GRAY CODES FOR NONCROSSING AND NONNESTING PARTITIONS OF CLASSICAL TYPES

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Abstract. In this paper we present Gray codes for the sets of noncrossing partitions associated with the classical Weyl groups, and for the set of nonnesting partitions of type B. An algorithm for the generation of type D nonnesting partitions is developed in which a Gray code is given for those partitions having a zero-block, while the remaining are arranged in lexicographic order.

1. Introduction

One of the fundamental topics on the area of combinatorial algorithms is the efficient generation of all objects in a specific combinatorial class in such a way that each item is generated exactly once, hence producing a listing of all objects in the considered class. A common approach to this problem has been the generation of the objects of a combinatorial class in such a way that two consecutive items differ in some pre–specified, usually small, way. Such generation is usually called a Gray code and, amongst the various applications of combination generation, Gray codes are especially valued since they usually involve recursive constructions which provide new insights into the structure of the combinatorial class [10].

The problem of finding a Gray code for a combinatorial class can be formulated as a Hamilton path/cycle problem: the vertices of the graph are the objects themselves, and two vertices are joined by an edge if they differ in a pre–specified way. This graph has a Hamilton path if and only if the required listing of the objects exist. A Hamilton cycle corresponds to a Gray code in which the first and last objects differ in the pre–specified way.

In [7], Huemer *et al.* defined a graph structure on the set of classical noncrossing partitions by declaring two partitions adjacent if they differ by the move of a single element from one block to another, and showed that this set has a Hamilton cycle. Recently, this result was also obtained for the set of all classical nonnesting partitions [6]. Classical noncrossing and nonnesting partitions are members of a broader class of objects, known as Coxeter-Catalan objects, associated with the symmetric group $\mathbb{S}_n$. Coxeter-Catalan combinatorics is an active field of research, having at its core the study of objects associated with a Coxeter group W and counted by the W-Catalan numbers, a generalization of the classical Catalan numbers. Two of these objects are the noncrossing partitions, associated to each finite Coxeter group, and the nonnesting partitions, defined for each

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crystallographic Coxeter group W. When W is one of the classical (finite) Coxeter groups, the sets of noncrossing and nonnesting partitions, denoted $NC(W)$ and $NN(W)$ respectively, have nice combinatorial descriptions in terms of permutation groups: the symmetric group is a representative for type $\mathbf{A}_{n-1}$, the hyperoctahedral group for type $\mathbf{B}_n$, and the even-signed permutation group for type $\mathbf{D}_n$, and their corresponding W-Catalan numbers are $\frac{1}{n+1}\binom{2n}{n}$, $\binom{2n}{n}$ and $\binom{2n}{n} - \binom{2n-2}{n-1}$, respectively. These two sets of objects are not only counted by the same numbers, but are deeply connected as they share many enumerative and combinatorial properties. Nevertheless, there are many gaps in our understanding of the relations between noncrossing and nonnesting partitions (see [1] for a comprehensive account of these objects).

In this paper we generalize the type **A** results of [6, 7] for Weyl groups of type **B** and **D**, constructing Hamilton cycles for the sets of noncrossing partitions of types **B** and **D**, and nonnesting partitions of type **B**, where now we declare two type **B** (or **D**) partitions adjacent if they differ by the move of at most two elements from one block to another. Although computational examples suggest that the set of type **D** nonnesting partitions is hamiltonian as well, we were only able to construct a Hamilton cycle on the subset formed by all those type **D** nonnesting partitions without zero-block. In [6] we designed an efficient algorithm for a lexicographic combinatorial generation of nonnesting set partitions of type **A**, using a characterization of such partitions in terms of arcs. This characterization is used in this paper to generate all nonnesting partitions of type **B** and all nonnesting partitions of type **D** with zero-block in lexicographic order. The concatenation of the Hamilton path formed by all type **D** nonnesting partitions without zero-block with the lexicographic ordering of those nonnesting partitions with zero-block gives a generating algorithm for all nonnesting partitions of type **D**.

The remainder of this paper is structured as follows. In Section 2 we review the usual combinatorial models for noncrossing and nonnesting partitions of types $\mathbf{A}, \mathbf{B}$ and **D**. The algorithm which will be the main tool for our constructions is presented in Section 3, and subsequently used to obtain Hamilton cycles for the sets of noncrossing partitions of types $\mathbf{A}, \mathbf{B}$ and **D**, nonnesting partitions of type **B**, and type **D** nonnesting partitions without zero-block. Our Hamilton cycle for type **A** noncrossing partitions is different from the one obtained in [7], and is needed for the construction of a Hamilton cycle in the set of noncrossing partitions of type **D**. The generation in lexicographic order of type **B** nonnesting partitions, and of those type **D** nonnesting partitions having zero-block are presented in Sections 3.4 and 3.5.

2. Preliminaries and notation

Let S be a finite non empty set. Throughout this paper, let

$$[n] = \{1, \ldots, n\},$$

$$[\pm n] = \{\overline{1}, \overline{2}, \ldots, \overline{n}, 1, 2, \ldots, n\}$$

for any positive integer n, where we set $\overline{i} := -i$.

A partition of S is a collection of mutually disjoint nonempty subsets of S, called blocks, whose union is the entire set S. The set of all partitions of S is denoted by $\Pi(S)$. When $S = [n]$ or $S = [\pm n]$, we simply write $\Pi(n)$ and $\Pi(\pm n)$ instead of $\Pi([n])$ and $\Pi([\pm n])$. A generic set partition with no further restriction is sometimes referred to as partition of type **A**, because the lattice of all set partitions of a set of n elements can be interpreted as the intersection lattice for the hyperplane arrangement corresponding to a root system of type $\mathbf{A}_{n-1}$, *i.e.* the symmetric group of n letters, $\mathbb{S}_n$.

Given partitions π, σ of S, their distance $D(\pi,\sigma)$ is defined as the minimum number of elements that must be deleted from S so that the two residual induced partitions are identical:

$$D(\pi,\sigma) = \min\{|A^c| : A \subseteq S,\, \pi_{|A} = \sigma_{|A}\},$$

where A^c is the complement of A in S and $\pi_{|A}$ is the partition of A induced by π, obtained by removing from π all the integers not in A. In other words, $D(\pi,\sigma)$ is equal to the minimum number of integers that must be moved between blocks of π, possibly creating a new block, so that the resulting partitions is σ. We call a pair (i,j) an arc of the partition π if i and $j \neq i$ occur in the same block and there is no other element k in the same block satisfying $i < k < j$. The first coordinate i of an arc (i,j) is called an opener and the second coordinate j is a closer of π. For example, $\pi = 125/34/6 \in \Pi(6)$ has three blocks $\{1,2,5\}, \{3,4\}$, and $\{6\}$, and set of arcs $\{(1,2),(2,5),(3,4)\}$ (when there is no ambiguity we simplify the partition notation by removing the parenthesis and the commas within each block of a partition). The set of openers and closers are, respectively, $\{1,2,3\}$ and $\{2,4,5\}$. The standard representation of a partition $\pi \in \Pi(n)$ is obtained by placing in a horizontal line the letters $1,2,\ldots,n$, in this order, and drawing an arc between the opener and the closer of each arc (i,j) of π.

Definition 1. A partition $\pi \in \Pi(n)$ is said to be noncrossing (resp. nonnesting) if it does not have two arcs (i,k) and (j,ℓ) such that $i < j < k < \ell$ (resp. $i < j, \ell < k$).

In other words, π is noncrossing (resp. nonnesting) if and only if the standard representation of π does not have two arcs which cross each other (resp. two arcs one of which nests in the other). We denote by $NC(n)$ the set of all noncrossing partitions of $[n]$ and by $NN(n)$ the set of all nonnesting partitions of $[n]$. See Figure 2.1 for the standard representations of $\pi = 125/34/6 \in NC(6)$ and $\sigma = 125/3/46 \in NN(6)$. Note that these partitions have distance $D(\pi,\sigma) = 1$.

A noncrossing partition π of $[n]$ can also be represented in a circular diagram, called the circular representation, obtained by placing clockwise around a circle the integers $1,2,\ldots,n$, and drawing a direct edge from vertex i to vertex j whenever (i,j) is an arc of π, or when i is the least and j is the greater element of a block. Then, π will be noncrossing if and only if for every pair of distinct blocks B, B' of π, the convex hulls of the vertices representing B and B' are disjoint. We may view the circular representation as obtained by bending round the horizontal line of the standard representation of π.

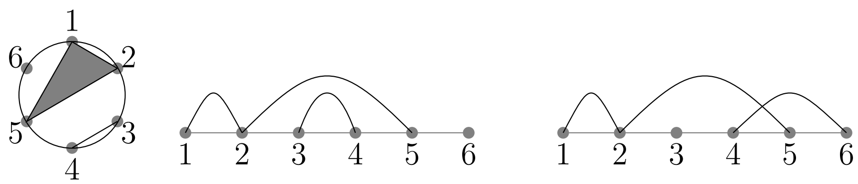

FIGURE 2.1. Circular and standard representation of $125/34/6 \in NC(6)$ and standard representation of $125/3/46 \in NN(6)$.

In the last years, these two classes of partitions have received great attention and have been generalized in many directions, both combinatorially and algebraically. One of these directions lead to the generalization of noncrossing partitions to each finite reflection group W, denoted $NC(W)$, by Bessis [3], Brady and Watt [4], where $NC(A_{n-1})$ is identified with $NC[n]$. In other direction, Postnikov [11, Remark 2] defined the set $NN(W)$ of nonnesting partitions for each crystallographic reflection group W, where $NN(A_{n-1})$ is identified with $NN(n)$.

In this paper we consider noncrossing and nonnesting partitions over the classical Weyl groups, which have combinatorial descriptions in terms of permutation groups: the symmetric group $\mathbb{S}_n$ is a representative for type $\mathbf{A}_{n-1}$, the hyperoctahedral group for type $\mathbf{B}_n$, and the even-signed permutation group for type $\mathbf{D}_n$. Next, we recall the combinatorial models for the noncrossing and nonnesting partitions of types $\mathbf{B}$ and $\mathbf{D}$ following [1, 2] and referring to [5, 8] for any undefined terminology and comprehensive references on Coxeter groups.

2.1. **Noncrossing and nonnesting partitions of types B and D.** The combinatorial models for noncrossing and nonnesting partitions of types $\mathbf{B}$ and $\mathbf{D}$ are based on the notion of a type $\mathbf{B}$ partition introduced by Reiner in [11]. A partition of type $\mathbf{B}_n$ is a partition π of the set $[\pm n]$ such that if B is a block of π then $-B = \{\bar{i} : i \in B\}$ is also a block of π, and there is at most one block, called the zero-block, which satisfies $B = -B$.

A partition of type $\mathbf{D}_n$ is a partition of type $\mathbf{B}_n$ with the additional property that the zero-block, when it is eventually present, has more than two elements. The set of all partitions of type $\mathbf{B}_n$ is denoted by $\Pi_B(n)$, and its subset consisting of all partitions of type $\mathbf{D}_n$ is denoted by $\Pi_D(n)$. The posets $\Pi_B(n)$ and $\Pi_D(n)$ are geometric lattices which are isomorphic to the intersection lattice of the $\mathbf{B}_n$ and $\mathbf{D}_n$ Coxeter hyperplane arrangement, respectively.

For example, $\pi = \bar{1}1/235/\bar{2}\,\bar{3}\,\bar{5}/4/\bar{4}$ is a partition of type $\mathbf{B}_5$, but not of type $\mathbf{D}_5$, with blocks $\{2,3,5\}$, $\{\bar{2},\bar{3},\bar{5}\}$, $\{4\}$, $\{\bar{4}\}$ and zero-block $\{\bar{1},1\}$. Its arcs are $\{(\bar{5},\bar{3})$, $(\bar{3},\bar{2})$, $(\bar{1},1),(2,3),(3,5)\}$, and its set of openers and closers are, respectively, $\{\bar{5},\bar{3},\bar{1},2,3\}$ and $\{\bar{3},\bar{2},1,3,5\}$.

If we fix the linearly ordered ground set

$$[\pm n] = \{1 < 2 < \cdots < n < \bar{1} < \bar{2} < \cdots < \bar{n}\},$$

which is isomorphic, through the map $i \mapsto i$ for $i \in [n]$ and $i \mapsto n + \bar{i}$ for $i \in \{\bar{1},\ldots,\bar{n}\}$, to

$$[2n] = \{1 < 2 < \ldots < n < n+1 < \cdots < 2n\},$$

we may define the set $NC(\pm n)$, of noncrossing partitions of $[\pm n]$, as the isomorphic image of $NC(2n)$. This allows us to define a $\mathbf{B}_n$ noncrossing partitions as an element of the intersection $NC(\pm n) \cap \Pi_B(n)$. As in type $\mathbf{A}$, we may depict a noncrossing partitions π of type $\mathbf{B}_n$ pictorially by placing the numbers $1,2,\ldots,n,\bar{1},\bar{2},\ldots,\bar{n}$ clockwise around a circle in this order, so that $\bar{n}$ is adjacent to 1, and for each block B of π drawing the convex hull $\rho(B)$ of the set of vertices labeled with the elements of B. Then π is noncrossing if and only if $\rho(B)$ and $\rho(B')$ have empty intersection for any two distinct blocks B and B' of π. Cutting the $2n$-gon between the integers $\bar{n}$ and 1 and stretching it along a line, we get a standard representation of the noncrossing partition π where no two arcs cross. See Figure 2.2 for an example. The set of all noncrossing partitions of type $\mathbf{B}_n$ is denoted by $NC_B(n)$.

Consider now the type $\mathbf{D}_n$ case. Let us label the vertices of a regular $(2n-2)$-gon as $2,3,\cdots,n,\bar{2},\bar{3},\cdots,\bar{n}$ clockwise, in this order, and label its centroid with both 1 and $\bar{1}$. Let $\pi \in \Pi_D(n)$ and for each block B of π let $\rho(B)$ be the convex hull of the set of vertices labeled with the elements of B. Two blocks B and B' of π are said to cross if $\rho(B) \neq \rho(B')$ and if the intersection of the relative interior of $\rho(B)$ and $\rho(B')$ is nonempty. Note that the case $\rho(B) = \rho(B')$ can occur only when B and B' are the singletons $\{1\}$ and $\{\bar{1}\}$, and that if π has a zero-block B, then B and the block containing 1 cross unless $\{1,\bar{1}\} \subseteq B$.

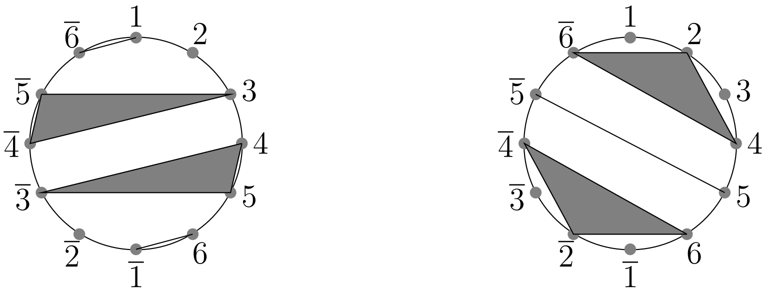

FIGURE 2.2. Two type $\mathbf{B}_6$ noncrossing partitions: $\pi = 1\overline{6}/\overline{1}6/45\overline{3}/\overline{4}\,\overline{5}3/2/\overline{2}$ without zero-block, and $\sigma = 5\overline{5}/24\overline{6}/\overline{2}\,\overline{4}6/3/\overline{3}$ with zero-block.

Thus, the zero-block of π, if it is eventually present, contains necessarily the integers 1 and $\overline{1}$, and at least one more pair i, $\bar{i}$, with $i \neq \pm 1$. A partition $\pi \in \Pi_D(n)$ is said to be noncrossing if no two of its blocks cross. The set of all type $\mathbf{D}_n$ noncrossing partitions is denoted by $NC_D(n)$. See Figure 2.3 for examples of $\mathbf{D}_7$ noncrossing partitions.

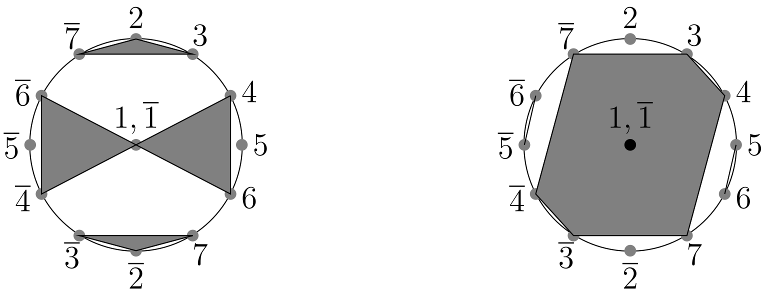

FIGURE 2.3. Two $\mathbf{D}_7$ noncrossing partitions: $\pi = \overline{1}46/1\overline{4}\,\overline{6}/23\overline{7}/\overline{2}\,\overline{3}7/5/\overline{5}$ without zero-block, and $\sigma = 1347\overline{1}\,\overline{3}\,\overline{4}\,\overline{7}/56/\overline{5}\,\overline{6}/2/\overline{2}$ with zero-block.

We turn now our attention to the construction of the combinatorial models for nonnesting partitions of type $\mathbf{B}_n$ and $\mathbf{D}_n$. Using the usual ordering, we can identify the set

$$[\pm n] \cup \{0\} = \{\overline{n} < \cdots < \overline{2} < \overline{1} < 0 < 1 < 2 < \cdots < n\}$$

with

$$[2n+1] = \{1 < 2 < \cdots < 2n+1\},$$

through the map $i \mapsto n+1-i$ for $i \in [\pm n]$ and $0 \mapsto n+1$. With this identification we may define the set of nonnesting partitions of $[\pm n] \cup \{0\}$ as the set of nonnesting partitions of $[2n+1]$: $NN([\pm n] \cup \{0\}) \cong NN(2n+1)$.

Given $\pi \in \Pi_B(n)$ let π_0 be the partition of $[\pm n] \cup \{0\}$ obtained from π by adding 0 to the zero-block if π has a zero-block, or by adding the singleton $\{0\}$ otherwise. We say that π is a type $\mathbf{B}_n$ nonnesting partition if $\pi_0 \in NN([\pm n] \cup \{0\})$. That is, π is nonnesting if and only if the standard representation of π_0 relative to the ground set

$$\overline{n} < \cdots < \overline{2} < \overline{1} < 0 < 1 < 2 < \cdots < n$$

is nonnesting. The presence of 0 in the ground set for nonnesting partitions of type $\mathbf{B}_n$ is necessary to correctly represent the arc between a positive number i an its negative (when it is eventually present). See Figure 2.4 for an example. Denote by $NN_B(n)$ the set of nonnesting partitions of type $\mathbf{B}_n$.

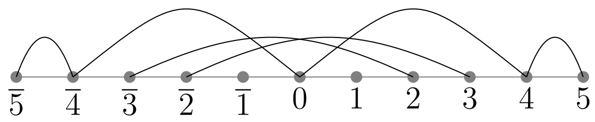

Figure 2.4. The nonnesting partition $4\overline{4}5\overline{5}/\overline{2}3/2\overline{3} \in NN_B(5)$.

Consider now the following partial order of the set $[\pm n]$:

$$[\pm n]' = \{\overline{n} < \cdots < \overline{2} < \overline{1}, 1 < 2 < \cdots < n\},$$

in which the integers 1 and $\overline{1}$ are not comparable. Using the obvious map, we identity this set with $[2n]$, and thus we can identify the set $NN([\pm n]')$ of nonnesting partitions of $[\pm n]'$ with the set $NN(2n)$, for which, according to Definition 1, an arc with 1 as closer and another one with $\overline{1}$ as closer are not considered nested. A partition $\pi \in \Pi_D(n)$ is said to be a $\mathbf{D}_n$ nonnesting partition if the zero-block, if present, contains the integers ± 1 and $\pi \in NN([\pm n]')$. Denote by $NN_D(n)$ the set of nonnesting partitions of type $\mathbf{D}_n$. An example of a nonnesting partition of type $\mathbf{D}_5$ is depicted in example 2.5.

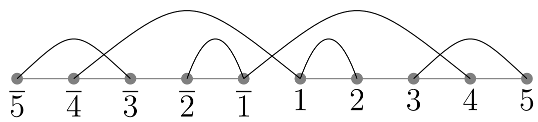

Figure 2.5. The nonnesting partition $\overline{4}12/\overline{1}\,\overline{2}4/35/\overline{35} \in NN_D(5)$.

3. Gray codes for noncrossing and nonnesting partitions

Let $T_A(n)$ denote one of the sets $NC(n)$ or $NN(n)$, and let $T_\psi(n)$ denote one of the sets $NC_\psi(n)$ or $NN_\psi(n)$, for $\psi = B$ or $\psi = D$. We can endow $T_A(n)$ (resp. $T_\psi(n)$) with a graph structure by declaring two partitions adjacent if their distance is 1 (resp. 1 or 2). A Hamilton path with distance 1 in $T_A(n)$ (resp. 2 in $T_\psi(n)$) corresponds to an exhaustive sequence of all partitions in $T_A(n)$ (resp. $T_\psi(n)$) such that the distance between two successive partitions is 1 (resp. 1 or 2), and thus it gives a Gray code for these objects. If this path is closed we have a Hamilton cycle. We use the same notation for the set of partitions and the corresponding graph. We point out that the distance between the partition $\pi = \{\pm 1, \pm 2, \ldots, \pm n\} \in \Pi(\pm n)$, with only one block, and any other type $\mathbf{B}$ or type $\mathbf{D}$ partition is at least 2, and thus there is no Gray code with distance 1 for the sets $NC_B(n)$, $NN_B(n)$, $NC_D(n)$ and $NN_D(n)$.

Given partitions π, σ in $T_A(n)$ or $T_\psi(n)$, we will write $\pi \sim \sigma$ to indicate that π and σ are adjacent. Moreover, to simplify notation, if π' is a partition of some set $S \subset [n]$, (resp. $S \subseteq [\pm n]$) we will often write $\pi = \pi'/sing$ to denote the partition of $[n]$ (resp. $[\pm n]$) where $\pi_{|S} = \pi'$ and $sing$ is the all singleton partition of $[n] \setminus S$ (resp. $[\pm n] \setminus S$), that is the partition of $[n] \setminus S$ (resp. $[\pm n] \setminus S$) where each block has only one element. In particular, $\pi = sing$ denotes the all singleton partition of $[n]$ (resp. $[\pm n]$).

The children of a partition $\pi \in T_A(n-1)$ are defined as the partitions in $T_A(n)$ obtained from π by adjoining the letter n to one of its blocks, or by adding the singleton block $\{n\}$. We denote by $\pi^* = \pi/n$ this last child of π and let $C(\pi)$ be the set of all children of π. Any partition in $T_A(n)$ has a unique parent in $T_A(n-1)$.

Similarly, for types $\mathbf{B}$ and $\mathbf{D}$ the children of a partition $\pi \in T_\psi(n-1)$ are defined as the partitions in $T_\psi(n)$ obtained from π by adjoining the letters n and $\overline{n}$ to some of its blocks, or by adding the zero-block $\{\pm n\}$ (only possible in type **B** if π has no zero-block), or the singletons blocks $\{n\}$ and $\{\overline{n}\}$. We denote by $\pi^* = \pi/n/\overline{n}$ this last child of π and let $C(\pi)$ be the set of all children of π. Any partition in $T_B(n)$ has a unique parent in $T_B(n-1)$, but this property is no longer valid in type **D**. Due to the restrictions on the cardinality of the zero-block, there are partitions in $T_D(n)$ which have no parent in $T_D(n-1)$. For instance, the $\mathbf{D}_3$ noncrossing partition $\pi = \{\pm 1, \pm 3\}/2/\overline{2}$ is not a child of any partition in $NC_D(2)$.

The following result is immediate from the definitions.

Lemma 1. *If π_1 and π_2 are children of the same partition $\pi \in T_\psi(n-1)$, then $\pi_1 \sim \pi_2$ for any $\psi = \mathbf{A}, \mathbf{B}$ or $\mathbf{D}$.*

The main tool for constructing the Gray codes is the following algorithm. It transforms a sequence of sets $C_1, \ldots, C_s$, where each set C_i is a collection of children of some partition π_i of $[n-1]$ (or $[\pm(n-1)]$) with two distinct elements π_i^* and $\pi_i^\diamond$ such that $\pi_{i-1}^* \sim \pi_i^*$ and $\pi_{i-1}^\diamond \sim \pi_i^\diamond$ for all i, into an ordered sequence of partitions of $[n]$ (or $[\pm n]$).

Algorithm 1.
Input: A sequence of sets $C_1, \ldots, C_s$ of partitions such that each element in C_i is a child of some partition π_i of $[n-1]$ (or $[\pm(n-1)]$), and there are $\pi_i^* \neq \pi_i^{\diamond_i} \in C_i$ and $\pi_{i+1}^* \neq \pi_{i+1}^{\diamond_i} \in C_{i+1}$ with $\pi_{i-1}^* \sim \pi_i^*$ and $\pi_i^{\diamond_i} \sim \pi_{i+1}^{\diamond_i}$ for each $i \in [s-1]$.

1. Start with π_1^* and transverse in any order all other elements of C_1, ending in $\pi_1^{\diamond_1}$. Let $\pi_1^{\bullet\bullet} := \pi_1^{\diamond_1}$.
2. For $i = 2$ to $s-1$ do
   Go to $\pi_i^\bullet$ and transverse, in any order, the remaining elements of C_i, ending in $\pi_i^{\bullet\bullet}$, where $\pi_i^\bullet = \begin{cases} \pi_i^*, & \text{if } \pi_{i-1}^{\bullet\bullet} = \pi_{i-1}^* \\ \pi_i^{\diamond_{i-1}}, & \text{if } \pi_{i-1}^{\bullet\bullet} = \pi_{i-1}^{\diamond_{i-1}} \end{cases}$ and
   $$\pi_i^{\bullet\bullet} = \begin{cases} \pi_i^*, & \text{if } \pi_i^\bullet = \pi_i^{\diamond_{i-1}} \\ \pi_i^{\diamond_i}, & \text{if } \pi_i^\bullet = \pi_i^* \end{cases}.$$
3. Go to $\pi_s^\bullet$ and transverse, in any order, all other elements of C_s, where
   $$\pi_s^\bullet = \begin{cases} \pi_s^*, & \text{if } \pi_{s-1}^{\bullet\bullet} = \pi_{s-1}^* \\ \pi_s^{\diamond_{s-1}}, & \text{if } \pi_{s-1}^{\bullet\bullet} = \pi_{s-1}^{\diamond_{s-1}} \end{cases}.$$
5. End.

Note that when we apply Algorithm 1 to the input sequence $C_1, \ldots, C_s$, then for each even (resp. odd) integer $i \in [s-1]$, the last partition of C_i to be placed into the output sequence is π_i^* (resp. $\pi_i^{\diamond_{s-1}}$).

3.1. **Type A noncrossing partitions.** The next result shows that among the children of two partitions with distance 1 in $NC(n)$ there are at least two pairs of children also with distance 1.

Lemma 2. *Let $\sigma, \pi \in NC(n-1)$ with $\sigma \sim \pi$ and $n \geq 3$. Then, there are children $\sigma^\diamond \in C(\sigma)$ and $\pi^\diamond \in C(\pi)$ such that $\sigma^\diamond \neq \sigma^*$, $\pi^\diamond \neq \pi^*$, and $\sigma^\diamond \sim \pi^\diamond$.*

Proof. If $\sigma = sing$ is the all singleton partition, then we must have $\pi = ij/sing$, for some $i < j$ in $[n-1]$. In this case, the partitions $\sigma^\diamond$ and $\pi^\diamond$, obtained from σ and π by placing the letter n in the blocks containing the letter i, satisfy the required condition. Notice also that $\sigma^\diamond$ is never equal to the partition $n(n-1)/sing$.

Assume now that neither σ nor π is the all singleton partition of $[n-1]$. Let $\sigma = B_1/B_2/\sigma'$ and $\pi = B_1'/B_2'/\sigma'$, such that $j \in B_1$ and $B_1' = B_1 \setminus \{j\}$ and $B_2' = B_2 \cup \{j\}$. If

$j \neq 1$ then let $\sigma^\diamond$ and $\pi^\diamond$ be obtained from σ and π by placing the letter n in the blocks containing the letter 1. Otherwise $j = 1$ and we let $\sigma^\diamond$ and $\pi^\diamond$ be obtained from σ and π by placing the letter n in the blocks containing the letter $n-1$. In any case the partitions $\sigma^\diamond$ and $\pi^\diamond$ satisfy the required conditions. □

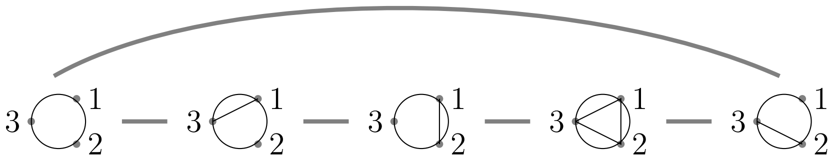

FIGURE 3.1. A Hamilton cycle in $NC(3)$

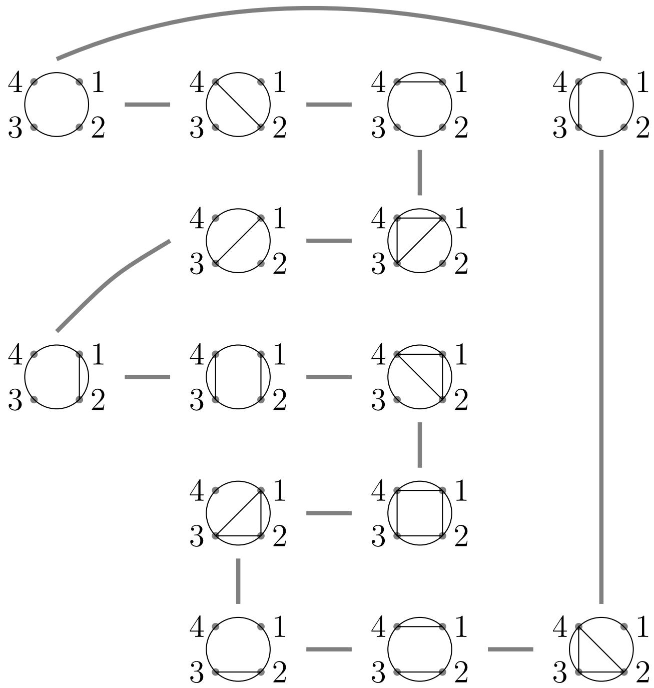

FIGURE 3.2. A Hamilton cycle in $NC(4)$

Theorem 3. *For $n \geq 2$ there is a Hamilton cycle $\pi_1, \dots, \pi_{s-1}, \pi_s$ in $NC_A(n)$ which starts with the all singleton partition $\pi_1 = sing$ and ends with the partitions $\pi_{s-1} = n(n-1)(n-2)/sing$ and $\pi_s = n(n-1)/sing$, where $s = \frac{1}{n+1}\binom{2n}{n}$.*

Proof. The proof is by induction on $n \geq 2$. The case $n = 2$ is trivial and the cases $n = 3, 4$ are depicted in Figures 3.1 and 3.2. Assume the result holds for $n - 1 \geq 3$ and let $\pi_1, \dots, \pi_{s-1}, \pi_s$ be a Hamilton cycle in $NC_A(n-1)$, with $\pi_1 = 1/2/\cdots/(n-1)$, $\pi_{s-1} = (n-1)(n-2)(n-3)/1/2/\cdots/(n-4)$ and $\pi_s = (n-1)(n-2)/1/2/\cdots/(n-3)$.

Let $\pi_1^\dagger = n(n-1)/sing \in C(\pi_1)$, $\pi_s^\dagger = n(n-1)(n-2)/sing \in C(\pi_s)$, and consider the sequence $C_1, \dots, C_s$, defined by $C_1 = C(\pi_1) \setminus \{\pi_1^\dagger\}$, $C_i = C(\pi_i)$, for $i = 2, \dots, s-1$, and $C_s = C(\pi_s) \setminus \{\pi_s^\dagger\}$. For $i \in [s-1]$, consider also the partitions $\pi_i^* \neq \pi_i^{\diamond_i} \in C_i$ and $\pi_{i+1}^* \neq \pi_{i+1}^{\diamond_i} \in C_{i+1}$ obtained in Lemma 2 applied to the pair $\pi_i \sim \pi_{i+1}$. The construction of a Hamilton cycle for $NC_A(n)$ results from the application of Algorithm 1 to the sequence $C_1, \ldots, C_s$, followed by the partitions $\pi_s^\dagger$ and $\pi_1^\dagger$, which satisfy $\pi_s^\dagger \sim \pi_1^\dagger$.

Lemmas 1 and 2 show that any two consecutive partitions in the sequence given by the algorithm above have distance 1, as well as the first and last partition. Moreover, this sequence exhausts all elements of $NC(n)$, since any partition on this set has a unique parent in $NC(n-1)$. It remains to show that $\pi_1^\dagger \neq \pi_1^{\diamond_1}$ and that $\pi_s^\dagger \neq \pi_s^{\diamond_{s-1}}$ when $s-1$ is odd, since by construction $\pi_s^\dagger \neq \pi_s^*$. The first inequality follows immediately from Lemma 2, since $\pi_1^{\diamond_1}$ is obtained by adding the letter n to the singleton containing some letter $i < n-1$. For the second inequality, notice that again by Lemma 2, if $n \geq 4$ then $\pi_{s-1}^{\diamond_{s-1}}$ and $\pi_s^{\diamond_{s-1}}$ are obtained from π_{s-1} and π_s by placing the letter n in the blocks containing the letter 1, and so we must have $\pi_s^\dagger \neq \pi_s^\diamond$. Note that when $n = 3$, we have $\pi_s^\diamond = \pi_s^\dagger$, but in this case $s-1$ is even and the sets $C(\pi_{s-1})$ and $C(\pi_s)$ are linked by π_{s-1}^* and π_s^*. $\square$

See Figure 3.2 for a Hamilton cycle in $NC(4)$ constructed by applying the algorithm described above, starting from the Hamilton cycle for $NC(3)$ given in Figure 3.1.

3.2. **Type B noncrossing partitions.** As in type **A**, we start by showing that among the children of two distinct partitions with distance less than or equal to 2 in $NC_B(n)$ there are at least two pairs of children also with distance less than or equal to 2.

Lemma 4. *Let $\sigma, \pi \in NC_B(n-1)$ with $\sigma \sim \pi$ and $n \geq 3$. Then, there are children $\sigma^\diamond \in C(\sigma)$ and $\pi^\diamond \in C(\pi)$ such that $\sigma^\diamond \neq \sigma^*$, $\pi^\diamond \neq \pi^*$, and $\sigma^\diamond \sim \pi^\diamond$.*

Proof. First assume that $\sigma = sing$ is the all singleton partition in $[\pm(n-1)]$. Then we must have either $\pi = \pm i/sing$ or $\pi = ij/\bar{i}\,\bar{j}/sing$, where in each case $sing$ is the appropriate all singleton partition and $|i| < |j| \in [n-1]$. Let $\sigma^\diamond$ and $\pi^\diamond$ be obtained from σ and π by adding the letters n and $\bar{n}$ to the blocks containing the letter $|i|$ and $\overline{|i|}$, respectively.

In any other case there must be an integer $j \in [n-1]$ such that j and/or $\bar{j}$ change blocks between partitions σ and π. Now, if $j \neq 1$ then define $\sigma^\diamond$ and $\pi^\diamond$ as the partitions obtained from σ and π adding the letters n and $\bar{n}$ to the blocks containing the letters $\bar{1}$ and 1, respectively. Otherwise, $\sigma^\diamond$ and $\pi^\diamond$ are obtained by adding the letters n and $\bar{n}$ to the blocks of σ and π containing the letter $n-1$ and $\overline{n-1}$, respectively.

In any case, the partitions $\sigma^\diamond$ and $\pi^\diamond$ are noncrossing and their distance is 2. $\square$

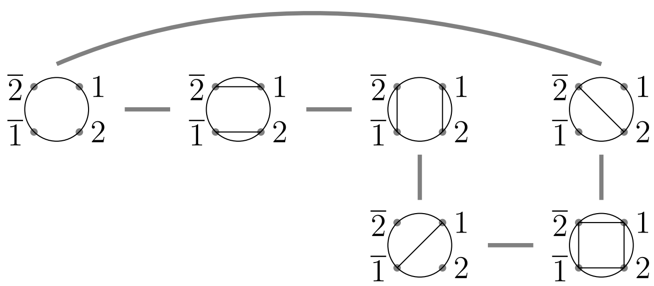

Figure 3.3. A Hamilton cycle in $NC_B(2)$

The construction of a Gray code for the noncrossing partitions of type **B**, given in the next result, follows the same lines used in type **A**.

Theorem 5. *For $n \geq 2$ there is a Hamilton cycle $\pi_1, \ldots, \pi_{s-1}, \pi_s$ in $NC_B(n)$ which starts with the all singleton partition $\pi_1 = sing$ and ends with the partitions $\pi_{s-1} = \{\pm n, \pm(n-1)\}/sing$ and $\pi_s = \pm n/\,sing$, where $s = \binom{2n}{n}$.*

Proof. The proof is by induction on $n \geq 3$. The cases $n = 2, 3$ are depicted in Figures 3.3 and 3.4. Assume the result holds for $n-1 \geq 3$ and let $\pi_1, \ldots, \pi_{s-1}, \pi_s$ be a Hamilton cycle in $NC_B(n-1)$, with $\pi_1 = 1/\overline{1}/\cdots/(n-1)/\overline{n-1}$, $\pi_{s-1} = \{\pm(n-2), \pm(n-1)\}/1/\overline{1}/\cdots/n-3/\overline{n-3}$ and $\pi_s = \pm(n-1)/1/\overline{1}/\cdots/n-2/\overline{n-2}$.

Consider the partitions $\pi_1^\dagger = \pm n/sing$ and $\pi_s^\dagger = \{\pm n, \pm(n-1)\}/sing$, children of π_1 and π_s respectively, and construct the sets $C_1, \ldots, C_s$, where $C_1 = C(\pi_1) \setminus \{\pi_1^\dagger\}$, $C_i = C(\pi_i)$, for $i = 2, \ldots, s-1$ and $C_s = C(\pi_s) \setminus \{\pi_s^\dagger\}$. For each $i = 1, \ldots, s-1$, consider also the partitions $\pi_i^* \neq \pi_i^{\diamond_i} \in C_i$ and $\pi_{i+1}^* \neq \pi_{i+1}^{\diamond_i} \in C_{i+1}$, obtained in Lemma 4 applied to the pair $\pi_i \sim \pi_{i+1}$.

Finally, consider the sequence obtained applying Algorithm 1 to the sequence $C_1, \ldots, C_s$, followed by $\pi_s^\dagger$ and $\pi_1^\dagger$.

We have $\pi_s^\dagger \sim \pi_1^\dagger$ and, as in the proof of Theorem 3, we can use Lemmas 1 and 4 to show that any other two consecutive partitions in the sequence obtained by the algorithm above have distance less than or equal to 2. Note also that the distance between the last partition $\pi_1^\dagger$ and the first partition π_1^* of the sequence is 1. Moreover, this sequence exhausts all elements of $NC_B(n)$, since any partition on this set has a unique parent in $NC_B(n-1)$. It remains to show that $\pi_1^\dagger \neq \pi_1^{\diamond_1}$ and, since the integer $s = \binom{2(n-1)}{n-1}$ is even (see for instance [9]), that $\pi_s^\dagger \neq \pi_s^{\diamond_{s-1}}$. The first inequality follows immediately from Lemma 4, since $\pi_1^{\diamond_1}$ is obtained by adding the letters n and $\overline{n}$ to the singletons containing some letter $i < n-1$ and $\overline{i}$. For the second inequality, notice that again by Lemma 4, $\pi_{s-1}^{\diamond_{s-1}}$ and $\pi_s^{\diamond_{s-1}}$ are obtained from π_{s-1} and π_s by placing the letters n and $\overline{n}$ in the blocks containing the letters $\overline{1}$ and 1, respectively, and so we must have $\pi_s^\dagger \neq \pi_s^{\diamond_{s-1}}$. ☐

3.3. **Type D noncrossing partitions.** For the construction of a Gray code for the set of all noncrossing partitions of type $\mathbf{D}_n$, we start by identifying the partitions in $NC_D(n)$ which have a parent in $NC_D(n-1)$.

Lemma 6. *A partition $\pi \in NC_D(n)$ is a child of some partition in $NC_D(n-1)$ if and only if its zero-block, when present, is not the set $\{\pm 1, \pm n\}$.*

Proof. Any partition $\pi \in NC_D(n)$ whose zero-block, when present, is not $\{\pm 1, \pm n\}$, is a child of the type $\mathbf{D}_{n-1}$ noncrossing partition obtained from π by removing the letters $\pm n$. On the other hand, if $\{\pm 1, \pm n\}$ is the zero-block of π, then it cannot be a child of a $\mathbf{D}_{n-1}$ noncrossing partition, since the zero-block of any such partition must have at least four elements, ± 1 and $\pm j$, for some $|j| \leq n-1$, and thus the distance between the children of any such partition and π is at least equal to 4, contradicting the result of Lemma 1. ☐

Lemma 7. *Let $\sigma, \pi \in NC_D(n-1)$ with $\sigma \sim \pi$ and $n \geq 3$. Then, there are children $\sigma^\diamond \in C(\sigma)$ and $\pi^\diamond \in C(\pi)$ such that $\sigma^\diamond \neq \sigma^*$, $\pi^\diamond \neq \pi^*$, and $\sigma^\diamond \sim \pi^\diamond$.*

Proof. First assume that $\sigma = sing$ is the all singleton partition in $[\pm(n-1)]$. Then we must have $\pi = ij/\overline{i}\,\overline{j}/sing$, where $|i| < |j| \in [n-1]$. Let $\sigma^\diamond$ and $\pi^\diamond$ be obtained from σ and π by adding the letters n and $\overline{n}$ to the blocks having the letter $|j|$ and $\overline{|j|}$, respectively.

If $\sigma = \{\pm 1, \pm(n-1)\}/sing$ and π has the same zero-block as σ, then we must have

$$\pi = \{\pm 1, \pm(n-1)\}/ij/\overline{i}\,\overline{j}/sing,$$

for some integers positive integers $i, j \in [2, n-1]$. In this case we let $\sigma^\diamond$ and $\pi^\diamond$ be the partitions obtained from σ and π by adding the letters n and $\overline{n}$ to the blocks having the letters $\overline{2}$ and 2, respectively.

In any other case there must be an integer $j \in [n-1]$ such that j and/or $\overline{j}$ change blocks between partitions σ and π. Now, if $j \neq 2$ then define $\sigma^\diamond$ and $\pi^\diamond$ as the partitions

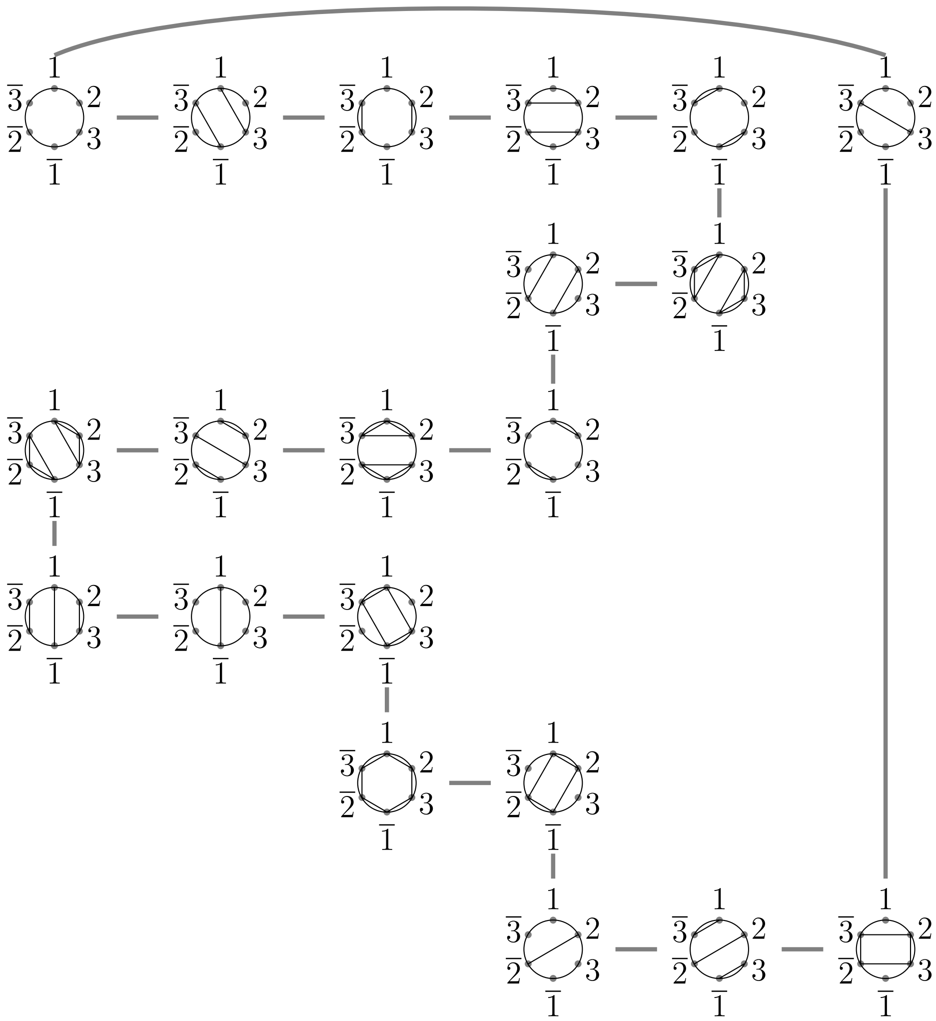

FIGURE 3.4. A Hamilton cycle in $NC_B(3)$

obtained from σ and π adding the letters n and $\overline{n}$ to the blocks having the letters $\overline{2}$ and 2, respectively. Otherwise, $\sigma^\diamond$ and $\pi^\diamond$ are obtained by adding the letters n and $\overline{n}$ to the blocks of σ and π having the letter $n-1$ and $\overline{n-1}$, respectively.

In all cases, the partitions $\sigma^\diamond$ and $\pi^\diamond$ are noncrossing and their distance is 2. □

We are now ready to start the construction of a Hamilton cycle in $NC_D(n)$ in which every two consecutive noncrossing partitions have distance 2. This construction will make use of the construction made for $\mathbf{A}_n$ noncrossing partitions given in Theorem 3.

Theorem 8. *For $n \geq 2$ there is a Hamilton path $\pi_1, \pi_2, \ldots, \pi_{s-1}, \pi_s$ in $NC_D(n)$ such that $\pi_1 = sing$, $\pi_2 = \{1, n\}/\{\overline{1}, \overline{n}\}/sing$, $\pi_1 \sim \pi_3$ and $\pi_s = \{\pm 1, \pm n\}/sing$, where $s = \binom{2n}{n} - \binom{2n-2}{n-1}$. Moreover, when $n \geq 4$, the zero-block of π_{s-1} is $\{\pm 1, \pm n\}$.*

Proof. The proof is by induction on $n \geq 3$. The cases $n = 2, 3$ are depicted in Figures 3.5 and 3.6. Assume the result holds for $n-1 \geq 3$ and let $\pi_1, \pi_2, \ldots, \pi_s$ be a Hamilton path in $NC_D(n-1)$, with $\pi_1 = 1/\overline{1}/\cdots/(n-1)/\overline{n-1}$, $\pi_2 = \{1, n-1\}/\{\overline{1}, \overline{n-1}\}/sing$, $\pi_1 \sim \pi_3$ and $\pi_s = \{\pm 1, \pm(n-1)\}/sing$. For the construction of a Hamilton path for $NC_D(n)$ we start by ordering the elements of $C(\pi_1)$: start with π_1^*, π_1' and transverse in any order the remaining children of π_1, ending in $\pi_1'' = \{n-1,n\}/\{\overline{n-1},\overline{n}\}/sing$, where

$$\pi_1' = \{1,n\}/\{\overline{1},\overline{n}\}/sing.$$

Next, apply Algorithm 1 to the sequence $C_2,\dots,C_s$, followed by $\pi_s^\dagger$, where $C_i = C(\pi_i)$, for $i = 2,\dots,s-1$, and $C_s = C(\pi_s)\setminus\{\pi_s^\dagger\}$ with $\pi_s^\dagger = \{\pm 1, \pm(n-1), \pm n\}/sing$, and where for each $i = 2,\dots,s-1$, the partitions $\pi_i^* \neq \pi_i^{\diamond_i} \in C_i$ and $\pi_{i+1}^* \neq \pi_{i+1}^{\diamond_i} \in C_{i+1}$ are the ones obtained by Lemma 7 applied to the pair $\pi_i \sim \pi_{i+1}$.

Since the set $C(\pi_1)$ has more than 4 elements, by Lemma 1 the distance between π_1^* and the third element considered in the first step of the algorithm above is at most 2. Note that $\pi_1'' \sim \pi_2^* = \{1,n-1\}/\{\overline{1},\overline{n-1}\}/sing$. For the algorithm to work we need to show that $\pi_s^\dagger \neq \pi_s^{\diamond_{s-1}}$ whenever $s-1$ is even, that is whenever the last child of π_{s-1} to be placed in the sequence is $\pi_{s-1}^{\diamond_{s-1}}$ (note that by construction, $\pi_s^\dagger \neq \pi_s^*$). When $n-1 = 3$ the integer $s-1$ is odd and thus the last child of π_{s-1} in the sequence is π_{s-1}^* as we can check in Figure 3.6. For $n-1 \geq 4$ the partitions π_{s-1} and π_s share the same zero-block $\{\pm 1, \pm(n-1)\}$, and thus by Lemma 7 we have $\pi_s^\dagger \neq \pi_s^{\diamond_{s-1}}$. Therefore, the construction above originates a sequence $\sigma_1,\dots,\sigma_t$ of type D noncrossing partitions of $[\pm n]$, with

$$\begin{aligned}
\sigma_1 &= \pi_1^* = sing,\\
\sigma_2 &= \{1,n\}/\{\overline{1},\overline{n}\}/sing,\\
\sigma_1 &\sim \sigma_3 \text{ and}\\
\sigma_t &= \pi_s^\dagger = \{\pm 1, \pm(n-1), \pm n\}/sing.
\end{aligned}$$

Next, use Theorem 3 and the obvious isomorphism $NC[n-2] \cong NC[2,n-1]$ to obtain a Hamilton cycle $\omega_1,\dots,\omega_q$ for type **A** noncrossing partitions of $[2,n-1] = \{2,3,\dots,n-1\}$, where $\omega_1 = 2/\cdots/n-1$ and $\omega_q = \{n-1,n-2\}/2/\cdots/n-3$. Define

$$\sigma_{t+\ell} := \{\pm 1, \pm n\}/\omega_\ell/\overline{\omega_\ell},$$

for each $\ell = 1,\dots,q$, where $q = \#NC(n-2) \geq 2$ since we are assuming $n-1 \geq 3$. Finally, consider the sequence

$$\sigma_1,\dots,\sigma_t,\sigma_{t+q},\dots,\sigma_{t+1}. \tag{3.1}$$

We claim that (3.1) is a Hamilton path for $NC_D(n)$.

First, note that by direct inspection we have

$$\sigma_t \sim \sigma_{t+q} = \{\pm 1, \pm n\}/\{n-1,n-2\}/\{\overline{n-1},\overline{n-2}\}/sing.$$

Moreover, the distance between any other two consecutive integers of the sequence (3.1) is 1 or 2 by Lemma 7 and Theorem 3.

Finally, by Lemma 6, any partition $\pi \in NC_D(n)$ whose zero-block, when present, is not $\{\pm 1, \pm n\}$, is a child of some type **D** noncrossing partition of $[n-1]$, and thus it must be one of the partitions $\sigma_1,\dots,\sigma_t,\pi_1^\dagger$. If $\{\pm 1, \pm n\}$ is the zero-block of π, then it is not a child of a type **D** partition of $[n-1]$, and any other block B of π must satisfy $B \subseteq [2,n-1]$ or $-B \subseteq [2,n-1]$. Therefore, the positive part of π, excluding the zero-block, is a type **A** noncrossing partition of $[2,n-1]$. It follows that (3.1) is an exhaustive list of the type **D** noncrossing partitions of $[n]$. $\square$

See Figure 3.6 for a Hamilton path in $NC_D(3)$ constructed by applying the algorithm described in the theorem above, starting from the Hamilton path for $NC_D(2)$ given in Figure 3.5. In this case, $n = 3$, there is only one partition with zero-block $\{\pm 1, \pm n\}$ since the set of type **A** noncrossing partitions $NC(n-2) = NC(1)$ has only one element.

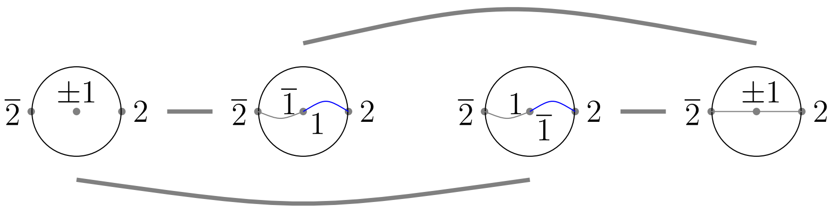

FIGURE 3.5. A Hamilton cycle in $NC_D(2)$

Corollary 9. *For $n \geq 2$ there is a Hamilton cycle in $NC_D(n)$.*

Proof. The case $n = 2$ is given in Figure 3.5. For $n \geq 3$ consider the Hamilton path $\pi_1, \pi_2, \pi_3, \ldots, \pi_{s-1}, \pi_s$ of $NC_D(n)$, given by the theorem above, where $\pi_1 = sing$, $\pi_2 = \{1, n\}/\{\overline{1}, \overline{n}\}/sing$, $\pi_1 \sim \pi_3$, and $\pi_s = \{\pm 1, \pm n\}/sing$. Since $\pi_s \sim \pi_2 \sim \pi_1$, it follows that the sequence

$$\pi_1, \pi_3, \ldots, \pi_{s-1}, \pi_s, \pi_2$$

is a Hamilton cycle in $NC_D(n)$. $\square$

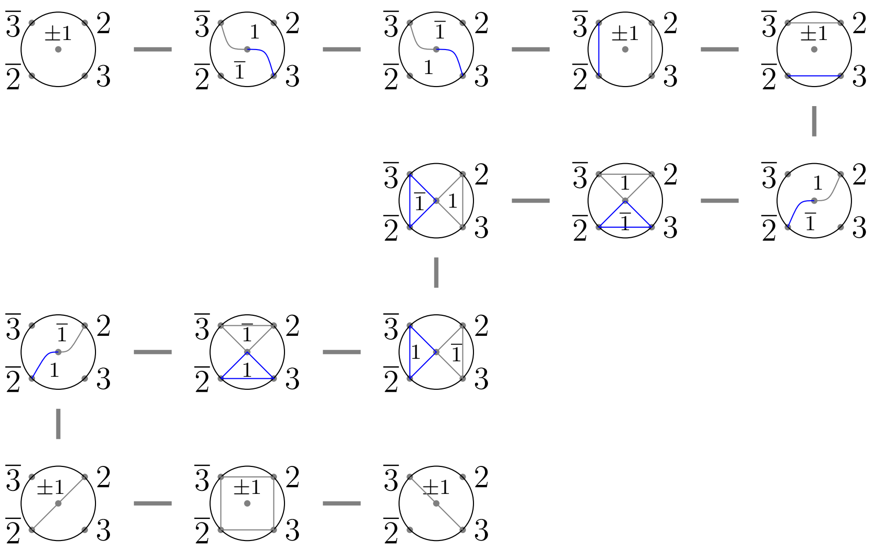

FIGURE 3.6. A Hamilton path in $NC_D(3)$

3.4. **Type B nonnesting partitions.** We now turn our attention to the exhaustive listing of nonnesting partitions of types $\mathbf{B}_n$ and $\mathbf{D}_n$, starting with the construction of a Hamilton cycle in type $\mathbf{B}$.

Lemma 10. *Let $\sigma, \pi \in NN_B(n-1)$ with $\sigma \sim \pi$ and $n \geq 3$. Then, there are children $\sigma^\diamond \in C(\sigma)$ and $\pi^\diamond \in C(\pi)$ such that $\sigma^\diamond \neq \sigma^*$, $\pi^\diamond \neq \pi^*$, and $\sigma^\diamond \sim \pi^\diamond$.*

Proof. If the letters $n-1$ and $\overline{n-1}$ do not change blocks between the partitions σ and π, then let $\sigma^\diamond$ and $\pi^\diamond$ be the partitions obtained from σ and π by placing the letters n and $\overline{n}$ in the blocks containing the letters $n-1$ and $\overline{n-1}$, respectively. Otherwise, set $\sigma^\diamond$ and $\pi^\diamond$ as the partitions obtained from σ and π by placing the letters n and $\overline{n}$ in the blocks containing the letters $n-2$ and $\overline{n-2}$, respectively. It is clear that $\sigma^\diamond$ and $\pi^\diamond$ satisfy the required conditions. $\square$

Theorem 11. *For $n \geq 2$ there is a Hamilton cycle $\pi_1, \ldots, \pi_{s-1}, \pi_s$ in $NN_B(n)$ which starts with the all singleton partition $\pi_1 = sing$ and ends with the partition $\pi_s = \{\pm 1\}/\, sing$, where $s = \binom{2n}{n}$.*

Proof. The proof is by induction over $n \geq 2$. The case $n = 2$ is depicted in Figure 3.7. Assuming the result for $n - 1 \geq 2$, let $\pi_1, \ldots, \pi_s$ be a Hamilton cycle for the type **B** nonnesting partitions of $[n-1]$, with $\pi_1 = sing$ and $\pi_s = \{\pm 1\}/\, sing$.

Apply Algorithm 1 to the sequence $C_1, \ldots, C_s$, where $C_i = C(\pi_i)$ for $i = 1, \ldots, s-1$ and $C_s = C(\pi_s) \setminus \{\pi_s^*\}$, with

$$\pi_s^* = \pm 1/n/\overline{n}/sing \in NC(n),$$

and for each $i = 1, \ldots, s-1$, the partitions $\pi_i^* \neq \pi_i^{\diamond_i} \in C_i$ and $\pi_{i+1}^* \neq \pi_{i+1}^{\diamond_i} \in C_{i+1}$ are the ones obtained by Lemma 10. Let $\sigma_1, \ldots, \sigma_{q-1}$ be the sequence of partitions obtained by this procedure, and add at its right end the partition $\sigma_q = \pi_s^*$.

Note that since the integer $s = \binom{2(n-1)}{n-1}$ is even (see for instance [9]), the first child of $C(\pi_s)$ to be placed in the sequence is $\pi_s^{\diamond_{s-1}}$. By Lemmas 1 and 10, any two consecutive partitions of $\sigma_1, \ldots, \sigma_q$ have distance at most 2, and $\sigma_q \sim \sigma_1 = \pi_1^* = sing$. Finally, since each partition in $NN_B(n)$ is a child of a unique partition in $NN_B(n-1)$, the sequence $\sigma_1, \ldots, \sigma_q$ is a complete list of all $\mathbf{B}_n$ nonnesting partitions. Thus it forms a Hamilton cycle with distance 2 for the set $NN_B(n)$. □

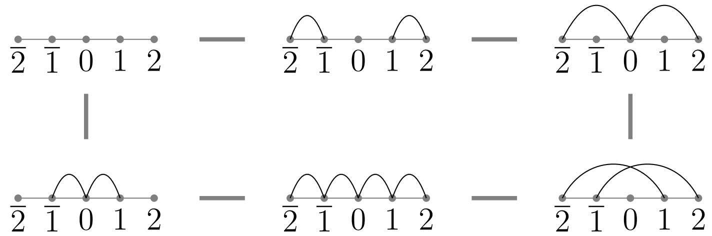

FIGURE 3.7. A Hamilton cycle in $NN_B(2)$

In [6], an algorithm $GenTot(n)$ was presented to generate all type **A** nonnesting partitions of $[n]$ in lexicographic order of their arcs, i.e. first according to the number of arcs and then, for partitions with the same number of arcs, according to their openers.

Using the identification of type $\mathbf{B}_n$ nonnesting partitions with type $\mathbf{B}_n$ partitions of $NN([\pm n] \cup \{0\}) \cong NN(2n+1)$, we may define a type **B** lexicographic order on the set $NN_B(n)$ as in type **A**, that is, we order the partitions first according to the number of arcs and then, for partitions with the same number of arcs, according to the openers of their arcs.

With this definition, we can use the $GenTot(n)$ algorithm of type $\mathbf{A}_n$ to generate in lexicographic order all type $\mathbf{B}_n$ nonnesting partitions as follows. First, apply $GenTot(2n+1)$ to list all nonnesting partitions of $[2n+1]$ in lexicographic order. Using the identification $[2n+1] \cong [\pm n] \cup \{0\}$, translate all partitions in the list into partitions of the set $NN([\pm n] \cup \{0\})$. Then, according to the definitions, the sublist formed by all those partitions with at most one zeroblock and for which for each arc (i, j) there is also the arc $(-j, -i)$ is the list of all type $\mathbf{B}_n$ nonnesting partitions arranged in lexicographic order.

Proposition 12. *The procedure above combinatorially generates all type **B** nonnesting partitions of $[n]$ in lexicographic order.*

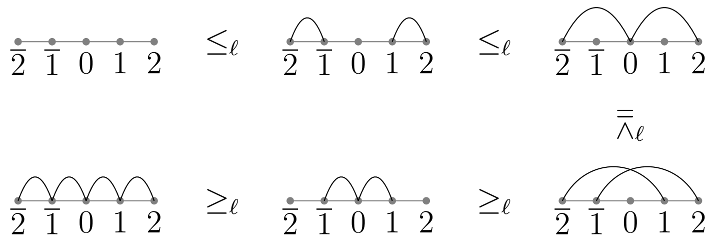

Figure 3.8. The elements of $NN_B(2)$ in lexicographic order.

3.5. **Type D nonnesting partitions.** We were not able to construct a Gray code for type $\mathbf{D}$ nonnesting partitions, but we present a construction for the generation of all partitions in $NN_D(n)$. This construction has two steps. First we construct a Gray code for all type $\mathbf{D}$ nonnesting partitions without zero-block, and then we give a list in lexicographic order of all those nonnesting partitions having zero-block. The union of these two lists generates all elements in $NN_D(n)$.

As for noncrossing partitions of type $\mathbf{D}_n$, we can identify the nonnesting partitions in $NN_D(n)$ which have a parent in $NN_D(n-1)$.

Lemma 13. *A partition $\pi \in NN_D(n)$ is a child of some partition in $NN_D(n-1)$ if and only if its zero-block, when present, is not the set $\{\pm 1, \pm n\}$.*

Proof. Analogous to the proof of Lemma 6. $\square$

Lemma 14. *Let $\sigma, \pi \in NN_D(n-1)$ with $\sigma \sim \pi$ and $n \geq 3$. Then, there are children $\sigma^\diamond \in C(\sigma)$ and $\pi^\diamond \in C(\pi)$ such that $\sigma^\diamond \neq \sigma^*$, $\pi^\diamond \neq \pi^*$, and $\sigma^\diamond \sim \pi^\diamond$.*

Proof. If the letters $n-1$ and $\overline{n-1}$ do not change blocks between the partitions σ and π, then let $\sigma^\diamond$ and $\pi^\diamond$ be the partitions obtained from σ and π by placing the letters n and $\overline{n}$ in the blocks containing the letters $n-1$ and $\overline{n-1}$, respectively. Otherwise, set $\sigma^\diamond$ and $\pi^\diamond$ as the partitions obtained from σ and π by placing the letters n and $\overline{n}$ in the blocks containing the letters $n-2$ and $\overline{n-2}$, respectively. It is clear that $\sigma^\diamond$ and $\pi^\diamond$ satisfy the required conditions. $\square$

Proposition 15. *For $n \geq 2$ there is a Hamilton cycle $\pi_1, \ldots, \pi_{s-1}, \pi_s$ in the subset of $NN_D(n)$ formed by those partitions without zero-block with $\pi_1 = sing$, $\pi_s = 12/\overline{1}\,\overline{2}/sing$ and $\pi_{s-1} = 12n/\overline{1}\,\overline{2}\overline{n}/sing$ if $n \geq 3$, or $\pi_{s-1} = 1\overline{2}/\overline{1}2$ if $n = 2$.*

Proof. The proof is by induction over $n \geq 3$. The cases $n = 2$ and $n = 3$ are depicted in Figures 3.9 and 3.10. Assuming the result for $n - 1 \geq 3$, let $\pi_1, \ldots, \pi_{s-1}, \pi_s$ be a Hamilton cycle for the type $\mathbf{D}_n$ nonnesting partitions of $[\pm(n-1)]$ without zero-block, with $\pi_1 = sing$, $\pi_{s-1} = 12(n-1)/\overline{1}\,\overline{2}(\overline{n-1})/sing$ and $\pi_s = 12/\overline{1}\,\overline{2}/sing$. Next, apply the first two steps of Algorithm 1 to the first $s-1$ sets of the sequence $C_1 = C(\pi_1), \ldots, C_s = C(\pi_s)$.

Following the application of Algorithm 1, if the sets $C(\pi_{s-1})$ and $C(\pi_s)$ are to be linked by $\pi_{s-1}^{\diamond_{s-1}}$ and $\pi_s^{\diamond_{s-1}}$, then by Lemma 14 we must have $\pi_s^{\diamond_{s-1}} = 12/\overline{1}\,\overline{2}/(n-2)n/(\overline{n-2})\overline{n}/sing$. Transverse the remaining children of π_s ending with $12n/\overline{1}\,\overline{2}\overline{n}/sing$ and $\pi_s^* = 12/\overline{1}\,\overline{2}/sing$. On the other hand, if $C(\pi_{s-1})$ and $C(\pi_s)$ are to be linked by π_{s-1}^* and π_s^*, then replace π_s^* with its sibling

$$\sigma = 12/\overline{1}\,\overline{2}/(n-1)n/(\overline{n-1})\overline{n}/sing \in C(\pi_s),$$

which satisfies $\pi_{s-1}^* \sim \sigma$. Transverse the remaining children of π_s ending with $12n/\overline{1}\,\overline{2}\overline{n}/sing$ and $\pi_s^* = 12/\overline{1}\,\overline{2}/sing$.

By Lemmas 1 and 14, any two consecutive partitions in the sequence generated by this algorithm have distance at most 2. Since by Lemma 13 any $\mathbf{D}_n$ nonnesting partition without zero-block is the child of a unique $\mathbf{D}_{n-1}$ nonnesting partition without zero-block, it follows that the sequence obtained by the procedure above is an exhaustive list of the elements of $NN_D(n)$ with no zero-block. □

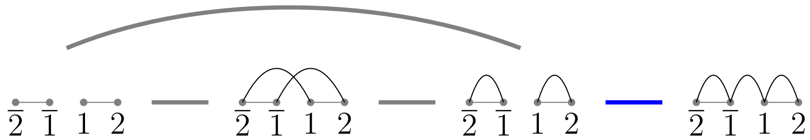

FIGURE 3.9. A Hamilton path in $NN_D(2)$ with a Hamilton cycle for the partitions without zero-block.

Since a type $\mathbf{D}_n$ nonnesting partition with zero-block must have an arc $(\overline{1}, 1)$, then no arc (i, j) linking a negative integer i to a positive integer j can exist in such partition, since otherwise we would have a nest: $i < \overline{1}, 1 < j$. Thus, if we restrict ourselves to the subset of $NN_D(n)$ formed by those nonnesting partition π having a zero-block, then any other block B of π satisfies $B \subset [2, n]$ or $-B \subset [2, n]$. This property can be used to order all type $\mathbf{D}_n$ nonnesting partition with zero-block, according to the openers of its positive arcs.

Apply the algorithm $GenTot(n)$ given in [6] to generate in lexicographic order the list of all type $\mathbf{A}$ nonnesting partitions of $[n]$, and consider the sublist $\pi_1, \ldots, \pi_k$ formed by those nonnesting partitions in which the integer 1 is in a non-singleton block. For each partition $\pi_i = B_1/\cdots/B_\ell$ in this sublist, where $1 \in B_1$, let $-\pi_i = -B_1/\cdots/-B_\ell$ be the nonnesting partition of $[-n, -1]$ obtained by negating all integers of π, and let

$$\pi_i' = B_1 \cup -B_1/B_2/\cdots/B_\ell/-B_2/\cdots/-B_\ell$$

be the partition of $[\pm n]$ obtained from the union of π_i with $-\pi_i$ with the blocks containing 1 and -1 merged. It follows that π_i' is a type $\mathbf{D}_n$ nonnesting partition with zero-block $B_1 \cup -B_1$. Moreover, from the discussion above, all type $\mathbf{D}_n$ nonnesting partition with zero-block arise from this process. It follows that $\pi_1', \ldots, \pi_k'$ is a list of all type $\mathbf{D}$ nonnesting partitions of $[n]$ that have zero-block, arranged in lexicographic order according to the openers of its positive arcs. Note also that by construction and definition of the lexicographic order, we must have $\pi_1' = \{\pm 1, \pm 2\}/sing$.

Proposition 16. *The procedure above combinatorially generates all type* $\mathbf{D}$ *nonnesting partitions of* $[n]$ *that have zero-block, in lexicographic order, starting with the partition* $\{\pm 1, \pm 2\}/sing$.

Concatenating the Hamilton path $\pi_1, \ldots, \pi_s$ formed by all nonnesting partitions of $NN_D(n)$ without zero-block given by Proposition 15 with the sequence $\pi_{s+1}, \ldots, \pi_{s+t}$, of all nonnesting partitions of $NN_D(n)$ with zero-block obtained in Proposition 16, we get the sequence

$$\pi_1, \ldots, \pi_s, \pi_{s+1}, \ldots, \pi_{s+t}$$

of all partitions in $NN_D(n)$, where

$$\pi_i \sim \pi_{i+1}, \qquad \text{for } i = 1, \ldots, s,$$

and

$$\pi_i \leq_\ell \pi_{i+1}, \qquad \text{for } i = s+1, \ldots, s+t-1 \text{ (lexicographic order).}$$

See Figure 3.10 for the list of partitions in $NN_D(3)$ with a Hamilton cycle for the partitions without zeroblock, and the partitions with zero-block arranged in lexicographic order.

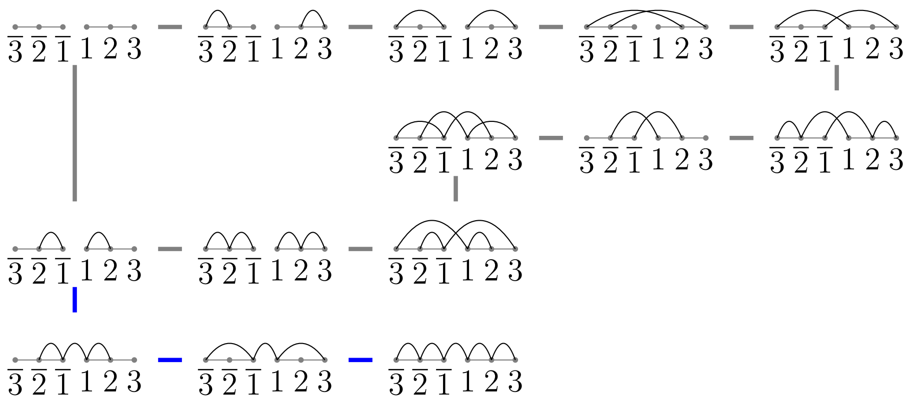

Figure 3.10. The list of partitions in $NN_D(3)$ with a Hamilton cycle for the partitions without zeroblock.

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