ON 2-STAR-PERMUTABILITY IN REGULAR MULTI-POINTED CATEGORIES

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ABSTRACT. 2-star-permutable categories were introduced in a joint work with Z. Janelidze and A. Ursini as a common generalisation of regular Mal'tsev categories and of normal subtractive categories. In the present article we first characterise these categories in terms of what we call star-regular pushouts. We then show that the 3×3 Lemma characterising normal subtractive categories and the Cuboid Lemma characterising regular Mal'tsev categories are special instances of a more general homological lemma for star-exact sequences. We prove that 2-star-permutability is equivalent to the validity of this lemma for a star-regular category.

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INTRODUCTION

The theory of *Mal'tsev categories* in the sense of A. Carboni, J. Lambek and M.C. Pedicchio [6] provides a beautiful example of the way how categorical algebra leads to a structural understanding of algebraic varieties (in the sense of universal algebra). Among regular categories, Mal'tsev categories are characterised by the property of 2-permutability of equivalence relations: given two equivalence relations R and S on the same object A, the two relational composites RS and SR are equal:

RS = SR.

In the case of a variety of universal algebras this property is actually equivalent to the existence of a ternary term p(x, y, z) satisfying the identities p(x, y, y) = x and p(x, x, y) = y [20]. In the pointed context, that is when the category has a zero object, there is also a suitable notion of 2-permutability, called "2-permutability at 0" [21]. In a variety this property can be expressed by requiring that, whenever for a given element x in an algebra A there is an element y with xRyS0 (here 0 is the unique constant in A), then there is also an element z in A with xSzR0. The validity of this property is equivalent to the existence of a binary term s(x, y) such that the identities s(x, 0) = x and s(x, x) = 0 hold true [21]. Among regular categories, the ones where the property of 2-permutability at 0 holds true are precisely the subtractive categories introduced in [16].

The aim of this paper is to look at regular Mal'tsev and at subtractive categories as special instances of the general notion of 2-star-permutable categories introduced in collaboration with Z. Janelidze and A. Ursini in [9]. This generalisation is achieved by working in the context of a *regular multi-pointed category*, i.e. a regular category equipped with an ideal \mathcal{N} of distinguished morphisms [7]. When

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 \mathcal{N} is the class of all morphisms, a situation which we refer to as the *total context*, regular multi-pointed categories are just regular categories, and 2-star-permutable categories are precisely the regular Mal'tsev categories. When \mathcal{N} is the class of all zero morphisms in a pointed category, we call this the *pointed context*, regular multi-pointed categories are regular pointed categories, and 2-star-permutable categories are the regular subtractive categories.

This paper follows the same line of research as in [9] which was mainly focused on the property of 3-star-permutability, a generalised notion which captures Goursat categories in the total context and, again, subtractive categories in the pointed context.

In this work we study two remarkable aspects of the property of 2-star-permutability. First we provide a characterisation of 2-star-permutable categories in terms of a special kind of pushouts (Proposition 2.4), that we call star-regular pushouts (Definition 2.2). Then we examine a homological diagram lemma of star-exact sequences, which can be seen as a generalisation of the 3×3 Lemma, whose validity is equivalent to 2-star-permutability. We call this lemma the Star-Upper Cuboid Lemma (Theorem 3.3). The validity of this lemma turns out to give at once a characterisation of regular Mal'tsev categories (extending a result in [11]) and, in the pointed context, a characterisation of those normal categories which are subtractive (this was first discovered in [17]).

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1. Star-regular categories

1.1. Regular categories and relations. A finitely complete category \mathbb{C} is said to be a *regular* category [1] when any kernel pair has a coequaliser and, moreover, regular epimorphisms are stable under pullbacks. In a regular category any morphism $f: X \to Y$ has a factorisation $f = m \cdot p$, where p is a regular epimorphism and m is a monomorphism. The corresponding (regular epimorphism, monomorphism) factorisation system is then stable under pullbacks.

A relation ρ from X to Y is a subobject $\langle \rho_1, \rho_2 \rangle \colon R \to X \times Y$. The opposite relation, denoted ρ° , is given by the subobject $\langle \rho_2, \rho_1 \rangle \colon R \to Y \times X$. We identify a morphism $f: X \to Y$ with the relation $(1_X, f): X \to X \times Y$ and write f° for the opposite relation. Given another relation σ from Y to Z, the composite relation of ρ and σ is a relation $\sigma \rho$ from X to Z. With this notation, we can write the above relation as $\rho = \rho_2 \rho_1^{\circ}$. The following properties are well known (see [5], for instance); we collect them in a lemma for future references.

Lemma 1.1. Let $f: X \to Y$ be any morphism in a regular category \mathbb{C} . Then:

- (a) ff°f = f and f°ff° = f°;
 (b) ff° = 1_Y if and only if f is a regular epimorphism.

A kernel pair of a morphism $f: X \to Y$, denoted by

$$(\pi_1,\pi_2)\colon \mathsf{Eq}(f) \rightrightarrows X,$$

is called an *effective equivalence relation*; we write it either as $\mathsf{Eq}(f) = f^{\circ}f$, or as $Eq(f) = \pi_2 \pi_1^{\circ}$, as mentioned above. When f is a regular epimorphism, then f is the coequaliser of π_1 and π_2 and the diagram

$$\mathsf{Eq}(f) \xrightarrow[\pi_2]{\pi_2} X \xrightarrow{f} Y$$

is called an *exact fork*. In a regular category any effective equivalence relation is the kernel pair of a regular epimorphism.

1.2. Star relations. We now recall some notions introduced in [10], which are useful to develop a unified treatment of pointed and non-pointed categorical algebra. Let \mathbb{C} denote a category with finite limits, and \mathcal{N} a distinguished class of morphisms that forms an *ideal*, i.e. for any composable pair of morphisms g, f, if either g or fbelongs to \mathcal{N} , then the composite $g \cdot f$ belongs to \mathcal{N} . An \mathcal{N} -kernel of a morphism $f: X \to Y$ is defined as a morphism $n_f: N_f \to X$ such that $f \cdot n_f \in \mathcal{N}$ and n_f is universal with this property (note that such n_f is automatically a monomorphism). A pair of parallel morphisms, denoted by $\sigma = (\sigma_1, \sigma_2) : S \rightrightarrows X$ with $\sigma_1 \in \mathcal{N}$, is called a *star*; it is called a monic star, or a *star relation*, when the pair (σ_1, σ_2) is jointly monomorphic.

Given a relation $\rho = (\rho_1, \rho_2) : R \Rightarrow X$ on an object X, we denote by $\rho^* : R^* \Rightarrow X$ the biggest subrelation of ρ which is a (monic) star. When \mathbb{C} has \mathcal{N} -kernels, it can be constructed by setting $\rho^* = (\rho_1 \cdot \mathbf{n}_{\rho_1}, \rho_2 \cdot \mathbf{n}_{\rho_1})$, where \mathbf{n}_{ρ_1} is the \mathcal{N} -kernel of ρ_1 . In particular, if we denote the discrete equivalence relation on an object X by $\Delta_X = (\mathbf{1}_X, \mathbf{1}_X) : X \Rightarrow X$, then $\Delta_X^* = (\mathbf{n}_{\mathbf{1}_X}, \mathbf{n}_{\mathbf{1}_X})$, where $\mathbf{n}_{\mathbf{1}_X}$ is the \mathcal{N} -kernel of $\mathbf{1}_X$.

The star-kernel of a morphism $f : X \to Y$ is a universal star $\sigma = (\sigma_1, \sigma_2) : S \rightrightarrows X$ with the property $f \cdot \sigma_1 = f \cdot \sigma_2$; it is easy to see that the star-kernel of f coincides with $\mathsf{Eq}(f)^* \rightrightarrows X$ whenever \mathcal{N} -kernels exist.

A category \mathbb{C} equipped with an ideal \mathcal{N} of morphisms is called a *multi-pointed* category [10]. If, moreover, every morphism admits an \mathcal{N} -kernel, then \mathbb{C} will be called a *multi-pointed category with kernels*.

Definition 1.2. [10] A regular multi-pointed category \mathbb{C} with kernels is called a *star-regular category* when every regular epimorphism in \mathbb{C} is a coequaliser of a star.

In the total context stars are pairs of parallel morphisms, \mathcal{N} -kernels are isomorphisms, star-kernels are kernel pairs and a star-regular category is precisely a regular category. In the pointed context, the first morphism σ_1 in a star $\sigma = (\sigma_1, \sigma_2) : S \Rightarrow X$ is the unique null morphism $S \to X$ and hence a star σ can be identified with a morphism (its second component σ_2). Then, \mathcal{N} -kernels and star-kernels become the usual kernels, and a star-regular category is the same as a normal category [18], i.e. a pointed regular category in which any regular epimorphism is a normal epimorphism.

1.3. Calculus of star relations. The calculus of star relations [9] can be seen as an extension of the usual calculus of relations (in a regular category) to the regular multi-pointed context. First of all note that for any relation $\rho: R \rightrightarrows X$ we have

$$\varrho^* = \varrho \Delta_X^*.$$

Inspired by this formula, for any relation ρ from X to an object Y, we define

$$\varrho^* = \varrho \Delta_X^*$$
 and ${}^*\varrho = \Delta_Y^* \varrho$.

Note that associativity of composition yields

$$^{*}(\varrho^{*}) = (^{*}\varrho)^{*}$$

and so we can write ρ^* for the above.

For any relation σ (from some object Y to Z), the associativity of composition also gives

$$(\sigma^*)\varrho = \sigma(^*\varrho),$$

and

$$\left(\sigma\varrho\right)^* = \sigma\varrho^*.$$

It is easy to verify that for any morphism $f: X \to Y$ we have

$$f^* = {}^*f^*$$
 and ${}^*f^\circ = {}^*f^{\circ*}$.

2. 2-STAR-PERMUTABILITY AND STAR-REGULAR PUSHOUTS

Recall that a finitely complete category \mathbb{C} is called a *Mal'tsev category* when any reflexive relation in \mathbb{C} is an equivalence relation [6, 5]. We recall the following well known characterisation of the regular categories which are Mal'tsev categories:

Proposition 2.1. A regular category \mathbb{C} is a Mal'tsev category if and only if the composition of effective equivalence relations in \mathbb{C} is commutative:

$$\mathsf{Eq}(f)\mathsf{Eq}(g) = \mathsf{Eq}(g)\mathsf{Eq}(f)$$

for any pair of regular epimorphisms f and g in \mathbb{C} with the same domain.

There are many known characterisations of regular Mal'tsev categories (see Section 2.5 in [2], for instance, and references therein). The one that will play a central role in the present work is expressed in terms of commutative diagrams of the form

$$C \xrightarrow{c} A$$

$$g \bigvee_{t} t \qquad f \bigvee_{s} B, \qquad (1)$$

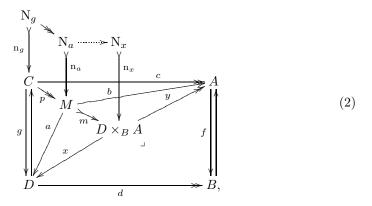
where f and g are split epimorphisms $(f \cdot s = 1_B, g \cdot t = 1_D), f \cdot c = d \cdot g, s \cdot d = c \cdot t,$ and c and d are regular epimorphisms. A diagram of type (1) is always a pushout; it is called a *regular pushout* [4] (alternatively, a *double extension* [15, 13]) when, moreover, the canonical morphism $\langle g, c \rangle \colon C \to D \times_B A$ to the pullback $D \times_B A$ of d and f is a regular epimorphism. Among regular categories, Mal'tsev categories can be characterized as those ones where any square (1) is a regular pushout: this easily follows from the results in [4], and a simple proof of this fact is given in [12].

Observe that a commutative diagram of type (1) is a regular pushout if and only if $cg^{\circ} = f^{\circ}d$ or, equivalently, $gc^{\circ} = d^{\circ}f$. This suggests to introduce the following notion:

Definition 2.2. A commutative diagram (1) is a *star-regular pushout* if it satisfies the identity $cg^{\circ*} = f^{\circ}d^*$ (or, equivalently, $gc^{\circ*} = d^{\circ}f^*$).

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Diagrammatically, the property of being a star-regular pushout can be expressed as follows. Consider the commutative diagram



where $(D \times_B A, x, y)$ is the pullback of (f, d), $m \cdot p$ is the (regular epimorphism, monomorphism) factorisation of the induced morphism $\langle g, c \rangle \colon C \to D \times_B A$. Then the identity $cg^{\circ} = ba^{\circ}$ allows one to identify $cg^{\circ*}$ with the relation $(a \cdot n_a, b \cdot n_a)$, while $f^{\circ}d = yx^{\circ}$ says that $f^{\circ}d^*$ can be identified with the relation $(x \cdot n_x, y \cdot n_x)$. Accordingly, diagram (1) is a star-regular pushout precisely when the dotted arrow from N_a to N_x is an isomorphism. Notice that in the total context the \mathcal{N} -kernels are isomorphisms, so that m is an isomorphism if and only if (1) is a regular pushout, as expected.

The "star-version" of the notion of Mal'tsev category can be defined as follows:

Definition 2.3. [9] A regular multi-pointed category with kernels \mathbb{C} is said to be a 2-star-permutable category if

$$\mathsf{Eq}(f)\mathsf{Eq}(g)^* = \mathsf{Eq}(g)\mathsf{Eq}(f)^*$$

for any pair of regular epimorphisms f and g in \mathbb{C} with the same domain.

One can check that the equality $\mathsf{Eq}(f)\mathsf{Eq}(g)^* = \mathsf{Eq}(g)\mathsf{Eq}(f)^*$ in the definition above can be actually replaced by $\mathsf{Eq}(f)\mathsf{Eq}(g)^* \leq \mathsf{Eq}(g)\mathsf{Eq}(f)^*$.

In the total context the property of 2-star-permutability characterises the regular categories which are Mal'tsev. In the pointed context this same property characterises the regular categories which are subtractive [16] (this follows from the characterisation of subtractivity given in Theorem 6.9 in [17]).

The next result gives a useful characterisation of 2-star-permutable categories. Given a commutative diagram of type (1), we write $g\langle \mathsf{Eq}(c) \rangle$ and $g\langle \mathsf{Eq}(c)^* \rangle$ for the direct images of the relations $\mathsf{Eq}(c)$ and $\mathsf{Eq}(c)^*$ along the split epimorphism g. The vertical split epimorphisms are such that both the equalities $g\langle \mathsf{Eq}(c) \rangle = \mathsf{Eq}(d)$ and $g\langle \mathsf{Eq}(c)^* \rangle = \mathsf{Eq}(d)^*$ hold true in \mathbb{C} .

Proposition 2.4. For a regular multi-pointed category with kernels \mathbb{C} the following statements are equivalent:

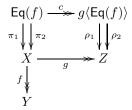
- (a) \mathbb{C} is a 2-star-permutable category;
- (b) any commutative diagram of the form (1) is a star-regular pushout.

Proof. (a) \Rightarrow (b) Given a pushout (1) we have

$$\begin{aligned} f^{\circ}d^{*} &= cc^{\circ}f^{\circ}d^{*} & (\text{Lemma 1.1(2)}) \\ &= cg^{\circ}d^{\circ}d^{*} & (f \cdot c = d \cdot g) \\ &= cg^{\circ}gc^{\circ}c^{*}g^{\circ} & (\mathsf{Eq}(d)^{*} = g\langle \mathsf{Eq}(c)^{*}\rangle) \\ &= cc^{\circ}cg^{\circ}g^{*}g^{\circ} & (\mathsf{Eq}(g)\mathsf{Eq}(c)^{*} = \mathsf{Eq}(c)\mathsf{Eq}(g)^{*} \text{ by Definition 2.3}) \\ &\leq cc^{\circ}cg^{\circ}gg^{\circ} & (g^{*} \leq g) \\ &= cg^{\circ}. & (\text{Lemma 1.1(1)}) \end{aligned}$$

Since cg°^*} is the largest star contained in cg° , it follows that $f^{\circ}d^* \leq cg^{\circ^*}$. The inclusion $cg^{\circ^*} \leq f^{\circ}d^*$ always holds, so that $cg^{\circ^*} = f^{\circ}d^*$.

(b) \Rightarrow (a) Let us consider regular epimorphisms $f: X \twoheadrightarrow Y$ and $g: X \twoheadrightarrow Z$. We want to prove that $\mathsf{Eq}(f)\mathsf{Eq}(g)^* = \mathsf{Eq}(g)\mathsf{Eq}(f)^*$. For this we build the following diagram



that represents the regular image of $\mathsf{Eq}(f)$ along g. The relation $g\langle \mathsf{Eq}(f) \rangle = (\rho_1, \rho_2)$ is reflexive and, consequently, ρ_1 is a split epimorphism. By assumption, we then know that the equality

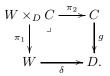
$$(A) \qquad \rho_1^\circ g^* = c \pi_1^{\circ *}$$

holds true. This implies that

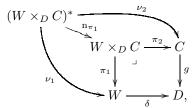
$$\begin{aligned} \mathsf{Eq}(f)\mathsf{Eq}(g)^* &= \pi_2 \pi_1^\circ g^\circ g^* \\ &= \pi_2 c^\circ \rho_1^\circ g^* \qquad (g \cdot \pi_1 = \rho_1 \cdot c) \\ &= \pi_2 c^\circ c \pi_1^\circ ^* \qquad (A) \\ &\leq \pi_2 c^\circ c \pi_2^\circ \pi_2 \pi_1^\circ ^* \qquad (\Delta_{\mathsf{Eq}(f)} \leq \pi_2^\circ \pi_2) \\ &= \mathsf{Eq}(g) \pi_2 \pi_1^\circ ^* \qquad (\pi_2 \langle \mathsf{Eq}(c) \rangle = \mathsf{Eq}(g)) \\ &= \mathsf{Eq}(g) \mathsf{Eq}(f)^*, \end{aligned}$$

where the equality $\pi_2 \langle \mathsf{Eq}(c) \rangle = \mathsf{Eq}(g)$ follows from the fact that the split epimorphisms π_2 and ρ_2 induce a split epimorphism from $\mathsf{Eq}(c)$ to $\mathsf{Eq}(g)$.

In the total context, Proposition 2.4 gives the characterisation of regular Mal'tsev categories through regular pushouts (see [4] and Proposition 3.4 of [12]), as expected. In the pointed context, condition (b) of Proposition 2.4 translates into the pointed version of the *right saturation* property [9] for any commutative diagram of type (1): the induced morphism $\bar{c} : \text{Ker}(g) \to \text{Ker}(f)$, from the kernel of g to the kernel of f is also a regular epimorphism. This can be seen by looking at diagram (2), where the \mathcal{N} -kernels now represent actual kernels, so that Ker(a) = Ker(x) = Ker(f). 2.1. The star of a pullback relation. Consider the pullback relation $\pi = (\pi_1, \pi_2)$ of a pair (g, δ) of morphisms as in the diagram



The star of the pullback relation π is defined as $\pi^* = \pi \Delta_W^*$. It can be described as the universal relation $\nu = (\nu_1, \nu_2)$ from W to C such that $\nu_1 \in \mathcal{N}$ and $\delta \cdot \nu_1 = g \cdot \nu_2$ as in the diagram



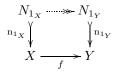
where n_{π_1} is the \mathcal{N} -kernel of π_1 , $\nu_1 = \pi_1 \cdot n_{\pi_1}$ and $\nu_2 = \pi_2 \cdot n_{\pi_1}$.

By using the composition of relations one has the equalities $\pi = \pi_2 \pi_1^\circ = g^\circ \delta$, so that

$$\pi^* = \pi_2 \pi_1^{\circ *} = g^{\circ} \delta^*.$$

In the total context, the star of a pullback relation is precisely that pullback relation. In the pointed context, the star of the pullback (relation) of (g, δ) is given by $\pi^* = (0, \ker(g))$.

A morphism $f: X \to Y$ in a multi-pointed category with kernels is said to be saturating [9] when the induced dotted morphism from the \mathcal{N} -kernel of 1_X to the \mathcal{N} -kernel of 1_Y making the diagram



commute is a regular epimorphism. All morphisms are saturating in the pointed context. This is also the case for any quasi-pointed category [3], namely a finitely complete category with an initial object 0 and a terminal object 1 such that the arrow $0 \rightarrow 1$ is a monomorphism. As in the pointed case, it suffices to choose for \mathcal{N} the class of morphisms which factor through the initial object 0. In this case we shall speak of the quasi-pointed context. In the total context, any regular epimorphism is saturating. The proof of the following result is straightforward:

Lemma 2.5. [9] Let \mathbb{C} be a regular multi-pointed category with kernels. For a morphism $f: X \to Y$ the following conditions are equivalent:

(a) f is saturating;

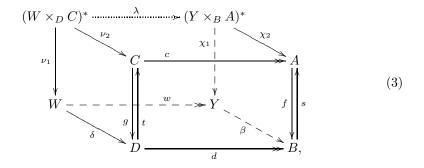
(b) $\Delta_Y^* = f^* f^\circ$.

The next result gives a characterisation of 2-star-permutable categories which will be useful in the following section.

Proposition 2.6. For a regular multi-pointed category \mathbb{C} with kernels and saturating regular epimorphisms the following statements are equivalent:

(a) \mathbb{C} is a 2-star-permutable category;

(b) for any commutative diagram



where the front square is of the form (1), $\beta \cdot w = d \cdot \delta$, w is a regular epimorphism, $((W \times_D C)^*, \nu_1, \nu_2)$ and $((Y \times_B A)^*, \chi_1, \chi_2)$ are stars of the corresponding pullback relations, then the comparison morphism $\lambda \colon (W \times_D C)^* \to (Y \times_B A)^*$ is also a regular epimorphism.

Proof. (a) \Rightarrow (b) To prove that the arrow λ in the cube above is a regular epimorphism, we must show that $\langle \chi_1, \chi_2 \rangle \lambda$ in the commutative diagram

is the (regular epimorphism, monomorphism) factorisation of the morphism $\langle w \cdot \nu_1, c \cdot \nu_2 \rangle$: $(W \times_D C)^* \to Y \times A$. That is, we must have $c\nu_2\nu_1^\circ w^\circ = \chi_2\chi_1^\circ$ or, equivalently, $cg^\circ \delta^* w^\circ = f^\circ \beta^*$, since $\nu_2\nu_1^\circ = \nu^* = g^\circ \delta^*$ and $\chi_2\chi_1^\circ = \chi^* = f^\circ \beta^*$ (see Section 2.1).

The front square of diagram (3) is a star-regular pushout by Proposition 2.4, which means that the equality

$$(B) \qquad cg^{\circ^*} = f^\circ d^*$$

holds true. Now, we always have

$$cg^{\circ}\delta^{*}w^{\circ} \leqslant f^{\circ}d\delta^{*}w^{\circ} \quad (\text{commutativity of the front face of (3)}) \\ = f^{\circ}\beta w^{*}w^{\circ} \quad (d \cdot \delta = \beta \cdot w) \\ = f^{\circ}\beta \Delta_{Y}^{*} \quad (\text{Lemma 2.5}) \\ = f^{\circ}\beta^{*}.$$

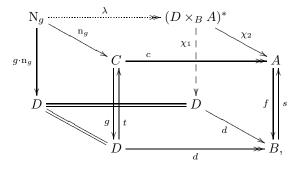
The other inequality follows from

$$cg^{\circ}\delta^{*}w^{\circ} \geq cg^{\circ*}\delta^{*}w^{\circ} \quad (g^{\circ} \geq g^{\circ*})$$

= $f^{\circ}d^{*}\delta^{*}w^{\circ}$ (B)
= $f^{\circ}d\delta^{*}w^{\circ}$ (* $\delta^{*} = \delta^{*}$; Section 1.3)
= $f^{\circ}\beta^{*}$. (as in the inequality above)

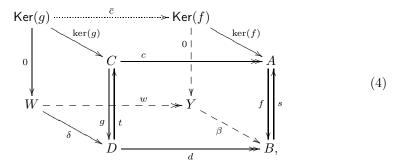
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(b) \Rightarrow (a) A commutative diagram of type (1) induces a commutative cube



where $\nu = (g \cdot \mathbf{n}_g, \mathbf{n}_g)$ is the star of the pullback (relation) of $(g, \mathbf{1}_D)$. By assumption, λ is a regular epimorphism which translates into the equality $cg^{\circ}\mathbf{1}_D^*\mathbf{1}_D = f^{\circ}d^*$, as observed in the first part of the proof. We get the equality $cg^{\circ *} = f^{\circ}d^*$, and this proves that diagram (1) is a star-regular pushout and, consequently, that \mathbb{C} is a 2-star-permutable category by Proposition 2.4.

In the total context, Proposition 2.6 is the "star version" of Proposition 3.6 in [12] (see also Proposition 4.1 in [4]). In the pointed context condition (b) of Proposition 2.6 also reduces to the pointed version of the right saturation property (in the sense of [9]). Indeed, in this context that condition says that, in the following commutative diagram



the induced arrow \overline{c} : $\operatorname{Ker}(g) \to \operatorname{Ker}(f)$ is a regular epimorphism.

We conclude this section with the pointed version of Propositions 2.4 and 2.6:

Corollary 2.7. (see Theorem 2.12 in [9]) For a pointed regular category \mathbb{C} the following statements are equivalent:

- (a) \mathbb{C} is a subtractive category;
- (b) any commutative diagram of the form (1) is right saturated, i.e. the comparison morphism \bar{c} : $\text{Ker}(g) \to \text{Ker}(f)$ is a regular epimorphism.

3. The Star-Cuboid Lemma

In [12] it was shown that regular Mal'tsev categories can be characterised through the validity of a homological lemma called the Upper Cuboid Lemma, a strong form of the denormalised 3×3 Lemma [4, 19, 11]. We are now going to extend this result to the star-regular context. We shall then observe that, in the pointed context, it gives back the classical Upper 3×3 Lemma characterising subtractive normal categories.

3.1. \mathcal{N} -trivial objects. An object X in a multi-pointed category is said to be \mathcal{N} -trivial when $1_X \in \mathcal{N}$. If a composite $f \cdot g$ belongs to \mathcal{N} and g is a strong epimorphism, then also f belongs to \mathcal{N} . This implies that \mathcal{N} -trivial objects are closed under strong quotients. One says that a multi-pointed category \mathbb{C} has enough trivial objects [8] when \mathcal{N} is a closed ideal [14], i.e. any morphism in \mathcal{N} factors through an \mathcal{N} -trivial object and, moreover, the class of \mathcal{N} -trivial objects is closed under subobjects and squares, where the latter property means that, for any \mathcal{N} -trivial object X, the object $X^2 = X \times X$ is \mathcal{N} -trivial. An equivalent way of expressing the existence of enough trivial objects is recalled in the following:

Proposition 3.1. [8] Let \mathbb{C} be a regular multi-pointed category with kernels. The following conditions are equivalent:

- (a) if $(\sigma_1, \sigma_2) : S \rightrightarrows X$ is a relation on X such that $\sigma_1 \cdot n \in \mathcal{N}$ and $\sigma_2 \cdot n \in \mathcal{N}$, then $n \in \mathcal{N}$;
- (b) \mathbb{C} has enough trivial objects.

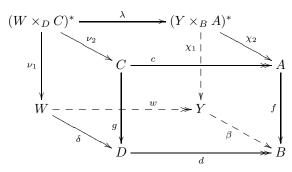
In the following we shall also assume that \mathcal{N} -trivial objects are closed under binary products. Remark that in the total and in the (quasi-)pointed contexts there are enough trivial objects, and \mathcal{N} -trivial objects are closed under binary products.

Under the presence of enough trivial objects the assumption that \mathcal{N} -trivial objects are closed under binary products is equivalent to the following condition:

(a') if $(\sigma_1, \sigma_2) : S \to X \times Y$ is a relation from X to Y such that $\sigma_1 \cdot n \in \mathcal{N}$ and $\sigma_2 \cdot n \in \mathcal{N}$, then $n \in \mathcal{N}$.

Whenever the category has enough trivial objects, condition (a') implies that starkernels "commute" with stars of pullback relations:

Lemma 3.2. Let \mathbb{C} be a multi-pointed category with kernels, enough trivial objects, and assume that \mathcal{N} -trivial objects are closed under binary products. Given a commutative cube



in \mathbb{C} , consider the star-kernels of c, d and w, and the induced morphisms $\overline{\delta} \colon \mathsf{Eq}(w)^* \to \mathsf{Eq}(d)^*$ and $\overline{g} \colon \mathsf{Eq}(c)^* \to \mathsf{Eq}(d)^*$. Then the following constructions are equivalent (up to isomorphism):

• taking the horizontal star-kernel of λ and then the induced morphisms $\mathsf{Eq}(\lambda)^* \to \mathsf{Eq}(w)^*$ and $\mathsf{Eq}(\lambda)^* \to \mathsf{Eq}(c)^*$;

 taking the star of the pullback (relation) of ḡ and δ̄ and then the induced morphisms (Eq(w)* ×_{Eq(d)*} Eq(c)*)* ⇒ (W ×_D C)*.

Proof. This follows easily by the usual commutation of kernel pairs with pullbacks and condition (a'). \Box

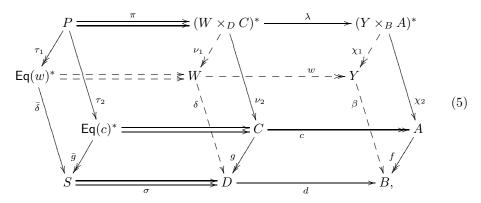
In a star-regular category, a (short) star-exact sequence is a diagram

$$\mathsf{Eq}(f)^* \xrightarrow[f_2]{f_1} X \xrightarrow{f} Y$$

where $Eq(f)^*$ is a star-kernel of f and f is a coequaliser of f_1 and f_2 (which, by star-regularity, is the same as to say that f is a regular epimorphism). In the total context, a star-exact sequence is just an exact fork, while in the (quasi-)pointed context it is a short exact sequence in the usual sense.

The Star-Upper Cuboid Lemma

Let $\mathbb C$ be a star-regular category. Consider a commutative diagram of morphisms and stars in $\mathbb C$



where the three diamonds are stars of pullback (relations) of regular epimorphisms along arbitrary morphisms (so that $P = (\mathsf{Eq}(w)^* \times_S \mathsf{Eq}(c)^*)^*$) and the two middle rows are star-exact sequences. Then the upper row is a star-exact sequence whenever the lower row is.

Note that, in the diagram (5) above, d is necessarily a regular epimorphism, $d \cdot \sigma_1 = d \cdot \sigma_2$ since \bar{g} is an epimorphism, and $\lambda \cdot \pi_1 = \lambda \cdot \pi_2$, because the pair of morphisms (χ_1, χ_2) is jointly monomorphic.

Theorem 3.3. Let \mathbb{C} be a star-regular category with saturating regular epimorphisms, enough trivial objects, and assume that \mathcal{N} -trivial objects are closed under binary products. The following conditions are equivalent:

- (a) \mathbb{C} is a 2-star-permutable category;
- (b) the Star-Upper Cuboid Lemma holds true in \mathbb{C} .

Proof. (a) \Rightarrow (b) Suppose that the lower row is a star-exact sequence. The fact that $\pi = \mathsf{Eq}(\lambda)^*$ follows from Lemma 3.2. As explained in Proposition 2.6, λ is a

regular epimorphism if and only if $cg^{\circ}\delta^*w^{\circ} \ge f^{\circ}\beta^*$. In fact we have

 $cg^{\circ}\delta^{*}w^{\circ} = cc^{\circ}cg^{\circ}gg^{\circ}\delta^{*}w^{\circ}$ (Lemma 1.1(1)) $\geqslant cc^{\circ}cg^{\circ}g^{*}g^{\circ}\delta^{*}w^{\circ}$ $(\mathsf{Eq}(g) \ge \mathsf{Eq}(g)^*)$ $= cg^{\circ}gc^{\circ}c^{*}g^{\circ}\delta^{*}w^{\circ}$ $(\mathsf{Eq}(c)\mathsf{Eq}(g)^* = \mathsf{Eq}(g)\mathsf{Eq}(c)^*;$ Definition 2.3) $= cg^{\circ}d^{\circ}d^{*}\delta^{*}w^{\circ}$ $(q\langle \mathsf{Eq}(c)^* \rangle = \mathsf{Eq}(d)^*$ by assumption) $= cg^{\circ}d^{\circ}d\delta^{*}w^{\circ}$ $(*\delta^* = \delta^*;$ Section 1.3) $= cc^{\circ}f^{\circ}\beta w^{*}w^{\circ}$ $(d \cdot q = f \cdot c, \, d \cdot \delta = \beta \cdot w)$ $= f^{\circ}\beta w^*w^{\circ}$ (Lemma 1.1(2)) $= f^{\circ} \beta \Delta_Y^*$ (Lemma 2.5) $= f^{\circ}\beta^*.$ (Section 1.3)

(b) \Rightarrow (a) Consider a commutative cube of the form (3). We construct a commutative diagram of type (5) by taking the star-kernels of c, w, d and λ , so that $\bar{g}, \bar{\delta}, \tau_1$ and τ_2 are the induced arrows between the star-kernels. By Lemma 3.2 we know that (τ_1, τ_2) is the star above the pullback (relation) of $(\bar{g}, \bar{\delta})$. By applying the Star-Upper Cuboid Lemma to this diagram we conclude that the upper row is a star-exact sequence and, consequently, λ is a regular epimorphism. By Proposition 2.6, \mathbb{C} is a 2-star-permutable category.

In the total context, Theorem 3.3 is precisely Theorem 4.3 in [12], which gives a characterisation of regular Mal'tsev categories through the Upper Cuboid Lemma, as expected. In the pointed context, the Star-Upper Cuboid Lemma gives the classical Upper 3×3 Lemma: in the pointed version of diagram (5), the back part is irrelevant (like in diagram (4)). Then the front part is a 3×3 diagram where all columns and the middle row are short exact sequences. The Star-Upper Cuboid Lemma claims that the upper row is a short exact sequence whenever the lower row is, i.e. the same as the Upper 3×3 Lemma. The pointed version of Theorem 3.3 is Theorem 5.4 of [18] which characterises normal subtractive categories. Note that in the pointed context, the Upper 3×3 Lemma is also equivalent to the Lower 3×3 Lemma as shown in [18].

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