Orthogonal polynomials on systems of non-uniform lattices from compatibility conditions

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Abstract

We deduce difference equations in the matrix form for Laguerre-Hahn orthogonal polynomials on systems of non-uniform lattices, the so-called compatibility conditions, involving the transfer matrices. As a consequence, we obtain closed form expressions for the recurrence relation coefficients of the Laguerre-Hahn polynomials of class zero.

Key words: Orthogonal polynomials; Divided-difference operator; Non-uniform lattices; Laguerre-Hahn class.
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1 Motivation

Systems of non-uniform lattices and the corresponding divided difference operators have been studied from many points of view (see [1,2,6,13,15] and references therein). In general terms, the difference calculus on non-uniform lattices generalizes the calculus on lattices of lower complexity, such as the linear

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and $q-$uniform lattices. More detailed information on the hierarchy of lattices/operators can be found, e.g., in [11,16] and [19, Sections 2,3]. In this paper we consider a general divided-difference operator [11, Eq. (1.1)] having the basic property of leaving a polynomial of degree $n-1$ when applied to a polynomial of degree n. Under some specifications, [11, Eq. (1.1)] is related to the Askey-Wilson operator [1, Section 5]. Our purpose is the analysis of the so-called Laguerre-Hahn class of orthogonal polynomials on systems of non-uniform lattices, that is, families of orthogonal polynomials for which the formal Stieltjes function satisfies a Riccati difference equation with polynomial coefficients.

For the best of our knowledge, Laguerre-Hahn orthogonal polynomials on systems of non-uniform lattices were firstly studied by A. Magnus in [11]. More recent literature includes [3,4]. The Laguerre-Hahn class contains the well-known classical and semi-classical orthogonal polynomials [7,16,19], as well as some of their rational spectral transformations [20, Section 5]. Our goal is to derive properties of orthogonal polynomials and their recurrence relation coefficients from the knowledge of the polynomial coefficients involved in the Riccati difference equation. Such a topic has been extensively studied in the literature, for several types of difference operators and families of orthogonal polynomials. For instance, a similar program was undertaken in [5], within the setting of continuous derivative, for the very classical Jacobi polynomials, and in [19], within the class of semi-classical orthogonal polynomials on systems of non-uniform lattices.

The main results of the present paper are given in Sections 3 and 4, where we deduce difference equations in the matrix form for the orthogonal polynomials (see Theorem 1), and the so-called compatibility conditions involving the transfer matrices (see Corollary 1). Under some constraints on the degrees of the polynomial coefficients of the Riccati equation, we obtain closed form expressions for the recurrence relation coefficients of the Laguerre-Hahn polynomials (see Theorem 2).

The remainder of the paper is organized as follows. In Section 2 we give the definitions and state the basic results to be used in the forthcoming sections: we follow the formalism and geometric interpretation from [11,13], and give some properties of the divided-difference operators, the related non-uniform lattices, and the corresponding Laguerre-Hahn families of orthogonal polynomials. In Section 3 we deduce the difference equations for the orthogonal polynomials as well as the compatibility conditions for the transfer matrices. In Section 4 we deduce the formulae for the recurrence relation coefficients of the orthogonal polynomials. Specializations to the Askey-Wilson operator, together with illustrative examples, are given in Sub-section 4.2.

2 Preliminary results

2.1 The operator $\mathbb{D}$, the related non-uniform lattice, and fundamental quantities

We consider the divided difference operator $\mathbb{D}$ given in [11, Eq.(1.1)], with the property that $\mathbb{D}$ leaves a polynomial of degree $n-1$ when applied to a polynomial of degree n. The operator $\mathbb{D}$, defined on the space of arbitrary functions, is given by

$$(\mathbb{D}f)(x) = \frac{f(y_2(x)) - f(y_1(x))}{y_2(x) - y_1(x)}\,, \tag{1}$$

where y_1 and y_2 are functions that satisfy

$$y_1(x) + y_2(x) = \text{polynomial of degree } 1\,, \tag{2}$$
$$(y_1(x))^2 + y_1(x)y_2(x) + (y_2(x))^2 = \text{polynomial of degree } 2\,, \tag{3}$$

the later condition being equivalent to $y_1(x)y_2(x) =$ polynomial of degree less or equal than 2. Conditions (2)–(3) define y_1 and y_2 as the two y-roots of a quadratic equation

$$\hat{a}y^2 + 2\hat{b}xy + \hat{c}x^2 + 2\hat{d}y + 2\hat{e}x + \hat{f} = 0\,, \quad \hat{a} \neq 0\,. \tag{4}$$

Identities involving y_1 and y_2, following from the fact that y_1, y_2 are the y-roots of (4):

$$y_1(x) + y_2(x) = -2(\hat{b}x + \hat{d})/\hat{a}\,, \tag{5}$$
$$y_1(x)y_2(x) = (\hat{c}x^2 + 2\hat{e}x + \hat{f})/\hat{a}\,, \tag{6}$$
$$(y_2(x) - y_1(x))^2 = 4\left((\hat{b}^2 - \hat{a}\hat{c})x^2 + 2(\hat{b}\hat{d} - \hat{a}\hat{e})x + \hat{d}^2 - \hat{a}\hat{f}\right)/\hat{a}^2\,, \tag{7}$$
$$y_1(x) = p(x) - \sqrt{r(x)}\,, \qquad y_2(x) = p(x) + \sqrt{r(x)}\,, \tag{8}$$

with p, r polynomials given by

$$p(x) = -\frac{\hat{b}x + \hat{d}}{\hat{a}}\,, \quad r(x) = \frac{\lambda}{\hat{a}^2}\left(x + \frac{\hat{b}\hat{d} - \hat{a}\hat{e}}{\lambda}\right)^2 + \frac{\tau}{\hat{a}\lambda} \quad \text{if } \lambda \neq 0\,, \tag{9}$$

where $\lambda = \hat{b}^2 - \hat{a}\hat{c}$, $\tau = \left((\hat{b}^2 - \hat{a}\hat{c})(\hat{d}^2 - \hat{a}\hat{f}) - (\hat{b}\hat{d} - \hat{a}\hat{e})^2\right)/\hat{a}$.

There are four primary classes of divided difference operators (1) and related lattices. Such a classification is done according to the two parameters λ and τ defined above, assuming $\hat{a}\hat{c} \neq 0$:

(i) $\lambda = \tau = 0$ - the linear lattice, related to the forward difference operator [17, Chapter 2, Section 12].
(ii) $\lambda \neq 0,\ \tau = 0$ - the q-linear lattice, related to the q-difference operator [8].
(iii) $\lambda = 0, \tau \neq 0$ - the quadratic lattice, related to the Wilson operator [1].
(iv) $\lambda\tau \neq 0$ - the q-quadratic lattice, related to the Askey-Wilson operator [1].

In the present paper we will consider the general case $\lambda\tau \neq 0$, and we shall operate with the divided difference operator $\mathbb{D}$ given in its general form (1), with y_1, y_2 given in (8), defined in terms of the polynomials p and r in (9). Throughout the paper we shall use the notation $\Delta_y = y_2 - y_1$. From (8), it follows that

$$\Delta_y = 2\sqrt{r}\,. \tag{10}$$

By defining the operators $\mathbb{E}_1$ and $\mathbb{E}_2$ (see [11]), acting on arbitrary functions f as

$$(\mathbb{E}_1 f)(x) = f(y_1(x))\,, \quad (\mathbb{E}_2 f)(x) = f(y_2(x))\,,$$

then the formula (1) is given by

$$(\mathbb{D}f)(x) = \frac{(\mathbb{E}_2 f)(x) - (\mathbb{E}_1 f)(x)}{(\mathbb{E}_2 x)(x) - (\mathbb{E}_1 x)(x)}\,.$$

The companion operator of $\mathbb{D}$ is defined as (see [11])

$$(\mathbb{M}f)(x) = \frac{(\mathbb{E}_1 f)(x) + (\mathbb{E}_2 f)(x)}{2}\,. \tag{11}$$

Some useful identities involving $\mathbb{D}$ and $\mathbb{M}$ are listed below (see [11]):

$$\mathbb{D}(gf) = \mathbb{D}g\,\mathbb{M}f + \mathbb{M}g\,\mathbb{D}f\,, \tag{12}$$

$$\mathbb{M}(gf) = \mathbb{M}g\,\mathbb{M}f + \frac{\Delta_y^2}{4}\,\mathbb{D}g\,\mathbb{D}f\,, \tag{13}$$

$$\mathbb{D}(1/f) = \frac{-\mathbb{D}f}{\mathbb{E}_1 f\,\mathbb{E}_2 f}\,, \tag{14}$$

$$\mathbb{M}(1/f) = \frac{\mathbb{M}f}{\mathbb{E}_1 f\,\mathbb{E}_2 f}\,. \tag{15}$$

Eq. (12) has the equivalent forms

$$\mathbb{D}(gf) = \mathbb{D}g\,\mathbb{E}_1 f + \mathbb{D}f\,\mathbb{E}_2 g\,, \quad \mathbb{D}(gf) = \mathbb{D}g\,\mathbb{E}_2 f + \mathbb{D}f\,\mathbb{E}_1 g\,. \tag{16}$$

Note that $\mathbb{M}f$ is a polynomial whenever f is a polynomial. Furthermore, if $\deg(f) = n$, then $\deg(\mathbb{M}f) = n$ [4, Lemma 1]. Indeed, let us emphasize that, throughout the text, unless stated in contrary, by a polynomial we mean a polynomial in the variable x.

Finally, we recall that for the $q-$quadratic class of lattices there is a parametric representation of the conic (4), $x = x(s), y = y(s)$, such that [13, pp. 254–255]

$$x(s) = x_c + \xi\sqrt{\hat{a}}\,(q^s + q^{-s})\,, \quad y(s) = y_c + \xi\sqrt{\hat{c}}\,(q^{s-1/2} + q^{-s+1/2})\,, \tag{17}$$

$x_c = (\hat{a}\hat{e} - \hat{b}\hat{d})/\lambda\,,\ y_c = (\hat{c}\hat{d} - \hat{b}\hat{e})/\lambda\,,\ \xi^2 = \tilde{f}/(4\lambda)\,,\ \tilde{f} = \hat{f} - \hat{a}y_c^2 - 2\hat{b}x_c y_c - \hat{c}x_c^2\,,$ and q defined through

$$q + q^{-1} = \frac{4\hat{b}^2}{\hat{a}\hat{c}} - 2\,. \tag{18}$$

In this case we have $y_1(x(s)) = y(s)$, $y_2(x(s)) = y(s+1)$. Thus, in the account of (8),

$$p(x(s)) - \sqrt{r(x(s))} = y(s)\,, \quad p(x(s)) + \sqrt{r(x(s))} = y(s+1)\,, \tag{19}$$

and (1) is given as

$$\mathbb{D}f(x(s)) = \frac{f(y(s+1)) - f(y(s))}{y(s+1) - y(s)}\,.$$

If $\hat{a} = \hat{c}, \hat{d} = \hat{e} = 0$, then $\tilde{f} = \hat{f}$, and we get

$$x(s) = \xi\sqrt{\hat{a}}(q^s + q^{-s})\,, \ \ y(s) = \xi\sqrt{\hat{a}}(q^{s-1/2} + q^{-s+1/2})\,, \tag{20}$$

hence,

$$y_1(x) = x(s - 1/2)\,, \quad y_2(x) = x(s + 1/2)\,, \tag{21}$$

thus (1) is given as

$$\mathbb{D}f(x(s)) = \frac{f(x(s+1/2)) - f(x(s-1/2))}{x(s+1/2) - x(s-1/2)}\,. \tag{22}$$

The fundamental quantities to be used in the main results of the paper depend on the lattice, namely, on the polynomials p, r as well as on the parameter q defined above. Let us note that, in the account of (19), it follows that

$$y(s+1) + y(s) = 2p(x(s))\,, \quad (y(s+1) - y(s))^2 = 4r(x(s))\,.$$

By writing

$$p(x) = p_1 x + p_0\,, \quad r(x) = r_2 x^2 + r_1 x + r_0\,,$$

the coefficients p_1, p_0, r_2, r_1, r_0 may be obtained through the linear systems

$$\begin{bmatrix} 2x(0) & 2 \\ 2x(1) & 2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_0 \end{bmatrix} = \begin{bmatrix} y(0) + y(1) \\ y(2) + y(1) \end{bmatrix}, \quad \begin{bmatrix} (x(0))^2 & x(0) & 1 \\ (x(1))^2 & x(1) & 1 \\ (x(2))^2 & x(2) & 1 \end{bmatrix} \begin{bmatrix} r_2 \\ r_1 \\ r_0 \end{bmatrix} = \begin{bmatrix} (y(1) - y(0))^2/4 \\ (y(2) - y(1))^2/4 \\ (y(3) - y(2))^2/4 \end{bmatrix}.$$

2.2 *Laguerre-Hahn orthogonal polynomials on non-uniform lattices*

We shall consider formal orthogonal polynomials related to a (formal) Stieltjes function defined by

$$S(x) = \sum_{n=0}^{+\infty} u_n x^{-n-1} \tag{23}$$

where $(u_n)_{n\geq 0}$, the sequence of moments, is such that $\det \left[u_{i+j}\right]_{i,j=0}^{n} \neq 0$, $n \geq 0$, and, without loss of generality, $u_0 = 1$. Throughout the paper we consider each P_n monic, and we will denote the sequence of monic orthogonal polynomials $\{P_n\}_{n\geq 0}$ by SMOP.

Monic orthogonal polynomials satisfy a three-term recurrence relation [18]

$$P_{n+1}(x) = (x - \beta_n)P_n(x) - \gamma_n P_{n-1}(x)\,, \quad n = 0, 1, 2, \dots\,, \tag{24}$$

with $P_{-1}(x) = 0,\ P_0(x) = 1$, and $\gamma_n \neq 0,\ n \geq 1,\ \gamma_0 = 1$.

The sequence $\{P_n^{(1)}\}_{n\geq 0}$, also referred to as the sequence of associated polynomials of the first kind, satisfies the three-term recurrence relation

$$P_n^{(1)}(x) = (x - \beta_n)P_{n-1}^{(1)}(x) - \gamma_n P_{n-2}^{(1)}(x)\,, \quad n = 1, 2, \dots \tag{25}$$

with $P_{-1}^{(1)}(x) = 0,\ P_0^{(1)}(x) = 1$.

In the framework of Hermite-Padé Approximation (see [12]), the polynomials P_n are the diagonal Padé denominators of (23), and the $P_{n-1}^{(1)}$'s are the numerator polynomial, thus, also determined through

$$S(x) - P_{n-1}^{(1)}(x)/P_n(x) = \mathcal{O}(x^{-2n-1})\,, \quad x \to \infty\,.$$

Definition 1 ([11]) *A SMOP $\{P_n\}_{n\geq 0}$ related to a Stieltjes function, S, is said to be Laguerre-Hahn if S satisfies a Riccati equation*

$$A(x)\mathbb{D}S(x) = B(x)\mathbb{E}_1 S(x)\mathbb{E}_2 S(x) + C(x)\mathbb{M}S(x) + D(x)\,, \tag{26}$$

where $A(x), B(x), C(x), D(x)$ are polynomials in x, $A \neq 0$.
If $B \equiv 0$, then $\{P_n\}_{n\geq 0}$ is said to be semi-classical.

Note that [11]

$$\deg(A) \leq m + 2\,,\ \deg(B) \leq m\,,\ \deg(C) \leq m + 1\,,\ \deg(D) \leq m\,, \tag{27}$$

where m is some nonnegative integer. When $B \equiv 0$ and $m = 0$ we get the so-called classical polynomials [7,15,16].

In the sequel we will use the following matrices:

$$\mathcal{P}_n = \begin{bmatrix} P_{n+1} & P_n^{(1)} \\ P_n & P_{n-1}^{(1)} \end{bmatrix}, \quad n \geq 0 . \tag{28}$$

In the account of (24) and (25), $\mathcal{P}_n$ satisfy the difference equation

$$\mathcal{P}_n = \mathcal{A}_n \mathcal{P}_{n-1} , \quad \mathcal{A}_n = \begin{bmatrix} x-\beta_n & -\gamma_n \\ 1 & 0 \end{bmatrix} , \quad n \geq 1 , \tag{29}$$

with initial condition $\mathcal{P}_0 = \begin{bmatrix} x-\beta_0 & 1 \\ 1 & 0 \end{bmatrix}$. The matrix $\mathcal{A}_n$ is called the transfer matrix.

3 Difference equations and compatibility conditions

Laguerre-Hahn polynomials related to Stieltjes functions such that (26) holds satisfy the difference equations, for all $n \geq 0$, [4, Th. 1]

$$\begin{cases} A\mathbb{D}P_{n+1} = (l_n + \Delta_y \pi_n)\mathbb{E}_1 P_{n+1} - C/2\,\mathbb{E}_2 P_{n+1} - B\mathbb{E}_2 P_n^{(1)} + \Theta_n \mathbb{E}_1 P_n , \\ A\mathbb{D}P_n^{(1)} = (l_n + \Delta_y \pi_n)\mathbb{E}_1 P_n^{(1)} + C/2\,\mathbb{E}_2 P_n^{(1)} + D\mathbb{E}_2 P_{n+1} + \Theta_n \mathbb{E}_1 P_{n-1}^{(1)} , \end{cases} \tag{30}$$

as well as [4, Th. 2]

$$A_{n+1}\mathbb{D}P_{n+1} = (l_n - C/2)\mathbb{M}P_{n+1} - B\mathbb{M}P_n^{(1)} + \Theta_n \mathbb{M}P_n , \tag{31}$$

$$A_{n+1}\mathbb{D}P_n^{(1)} = (l_n + C/2)\mathbb{M}P_n^{(1)} + D\mathbb{M}P_{n+1} + \Theta_n \mathbb{M}P_{n-1}^{(1)} \tag{32}$$

with

$$A_{n+1} = A + \frac{\Delta_y^2}{2}\,\pi_n , \tag{33}$$

and l_n, π_n, Θ_n polynomials such that

$$\begin{gathered} \deg(\Theta_n) \leq \max\{\deg(A)-2, \deg(B)-2, \deg(C)-1\} , \\ \deg(l_n) \leq \max\{\deg(A)-1, \deg(B)-1, \deg(C)\} , \\ \deg(\pi_n) \leq \max\{\deg(B)-2, \deg(C)-1\} , \end{gathered}$$

satisfying the conditions

$$\pi_{-1} = 0,\ \pi_0 = -D/2, \tag{34}$$

$$\Theta_{-1} = D,\ \Theta_0 = A - \frac{\Delta_y^2}{4} D - (l_0 - C/2)\mathbb{M}(x-\beta_0) + B, \tag{35}$$

$$l_{-1} = C/2,\ l_0 = -\mathbb{M}(x-\beta_0)D - C/2\,. \tag{36}$$

Remark 1 *Eqs. (30) are equivalent to*

$$\begin{cases} A\mathbb{D}P_{n+1} = (l_n - \Delta_y \pi_n)\mathbb{E}_2 P_{n+1} - C/2\,\mathbb{E}_1 P_{n+1} - B\mathbb{E}_1 P_n^{(1)} + \Theta_n \mathbb{E}_2 P_n\,, \\ A\mathbb{D}P_n^{(1)} = (l_n - \Delta_y \pi_n)\mathbb{E}_2 P_n^{(1)} + C/2\,\mathbb{E}_1 P_n^{(1)} + D\mathbb{E}_1 P_{n+1} + \Theta_n \mathbb{E}_2 P_{n-1}^{(1)}\,, \end{cases} \tag{37}$$

for all $n \geq 0$.

Taking into account the notations above, we obtain the result that follows.

Theorem 1 *Let* S *be a Stieltjes function satisfying (26), and let* $\{\mathcal{P}_n\}_{n\geq 0}$*, be the corresponding sequence given by (28). The following relations hold, for all* $n \geq 0$*:*

$$A\,\mathbb{D}\mathcal{P}_n = \mathcal{B}_{1,n}\,\mathbb{E}_1\mathcal{P}_n - \mathbb{E}_2\mathcal{P}_n\,\mathcal{C}\,, \tag{38}$$

$$A\,\mathbb{D}\mathcal{P}_n = \mathcal{B}_{2,n}\,\mathbb{E}_2\mathcal{P}_n - \mathbb{E}_1\mathcal{P}_n\,\mathcal{C}\,, \tag{39}$$

with the matrices $\mathcal{B}_{j,n}$*,* $j = 1, 2$*, and* $\mathcal{C}$ *given by*

$$\mathcal{B}_{j,n} = \begin{bmatrix} l_n + (-1)^{j+1}\Delta_y \pi_n & \Theta_n \\ -\frac{\Theta_{n-1}}{\gamma_n} & l_{n-1} + (-1)^{j+1}\Delta_y \pi_{n-1} + \frac{\Theta_{n-1}}{\gamma_n}\,\mathbb{E}_j(x-\beta_n) \end{bmatrix},$$

$$\mathcal{C} = \begin{bmatrix} C/2 & -D \\ B & -C/2 \end{bmatrix},$$

where the l_n *and* Θ_n*'s are the polynomials in (30).*

PROOF. Let us deduce (38). Taking the relations (30) with $n-1$, we have

$$A\mathbb{D}P_n = (l_{n-1} + \Delta_y \pi_{n-1})\mathbb{E}_1 P_n - C/2\,\mathbb{E}_2 P_n - B\mathbb{E}_2 P_{n-1}^{(1)} + \Theta_{n-1}\mathbb{E}_1 P_{n-1}\,, \quad n \geq 1\,.$$

Using the three-term recurrence relation, $P_{n-1} = \frac{(x-\beta_n)}{\gamma_n} P_n - \frac{P_{n+1}}{\gamma_n}$, in the above equation, we get

$$A\mathbb{D}P_n = -\frac{\Theta_{n-1}}{\gamma_n}\mathbb{E}_1 P_{n+1} + \left(l_{n-1} + \Delta_y \pi_{n-1} + \frac{\Theta_{n-1}}{\gamma_n}\mathbb{E}_1(x-\beta_n) \right)\mathbb{E}_1 P_n - C/2\,\mathbb{E}_2 P_n - B\mathbb{E}_2 P_{n-1}^{(1)}\,. \tag{40}$$

In the same manner we get

$$A\mathbb{D}P_{n-1}^{(1)} = -\frac{\Theta_{n-1}}{\gamma_n}\mathbb{E}_1 P_n^{(1)} + \left(l_{n-1} + \Delta_y \pi_{n-1} + \frac{\Theta_{n-1}}{\gamma_n}\mathbb{E}_1(x-\beta_n)\right)\mathbb{E}_1 P_{n-1}^{(1)} + C/2\,\mathbb{E}_2 P_{n-1}^{(1)} + D\mathbb{E}_2 P_n\,. \quad (41)$$

Now we write (30), (40), and (41), in the matrix form, thus getting (38), which holds for all $n \geq 1$. From the initial conditions (34)–(36) there follows that (38) also holds for $n = 0$.

Starting with (37) and proceeding in the same way as above we get (39).

Corollary 1 *The following compatibility conditions hold for the transfer matrices $\mathcal{A}_n$, for all $n \geq 1$:*

$$A\,\mathbb{D}\mathcal{A}_n = \mathcal{B}_{1,n}\,\mathbb{E}_1\mathcal{A}_n - \mathbb{E}_2\mathcal{A}_n\,\mathcal{B}_{1,n-1}\,, \quad (42)$$

$$A\,\mathbb{D}\mathcal{A}_n = \mathcal{B}_{2,n}\,\mathbb{E}_2\mathcal{A}_n - \mathbb{E}_1\mathcal{A}_n\,\mathcal{B}_{2,n-1}\,. \quad (43)$$

PROOF. Let us deduce (42). From (29) we have $\mathbb{D}\mathcal{P}_n = \mathbb{D}(\mathcal{A}_n\mathcal{P}_{n-1})$. Thus, in the account of (16), we get

$$\mathbb{D}\mathcal{P}_n = \mathbb{D}\mathcal{A}_n\,\mathbb{E}_1\mathcal{P}_{n-1} + \mathbb{E}_2\mathcal{A}_n\,\mathbb{D}\mathcal{P}_{n-1}\,.$$

Using the equality above in (38) we get

$$A\mathbb{D}\mathcal{A}_n\,\mathbb{E}_1\mathcal{P}_{n-1} + \mathbb{E}_2\mathcal{A}_n\,(\mathcal{B}_{1,n-1}\mathbb{E}_1\mathcal{P}_{n-1} - \mathbb{E}_2\mathcal{P}_{n-1}\mathcal{C}) = \mathcal{B}_{1,n}\,\mathbb{E}_1\mathcal{P}_n - \mathbb{E}_2\mathcal{P}_n\,\mathcal{C}\,. \quad (44)$$

The use of (29), $\mathcal{P}_n = \mathcal{A}_n\mathcal{P}_{n-1}$, in the above equation yields, after cancelations,

$$A\mathbb{D}\mathcal{A}_n\,\mathbb{E}_1\mathcal{P}_{n-1} + \mathbb{E}_2\mathcal{A}_n\mathcal{B}_{1,n-1}\mathbb{E}_1\mathcal{P}_{n-1} = \mathcal{B}_{1,n}\,\mathbb{E}_1\mathcal{A}_n\mathbb{E}_1\mathcal{P}_{n-1}\,.$$

Note that [18] $P_n^{(1)}P_n - P_{n+1}P_{n-1}^{(1)} = \prod_{k=0}^{n}\gamma_k,\ n \geq 0$. Thus, $\det(\mathbb{E}_1\mathcal{P}_{n-1}) = -\prod_{k=0}^{n-1}\gamma_k$ and, consequently, as $\gamma_n \neq 0,\ n \geq 0$, $\mathbb{E}_1\mathcal{P}_{n-1}$ is invertible. Therefore, we get

$$A\mathbb{D}\mathcal{A}_n + \mathbb{E}_2\mathcal{A}_n\mathcal{B}_{1,n-1} = \mathcal{B}_{1,n}\,\mathbb{E}_1\mathcal{A}_n\,,$$

thus obtaining (42).

Eq. (43) is deduced in a similar way.

As a consequence of the compatibility conditions (42)–(43), we obtain the following relations for the polynomials π_n, l_n, Θ_n.

Corollary 2 *The polynomials π_n, l_n, Θ_n satisfy, for all $n \geq 0$, the following relations:*

$$\pi_{n+1} + \pi_n = -\frac{\Theta_n}{2\gamma_{n+1}} - \sum_{k=0}^{n} \frac{\Theta_{k-1}}{\gamma_k}\,, \tag{45}$$

$$l_{n+1} + l_n + \mathbb{M}(x - \beta_{n+1})\frac{\Theta_n}{\gamma_{n+1}} = 0\,, \tag{46}$$

$$-A + \mathbb{M}(x - \beta_{n+1})(l_{n+1} - l_n) - \frac{\Delta_y^2}{2}(\pi_{n+1} + \pi_n) + \Theta_{n+1} = \frac{\gamma_{n+1}}{\gamma_n}\Theta_{n-1}\,, \tag{47}$$

$$l_{n+1} + l_n = 2\mathbb{M}(x - \beta_{n+1})(\pi_{n+1} - \pi_n)\,, \tag{48}$$

together with the initial conditions (34)–(36).

PROOF. The equation (42) has two non-trivial entries, on positions $(1,1), (1,2)$. Respectively, for all $n \geq 1$, we have

$$A = (l_n + \Delta_y\,\pi_n)\mathbb{E}_1(x - \beta_n) + \Theta_n - (l_{n-1} + \Delta_y\,\pi_{n-1})\mathbb{E}_2(x - \beta_n) - \gamma_n\frac{\Theta_{n-2}}{\gamma_{n-1}} \tag{49}$$

$$0 = -(l_n + \Delta_y\,\pi_n)\gamma_n - \mathbb{E}_2(x - \beta_n)\Theta_{n-1} + \gamma_n\Big(l_{n-2} + \Delta_y\,\pi_{n-2} + \frac{\Theta_{n-2}}{\gamma_{n-1}}\mathbb{E}_1(x - \beta_{n-1})\Big)\,. \tag{50}$$

Recall that $\Delta_y = \mathbb{E}_2 x - \mathbb{E}_1 x$. Thus, after some computations, from (49) we get

$$A = \mathbb{M}(x - \beta_n)(l_n - l_{n-1}) + \Theta_n - \gamma_n\frac{\Theta_{n-2}}{\gamma_{n-1}} - \frac{\Delta_y^2}{2}(\pi_n + \pi_{n-1}) + \Delta_y\left(-\frac{1}{2}(l_n + l_{n-1}) + \mathbb{M}(x - \beta_n)(\pi_n - \pi_{n-1})\right)\,. \tag{51}$$

As the left-hand side of (51) is a polynomial, the following equations must hold, for all $n \geq 1$:

$$\mathbb{M}(x - \beta_n)(l_n - l_{n-1}) + \Theta_n - \gamma_n\frac{\Theta_{n-2}}{\gamma_{n-1}} - \frac{\Delta_y^2}{2}(\pi_n + \pi_{n-1}) = A\,, \tag{52}$$

$$-\frac{1}{2}(l_n + l_{n-1}) + \mathbb{M}(x - \beta_n)(\pi_n - \pi_{n-1}) = 0\,. \tag{53}$$

Hence, we get (47) and (48) for all $n \geq 0$.

Eqs. (45) and (46) are obtained from Eq. (50), using similar computations as above.

Remark 2 *Eqs. (45)–(47) also appear in [4]. There, they were deduced in a different way than above.*

Remark 3 *From (46) and (48) it follows that*

$$\pi_{n+1} - \pi_n = -\frac{\Theta_n}{2\gamma_{n+1}}\,, \tag{54}$$

which, when summed with (45), yields

$$\pi_{n+1} = -\frac{1}{2}\sum_{k=0}^{n+1}\frac{\Theta_{k-1}}{\gamma_k}\,, \quad n \geq 0\,. \tag{55}$$

Note that, in the account of (33), Eq. (47) can be written in the equivalent form

$$A - A_{n+2} - A_{n+1} + \mathbb{M}(x - \beta_{n+1})(l_{n+1} - l_n) + \Theta_{n+1} = \frac{\gamma_{n+1}}{\gamma_n}\Theta_{n-1}\,. \tag{56}$$

Eqs. (45)–(48) and (55)–(56) will play a fundamental role in the sequel.

4 Laguerre-Hahn orthogonal polynomials from compatibility conditions

4.1 Laguerre-Hahn orthogonal polynomials of class $m = 0$ in (27)

We consider the Laguerre-Hahn families of the so-called class zero, that is, $m = 0$ in (27). Hence, we consider SMOP's $\{P_n\}_{n\geq 0}$ whose Stietjes function satisfies $A(x)\mathbb{D}S(x) = B(x)\mathbb{E}_1S(x)\mathbb{E}_2S(x) + C(x)\mathbb{M}S(x) + D(x)$ with

$$\deg(A) \leq 2\,,\ \deg(B) = 0\,,\ \deg(C) \leq 1\,,\ \deg(D) = 0\,. \tag{57}$$

Recall the difference equations (31)–(32),

$$\begin{aligned} A_{n+1}\mathbb{D}P_{n+1} &= (l_n - C/2)\mathbb{M}P_{n+1} - B\mathbb{M}P_n^{(1)} + \Theta_n\mathbb{M}P_n\,, \\ A_{n+1}\mathbb{D}P_n^{(1)} &= (l_n + C/2)\mathbb{M}P_n^{(1)} + D\mathbb{M}P_{n+1} + \Theta_n\mathbb{M}P_{n-1}^{(1)}\,. \end{aligned}$$

As we are taking $m = 0$ in (27), then $\deg(l_n) = 1, \deg(\pi_n) = \deg(\Theta_n) = 0$.

We will use the following notations:

$$\begin{aligned} &A(x) = a_2x^2 + a_1x + a_0\,,\ B = b_0\,,\ C(x) = c_1x + c_0\,,\ D(x) = d_0\,, \\ &l_n(x) = \ell_{n,1}x + \ell_{n,0}\,,\ \pi_n(x) = \pi_n\,,\ \Theta_n(x) = \Theta_n\,,\ \ \pi_n, \Theta_n \text{ constants}\,, \end{aligned}$$

and

$$A_n(x) = a_{n,2}x^2 + a_{n,1}x + a_{n,0}\,. \tag{58}$$

In the account of (1), (11), (23) and according to the previous notations, we have

$$d_0 = -(a_2 + c_1 p_1)/(p_1^2 - r_2)\,. \tag{59}$$

In the next lemma we show that some quantities, to be used in the sequel, depend only on the lattice as well as on the coefficients of the Riccati equation. Recall that p_1 and r_2 are the leading coefficients of $p(x), r(x)$, respectively, defined in (9), and q is defined through (18).

Lemma 1 *Under the previous notations, the quantities $\ell_{n,1}$, Θ_n/γ_{n+1} and π_n are given, for all $n \geq 0$, by*

$$\ell_{n+1,1} = \left(\frac{q^{n+1} - q^{-(n+1)}}{q^{-1} - q}\right)\left(\ell_{0,1} + p_1\frac{\Theta_0}{\gamma_1}\right) + \left(\frac{q^n - q^{-n}}{q^{-1} - q}\right)\ell_{0,1}\,, \tag{60}$$

$$\frac{\Theta_{n+1}}{\gamma_{n+2}} = \left(\frac{q^{-(n+1)} - q^{n+1}}{q^{-1} - q}\right)\left(\frac{(1+q)(1+q^{-1})}{p_1}\ell_{0,1} + \frac{\Theta_0}{\gamma_1}\right) + \left(\frac{q^{-(n+2)} - q^{n+2}}{q^{-1} - q}\right)\frac{\Theta_0}{\gamma_1}\,, \tag{61}$$

$$\pi_{n+1} = -\frac{d_0}{2} - \frac{\Theta_0}{2\gamma_1} - \frac{1}{2(q^{-1} - q)}\left(q^{-1}\frac{1 - q^{-n}}{1 - q^{-1}} - q\frac{1 - q^n}{1 - q}\right)\left(\frac{(1+q)(1+q^{-1})}{p_1}\ell_{0,1} + \frac{\Theta_0}{\gamma_1}\right) - \frac{1}{2(q^{-1} - q)}\left(q^{-2}\frac{1 - q^{-n}}{1 - q^{-1}} - q^2\frac{1 - q^n}{1 - q}\right)\frac{\Theta_0}{\gamma_1}\,, \tag{62}$$

with

$$\ell_{0,1} = -p_1 d_0 - \frac{c_1}{2}\,,\quad \frac{\Theta_0}{\gamma_1} = \frac{-a_2 + c_1 p_1 + 2d_0(r_2 + p_1^2)}{p_1^2 - r_2}\,,\quad \pi_0 = -\frac{d_0}{2}\,. \tag{63}$$

PROOF. Recall Eqs. (46) and (56),

$$l_{n+1} + l_n + \mathbb{M}(x - \beta_{n+1})\frac{\Theta_n}{\gamma_{n+1}} = 0\,,$$

$$A - A_{n+2} - A_{n+1} + \mathbb{M}(x - \beta_{n+1})(l_{n+1} - l_n) + \Theta_{n+1} = \frac{\gamma_{n+1}}{\gamma_n}\Theta_{n-1}\,,$$

as well as (54), $\pi_{n+1} - \pi_n = -\dfrac{\Theta_n}{2\gamma_{n+1}}$. Recall that, in the account of (5), (8) and (9), we have

$$\mathbb{M}(x - \beta_{n+1}) = p(x) - \beta_{n+1}\,. \tag{64}$$

Collecting the coefficients of x in (46) we have

$$\ell_{n+1,1} = -\ell_{n,1} - p_1 \frac{\Theta_n}{\gamma_{n+1}}\,. \tag{65}$$

Let us now obtain $\Theta_{n+1}/\gamma_{n+2}$ as a linear combination of $\ell_{n,1}$ and Θ_n/γ_{n+1}. Starting with the definition of A_n in (33) we have $a_{n+1,2}-a_{n,2} = 2r_2(\pi_n-\pi_{n-1})$, thus, in the account of (54),

$$a_{n+1,2} = a_{n,2} - r_2\Theta_{n-1}/\gamma_n\,. \tag{66}$$

Collecting the coefficients of x^2 in (56) we get

$$a_2 - a_{n+2,2} - a_{n+1,2} + p_1(\ell_{n+1,1} - \ell_{n,1}) = 0\,. \tag{67}$$

Taking into account (66), we get

$$a_2 - 2a_{n+1,2} + r_2\frac{\Theta_n}{\gamma_{n+1}} + p_1(\ell_{n+1,1} - \ell_{n,1}) = 0\,,$$

thus, in the account of (65), it follows that

$$a_{n+1,2} = \left(\frac{r_2 - p_1^2}{2}\right)\frac{\Theta_n}{\gamma_{n+1}} - p_1\ell_{n,1} + \frac{a_2}{2}\,. \tag{68}$$

From (66) and (68) we get

$$a_{n,2} = \left(\frac{r_2 - p_1^2}{2}\right)\frac{\Theta_n}{\gamma_{n+1}} + r_2\frac{\Theta_{n-1}}{\gamma_n} - p_1\ell_{n,1} + \frac{a_2}{2}\,. \tag{69}$$

Now, from (68) and (69) we get

$$\left(\frac{r_2 - p_1^2}{2}\right)\frac{\Theta_{n-1}}{\gamma_n} - p_1\ell_{n-1,1} = \left(\frac{r_2 - p_1^2}{2}\right)\frac{\Theta_n}{\gamma_{n+1}} + r_2\frac{\Theta_{n-1}}{\gamma_n} - p_1\ell_{n,1}\,.$$

Let us write the above equation for $n+1$ and use (65). We get, after basic computations,

$$\frac{\Theta_{n+1}}{\gamma_{n+2}} = \frac{-4p_1}{r_2 - p_1^2}\ell_{n,1} + \left(1 - \frac{2(r_2 + p_1^2)}{r_2 - p_1^2}\right)\frac{\Theta_n}{\gamma_{n+1}}\,. \tag{70}$$

Now let us write Eqs. (65) and (70) in the matrix form,

$$\begin{bmatrix} \ell_{n+1,1} \\ \Theta_{n+1}/\gamma_{n+2} \end{bmatrix} = \mathcal{X}\begin{bmatrix} \ell_{n,1} \\ \Theta_n/\gamma_{n+1} \end{bmatrix}, \quad \mathcal{X} = \begin{bmatrix} -1 & -p_1 \\ \dfrac{-4p_1}{r_2 - p_1^2} & 1 - \dfrac{2(r_2 + p_1^2)}{r_2 - p_1^2} \end{bmatrix}. \tag{71}$$

Iteration in (71) yields, for all $n \geq 0$,

$$\ell_{n+1,1} = \left(\mathcal{X}^{n+1}\right)_{(1,1)} \ell_{0,1} + \left(\mathcal{X}^{n+1}\right)_{(1,2)} \Theta_0/\gamma_1\,, \tag{72}$$

$$\Theta_{n+1}/\gamma_{n+2} = \left(\mathcal{X}^{n+1}\right)_{(2,1)} \ell_{0,1} + \left(\mathcal{X}^{n+1}\right)_{(2,2)} \Theta_0/\gamma_1\,, \tag{73}$$

where $(\mathcal{X}^{n+1})_{(i,j)}$ denotes the element on position (i,j) of the matrix $\mathcal{X}^{n+1}$.

The set of eigenvalues of $\mathcal{X}$ is given by $\sigma(\mathcal{X}) = \{\lambda_1, \lambda_2\}$, where $\lambda_1 + \lambda_2 = -\dfrac{2(r_2 + p_1^2)}{r_2 - p_1^2}$. In the notation of (18), $\lambda_1 + \lambda_2 = q + q^{-1}$. As $\det(\mathcal{X}) = 1$, we have $\lambda_1 \lambda_2 = 1$, thus $\{\lambda_1, \lambda_2\} = \{q, q^{-1}\}$. Hence, we can assume, without loss of generality, that $\lambda_1 = q, \lambda_2 = q^{-1}$. After determining the eigenvalues of $\mathcal{X}$ we get $\mathcal{X} = \mathcal{V}\mathcal{D}\mathcal{V}^{-1}$, with $\mathcal{V}, \mathcal{D}$ given as

$$\mathcal{V} = \begin{bmatrix} \dfrac{-p_1}{1+q} & \dfrac{-p_1}{1+q^{-1}} \\ 1 & 1 \end{bmatrix}, \quad \mathcal{D} = \begin{bmatrix} q & 0 \\ 0 & q^{-1} \end{bmatrix}. \tag{74}$$

Note that $\mathcal{X}^{n+1} = \mathcal{V}\mathcal{D}^{n+1}\mathcal{V}^{-1}$. Thus, after simplifications, from (72)–(74) we get (60)–(61). To get (62) we use (61) combined with (55) and the initial conditions (34).

Finally, to get $\ell_{0,1}$ and π_0 we use (36) and (34), respectively. The quantity Θ_0/γ_1 follows from equating the coefficients of x^2 in Eq. (32) with $n = 1$ and using (46) with $n = 0$.

The equations given in Lemma 1 will now be used to determine the three-term recurrence relation coefficients of the Laguerre-Hahn polynomials.

Theorem 2 *Let $\{P_n\}_{n\geq 0}$ be the SMOP related to the Stieltjes function S satisfying $A(x)\mathbb{D}S(x) = B(x)\mathbb{E}_1 S(x)\mathbb{E}_2 S(x) + C(x)\mathbb{M}S(x) + D(x)$ under the degrees (57). Let the recurrence relation (24) hold,*

$$P_{n+1}(x) = (x - \beta_n)P_n(x) - \gamma_n P_{n-1}(x)\,, \quad n = 0, 1, 2, \ldots\,,$$

with $\beta_n \neq 0\,,\ \ n = 0, 1, 2, \ldots\,,\ \gamma_n \neq 0, \ \ n = 1, 2, \ldots\,.$
The following identities take place, for all $n \geq 0$:

$$\beta_{n+2} = s_n \beta_{n+1} + t_n\,, \tag{75}$$

$$\gamma_{n+2} = \frac{\gamma_1 \mu_0 D + \displaystyle\sum_{k=0}^{n} \mu_k \left(A(x_{k+1}) + 2r(x_{k+1})\lambda_k\right)}{\mu_n \mu_{n+1}}\,, \tag{76}$$

where $x_{k+1} = (\beta_{k+1} - p_0)/p_1$, and

$$s_n = \frac{\eta_n + p_1\mu_n}{p_1\mu_{n+1} - \eta_{n+1}}\,, \tag{77}$$

$$t_n = \frac{(p_1r_1 - p_0r_2)(\eta_{n+1} + \eta_n) + p_0p_1r_2(\mu_{n+1} - \mu_n) + 2r_2a_1 - 2r_1a_2}{r_2(p_1\mu_{n+1} - \eta_{n+1})}\,, \tag{78}$$

$$\lambda_n = \pi_{n+1} + \pi_n\,, \tag{79}$$

with, for all $n \geq 1$,

$$\eta_n = \left(\frac{q^{n+1} - q^{-(n+1)}}{q^{-1} - q}\right)\left(\ell_{0,1} + p_1\frac{\Theta_0}{\gamma_1}\right) - \left(\frac{q^n - q^{-n}}{q^{-1} - q}\right)p_1\frac{\Theta_0}{\gamma_1} - \left(\frac{q^{n-1} - q^{-(n-1)}}{q^{-1} - q}\right)\ell_{0,1}\,,$$

$$\mu_n = \left(\frac{q^{-n} - q^n}{q^{-1} - q}\right)\left(\frac{(1+q)(1+q^{-1})}{p_1}\ell_{0,1} + \frac{\Theta_0}{\gamma_1}\right) + \left(\frac{q^{-(n+1)} - q^{n+1}}{q^{-1} - q}\right)\frac{\Theta_0}{\gamma_1}\,.$$

and

$$\eta_0 = -2\ell_{0,1} - p_1\Theta_0/\gamma_1\,, \quad \mu_0 = \Theta_0/\gamma_1\,.$$

The quantities π_n, $\ell_{0,1}$ *and* Θ_0/γ_1 *are given in Lemma 1. As a consequence of (75),*

$$\beta_{n+2} = \prod_{k=0}^{n} s_k\left(\beta_1 + \sum_{k=0}^{n}\left(\prod_{j=0}^{k} s_j\right)^{-1} t_k\right)\,, \quad n \geq 0\,. \tag{80}$$

The parameter β_0 *is determined through*

$$\beta_0 = \frac{a_1 + p_1c_0 + p_0c_1 + 2p_0p_1d_0 - r_1d_0}{c_1 + 2p_1d_0}\,, \tag{81}$$

and β_1, γ_1 *are given by*

$$\beta_1 = \frac{(2p_0p_1 - r_1)\Theta_0/\gamma_1 + a_1 - p_1c_0 - p_0c_1 - 2d_0(2p_0p_1 + r_1 - p_1\beta_0)}{2p_1\Theta_0/\gamma_1 - c_1 - 2p_1d_0}\,, \tag{82}$$

$$\gamma_1 = \frac{(a_0 + b_0 + p_0c_0 + p_0^2d_0 - r_0d_0 - c_0\beta_0 - 2p_0d_0\beta_0 + d_0\beta_0^2)\,(p_1^2 - r_2)}{-a_2 + c_1p_1 + 2d_0(r_2 + p_1^2)} \tag{83}$$

PROOF. Let us deduce the equation for the β_n's.

By collecting the coefficient of the x^2 and x terms in (47) we obtain, respectively,

$$-a_2 + p_1(\ell_{n+1,1} - \ell_{n,1}) - 2r_2(\pi_{n+1} + \pi_n) = 0\,,$$
$$-a_1 + p_1(\ell_{n+1,0} - \ell_{n,0}) + (p_0 - \beta_{n+1})(\ell_{n+1,1} - \ell_{n,1}) - 2r_1(\pi_{n+1} + \pi_n) = 0\,.$$

By eliminating the $\pi_{n+1} + \pi_n$ term between the two above equations we obtain

$$a_1r_2 - a_2r_1 + (\ell_{n+1,1} - \ell_{n,1})(p_1r_1 - r_2(p_0 - \beta_{n+1})) - p_1r_2(\ell_{n+1,0} - \ell_{n,0}) = 0\,. \tag{84}$$

Using the coefficient of the x^0 term of (46),

$$\ell_{n+1,0} = -\ell_{n,0} - (p_0 - \beta_{n+1})\frac{\Theta_n}{\gamma_{n+1}}\,, \tag{85}$$

in (84) we obtain

$$\ell_{n,0} = f_n\beta_{n+1} + g_n\,, \tag{86}$$

where

$$f_n = \frac{p_1\mu_n - \eta_n}{2p_1}\,, \quad g_n = \frac{a_2r_1 - a_1r_2 - (p_1r_1 - r_2p_0)\eta_n - p_0p_1r_2\mu_n}{2p_1r_2}\,,$$
$$\mu_n = \Theta_n/\gamma_{n+1},\; \eta_n = \ell_{n+1,1} - \ell_{n,1}\,.$$

Now, substituting (86) into (85) we obtain

$$\beta_{n+2} = s_n\beta_{n+1} + t_n\,,$$

where

$$s_n = \frac{\mu_n - f_n}{f_{n+1}}\,, \quad t_n = -\frac{g_{n+1} + g_n + p_0\mu_n}{f_{n+1}}\,.$$

Thus, we get (75) with s_n, t_n given by (77)–(78).

Equation (80) follows from the general form for solutions of recurrences of type $z_{n+1} = x_nz_n + y_n,\ n \geq 0$ (see, e.g., [14, Lemma 3.3]).

Let us now deduce the equation for the γ_n's.

Evaluating the identity (47) at $x_{n+1} = (\beta_{n+1} - p_0)/p_1$ we get, taking into account (64),

$$-A(x_{n+1}) - 2r(x_{n+1})\lambda_n + \gamma_{n+2}\mu_{n+1} = \gamma_{n+1}\mu_{n-1}\,, \quad \lambda_n = \pi_{n+1} + \pi_n\,. \tag{87}$$

Multiplying (87) by μ_n we get

$$T_{n+1} = T_n + \mu_n\left(A(x_{n+1}) + 2r(x_{n+1})\lambda_n\right)\,, \quad T_n = \gamma_{n+1}\mu_{n-1}\mu_n\,, \;\; n \geq 0\,.$$

Iteration yields

$$T_{n+1} = T_0 + \sum_{k=0}^{n}\mu_k\left(A(x_{k+1}) + 2r(x_{k+1})\lambda_k\right)\,.$$

Thus, we get (76), where we used $\mu_{-1} = D$ (see (35)).

To obtain (81), we use the coefficients of x from (35) with the initial conditions (36). The coefficient of x^0 in (35) gives us Θ_0. This, combined with the Θ_0/γ_1 from (63) yields (83). Equation (82) follows from equating the coefficient of x in (32) with $n = 1$.

4.2 Specializations: the Askey-Wilson divided-difference calculus

Let us define the base $q = e^{2i\eta}$ and consider the projection map from the unit circle $\{z = e^{i\theta},\ \theta \in [-\pi, \pi[\}$ onto $[-1, 1]$ by $x = \frac{1}{2}(z + z^{-1})$. Consider the symmetrised and canonical form of the lattice defined through (4),

$$\hat{a} = \hat{c}\,,\ \text{arbitrary and non-zero},\ \hat{b} = -\hat{a}\cos(\eta)\,,\ \hat{d} = \hat{e} = 0\,,\ \hat{f} = -\hat{a}\sin^2(\eta)\,, \tag{88}$$

and $\theta = 2s\eta$. Then we get (20) given by

$$x(s) = \frac{1}{2}(q^s + q^{-s}) \tag{89}$$

and we obtain, from (22), the Askey-Wilson operator (see [9, Eq. (12.1.12)], [19, Sec. 2])

$$\mathbb{D}f(x) = \frac{f(\frac{1}{2}(q^{1/2}z + q^{-1/2}z^{-1})) - f(\frac{1}{2}(q^{-1/2}z + q^{1/2}z^{-1}))}{\frac{1}{2}(q^{1/2} - q^{-1/2})(z - z^{-1})}\,.$$

Using (8) and (21) combined with (89) or by plugging the data (88) into the definition of $p(x)$, $r(x)$ in (9) we get

$$p(x) = \frac{1}{2}(q^{1/2} + q^{-1/2})x\,, \quad r(x) = \frac{1}{4}(q^{1/2} - q^{-1/2})^2(x^2 - 1)\,, \tag{90}$$

hence, in the previous notations,

$$p_1 = \frac{1}{2}(q^{1/2} + q^{-1/2}),\ p_0 = 0,\ r_2 = \frac{1}{4}(q^{1/2} - q^{-1/2})^2,\ r_1 = 0,\ r_0 = -r_2\,.$$

The quantities s_n, t_n in Theorem 2 simplify as

$$s_n = -\frac{q\left(-(q+1)\left(q^{2n} - q\right)\ell_{0,1} - p_1 q\left(q^{2n} - 1\right)\Theta_0/\gamma_1\right)}{(q+1)\left(q^{2n+3} - 1\right)\ell_{0,1} + p_1\left(q^{2n+4} - 1\right)\Theta_0/\gamma_1}\,, \tag{91}$$

$$t_n = \frac{a_1(q - q^{-1})}{(q^{n+1} - q^{-n-1})\ell_{0,1} + (q^{n+2} - q^{-n-2})(\ell_{0,1} + p_1\Theta_0/\gamma_1)}\,. \tag{92}$$

4.2.1 The Askey-Wilson polynomials

The Askey-Wilson polynomials, henceforth denoted by P_n, are orthogonal with respect to the weight [1] (see also [10])

$$w(x; \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}) = \frac{h(x,1)h(x,-1)h(x,q^{1/2})h(x,-q^{1/2})}{\sqrt{1-x^2}\,h(x,\alpha_1)h(x,\alpha_2)h(x,\alpha_3)h(x,\alpha_4)}\,,$$

where

$$h(x,\alpha) = \prod_{k=0}^{+\infty}(1 - 2\alpha x q^k + \alpha^2 q^{2k})\,, \quad x = \cos(\theta)\,.$$

Let us denote by σ_j the j-th elementary symmetric polynomial of $\alpha_1, \ldots, \alpha_4$, that is,

$$\sigma_1 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\,,\ \sigma_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_2\alpha_3 + \alpha_2\alpha_4 + \alpha_3\alpha_4$$
$$\sigma_3 = \alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 + \alpha_2\alpha_3\alpha_4\,,\ \sigma_4 = \alpha_1\alpha_2\alpha_3\alpha_4\,.$$

We have $A\mathbb{D}w = C\mathbb{M}w$, with the polynomials $A(x) = a_2x^2 + a_1x + a_0$, $C(x) = c_1x + c_0$, where [19]

$$a_2 = 2(1 + \sigma_4 q^{-2})\,,\ a_1 = -(q^{-1/2}\sigma_1 + q^{-3/2}\sigma_3)\,,\ a_0 = -1 + q^{-1}\sigma_2 - q^{-2}\sigma_4\,, \tag{93}$$

$$c_1 = 4\frac{q^{-2}\sigma_4 - 1}{q^{1/2} - q^{-1/2}}\,,\ c_0 = 2\frac{q^{-1/2}\sigma_1 - q^{-3/2}\sigma_3}{q^{1/2} - q^{-1/2}}\,. \tag{94}$$

Thus, $A\mathbb{D}S = C\mathbb{M}S + D$ (see [19, Prop. 4.1]), with

$$D = d_0 = -(a_2 + c_1 p_1)\,. \tag{95}$$

The recurrence relation coefficients β_n of P_n are determined through (75) or (80), with the data (91)–(92). The γ_n are determined trough (76), now given by

$$\gamma_{n+2} = \frac{\gamma_1\mu_0 D + \sum_{k=0}^{n}\mu_k\left(A(\beta_{k+1}/p_1) + 2r(\beta_{k+1}/p_1)\lambda_k\right)}{\mu_n\mu_{n+1}}\,, \tag{96}$$

with the quantities μ_n, λ_n given in Theorem 2 with initial conditions (cf. (63)),

$$\ell_{0,1} = -p_1 d_0 - c_1/2\,,\quad \Theta_0/\gamma_1 = -a_2 + c_1 p_1 + d_0(q + q^{-1})\,.$$

Here it was used $p_1^2 - r_2 = 1, r_2 + p_1^2 = \frac{1}{2}(q + q^{-1})$. Thus, we recover

$$\beta_n = \left[\sigma_1(q + \sigma_4(q^{2n} - q^n - q^{n-1})) + \sigma_3(1 - q^n - q^{n+1} + \sigma_4 q^{2n-1})\right]$$
$$\times \frac{q^{n-1}}{2(1 - \sigma_4 q^{2n})(1 - \sigma_4 q^{2n-2})}\,,\quad n \geq 0\,, \tag{97}$$

$$\gamma_n = \frac{1}{4}\frac{(1 - q^n)(1 - \sigma_4 q^{n-2})G_n}{(1 - \sigma_4 q^{2n-3})(1 - \sigma_4 q^{2n-2})^2(1 - \sigma_4 q^{2n-1})}\,,\quad n \geq 1\,,\ \gamma_0 = 1\,, \tag{98}$$

where

$$G_n = (1 - \alpha_1\alpha_2 q^{n-1})(1 - \alpha_1\alpha_3 q^{n-1})(1 - \alpha_1\alpha_4 q^{n-1})$$
$$\times (1 - \alpha_2\alpha_3 q^{n-1})(1 - \alpha_2\alpha_4 q^{n-1})(1 - \alpha_3\alpha_4 q^{n-1})\,.$$

4.2.2 The associated of Askey-Wilson polynomials

Firstly, let us adopt a new notation. Henceforth we denote the recurrence relation coefficients of the Askey-Wilson polynomials by $\tilde{\gamma}_n, \tilde{\beta}_n, n \geq 0$, with the convention $\tilde{\gamma}_0 = 1$.

The Stieltjes function related to the associated of Askey-Wilson polynomials, $\{P_n^{(1)}\}_{n\geq 0}$, is defined through

$$\tilde{\gamma}_1 S^{(1)}(x) = -\frac{1}{S(x)} + (x - \tilde{\beta}_0)\,, \tag{99}$$

where S is the Stieltjes function related to the Askey-Wilson polynomials. Thus, using (12)–(15), we get

$$A_1 \mathbb{D} S^{(1)} = B_1 \mathbb{E}_1 S^{(1)} \mathbb{E}_2 S^{(1)} + C_1 \mathbb{M} S^{(1)} + D_1\,, \tag{100}$$

with (see also [20])

$$\begin{aligned} A_1 &= A - 2r(x)D\,,\ B_1 = \tilde{\gamma}_1 D\,,\ C_1 = -C - 2D\mathbb{M}(x - \tilde{\beta}_0)\,, \\ D_1 &= \frac{1}{\tilde{\gamma}_1}\left(A + C\mathbb{M}(x - \tilde{\beta}_0) + D\mathbb{E}_1(x - \tilde{\beta}_0)\mathbb{E}_2(x - \tilde{\beta}_0)\right)\,. \end{aligned}$$

Here, A, C, D are the polynomials in sub-section 4.2.1.

The polynomials A_1, B_1, C_1, D_1 satisfy the bounds (57). Indeed, it only remains to analyse $\deg(D_1)$. Substituting d_0 and taking into account the coefficient of x in the second equation of (35), we get $\deg(D_1) = 0$.

Thus, the data to be used in Theorem 2, that is, the coefficients of the Riccati equation, are as follows:

$$A_1(x) = a_2^{(1)} x^2 + a_1^{(1)} x + a_0^{(1)}, B_1(x) = b_0^{(1)}, C_1(x) = c_1^{(1)} x + c_0^{(1)}, D_1(x) = d_0^{(1)}\,,$$

with

$$a_2^{(1)} = a_2(1 + 2r_2) + 2r_2 c_1 p_1\,,\ a_1^{(1)} = a_1 + 2r_1(a_2 + p_1 c_1)\,, \tag{101}$$

$$a_0^{(1)} = a_0 + 2r_0(a_2 + c_1 p_1)\,,\ b_0^{(1)} = \tilde{\gamma}_1 d_0\,, \tag{102}$$

$$c_1^{(1)} = -c_1 - 2d_0 p_1\,,\ c_0^{(1)} = -c_0 - 2d_0(p_0 - \tilde{\beta}_0)\,, \tag{103}$$

$$d_0^{(1)} = \frac{1}{\tilde{\gamma}_1}\left(a_0 + c_0(p_0 - \tilde{\beta}_0) + d_0(p_0^2 - r_0 - 2p_0\tilde{\beta}_0 + \tilde{\beta}_0^2)\right)\,. \tag{104}$$

Here, the a's, b's, c's and d_0 are given above (see (93)–(95)). The recurrence relation coefficients β_n, γ_n of $P_n^{(1)}$ are determined through (80) with the data (91)–(92) and (76), now given by

$$\gamma_{n+2} = \frac{\gamma_1 \mu_0 D_1 + \sum_{k=0}^{n} \mu_k \left(A_1(\beta_{k+1}/p_1) + 2r(\beta_{k+1}/p_1)\lambda_k\right)}{\mu_n \mu_{n+1}}\,,$$

with the quantities μ_n, λ_n given in Theorem 2 with initial conditions (cf. (63)),

$$\ell_{0,1} = -p_1 d_0^{(1)} - c_1^{(1)}/2\,, \quad \Theta_0/\gamma_1 = -a_2^{(1)} + c_1^{(1)} p_1 + d_0^{(1)}(q+q^{-1})\,.$$

Recall that $p_1^2 - r_2 = 1, r_2 + p_1^2 = \frac{1}{2}(q+q^{-1})$. Thus, we recover

$$\beta_n = \tilde{\beta}_{n+1}\,, \quad \gamma_n = \tilde{\gamma}_{n+1}\,, \quad n \geq 1\,. \tag{105}$$

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