curves and ∂D is disjoint from ∂H. This is a contradiction to Lemma 1, which says ∂T is incompressible in $E_T(s)$ and in $E_H(T)$. Assume now D is a boundary compressing disk of F_1 in $E(K_i)$. If D is in T then we have a contradiction to Lemma 1(b) for s being essential in T. If D is not in T then, by using an innermost curve argument, we can assume that a intersects ∂H at two points and that $D \cap \partial H$ is an arc separating from D a disk O in H with boundary an arc in ∂H and an arc that we can assume in ∂T having ends in $\partial H \cap \partial T$. Hence, O contradicts Lemma 2(b). Therefore, we have that F_1 is essential in $E(K_i)$.

Suppose $g = 2$. By a small isotopy of a neighborhood of ∂F_2 we can assume that the component of $\partial F_2 \cap X$ is in H and that $\partial F_2 \cap Y$ is in H_i. Suppose D is a compressing disk of F_2 in $E(K_i)$. If D is disjoint from ∂H then D is a compressing disk for X or Y in $E(K_i)$, which is a contradiction to Lemma 1(a). Then, assume D intersects ∂H at a minimal collection of arcs. Consider an outermost arc α of $D \cap \partial H$ in D and let O be the respective outermost disk, with $O \cap F_2 = \beta$ an arc in X or in Y. Without loss of generality, suppose β is in X. If α or β does not co-bound a disk in ∂H or X, respectively, with $\partial_2 X$ we have a contradiction to Lemma 2(b). Otherwise, ∂O bounds a disk O' in $\partial H \cup \partial T$ and using the ball bounded by $O \cup O'$ we can isotope D reducing $|D \cap \partial H|$ which is a contradiction to its minimality.
Suppose now that D is a boundary compressing disk for F_2 in $E(K_i)$. As the two

components of ∂F_2 are in opposite sides of ∂H by an innermost curve argument we can prove that a intersects ∂H at a single point. Hence, $D \cap \partial H$ is an arc with one end in a and one end in b and, possibly, arcs with both ends in b, as in Figure 7.

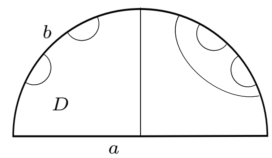


FIGURE 7: Arcs of $D \cap \partial H$ when D is a boundary compressing disk of F_2.

If D is in $T \cup T_i$ then the arcs of $D \cap \partial H$ with both ends in b are in the annulus O_2. Hence, each arc of $D \cap \partial H$ with ends in b co-bounds a disk in O_2 with $\partial_2 X$. Consider an outermost of such arcs in O_2, and the respective outermost disk O. By cutting and pasting D along O we contradict Lemma 1(b) or we can reduce $|D \cap \partial H|$ contradicting its minimality. Therefore, in this case, $D \cap \partial H$ is an arc with an end in a. This arc cuts D into two disks, one in T and the other in T_i, contradicting Lemma 1(b). If D is in $E_H(T) \cup E_{H_i}(T_i)$ we consider an outermost arc α between the arcs of $D \cap \partial H$ in D and the respective outermost disk, also denoted by O. If the arc β, that is $\partial O \cap F_2$, co-bounds a disk in F_2 with $F_2 \cap \partial H$, using an argument as before, we can reduce $|D \cap \partial H|$ contradicting its minimality. Otherwise, the disk O is in contradiction to Lemma 2(b). Hence, $D \cap \partial H$ is only an arc with an end in a, and the disk separated by this arc in D is also in contradiction to Lemma 2(b). Consequently, F_2 is essential in $E(K_i)$. □

To prove the surfaces F_g, $g \geq 3$, are essential in the complement of the knots K_i, we use branched surface theory. First, we start by revising the definitions and result relevant to this paper from Oertel's work in [17], and also Floyd and Oertel's work in [6].

A *branched surface* B with generic branched locus is a compact space locally modeled on Figure 8(a). Hence, a union of finitely many compact smooth surfaces in a 3-manifold M, glued together to form a compact subspace of M respecting the local model, is a branched surface. We denote by $N = N(B)$ a fibered regular neighborhood of B (embedded) in M, locally modelled on Figure 8(b). The boundary of N is the union of three compact surfaces $\partial_h N$, $\partial_v N$ and $\partial M \cap \partial N$, where a fiber of N meets $\partial_h N$ transversely at its endpoints and either is disjoint from $\partial_v N$ or meets $\partial_v N$ in a closed interval in its interior. We say that a surface S is *carried* by B if it can be isotoped into N so that it is transverse to the fibers. Furthermore, S is carried by B with *positive weights* if S intersects every fiber of N. If we associate a weight $w_i \geq 0$ to each component on the complement of the branch locus in B we say that we have an *invariant measure* provided that the weights satisfy *branch equations* as in Figure 8(c). Given an invariant measure on B we can define a surface carried by B, with respect to the number of intersections between the fibers and the surface. We also note that if all weights are positive then the surface carried can be isotoped to be transverse to all fibers of N, and hence is carried with positive weights by B.

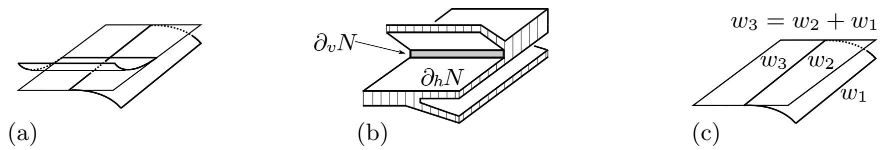


FIGURE 8: Local model for a branched surface, in (a), and its regular neighborhood, in (b).

A *disc of contact* is a disc D embedded in N transverse to fibers and with $\partial D \subset \partial_v N$. A *half-disc of contact* is a disc D embedded in N transverse to fibers with ∂D being the union of an arc in $\partial M \cap \partial N$ and an arc in $\partial_v N$. A *monogon* in the closure of $M - N$ is a disc D with $D \cap N = \partial D$ which intersects $\partial_v N$ in a single fiber. (See Figure 9.)

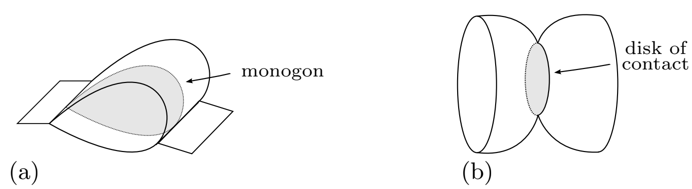


FIGURE 9: Illustration of a monogon and a disk of contact on a branched surface.

A branched surface embedded B in M is said *incompressible* if it satisfies the following three properties:

(i) B has no disk of contact or half-disks of contact;
(ii) $\partial_h N$ is incompressible and boundary incompressible in the closure of $M - N$, where a boundary compressing disk is assumed to have boundary defined by an arc in ∂M and an arc in $\partial_h N$;
(iii) There are no monogons in the closure of $M - N$.

The following theorem proved by Floyd and Oertel in [6] let us infer if a surface carried by a branched surface is essential.

Theorem 2 (Floyd and Oertel, [6]). *A surface carried with positive weights by an incompressible branched surface is essential.*

We now prove that the remaining surfaces F_g are essential.

Lemma 6. *The surfaces F_g, $g \geq 3$, are essential in the exterior of the knot K_i, $i \in \mathbb{N}$.*

Proof. To prove the statement of this theorem, we construct a branched surface that carries F_g, $g \geq 3$, and show that it is incompressible in the exterior of K_i, $i \in \mathbb{N}$.

Let us consider the puntured torus X in H, the annulus A in H_i, the annulus O_1 and the punctured torus Y in H_i. Note that the boundaries of $\partial_i X$, $\partial_i Y$ and $\partial_i A$, for $i = 1, 2$, are the same. Consider the union $X \cup A \cup Y$ and isotope $\partial_1 Y$, in this union, into the interior of $A \cap H_i$. Now we add O_1 to the previous union and denote the resulting space by B. We smooth the space B on the intersections of the surfaces X,

A, Y and O_1 as follows: the annulus O_1 in its intersection with $X \cup A$ is smoothed in the direction of X; the punctured torus Y on its boundary $\partial_1 Y$ is smoothed in the direction of $\partial_2 A$, and on its boundary $\partial_2 Y$ is smoothed in the direction of X. We keep denoting the resulting topological space by B. From the construction, the space B is a branched surface with sections denoted naturally by X, O_1, A', A and Y, as illustrated in Figure 3. We denote a regular neighborhood of B by $N(B)$.

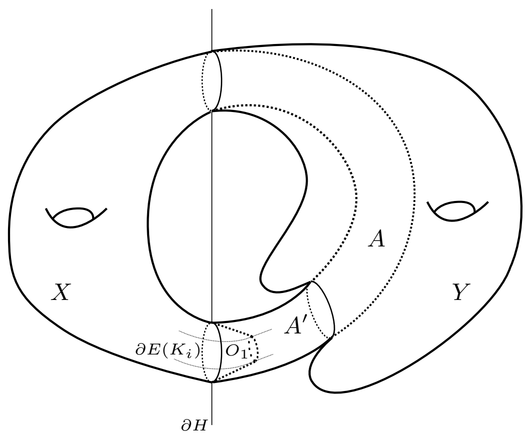


FIGURE 10: The branched surface B in $E(K_i)$.

Given the invariant measure on B defined by $w_X = g-1$, $w_{O_1} = 2$, $w_{A'} = g-3$, $w_A = g-2$ and $w_Y = 1$, the weights for the sections X, O_1, A', A and Y respectively, we have that, for each $g \geq 3$, the surface F_g is carried with positive weights by B. To prove that F_g, $g \geq 3$, is essential in the complement of $E(K_i)$ we show that B is an incompressible branched surface in $E(K_i)$ and use Theorem 2.

The space $N(B)$ decomposes $E(K_i)$ into three components: a component cut from $E(K_i)$ by $X \cup O_1 \cup A' \cup A$ that we denote E_X; a component cut from $E(K_i)$ by $Y \cup A$ that we denote by E_Y; a component cut from $E(K_i)$ by $Y \cup A' \cup X$ that we denote by E_B. Note that $\partial E_X \cap \partial N(B)$ is ambient isotopic to ∂T in $E(K_i)$. Hence, from Lemma 1(b), we have that $\partial_h N(B)$ is incompressible and boundary incompressible in E_X, and also that there are no monogons in E_X. Similarly, $\partial E_Y \cap \partial N(B)$ is ambient isotopic to ∂T_i in $E(K_i)$. As $\partial_2 Y$ corresponds to the only component of $\partial_v N(B)$ in E_Y, a monogon in E_Y corresponds to the arc s_i being trivial in T_i. Therefore, from Lemma 1, there are no monogons in E_Y, and $\partial_h N(B)$ is incompressible and boundary incompressible in E_Y. At last, we consider the component E_B, which corresponds to gluing $E_H(T)$ and $E_{H_i}(T_i)$ along their boundaries as before. Suppose there is a compressing disk D for ∂E_B in E_B. Note also that ∂E_B is a ambient homotopic to $X \cup Y$ identified along their boundaries. As in the proof of Lemma 5, we assume $|D \cap \partial H|$ to be minimal and that the intersection $D \cap \partial H$ contains no simple closed curves. If D is disjoint from ∂H then ∂D is a compressing disk for X in H or for Y in H_i, which contradicts Lemma 1(a). Then, D intersects ∂H in a collection of arcs, as in Figure 11.

Consider an outermost arc in D between the arcs of $D \cap \partial H$ and denote it by α. Let O be the outermost disk cut from D by α, and let $\partial O = \alpha \cup \beta$ where β is an arc in ∂E_B. Consequently, the disk O is in H or in H_i. If β co-bounds a disk in

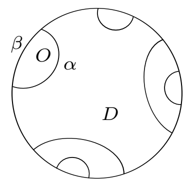


FIGURE 11: The disk D together with the arcs $D \cap \partial H$.

$\partial N(B)$ with an arc in ∂H we can reduce $|D \cap \partial H|$ and contradict its minimality. Then, O is a compressing disk of $\partial E_H(T)$ in $E_H(T)$. As ∂O has boundary defined by the union of an arc in ∂T and an arc in ∂H, we have a contradiction to Lemma 2(b). □

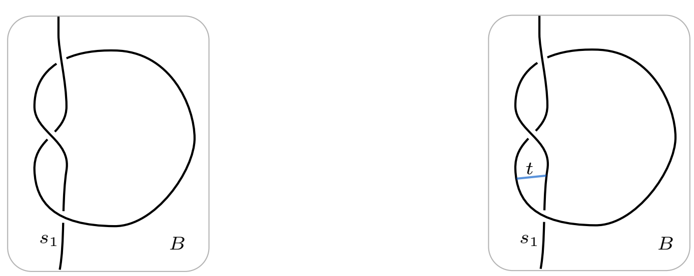
Proof of Theorem 1. From Lemma 5 and 6 we have that the surfaces F_g, $g \in \mathbb{N}$, are essential in the complements of the knots K_i, $i \in \mathbb{N}$. Together with Lemma 1, we obtain the statement of the theorem. □

The proof Corollary 1.1 now follows naturally.

Proof of Corollary 1.1. In Theorem 1 we proved that the knots K_i, $i \in \mathbb{N}$, are an infinite collection of prime knots with meridional essential surfaces in their complements for each positive genus and two boundary components. Hence, considering the knots K_i connected sum with some other knot, we have infinitely many knots with meridional essential surfaces of genus g and two boundary components for all $g \geq 0$. □

4. APPENDIX

In this appendix we give an example of a 2-string essential free tangle with both strings knotted.
For a string s in a ball B we can consider the knot obtained by capping off s along ∂B, that is by gluing to s an arc in ∂B along the respective boundaries. We denote this knot by $K(s)$. The string s is said to be *knotted* if the knot $K(s)$ is not trivial. Let s_1 be an arc in a ball B such that $K(s_1)$ is a trefoil, and consider also an unknotting tunnel t for $K(s_1)$, as in Figure 12.



(a) (b)

FIGURE 12: The string s_1 when capped off along ∂B is a trefoil knot with an unknotting tunnel t.

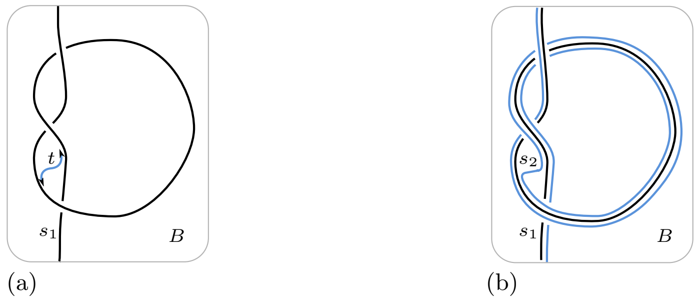


Figure 13: Construction of a 2-string essential free tangle, with both strings knotted, from s_1 and the unknotting tunnel t.

If we slide ∂t along s_1 into ∂B, as illustrated in Figure 13(a), we get a new string that we denote by s_2, as in Figure 13(b).
The knot $K(s_2)$ is the $(3,-4)$-torus knot, and hence knotted. The tangle $(B, s_1 \cup s_2)$ is free by construction. In fact, as t is an unknotting tunnel of $K(s_1)$, the complement of $N(s_1) \cup N(t)$ in B is a handlebody. Henceforth, by an ambient isotopy, the complement of $N(s_1) \cup N(s_2)$ is also a handlebody. As the tangle $(B, s_1 \cup s_2)$ is free and both strings are knotted then it is necessarily essential. Otherwise, the complement of $N(s_1) \cup N(s_2)$ in B is not a handlebody as it is obtained by gluing two non-trivial knot complements along a disk in their boundaries, which is a contradiction to the tangle $(B, s_1 \cup s_2)$ being free.

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