ON 2-STAR-PERMUTABILITY IN REGULAR MULTI-POINTED CATEGORIES

MARINO GRAN AND DIANA RODELO

Abstract. 2-star-permutable categories were introduced in a joint work with Z. Janelidze and A. Ursini as a common generalisation of regular Mal’tsev categories and of normal subtractive categories. In the present article we first characterise these categories in terms of what we call star-regular pushouts. We then show that the 3 × 3 Lemma characterising normal subtractive categories and the Cuboid Lemma characterising regular Mal’tsev categories are special instances of a more general homological lemma for star-exact sequences. We prove that 2-star-permutability is equivalent to the validity of this lemma for a star-regular category.

Mathematics Subject Classification 2010: 18C05, 08C05, 18B10, 18E10

Introduction

The theory of Mal’tsev categories in the sense of A. Carboni, J. Lambek and M.C. Pedicchio [6] provides a beautiful example of the way how categorical algebra leads to a structural understanding of algebraic varieties (in the sense of universal algebra). Among regular categories, Mal’tsev categories are characterised by the property of 2-permutability of equivalence relations: given two equivalence relations $R$ and $S$ on the same object $A$, the two relational composites $RS$ and $SR$ are equal: $RS = SR$.

In the case of a variety of universal algebras this property is actually equivalent to the existence of a ternary term $p(x, y, z)$ satisfying the identities $p(x, y, y) = x$ and $p(x, x, y) = y$ [20]. In the pointed context, that is when the category has a zero object, there is also a suitable notion of 2-permutability, called “2-permutability at 0” [21]. In a variety this property can be expressed by requiring that, whenever for a given element $x$ in an algebra $A$ there is an element $y$ with $xRyS0$ (here 0 is the unique constant in $A$), then there is also an element $z$ in $A$ with $xSzR0$. The validity of this property is equivalent to the existence of a binary term $s(x, y)$ such that the identities $s(x, 0) = x$ and $s(x, x) = 0$ hold true [21]. Among regular categories, the ones where the property of 2-permutability at 0 holds true are precisely the subtractive categories introduced in [14].

The aim of this paper is to look at regular Mal’tsev and at subtractive categories as special instances of the general notion of 2-star-permutable categories introduced in collaboration with Z. Janelidze and A. Ursini in [9]. This generalisation is achieved by working in the context of a regular multi-pointed category, i.e. a regular category equipped with an ideal $\mathcal{N}$ of distinguished morphisms [7]. When

Key words and phrases. Regular multi-pointed category; star relation; Mal’tsev category; subtractive category; varieties of algebras; homological diagram lemma.
\( \mathcal{N} \) is the class of all morphisms, a situation which we refer to as the \textit{total context}, regular multi-pointed categories are just regular categories, and 2-star-permutable categories are precisely the regular Mal’tsev categories. When \( \mathcal{N} \) is the class of all zero morphisms in a pointed category, we call this the \textit{pointed context}, regular multi-pointed categories are regular pointed categories, and 2-star-permutable categories are the regular subtractive categories.

This paper follows the same line of research as in [9] which was mainly foc used on the property of 3-star-permutability, a generalised notion which captures Goursat categories in the total context and, again, subtractive categories in the pointed context.

In this work we study two remarkable aspects of the property of 2-star-permutability. First we provide a characterisation of 2-star-permutable categories in terms of a special kind of pushouts (Proposition 2.4), that we call \textit{star-regular pushouts} (Definition 2.2). Then we examine a homological diagram lemma of star-exact sequences, which can be seen as a generalisation of the \( 3 \times 3 \) Lemma, whose validity is equivalent to 2-star-permutability. We call this lemma the Star-Upper Cuboid Lemma (Theorem 3.3). The validity of this lemma turns out to give at once a characterisation of regular Mal’tsev categories (extending a result in [11]) and, in the pointed context, a characterisation of those normal categories which are subtractive (this was first discovered in [17]).

\textbf{Acknowledgement.} The authors are grateful to Zurab Janelidze for some useful conversations on the subject of the paper.

1. Star-regular categories

1.1. Regular categories and relations. A finitely complete category \( \mathcal{C} \) is said to be a \textit{regular category} [1] when any kernel pair has a coequaliser and, moreover, regular epimorphisms are stable under pullbacks. In a regular category any morphism \( f: X \to Y \) has a factorisation \( f = m \cdot p \), where \( p \) is a regular epimorphism and \( m \) is a monomorphism. The corresponding (regular epimorphism, monomorphism) factorisation system is then stable under pullbacks.

A relation \( \varrho \) from \( X \) to \( Y \) is a subobject \( \langle \varrho_1, \varrho_2 \rangle: R \rightrightarrows X \times Y \). The opposite relation, denoted by \( \varrho^o \), is given by the subobject \( \langle \varrho_2, \varrho_1 \rangle: R \rightrightarrows Y \times X \). We identify a morphism \( f : X \to Y \) with the relation \( \langle 1_X, f \rangle: X \rightrightarrows X \times Y \) and write \( f^o \) for the opposite relation. Given another relation \( \sigma \) from \( Y \) to \( Z \), the composite relation of \( \varrho \) and \( \sigma \) is a relation \( \sigma \varrho \) from \( X \) to \( Z \). With this notation, we can write the above relation as \( \varrho = \varrho_2 \varrho_1 \). The following properties are well known (see [5], for instance): we collect them in a lemma for future references.

\textbf{Lemma 1.1.} Let \( f : X \to Y \) be any morphism in a regular category \( \mathcal{C} \). Then:

(a) \( f f^o f = f \) and \( f^o f f^o = f^o \);

(b) \( f f^o = 1_Y \) if and only if \( f \) is a regular epimorphism.

A kernel pair of a morphism \( f : X \to Y \), denoted by

\[ (\pi_1, \pi_2): \text{Eq}(f) \rightrightarrows X, \]

is called an \textit{effective equivalence relation}; we write it either as \( \text{Eq}(f) = f^o f \), or as \( \text{Eq}(f) = \pi_2 \pi_1^o \), as mentioned above. When \( f \) is a regular epimorphism, then \( f \) is
the coequaliser of $\pi_1$ and $\pi_2$ and the diagram

$$\text{Eq}(f) \xrightarrow{\pi_1} X \xrightarrow{f} Y$$

is called an exact fork. In a regular category any effective equivalence relation is the kernel pair of a regular epimorphism.

1.2. Star relations. We now recall some notions introduced in [10], which are useful to develop a unified treatment of pointed and non-pointed categorical algebra. Let $\mathbb{C}$ denote a category with finite limits, and $\mathcal{N}$ a distinguished class of morphisms that forms an ideal, i.e. for any composable pair of morphisms $g, f$, if either $g$ or $f$ belongs to $\mathcal{N}$, then the composite $g \cdot f$ belongs to $\mathcal{N}$. An $\mathcal{N}$-kernel of a morphism $f : X \to Y$ is defined as a morphism $n_f : N_f \to X$ such that $f \cdot n_f \in \mathcal{N}$ and $n_f$ is universal with this property (note that such $n_f$ is automatically a monomorphism). A pair of parallel morphisms, denoted by $\sigma = (\sigma_1, \sigma_2) : S \rightrightarrows X$ with $\sigma_1 \in \mathcal{N}$, is called a star; it is called a monic star, or a star relation, when the pair $(\sigma_1, \sigma_2)$ is jointly monomorphic.

Given a relation $\varrho = (\varrho_1, \varrho_2) : R \rightrightarrows X$ on an object $X$, we denote by $\varrho^* : R^* \rightrightarrows X$ the biggest subrelation of $\varrho$ which is a (monic) star. When $\mathbb{C}$ has $\mathcal{N}$-kernels, it can be constructed by setting $\varrho^* = (\varrho_1 \cdot n_{\varrho_1}, \varrho_2 \cdot n_{\varrho_1})$, where $n_{\varrho_1}$ is the $\mathcal{N}$-kernel of $\varrho_1$. In particular, if we denote the discrete equivalence relation on an object $X$ by $\Delta_X = (1_X, 1_X) : X \rightrightarrows X$, then $\Delta_X^* = (n_{1_X}, n_{1_X})$, where $n_{1_X}$ is the $\mathcal{N}$-kernel of $1_X$.

The star-kernel of a morphism $f : X \to Y$ is a universal star $\sigma = (\sigma_1, \sigma_2) : S \rightrightarrows X$ with the property $f \cdot \sigma_1 = f \cdot \sigma_2$; it is easy to see that the star-kernel of $f$ coincides with $\text{Eq}(f)^* \rightrightarrows X$ whenever $\mathcal{N}$-kernels exist.

A category $\mathbb{C}$ equipped with an ideal $\mathcal{N}$ of morphisms is called a multi-pointed category [10]. If, moreover, every morphism admits an $\mathcal{N}$-kernel, then $\mathbb{C}$ will be called a multi-pointed category with kernels.

Definition 1.2. [10] A regular multi-pointed category $\mathbb{C}$ with kernels is called a star-regular category when every regular epimorphism in $\mathbb{C}$ is a coequaliser of a star.

In the total context stars are pairs of parallel morphisms, $\mathcal{N}$-kernels are isomorphisms, star-kernels are kernel pairs and a star-regular category is precisely a regular category. In the pointed context, the first morphism $\sigma_1$ in a star $\sigma = (\sigma_1, \sigma_2) : S \rightrightarrows X$ is the unique null morphism $S \to X$ and hence a star $\sigma$ can be identified with a morphism (its second component $\sigma_2$). Then, $\mathcal{N}$-kernels and star-kernels become the usual kernels, and a star-regular category is the same as a normal category [18], i.e. a pointed regular category in which any regular epimorphism is a normal epimorphism.

1.3. Calculus of star relations. The calculus of star relations [9] can be seen as an extension of the usual calculus of relations (in a regular category) to the regular multi-pointed context. First of all note that for any relation $\varrho : R \rightrightarrows X$ we have

$$\varrho^* = \varrho \Delta_X^*.$$

Inspired by this formula, for any relation $\varrho$ from $X$ to an object $Y$, we define

$$\varrho^* = \varrho \Delta_X^* \text{ and } \varrho^* = \Delta_Y^* \varrho.$$
Note that associativity of composition yields
\((\ast(\ast)) = (\ast\ast)\)
and so we can write \(\ast\ast\) for the above.

For any relation \(\sigma\) (from some object \(Y\) to \(Z\)), the associativity of composition also gives
\[(\ast\ast) = \ast(\ast),\]
and
\[(\ast\ast) = \ast.\]

It is easy to verify that for any morphism \(f : X \to Y\) we have
\(f^* = \ast f\) and \(\ast f^* = f^*\).

2. 2-star-permutability and star-regular pushouts

Recall that a finitely complete category \(C\) is called a Mal’tsev category when any reflexive relation in \(C\) is an equivalence relation \([6, 5]\). We recall the following well known characterisation of the regular categories which are Mal’tsev categories:

**Proposition 2.1.** A regular category \(C\) is a Mal’tsev category if and only if the composition of effective equivalence relations in \(C\) is commutative:

\[\text{Eq}(f)\text{Eq}(g) = \text{Eq}(g)\text{Eq}(f)\]

for any pair of regular epimorphisms \(f\) and \(g\) in \(C\) with the same domain.

There are many known characterisations of regular Mal’tsev categories (see Section 2.5 in \([2]\), for instance, and references therein). The one that will play a central role in the present work is expressed in terms of commutative diagrams of the form

\[
\begin{array}{ccc}
C & \xrightarrow{c} & A \\
\downarrow{g} & \searrow{f} & \downarrow{s} \\
D & \xrightarrow{d} & B,
\end{array}
\]

(1)

where \(f\) and \(g\) are split epimorphisms \((f \cdot s = 1_B, g \cdot t = 1_D)\), \(f \cdot c = d \cdot g, s \cdot d = c \cdot t\), and \(c\) and \(d\) are regular epimorphisms. A diagram of type (1) is always a pushout; it is called a regular pushout \([4]\) (alternatively, a double extension \([15,13]\)) when, moreover, the canonical morphism \((g, c) : C \to D \times_B A\) to the pullback \(D \times_B A\) of \(d\) and \(f\) is a regular epimorphism. Among regular categories, Mal’tsev categories can be characterized as those ones where any square (1) is a regular pushout: this easily follows from the results in \([4]\), and a simple proof of this fact is given in \([12]\).

Observe that a commutative diagram of type (1) is a regular pushout if and only if \(c g^o = f^o d\) or, equivalently, \(g c^o = d f^o\). This suggests to introduce the following notion:

**Definition 2.2.** A commutative diagram (1) is a star-regular pushout if it satisfies the identity \(c g^o = f^o d\) (or, equivalently, \(g c^o = d f^o\)).
Diagrammatically, the property of being a star-regular pushout can be expressed as follows. Consider the commutative diagram

\[
\begin{array}{cccc}
N_g & \rightarrow & N_x \\
\downarrow & & \downarrow \\
C & \rightarrow & A \\
\end{array}
\]

where \((D \times_B A, x, y)\) is the pullback of \((f, d), m \cdot p\) is the (regular epimorphism, monomorphism) factorisation of the induced morphism \((g, c) : C \rightarrow D \times_B A\). Then the identity \(cg^\circ = ba^\circ\) allows one to identify \(cg^\circ\) with the relation \((a \cdot n_a, b \cdot n_a)\), while \(f^\circ d = xy^\circ\) says that \(f^\circ d^*\) can be identified with the relation \((x \cdot n_x, y \cdot n_x)\). Accordingly, diagram (1) is a star-regular pushout precisely when the dotted arrow from \(N_a\) to \(N_x\) is an isomorphism. Notice that in the total context the \(N\)-kernels are isomorphisms, so that \(m\) is an isomorphism if and only if (1) is a regular pushout, as expected.

The “star-version” of the notion of Mal’tsev category can be defined as follows:

**Definition 2.3.** A regular multi-pointed category with kernels \(C\) is said to be a 2-star-permutable category if

\[
\text{Eq}(f)\text{Eq}(g)^* = \text{Eq}(g)\text{Eq}(f)^*
\]

for any pair of regular epimorphisms \(f\) and \(g\) in \(C\) with the same domain.

One can check that the equality \(\text{Eq}(f)\text{Eq}(g)^* = \text{Eq}(g)\text{Eq}(f)^*\) in the definition above can be actually replaced by \(\text{Eq}(f)\text{Eq}(g)^* \leq \text{Eq}(g)\text{Eq}(f)^*\).

In the total context the property of 2-star-permutability characterises the regular categories which are Mal’tsev. In the pointed context this same property characterises the regular categories which are subtractive [16] (this follows from the characterisation of subtractivity given in Theorem 6.9 in [17]).

The next result gives a useful characterisation of 2-star-permutable categories. Given a commutative diagram of type (1), we write \(g(\text{Eq}(c))\) and \(g(\text{Eq}(c)^*)\) for the direct images of the relations \(\text{Eq}(c)\) and \(\text{Eq}(c)^*\) along the split epimorphism \(g\). The vertical split epimorphisms are such that both the equalities \(g(\text{Eq}(c)) = \text{Eq}(d)\) and \(g(\text{Eq}(c)^*) = \text{Eq}(d)^*\) hold true in \(C\).

**Proposition 2.4.** For a regular multi-pointed category with kernels \(C\) the following statements are equivalent:

(a) \(C\) is a 2-star-permutable category;

(b) any commutative diagram of the form (1) is a star-regular pushout.
Proof. (a) ⇒ (b) Given a pushout \([1]\) we have

\[
\begin{align*}
  f \circ d^* &= cc^* f \circ d^* \quad \text{(Lemma [1,12])} \\
  &= cg^* d^* \quad (f \cdot c = d \cdot g) \\
  &= cg^* gc^* g^o \quad (\text{Eq}(d)^* = g(\text{Eq}(c)^*)) \\
  &= cc^* cg^* g^o \quad (\text{Eq}(g)\text{Eq}(c)^* = \text{Eq}(c)\text{Eq}(g)^* \text{ by Definition [2,3]}) \\
  &\leq cc^* cg^* g g^o \quad (g^* \leq g) \\
  &= cg^*. \quad \text{(Lemma [1,11])}
\end{align*}
\]

Since \(cg^o\) is the largest star contained in \(cg^o\), it follows that \(f \circ d^* \leq cg^o\). The inclusion \(cg^o \leq f \circ d^*\) always holds, so that \(cg^o = f \circ d^*\).

(b) ⇒ (a) Let us consider regular epimorphisms \(f: X \to Y\) and \(g: X \to Z\). We want to prove that \(\text{Eq}(f)\text{Eq}(g)^* = \text{Eq}(g)\text{Eq}(f)^*\). For this we build the following diagram

\[
\begin{array}{ccc}
  \text{Eq}(f) & \xrightarrow{c} & g(\text{Eq}(f)) \\
  \pi_1 \downarrow & & \downarrow \pi_2 \\
  X & \xrightarrow{\rho_1} & \rho_2 \\
  f \downarrow & & \downarrow g \\
  Y & \xrightarrow{\pi_2} & Z
\end{array}
\]

that represents the regular image of \(\text{Eq}(f)\) along \(g\). The relation \(g(\text{Eq}(f)) = (\rho_1, \rho_2)\) is reflexive and, consequently, \(\rho_1\) is a split epimorphism. By assumption, we then know that the equality

\[
(A) \quad \rho_1^o g^* = c\pi_1^o
\]

holds true. This implies that

\[
\begin{align*}
  \text{Eq}(f)\text{Eq}(g)^* &= \pi_2\pi_1^o g^o \quad (\text{Eq}(f) = \rho_1 \cdot c) \\
  &= \pi_2 c^o \rho_1^o g^* \\
  &= \pi_2 c^o c\pi_1^o \quad (A) \\
  &\leq \pi_2 c^o c\pi_1^o \pi_2\pi_1^o \quad (\Delta_{\text{Eq}(f)} \leq \pi_2^o \pi_2) \\
  &= \text{Eq}(g)\pi_2\pi_1^o \quad (\pi_2(\text{Eq}(c)) = \text{Eq}(g)) \\
  &= \text{Eq}(g)\text{Eq}(f)^*,
\end{align*}
\]

where the equality \(\pi_2(\text{Eq}(c)) = \text{Eq}(g)\) follows from the fact that the split epimorphisms \(\pi_2\) and \(\rho_2\) induce a split epimorphism from \(\text{Eq}(c)\) to \(\text{Eq}(g)\).

In the total context, Proposition [24] gives the characterisation of regular Mal’tsev categories through regular pushouts (see [4] and Proposition 3.4 of [12]), as expected. In the pointed context, condition (b) of Proposition [24] translates into the pointed version of the right saturation property [9] for any commutative diagram of type \([1]\): the induced morphism \(\tilde{c}: \text{Ker}(g) \to \text{Ker}(f)\), from the kernel of \(g\) to the kernel of \(f\) is also a regular epimorphism. This can be seen by looking at diagram \([2]\), where the \(N\)-kernels now represent actual kernels, so that \(\text{Ker}(a) = \text{Ker}(x) = \text{Ker}(f)\). \(\square\)
2.1. **The star of a pullback relation.** Consider the pullback relation \( \pi = (\pi_1, \pi_2) \) of a pair \((g, \delta)\) of morphisms as in the diagram

\[
\begin{array}{ccc}
W \times_D C & \overset{\pi_2}{\rightarrow} & C \\
\downarrow \pi_1 & & \downarrow g \\
W & \overset{\delta}{\rightarrow} & D.
\end{array}
\]

The **star of the pullback relation** \( \pi \) is defined as \( \pi^* = \pi \Delta^*_W \). It can be described as the universal relation \( \nu = (\nu_1, \nu_2) \) from \( W \) to \( C \) such that \( \nu_1 \in N \) and \( \delta \cdot \nu_1 = g \cdot \nu_2 \) as in the diagram

\[
\begin{array}{ccc}
(W \times_D C)^* & \overset{\nu_2}{\rightarrow} & C \\
\downarrow \nu_1 & & \downarrow \nu_2 \\
(W \times_D C) & \overset{\pi_2}{\rightarrow} & C \\
\downarrow \pi_1 & & \downarrow \delta \\
W & \overset{\delta}{\rightarrow} & D.
\end{array}
\]

where \( n_{\pi_1} \) is the \( N \)-kernel of \( \pi_1 \), \( \nu_1 = \pi_1 \cdot n_{\pi_1} \) and \( \nu_2 = \pi_2 \cdot n_{\pi_1} \).

By using the composition of relations one has the equalities \( \pi = \pi_2 \pi_1^* = g^* \delta^* \), so that

\[
\pi^* = \pi_2 \pi_1^* = g^* \delta^*.
\]

In the total context, the star of a pullback relation is precisely that pullback relation. In the pointed context, the star of the pullback (relation) of \((g, \delta)\) is given by \( \pi^* = (0, \ker(g)) \).

A morphism \( f: X \rightarrow Y \) in a multi-pointed category with kernels is said to be **saturating** [9] when the induced dotted morphism from the \( N \)-kernel of \( 1_X \) to the \( N \)-kernel of \( 1_Y \) making the diagram

\[
\begin{array}{ccc}
N_{1_X} & \overset{f^*}{\leftarrow} & N_{1_Y} \\
\downarrow n_{1_X} & & \downarrow n_{1_Y} \\
X & \overset{f}{\rightarrow} & Y
\end{array}
\]

commute is a regular epimorphism. All morphisms are saturating in the pointed context. This is also the case for any **quasi-pointed category** [3], namely a finitely complete category with an initial object \( 0 \) and a terminal object \( 1 \) such that the arrow \( 0 \rightarrow 1 \) is a monomorphism. As in the pointed case, it suffices to choose for \( N \) the class of morphisms which factor through the initial object \( 0 \). In this case we shall speak of the **quasi-pointed context**. In the total context, any regular epimorphism is saturating. The proof of the following result is straightforward:

**Lemma 2.5.** [9] *Let \( \mathcal{C} \) be a regular multi-pointed category with kernels. For a morphism \( f: X \rightarrow Y \) the following conditions are equivalent:*

(a) \( f \) is saturating;
(b) \( \Delta_Y^* = f^*f^* \).

The next result gives a characterisation of 2-star-permutable categories which will be useful in the following section.

**Proposition 2.6.** For a regular multi-pointed category \( \mathcal{C} \) with kernels and saturating regular epimorphisms the following statements are equivalent:
(a) \( C \) is a 2-star-permutable category;
(b) for any commutative diagram

\[
\begin{array}{c}
\begin{array}{ccc}
(W \times_D C)^* & \xrightarrow{\lambda} & (Y \times_B A)^* \\
\nu_1 & \searrow & \chi_1 \\
W & \xrightarrow{c} & A \\
\downarrow & & \downarrow \chi_2 \\
D & \xrightarrow{w} & Y \\
\delta & \searrow & \beta \\
\downarrow & & \downarrow f \\
B, & \xrightarrow{s} & B.
\end{array}
\end{array}
\]

where the front square is of the form \((1)\), \(\beta \cdot w = d \cdot \delta\), \(w\) is a regular epimorphism, \(((W \times_D C)^*, \nu_1, \nu_2)\) and \(((Y \times_B A)^*, \chi_1, \chi_2)\) are stars of the corresponding pullback relations, then the comparison morphism \(\lambda: (W \times_D C)^* \to (Y \times_B A)^*\) is also a regular epimorphism.

Proof. \((a) \Rightarrow (b)\) To prove that the arrow \(\lambda\) in the cube above is a regular epimorphism, we must show that \(\langle \chi_1, \chi_2 \rangle \lambda\) in the commutative diagram

\[
\begin{array}{c}
\begin{array}{ccc}
(W \times_D C)^* & \xrightarrow{\lambda} & (Y \times_B A)^* \\
\nu_1 & \searrow & \chi_1 \\
W \times C & \xrightarrow{w \times c} & Y \times A \\
\downarrow & & \downarrow \langle \chi_1, \chi_2 \rangle \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & 
\end{array}
\end{array}
\]

is the (regular epimorphism, monomorphism) factorisation of the morphism \(\langle w \cdot \nu_1, c \cdot \nu_2 \rangle: (W \times_D C)^* \to Y \times A\). That is, we must have \(c \nu_1 \nu_2 = \chi_2 \chi_1\) or, equivalently, \(cg \delta^* w = f^\circ \beta^*\), since \(\nu_2 \nu_1 = \nu^* = g^\circ \delta^*\) and \(\chi_2 \chi_1 = \chi^* = f^\circ \beta^*\) (see Section 2.1).

The front square of diagram \((3)\) is a star-regular pushout by Proposition \(2.4\), which means that the equality

\[(B) \quad cg^* = f^\circ d^*\]

holds true. Now, we always have

\[
\begin{align*}
ge g^\circ \delta^* w^o & \leq f^\circ d^* w^o & \text{(commutativity of the front face of (3))} \\
& = f^\circ \beta^* w^o & (d \cdot \delta = \beta \cdot w) \\
& = f^\circ \beta \Delta Y & \text{(Lemma 2.5)} \\
& = f^\circ \beta^*. \\
\end{align*}
\]

The other inequality follows from

\[
\begin{align*}
ge g^\circ \delta^* w^o & \geq cg^\circ \delta^* w^o & \text{(\(g^\circ \geq g^\circ\))} \\
& = f^\circ d^* w^o & \text{(B)} \\
& = f^\circ d^* w^o & (* \delta^* = \delta^*; \text{Section 1.3}) \\
& = f^\circ \beta^*. & \text{(as in the inequality above)}
\end{align*}
\]
(b) ⇒ (a) A commutative diagram of type (1) induces a commutative cube

\[
\begin{array}{c}
\text{N}_g \ \xrightarrow{\lambda} \ (D \times_B A)^* \\
g \cdot n_g \downarrow \downarrow \ \vdash \lambda \rightarrow \rightarrow \\
C \ \xrightarrow{c} \ C \\
g \downarrow t \downarrow \downarrow f \downarrow s \\
D \ \xrightarrow{d} \ D \\
\end{array}
\]

where \( \nu = (g \cdot n_g, n_g) \) is the star of the pullback (relation) of \((g, 1_D)\). By assumption, \( \lambda \) is a regular epimorphism which translates into the equality \( cg^o 1^*_D 1_D = f^o d^* \), as observed in the first part of the proof. We get the equality \( cg \circ 1^* = f \circ d^* \), and this proves that diagram (1) is a star-regular pushout and, consequently, that \( \mathcal{C} \) is a 2-star-permutable category by Proposition 2.4.

\( \square \)

In the total context, Proposition 2.6 is the “star version” of Proposition 3.6 in [12] (see also Proposition 4.1 in [4]). In the pointed context condition (b) of Proposition 2.6 also reduces to the pointed version of the right saturation property (in the sense of [9]). Indeed, in this context that condition says that, in the following commutative diagram

\[
\begin{array}{c}
\text{Ker}(g) \ \xrightarrow{\overline{c}} \ \text{Ker}(f) \\
0 \downarrow \downarrow \downarrow \downarrow 0 \\
W \ \xrightarrow{\delta} \ Y \\
D \ \xrightarrow{d} \ B, \\
\end{array}
\]

the induced arrow \( \overline{c} : \text{Ker}(g) \rightarrow \text{Ker}(f) \) is a regular epimorphism.

We conclude this section with the pointed version of Propositions 2.4 and 2.6.

**Corollary 2.7.** (see Theorem 2.12 in [9]) For a pointed regular category \( \mathcal{C} \) the following statements are equivalent:

(a) \( \mathcal{C} \) is a subtractive category;

(b) any commutative diagram of the form (1) is right saturated, i.e. the comparison morphism \( \overline{c} : \text{Ker}(g) \rightarrow \text{Ker}(f) \) is a regular epimorphism.

### 3. The Star-Cuboid Lemma

In [12] it was shown that regular Mal’tsev categories can be characterised through the validity of a homological lemma called the Upper Cuboid Lemma, a strong form of the denormalised \( 3 \times 3 \) Lemma [4, 19, 11]. We are now going to extend this result to the star-regular context. We shall then observe that, in the pointed
context, it gives back the classical Upper $3 \times 3$ Lemma characterising subtractive normal categories.

### 3.1. $\mathcal{N}$-trivial objects

An object $X$ in a multi-pointed category is said to be $\mathcal{N}$-trivial when $1_X \in \mathcal{N}$. If a composite $f \circ g$ belongs to $\mathcal{N}$ and $g$ is a strong epimorphism, then also $f$ belongs to $\mathcal{N}$. This implies that $\mathcal{N}$-trivial objects are closed under strong quotients. One says that a multi-pointed category $\mathcal{C}$ has enough trivial objects when $\mathcal{N}$ is a closed ideal, i.e. any morphism in $\mathcal{N}$ factors through an $\mathcal{N}$-trivial object and, moreover, the class of $\mathcal{N}$-trivial objects is closed under subobjects and squares, where the latter property means that, for any $\mathcal{N}$-trivial object $X$, the object $X^2 = X \times X$ is $\mathcal{N}$-trivial. An equivalent way of expressing the existence of enough trivial objects is recalled in the following:

**Proposition 3.1.** [8] Let $\mathcal{C}$ be a regular multi-pointed category with kernels. The following conditions are equivalent:

(a) if $(\sigma_1, \sigma_2) : S \rightrightarrows X$ is a relation on $X$ such that $\sigma_1 \cdot n \in \mathcal{N}$ and $\sigma_2 \cdot n \in \mathcal{N}$, then $n \in \mathcal{N}$;

(b) $\mathcal{C}$ has enough trivial objects.

In the following we shall also assume that $\mathcal{N}$-trivial objects are closed under binary products. Remark that in the total and in the (quasi-)pointed contexts there are enough trivial objects, and $\mathcal{N}$-trivial objects are closed under binary products.

Under the presence of enough trivial objects the assumption that $\mathcal{N}$-trivial objects are closed under binary products is equivalent to the following condition:

(a') if $(\sigma_1, \sigma_2) : S \rightrightarrows X \times Y$ is a relation from $X$ to $Y$ such that $\sigma_1 \cdot n \in \mathcal{N}$ and $\sigma_2 \cdot n \in \mathcal{N}$, then $n \in \mathcal{N}$.

Whenever the category has enough trivial objects, condition (a') implies that star-kernels “commute” with stars of pullback relations:

**Lemma 3.2.** Let $\mathcal{C}$ be a multi-pointed category with kernels, enough trivial objects, and assume that $\mathcal{N}$-trivial objects are closed under binary products. Given a commutative cube

\[
\begin{array}{ccc}
(W \times_D C)^* & \xrightarrow{\lambda} & (Y \times_B A)^* \\
\nu_1 \downarrow & & \downarrow \nu_2 \\
C & \xrightarrow{c} & A \\
\downarrow \chi_1 & & \downarrow \chi_2 \\
W & \xrightarrow{g} & Y \\
\delta \downarrow & & \downarrow \beta \\
D & \xrightarrow{w} & B \\
\end{array}
\]

in $\mathcal{C}$, consider the star-kernels of $c$, $d$ and $w$, and the induced morphisms $\overline{\tau} : \text{Eq}(w)^* \to \text{Eq}(d)^*$ and $\overline{\varphi} : \text{Eq}(c)^* \to \text{Eq}(d)^*$. Then the following constructions are equivalent (up to isomorphism):

- taking the horizontal star-kernel of $\lambda$ and then the induced morphisms $\text{Eq}(\lambda)^* \to \text{Eq}(w)^*$ and $\text{Eq}(\lambda)^* \to \text{Eq}(c)^*$;
• taking the star of the pullback (relation) of $\bar{\pi}$ and $\bar{\delta}$ and then the induced morphisms $(\text{Eq}(w)^* \times_{\text{Eq}(d)^*} \text{Eq}(c)^*)^* \Rightarrow (W \times_D C)^*$.

Proof. This follows easily by the usual commutation of kernel pairs with pullbacks and condition (a').

In a star-regular category, a (short) star-exact sequence is a diagram

$$
\text{Eq}(f)^* \xrightarrow{f_1} X \xrightarrow{f} Y
$$

where $\text{Eq}(f)^*$ is a star-kernel of $f$ and $f$ is a coequaliser of $f_1$ and $f_2$ (which, by star-regularity, is the same as to say that $f$ is a regular epimorphism). In the total context, a star-exact sequence is just an exact fork, while in the (quasi-)pointed context it is a short exact sequence in the usual sense.

The Star-Upper Cuboid Lemma
Let $C$ be a star-regular category. Consider a commutative diagram of morphisms and stars in $C$

where the three diamonds are stars of pullback (relations) of regular epimorphisms along arbitrary morphisms (so that $P = (\text{Eq}(w)^* \times S \text{Eq}(c)^*)^*$) and the two middle rows are star-exact sequences. Then the upper row is a star-exact sequence whenever the lower row is.

Note that, in the diagram (5) above, $d$ is necessarily a regular epimorphism, $d \cdot \sigma_1 = d \cdot \sigma_2$ since $\bar{g}$ is an epimorphism, and $\lambda \cdot \pi_1 = \lambda \cdot \pi_2$, because the pair of morphisms $(\chi_1, \chi_2)$ is jointly monomorphic.

Theorem 3.3. Let $C$ be a star-regular category with saturating regular epimorphisms, enough trivial objects, and assume that $\mathcal{N}$-trivial objects are closed under binary products. The following conditions are equivalent:

(a) $C$ is a 2-star-permutable category;

(b) the Star-Upper Cuboid Lemma holds true in $C$.

Proof. (a) $\Rightarrow$ (b) Suppose that the lower row is a star-exact sequence. The fact that $\pi = \text{Eq}(\lambda)^*$ follows from Lemma 3.2. As explained in Proposition 2.6, $\lambda$ is a
regular epimorphism if and only if \(cg^o\delta^*w^o \geq f^o\beta^*.\) In fact we have

\[\begin{align*}
&cg^o\delta^*w^o = cc^o cg^o g^o \delta^*w^o \quad \text{(Lemma 1.1 (1))} \\
&\geq cc^o cg^o g^o \delta^*w^o \quad \text{(Eq}(g) \geq \text{Eq}(g)^*) \\
= cg^o gc^o c^o g^o \delta^*w^o \quad \text{(Eq}(c)\text{Eq}(g)^*) = \text{Eq}(g)\text{Eq}(c)^*; \text{ Definition 2.3)} \\
= cg^o d^o d^o \delta^*w^o \quad (g\text{Eq}(c)^*) = \text{Eq}(d)^* \text{ by assumption} \\
= cg^o d^o d^o \delta^*w^o \quad (* \delta^* = \delta^*; \text{ Section 1.3)} \\
= cc^o f^o \beta w^o w^o \quad (d \cdot g = f \cdot c, d \cdot \delta = \beta \cdot w) \\
= f^o \beta w^o w^o \quad \text{(Lemma 1.1 (2))} \\
= f^o \beta \Delta_{\gamma} \quad \text{(Lemma 2.3)} \\
= f^o \beta^*. \quad \text{(Section 1.3)}
\end{align*}\]

(b) \(\Rightarrow\) (a) Consider a commutative cube of the form (6). We construct a commutative diagram of type (5) by taking the star-kernels of \((b)\Rightarrow\) row is a star-exact sequence and, consequently, \(\lambda\) is a regular epimorphism. By Proposition 2.6, we know that \((\tau_1, \tau_2)\) is the star above the pullback (relation) of \((\bar{g}, \delta)\). By applying the Star-Upper Cuboid Lemma to this diagram we conclude that the upper row is a short exact sequence and, consequently, \(\lambda\) is a regular epimorphism. By Proposition 2.6 \(C\) is a 2-star-permutable category.

In the total context, Theorem 5.3 is precisely Theorem 4.3 in [12], which gives a characterisation of regular Mal’tsev categories through the Upper Cuboid Lemma, as expected. In the pointed context, the Star-Upper Cuboid Lemma gives the classical Upper 3 \times 3 Lemma: in the pointed version of diagram (5), the back part is irrelevant (like in diagram (4)). Then the front part is a 3 \times 3 diagram where all columns and the middle row are short exact sequences. The Star-Upper Cuboid Lemma claims that the upper row is a short exact sequence whenever the lower row is, i.e. the same as the Upper 3 \times 3 Lemma. The pointed version of Theorem 5.3 is Theorem 5.4 of [18] which characterises normal subtractive categories. Note that in the pointed context, the Upper 3 \times 3 Lemma is also equivalent to the Lower 3 \times 3 Lemma as shown in [18].

References


Institut de Recherche en Mathématique et Physique, Université catholique de Louvain, Chemin du Cyclotron 2, 1348 Louvain-la-Neuve, Belgium

Departamento de Matemática, Faculdade de Ciências e Tecnologia, Universidade do Algarve, Campus de Gambelas, 8005–139 Faro, Portugal, and, CMUC, Universidade de Coimbra, 3001-454 Coimbra, Portugal, dordeloe@ualg.pt