ON THE LOCALNESS OF THE EMBEDDING OF ALGEBRAS

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Dedicated to Manuela Sobral

Abstract. Let $N : \mathcal{A} \to \mathcal{B}$ be a faithful functor between categories. Given an object B of $\mathcal{B}$, we may ask whether there is an embedding $B \to NA$ with $A \in \mathcal{A}$. In some cases the answer is well known. For instance, an abelian semigroup may be embedded in an abelian group if and only if it is cancellative. And every Lie algebra over a field K is embeddable in an associative K-algebra with identity. Many other examples are known. This paper concentrates on the localness of the embeddability. That is, it studies conditions under which the following property holds: $B \in \mathcal{B}$ is embeddable in NA for some object A of $\mathcal{A}$, whenever every finitely generated subobject of $\mathcal{B}$ is so.

1. Introduction

The following problem has been investigated for various algebraic categories: Let $N : \mathcal{A} \to \mathcal{B}$ be a faithful functor; given an object B of $\mathcal{B}$, determine if there is a monomorphism $B \hookrightarrow NA$ with $A \in \mathcal{A}$. The following two results on this subject are well known:

(a) An abelian semigroup may be embedded in an abelian group if and only if it is cancellative.
(b) Poincaré-Birkhoff-Witt Theorem: Every Lie algebra over a field K is embeddable in an associative K-algebra with identity.

There are many other examples on the embeddability of algebras in the literature. J. MacDonald studied the subject from a categorical point of view

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[7, 8, 9]; in particular, he obtained a categorical generalization of the Poincaré-Birkhoff-Witt Theorem. In [6] P. T. Johnstone gave a new approach to the characterization of the semigroups which can be embedded in a group, unifying previous existing results. More generally, he obtained a characterization of the categories which can be embedded in a groupoid.

However these studies have very different aspects, and a general categorical treatment of this problem that could encompass a larger number of known results of the above type seems to be very difficult. This paper devotes just to its localness facet. That is, the aim of this paper is to study conditions under which, for a faithful functor $N : \mathcal{A} \to \mathcal{B}$, and an object B of $\mathcal{B}$, we have:

(E) *B is embeddable in an object of the form NA, with $A \in \mathcal{A}$, whenever every finitely generated subobject of B is so.*

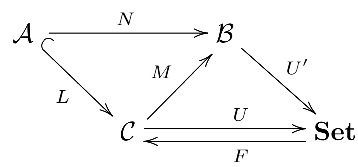
This kind of result was already achieved by B. H. Neumann in [11], and the present paper was inspired by his work. The main result, stated in Theorem 3.3 of Section 3, provides a categorical approach to the embedding theorem of Neumann.

Let Σ and Σ' be finitary signatures such that $\Sigma' \subseteq \Sigma$, and let Q be a set of quasi-identities with respect to Σ. (By a quasi-identity we mean a formula of the form $(u_i = v_i,\, i = 1, \dots, k) \Rightarrow (u = v)$, where $u_i = v_i$ and $u = v$ are identities.) Let $\mathcal{A} = \mathrm{Alg}(\Sigma, Q)$ be the category of Σ-algebras which satisfy the quasi-identities of Q, let $\mathcal{B} = \mathrm{Alg}(\Sigma')$ be the category of Σ'-algebras, and let $N : \mathcal{A} \to \mathcal{B}$ be the faithful functor which forgets the operations indicated by Σ but not by Σ'. It follows from Neumann's result that property (E) holds for N. This case is Leading Example of Section 2. The three definitions and the three lemmas of that section capture properties of the example which are going to play a role in the proof of the main result.

2. Leading Example

Leading Example. Let $\mathcal{A}$ be a quasi-variety of the form $\mathrm{Alg}(\Sigma, Q)$, where Σ is a finitary signature and Q is a set of quasi-identities, and let $\mathcal{B} = \mathrm{Alg}(\Sigma')$ for a signature Σ' contained in Σ. Let $N : \mathcal{A} \to \mathcal{B}$ be the natural forgetful functor. Moreover, let us consider $\mathcal{C} = \mathrm{Alg}(\Sigma)$. Then the faithful functor N is the composition of the inclusion functor $L : \mathcal{A} \hookrightarrow \mathcal{C}$ with a faithful functor M:

(2.1)



In the diagram, U' and U denote the usual forgetful functors, and F denotes the left adjoint of U. Thus, the two upper triangles commute.

In this section, we present some definitions and prove some lemmas that guarantee that the above leading example is encompassed by the hypotheses of the theorem of the next section, whose proof is of a categorical type.

Let I be a set of identities with respect to Σ', and contained in Q. Note that, whenever the functor $N : \mathrm{Alg}(\Sigma, Q) \to \mathrm{Alg}(\Sigma')$ satisfies (E), the same is true for the forgetful functor $N' : \mathrm{Alg}(\Sigma, Q) \to \mathrm{Alg}(\Sigma', I)$. This follows from the fact that $\mathrm{Alg}(\Sigma', I)$ is a full subcategory of $\mathrm{Alg}(\Sigma')$ closed under subobjects and the inclusion functor of $\mathrm{Alg}(\Sigma', I)$ into $\mathrm{Alg}(\Sigma')$ preserves monomorphisms.

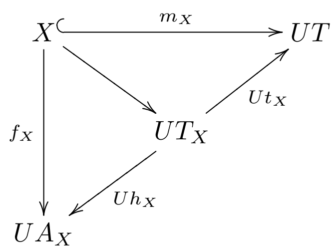
General Assumptions. From now on, we assume that $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are arbitrary categories, with $\mathcal{A}$ a full subcategory of $\mathcal{C}$, and $N : \mathcal{A} \to \mathcal{B}$ and $M : \mathcal{C} \to \mathcal{B}$ are faithful functors such that $N = ML$ for L the inclusion functor from $\mathcal{A}$ to $\mathcal{C}$. In addition, $U : \mathcal{C} \to \mathbf{Set}$ and $U' : \mathcal{B} \to \mathbf{Set}$ are functors such that $U'M = U$. Moreover, we assume that:

- $\mathcal{C}$ is cocomplete, finitely complete, and has intersections;
- $\mathcal{B}$ has intersections;
- $U : \mathcal{C} \to \mathbf{Set}$ and $U' : \mathcal{B} \to \mathbf{Set}$ are faithful, and preserve monomorphisms and intersections.

We recall that, given an object C of $\mathcal{C}$ and a subset X of UC, with inclusion map $m : X \to UC$, we obtain the *subobject of C generated by X* by taking the intersection of all subobjects $n_A : A \to C$ of C such that $m : X \to UC$ factorizes through Un_A. When a subobject of C is generated by a finite set we say that it is *finitely generated.*

Definition 2.1. *Let T be an object of $\mathcal{C}$. An equivalence relation P on UT is said to be* U-separated *if, for every finite subset X of UT there exists a commutative diagram of the form*

(2.2)



in which m_X is the inclusion map, t_X is a monomorphism, and $\ker(f_X) = P \cap (X \times X)$.

An epimorphism $c : UT \to C$ of **Set** *is said to be* U-separated *if it is the coequalizer of some U-separated equivalence relation.*

Lemma 2.1. *In the context of Leading Example, we have:*

(a) for every object T of $\mathcal{C}$, every U-separated equivalence relation on UT is a congruence on T;

(b) U creates U-separated epimorphisms, that is, if $c : UT \to C$ is a U-separated epimorphism, then there is a unique morphism $\bar{c} : T \to \bar{C}$ in $\mathcal{C}$ with $U\bar{c} = c$.

Proof. (a) We use the data of diagram (2.2), in which we assume, without loss of generality, that T_X is a subalgebra of T, and that t_X is the corresponding inclusion map. We have to show that, given $(x_1, y_1), \ldots, (x_n, y_n) \in P$ and $\theta \in \Sigma_n$, the pair $(\theta(x_1, \ldots, x_n), \theta(y_1, \ldots, y_n))$ belongs to P. For, we take

$$X = \{x_1, \ldots, x_n, y_1, \ldots, y_n, \theta(x_1, \ldots, x_n), \theta(y_1, \ldots y_n)\},$$

and note that, according to our conditions on diagram (2.2), it suffices to prove the equality $f_X(\theta(x_1, \ldots, x_n)) = f_X(\theta(y_1, \ldots y_n))$. We have:

$$\begin{aligned} f_X(\theta(x_1, \ldots, x_n)) &= h_X(\theta(x_1, \ldots, x_n)) \ \ \text{(since } h_X \text{ restricted to } X \text{ gives } f_X) \\ &= \theta(h_X(x_1), \ldots, h_X(x_n)) \ \ \text{(since } h_X \text{ is a homomorphism of algebras)} \\ &= \theta(f_X(x_1), \ldots, f_X(x_n)) \ \ \text{(again, since } h_X \text{ restricted to } X \text{ gives } f_X) \\ &= \theta(f_X(x_1'), \ldots, f_X(x_n')) \ \ \text{(since } (x_1, y_1, \ldots, (x_n, y_n) \in P) \\ &= f_X(\theta(y_1, \ldots y_n)) \ \ \text{(using the same arguments as before).} \end{aligned}$$

(b) Just apply (a) to the equivalence relation P on UT determined by c, and take $\bar{c} : T \to \bar{C}$ to be the canonical homomorphism $T \to T/P$; the uniqueness is obvious. □

Remark 2.1. *A more standard formulation of Lemma 2.1(b) would be to say that U creates coequalizers of U-separated equivalence relations, which could also be called* locally effective. *However, that reformulation would be unnecessarily restrictive in the categorical context of Section 3.*

Definition 2.2. *We say that U* locally detects $\mathcal{C}$-morphisms *if we have*

$$(\ UB \xrightarrow{g} UC\ ) = U(\ B \xrightarrow{h} C\ ),\ \textit{for some } \mathcal{C}\textit{-morphism } h,$$

whenever $g : UB \to UC$ is a map satisfying the following "local" condition:

For every finite set $X \subseteq UB$, there exists a $\mathcal{C}$-object D, a monomorphism $d : X \to UD$ and a $\mathcal{C}$-morphism $\bar{g} : D \to C$ such that:

(i) *The diagram*

(2.3)
$$\begin{array}{ccccc} X & \hookrightarrow & UB & \xrightarrow{g} & UC \\ & \searrow^{d} & & \nearrow_{U\bar{g}} & \\ & & UD & & \end{array}$$

is commutative.

(ii) *The family of all morphisms*

$$UD \xrightarrow{Uf} UA$$

for which there are a subobject of B, $s : S \to B$, with the inclusion map $X \hookrightarrow UB$ factorized through Us as $Us \cdot k$, and a $\mathcal{C}$-monomorphism $m : S \to A$ making the diagram

(2.4)
$$\begin{array}{ccccc} X & \xrightarrow{k} & US & \xrightarrow{Um} & UA \\ & \searrow^{d} & & \nearrow_{Uf} & \\ & & UD & & \end{array}$$

commute, separates every pair of points of UD separated by $U\bar{g}$, that is, $(Uf(u) = Uf(v)$ for all $f) \Rightarrow U\bar{g}(u) = U\bar{g}(v)$.

Lemma 2.2. *(a) In the context of Leading Example, U locally detects $\mathcal{C}$-morphisms (and, analogously, U' locally detects $\mathcal{B}$-morphisms).*

*(b) For every faithful functor $U : \mathcal{C} \to$ **Set**, if U locally detects $\mathcal{C}$-morphisms, then it reflects isomorphisms.*

Proof. (a) Let $g : UB \to UC$ be a map with $B, C \in \mathcal{C}$ and under the conditions of Definition 2.2. Let $\theta \in \Sigma_n$, and let $b_1, \ldots, b_n \in B$. We want to show that $g(\theta_B(b_1, \ldots, b_n)) = \theta_C(g(b_1), \ldots, g(b_n))$.

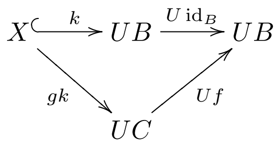
Let $X = \{b_1, \ldots, b_n, \theta_B(b_1, \ldots, b_n)\} \subseteq B$, and consider a diagram as in (2.3). In order to simplify the writing, we assume, without loss of generality, that d is an inclusion map. Then $u = \theta_B(b_1, \ldots, b_n)$ and $v = \theta_D(b_1, \ldots, b_n)$ belong to D. We show that $\bar{g}(u) = \bar{g}(v)$. For that, taking into account the condition (ii) on the map $\bar{g}$, it suffices to show that for every homomorphism $f : D \to A$ with Uf belonging to the family described in (ii), $f(u) = f(v)$. Again, for the sake of simplicity, we assume that k is an inclusion. Since $\theta_S(b_1, b_2, \ldots, b_n) = \theta_B(b_1, b_2, \ldots, b_n)$ and (2.4) is commutative, we have that $f(u) = f(\theta_B(b_1, b_2, \ldots, b_n)) = m(\theta_S(b_1, b_2, \ldots, b_n))$. As $m : S \to A$ is a homomorphism, we get $f(u) = \theta_A(m(b_1), m(b_2), \ldots, m(b_n))$. Using the commutativity of (2.4) again, and the fact that $f : D \to A$ is a homomorphism, we conclude then that

$$f(u) = \theta_A(f(b_1), f(b_2), \ldots, f(b_n)) = f(\theta_D(b_1, b_2, \cdots, b_n)) = f(v).$$

Consequently, $\bar{g}(u) = \bar{g}(v)$, and it follows that

$$\begin{aligned} g(\theta_B(b_1, \ldots, b_n)) &= \bar{g}(\theta_B(b_1, \ldots, b_n)) = \bar{g}(\theta_D(b_1, \ldots, b_n)) \\ &= \theta_C(\bar{g}(b_1), \ldots, \bar{g}(b_n)) = \theta_C(g(b_1), \ldots, g(b_n)). \end{aligned}$$

(b) Let $f : C \to B$ be a $\mathcal{C}$-morphism such that Uf is an isomorphism, and let $g : UB \to UC$ be the inverse of Uf. For every finite set $X \subseteq UB$, let $k : X \to UB$ be the inclusion map. Then, we obtain a triangle as in (2.3) of Definition 2.2, by putting $D = C$, $d = gk$ and $\bar{g} = \mathrm{id}_C$. Moreover, for every pair of elements u and v of UC, with $U\mathrm{id}_C(u) \neq U\mathrm{id}_C(v)$, we have $Uf(u) \neq Uf(v)$; thus, the commutative triangle



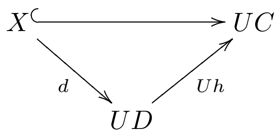
assures that (ii) of Definition 2.2 is also satisfied.

Consequently, as U locally detects $\mathcal{C}$-morphisms, we have that $g = Uh$ for some $h : B \to C$. Since U is faithful, we conclude that f is an isomorphism with $f^{-1} = h$. □

Definition 2.3. *Let $U : \mathcal{C} \to$* **Set** *be a faithful functor and let $\mathcal{A}$ be a full subcategory of $\mathcal{C}$. An object C of $\mathcal{C}$ is said to have a* local $\mathcal{A}$-behaviour *if, for every finite set $X \subseteq UC$, there is a $\mathcal{C}$-morphism $h : D \to C$ and a monomorphism $d : X \to UD$ satisfying the following conditions:*

(i) *The diagram*

(2.5)



is commutative (where the unnamed arrow $\hookrightarrow$ is the inclusion map).

(ii) *For every finite set Z such that d factors through Z into two monomorphisms,*

$$X \longrightarrow Z \xrightarrow{s} UD$$

there is some $\mathcal{C}$-morphism $f : D \to A$, with $A \in \mathcal{A}$, such that $\ker(Uf \cdot s) = \ker(Uh \cdot s)$.

Lemma 2.3. *For $\mathcal{A}$ and $\mathcal{C}$ as in Leading Example, every $C \in \mathcal{C}$ with a local $\mathcal{A}$-behaviour belongs to $\mathcal{A}$.*

PROOF. Let C be an object of $\mathcal{C}$ with local $\mathcal{A}$-behaviour. Consider a quasi-identity of Q,

$$(u_i(x) = v_i(x),\ i = 1, \dots, k) \Rightarrow (u(x) = v(x)), \tag{2.6}$$

where $x = (x_1, \dots, x_n)$, and $u_i(x)$, $v_i(x)$, $u(x)$ and $v(x)$ are terms on the variables $x_1, \dots, x_n$. Given $C \in \mathcal{C}$, let $c_1, \dots, c_n \in C$, and put $c = (c_1, \dots, c_n)$. We write

$$u^C(c)$$

for denoting the element of C obtained from $u(x)$ by replacing every x_i by c_i, and every operation symbol $\theta \in \Sigma$ by the operation θ_C.

Suppose that

$$u_i^C(c) = v_i^C(c),\ i = 1, \dots, k.$$

We want to prove that then $u^C(c) = v^C(c)$. Put

$$X = \{c_1, \dots, c_n\} \cup \{u_i^C(c),\ i = 1, \dots, k\} \cup \{u^C(c),\ v^C(c)\}.$$

By hypothesis we have a commutative diagram as in (2.5). Without loss of generality, we may assume that d is an inclusion map. Then,

$$h(u_i^D(c)) = u_i^C(h(c_1),\ \dots,\ h(c_n)) = u_i^C(c_1,\ \dots,\ c_n) = u_i^C(c),$$

and, analogously, $h(v_i^D(c)) = v_i^C(c)$. Consequently, $h(u_i^D(c)) = h(v_i^D(c))$. Consider the subset

$$Z = X \cup \{u_i^D(c),\ i = 1, \dots, k\} \cup \{v_i^D(c),\ i = 1, \dots, k\} \cup \{u^D(c),\ v^D(c)\}$$

of UD, and let $f : D \to A$ be as in (ii) of Definition 2.3. By hypothesis, $\ker(Uf \cdot s) = \ker(Uh \cdot s)$, and so the equality $h(u_i^D(c)) = h(v_i^D(c))$ implies

$$f(u_i^D(c)) = f(v_i^D(c)).$$

And then, since f is a homomorphism,

$$u_i^A(f(c_1),\ \dots,\ f(c_n)) = v_i^A(f(c_1),\ \dots,\ f(c_n)),\ i = 1, \dots, k.$$

Hence, since A satisfies the given quasi-identity (2.6),

$$u^A(f(c_1),\ \dots,\ f(c_n)) = v^A(f(c_1),\ \dots,\ f(c_n)).$$

This is the same as $f(u^D(c)) = f(v^D(c))$. But, again by the fact that $\ker(Uf \cdot s) = \ker(Uh \cdot s)$, this implies that

$$h(u^D(c)) = h(v^D(c)).$$

That is, taking into account the commutativity of (2.5), where h is a homomorphism and the other two maps are inclusion maps,

$$u^C(c) = v^C(c).$$

□

3. Main result

Before stating and proving the main result, we need some properties on nontrivial right adjoints over **Set**. (By nontrivial we mean that there is some $C \in \mathcal{C}$ such that UC has at least two elements.) In particular, we will see that, for faithful nontrivial adjunctions $(F, U, \eta, \varepsilon) : \mathbf{Set} \to \mathcal{C}$, if $f : X \to Y$ is a monomorphism of **Set**, then the square

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & UFX \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle UFf} \\ Y & \xrightarrow{\eta_Y} & UFY \end{array} \tag{3.1}$$

is a pullback. In the terminology of [5], this means that every monomorphism $f : X \to Y$ is *split over the identity morphism* $\mathrm{id}_Y : Y \to Y$.

Part (a) of the following lemma is showed in Manes [10] (Proposition 5.2 and Proposition 5.42).

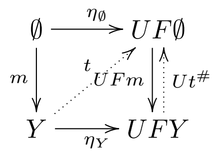
Lemma 3.1. *Let $(F, U, \eta, \varepsilon) : \mathbf{Set} \to \mathcal{C}$ be a nontrivial adjunction with U faithful. Then:*

(a) The unit η is pointwise injective and F preserves monomorphisms.

(b) Every monomorphism $f : X \to Y$ of **Set** *is split over the identity morphism* id_Y.

Proof. (a) Given $X \in \mathbf{Set}$, and two different elements $x, y \in X$, let C be an object of $\mathcal{C}$ such that UC has at least two elements, a and b. Define $h : X \to UC$ by $h(x) = a$ and $h(z) = b$ for all $z \neq x$. Now let $h^\#$ be the morphism in $\mathcal{C}$ such that $Uh^\# \cdot \eta_X = h$. Since $h(x) \neq h(y)$, then $\eta_X(x) \neq \eta_X(y)$.

Let now $m : X \to Y$ be an injective map. If $X \neq \emptyset$, then m is a split monomorphism, thus the same is true for Fm. If $X = UFX = \emptyset$, UFm is a monomorphism since it has empty domain, and then Fm is a monomorphism since U being faithful reflects isomorphisms. If $X = \emptyset$ and $UFX \neq \emptyset$, consider the diagram

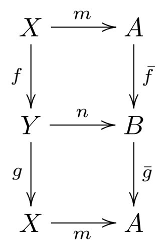


where t is any map from Y to $UF\emptyset$. Then we have $U(t^{\#}Fm)\eta_{\emptyset} = Ut^{\#} \cdot \eta_Y \cdot m = tm = \eta_{\emptyset}$. Thus $t^{\#}Fm = \mathrm{id}_{F\emptyset}$, so Fm is a monomorphism.

(b) Given a monomorphism $f : X \to Y$ in **Set**, we want to prove that the above square (3.1) is a pullback.

Assume that $X \neq \emptyset$. Then, f and UFf are split monomorphisms, and η_Y is a monomorphism, by (a).

Then the result immediately follows from a more general one: In a commutative diagram of the form



with n a monomorphism, $gf = \mathrm{id}_X$ and $\bar{g}\bar{f} = \mathrm{id}_A$, the upper square is a pullback.

To prove this, let $Y \xleftarrow{y} P \xrightarrow{a} A$ form the pullback of $Y \xrightarrow{n} B \xleftarrow{\bar{f}} A$, and let $x : X \to P$ be the unique morphism such that $ax = m$ and $yx = f$. The morphism x is a monomorphism since f is so. To conclude that x is an isomorphism, and then the desired result, we show that $xgy = \mathrm{id}_P$. We have that:

$$axgy = mgy = \bar{g}ny = \bar{g}\bar{f}a = a. \tag{3.2}$$

Moreover,

$$nyxgy = \bar{f}axgy = \bar{f}a = ny, \tag{3.3}$$

by using (3.2) in the second equality. Thus, since n is a monomorphism,

$$yxgy = y. \tag{3.4}$$

Now, because a and y are the projections of a pullback, we conclude from (3.2) and (3.4) that $xgy = \mathrm{id}_P$.

Let now $X = \emptyset$. We want to prove that the square

$$\begin{array}{ccc} \emptyset & \xrightarrow{\eta_\emptyset} & UF\emptyset \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle UFf} \\ Y & \xrightarrow[\eta_Y]{} & UFY \end{array} \tag{3.5}$$

is a pullback. We know that every right adjoint U over **Set** is representable. More precisely, U is naturally isomorphic to the hom-functor $\hom(F1, -)$, where $1 = \{*\}$. Moreover, putting $U = \hom(F1, -)$, we have that the unit η is defined, for every set X and every $x \in X$, by $\eta_X(x) = (F\bar{x} : F1 \to FX)$, where $\bar{x} : 1 \to X$ takes $*$ to x. Now, if the pullback of η_Y and UFf is not empty, then there are some $y \in Y$ and some morphism $\alpha : F1 \to F\emptyset$ such that $\eta_Y(y)$ coincides with $\hom(F1, Ff)(\alpha)$. That is, $(F1 \stackrel{F\bar{y}}{\to} FY) = (F1 \stackrel{\alpha}{\to} F\emptyset \stackrel{Ff}{\to} FY)$. But $\bar{y}$ is a monomorphism, thus so is $F\bar{y}$, by (a). Hence $\alpha : F1 \to F\emptyset$ is also a monomorphism. Since $\emptyset$ is an initial object in **Set** and F is a left adjoint, $F\emptyset$ is initial in $\mathcal{C}$. Hence, being a monomorphism, α is also an isomorphism, that is, $F1$ is initial in $\mathcal{C}$ too. This means that $UC = \hom(F1, C)$ is a singleton for every $C \in \mathcal{C}$, which contradicts the fact that U is nontrivial. □

Lemma 3.2. *Let $(F, U, \eta, \varepsilon)$: **Set** $\to \mathcal{C}$ be a nontrivial adjunction with U faithful and preserving directed colimits.*

(a) Given sets X and Z, with X finite, and a monomorphism $m : X \to UFZ$, there exists a finite subset E of Z and a monomorphism $n : X \to UFE$ such that $UFd \cdot n = m$, where d is the inclusion map of E into Z.

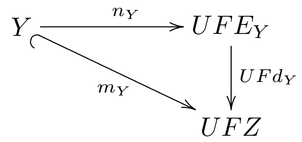
(b) If F preserves intersections, then, given sets X and Z as in (a), there is the smallest set E as in (a), denoted below by E_X.

(c) Under the assumption of (b), let X and Y be finite sets such that $X \subseteq Z$ and $Y \subseteq UFZ$, with $j : X \to Z$ and $m_Y : Y \to UFZ$ the corresponding inclusion maps. Let $s : Y \to UFX$ be a morphism such that η_X factors through s and $UFj \cdot s = m_Y$. Then $E_Y = X$.

Proof. (a) Let Z be a set, and let Z_i, $i \in I$, be the family of all finite subsets of Z, with $d_i : Z_i \to Z$ the corresponding inclusion maps. Then, by hypothesis, the maps $UFd_i : UFZ_i \to UFZ$ form a directed colimit (in fact, a directed union). Hence, for $m : X \to UFZ$ with X finite, there is some $i \in I$ and a map $n : X \to UFZ_i$ such that $UFd_i \cdot n = m$. The morphism n is clearly monomorphic.

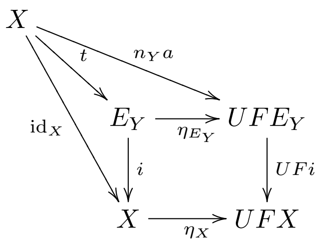
(b) We take the intersection E_X of all subsets E of Z with that property. Since UF preserves intersections, $m : X \to UFZ$ factors through the corresponding monomorphism $UFd_X : UFE_X \to UFZ$.

(c) Consider the diagram



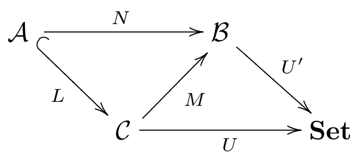
where E_Y is the smallest subset of Z such that there is $n_Y : Y \to UFE_Y$ with $m_Y = UFd_Y \cdot n_Y$, and where d_Y is the inclusion of E_Y into Z.

Thus, E_Y is contained in X; let $i : E_Y \to X$ be the inclusion map. And let $a : X \to Y$ be the morphism such that $sa = \eta_X$. Then we have that $UFj \cdot UFi \cdot n_Y \cdot a = UFd_Y \cdot n_Y \cdot a = UFj \cdot s \cdot a = UFj \cdot \eta_X$. Since j is a monomorphism, so is UFj, and, thus, $UFi \cdot n_Y \cdot a = \eta_X$. By (b) of Lemma 3.1, the square part of the commutative diagram



is a pullback. Hence, there is a map $t : X \to E_Y$ with $it = \mathrm{id}_X$. Since i is an inclusion, we conclude that $X = E_Y$. □

Theorem 3.3. *Let*



be a commutative diagram of categories and functors satisfying General Assumptions of Section 2. Moreover, assume that:

(H0) *U is nontrivial, preserves directed colimits, and has an intersection preserving left adjoint F;*
(H1) *U creates U-separated epimorphisms;*
(H2) *U' locally detects $\mathcal{B}$-morphisms;*

(H3) *$\mathcal{A}$ contains all objects in $\mathcal{C}$ with local $\mathcal{A}$-behaviour.*

Then, in $\mathcal{B}$, an object B is a subobject of some object NA with $A \in \mathcal{A}$, whenever every finitely generated subobject of B is so.

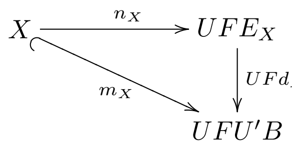
Proof. Let $B \in \mathcal{B}$ be such that every finitely generated subobject of B is a subobject of some object NA with $A \in \mathcal{A}$.

1. *Involving an inverse limit of nonempty finite sets.* Let

$$m_X : X \hookrightarrow UFU'B, \quad X \in \mathcal{F},$$

denote all inclusions of a finite subset X into $UFU'B$. For every $X \in \mathcal{F}$, let E_X be the smallest (finite) subset of $U'B$ such that the diagram

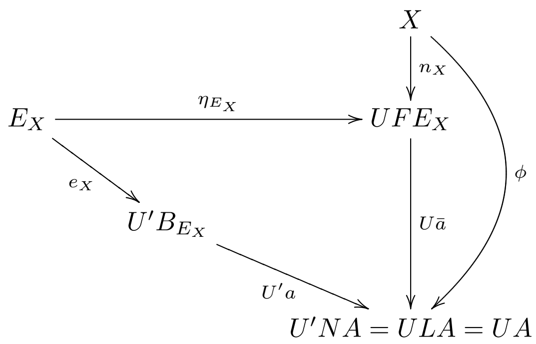
(3.6)



commutes, where d_X is the corresponding inclusion map. (The existence of this set E_X is assured by (b) of Lemma 3.2.)

For each E_X, let $r_X : B_{E_X} \to B$ be the subobject of B generated by E_X, and let $e_X : E_X \to U'B_{E_X}$ be the injective map such that $U'r_X \cdot e_X = d_X$. By hypothesis, there is some $\mathcal{B}$-monomorphism $a : B_{E_X} \to NA$, with $A \in \mathcal{A}$. Consider the diagram

(3.7)



where $\bar{a} : FE_X \to LA = A$ is the unique morphism of $\mathcal{C}$ such that $U\bar{a} \cdot \eta_{E_X} = U'a \cdot e_X$, and $\phi = U\bar{a} \cdot n_X$.

Let us now define a functor

$$\mathcal{F}^{\mathrm{op}} \xrightarrow{K} \mathbf{Set}$$

from the dual of the directed category $\mathcal{F}$, formed by all finite subsets of $U'B$ and inclusions between them, to **Set**. For every X, KX is the set of all kernel pairs of maps of the form $\phi = U\bar{a} \cdot n_X$, where $A \in \mathcal{A}$, and the morphisms $n_X : X \hookrightarrow UFE_X$ and $\bar{a} : FE_X \to LA$ are as in (3.7). Moreover, given an inclusion $i : X \hookrightarrow Y$, Ki sends each kernel pair of a morphism $\phi' = U\hat{a} \cdot n_Y : Y \to ULA'$, obtained by a way analogous to ϕ in (3.7), to the kernel pair of the morphism $\phi = \phi' i : X \to ULA'$. (This last morphism is of the form $U\bar{a} \cdot n_X$, with $\bar{a} = \hat{a} \cdot Fu$ for $u : E_X \to E_Y$ the inclusion map.) Cearly, all KX are finite. Moreover, since, for every $X \in \mathcal{F}$, there is at least one object A of $\mathcal{A}$ as in (3.7), all KX are nonempty. Consequently, by the well-known result that states that the projective limit of nonempty compact spaces is nonempty, we conclude that the limit of K is nonempty (being a finite set, KX is a compact discrete space). Let

$$(P_X)_{X \in \mathcal{F}}$$

be an element of this limit. For every $X \in \mathcal{F}$, let $P_X \underset{\pi_2^X}{\overset{\pi_1^X}{\rightrightarrows}} X$ denote the corresponding projections. In particular, every P_X is the kernel pair of a morphism

$$X \xrightarrow{f_X} UA_X \ , \quad \text{with} \quad A_X \in \mathcal{A},$$

which is obtained as ϕ in (3.7). Put $c_X = \mathrm{coeq}(m_X \pi_1^X, m_X \pi_2^X)$

$$\begin{array}{ccccccc} P_X & \underset{\pi_2^X}{\overset{\pi_1^X}{\rightrightarrows}} & X & \overset{m_X}{\hookrightarrow} & UFU'B & \xrightarrow{c_X} & C_X \\ & & {\scriptstyle f_X}\downarrow & & & & \\ & & UA_X & & & & \end{array}$$

and let $c : UFU'B \to C$ be the cointersection in **Set** of all these c_X.

We are going to show that the morphism

$$U'B \xrightarrow{\eta_{U'B}} UFU'B \xrightarrow{c} C$$

is the underlying map of a monomorphism $B \to N\bar{A}$ in $\mathcal{B}$ with $\bar{A} \in \mathcal{A}$, which proves the theorem.

2. *Applying (H1).* The fact that $(P_X)_{X \in \mathcal{F}}$ belongs to the limit of K assures that for every $X, Y \in \mathcal{F}$, with $X \subseteq Y$, $P_X = P_Y \cap (X \times X)$. Moreover, we have an accordingly chosen family of maps

$$f_X : X \to UA_X \quad (X \in \mathcal{F})$$

satisfying the conditions of Definition 2.1: put $T = FU'B$, $T_X = FE_X$, $t_X = Fd_X$ and $h_X = \bar{a}$.

Thus, the epimorphism $c : UFU'B \to C$ of **Set** is U-separated, and then, by the hypothesis (H1), there is a unique epimorphism $\bar{c} : FU'B \to \bar{C}$ in $\mathcal{C}$ such that $U\bar{c} = c$.

3. *Applying (H2).* We show now that the morphism

$$U'B \xrightarrow{\eta_{U'B}} UFU'B \xrightarrow{U\bar{c}} U\bar{C} = U'M\bar{C}$$

may be lifted to a morphism of $\mathcal{B}$. Since U' locally detects $\mathcal{B}$-morphisms, it suffices to show that the morphism $U\bar{c} \cdot \eta_{U'B} : U'B \to U'M\bar{C}$ is under the conditions of Definition 2.2. Let X be a finite subset of $U'B$, with $j : X \hookrightarrow U'B$ the inclusion map. Then, we have the following commutative triangle:

$$\begin{array}{ccccccc} X & \overset{j}{\hookrightarrow} & U'B & \xrightarrow{\eta_{U'B}} & UFU'B & \xrightarrow{U\bar{c}} & U\bar{C} = U'M\bar{C} \\ & {\scriptstyle \eta_X} \searrow & & & & \nearrow {\scriptstyle U'M(\bar{c}\cdot Fj)} & \\ & & UFX = U'MFX & & & & \end{array}$$

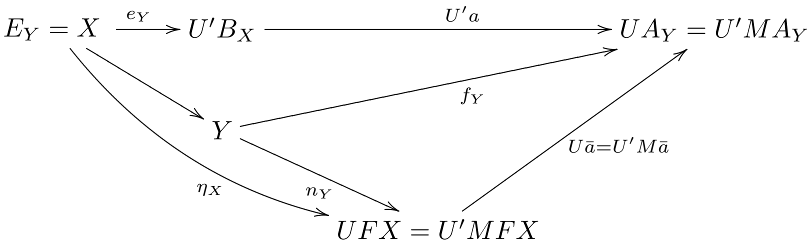
This diagram plays, for U', the role of (2.3) of condition (i) of Definition 2.2.

We show that also condition (ii) is satisfied. Indeed, let u and v be two elements of $UFX = U'MFX$, and assume that $U'f(u) = U'f(v)$, for every $\mathcal{B}$-morphism $f : MFX \to A$ for which there is a subobject $s : S \to B$ of B, with $(X \xrightarrow{j} U'B) = (X \xrightarrow{k} U'S \xrightarrow{U's} U'B)$, and a $\mathcal{B}$-monomorphism $m : S \to A$ making the diagram

$$\begin{array}{ccccc} X & \xrightarrow{k} & U'S & \xrightarrow{U'm} & U'A \\ & {\scriptstyle \eta_X} \searrow & & \nearrow {\scriptstyle U'f} & \\ & & U'MFX & & \end{array} \tag{3.8}$$

commute. We want to prove that then $U'M(\bar{c}Fj)(u) = U'M(\bar{c}Fj)(v)$, that is, $U(\bar{c}Fj)(u) = U(\bar{c}Fj)(v)$.

Let Y be the finite subset of $UFU'B$ that is the image of $\eta_X[X] \cup \{u, v\}$ under UFj, i.e., $Y = (UFj)[\eta_X[X] \cup \{u, v\}]$. Lemma 3.2(c) assures that $E_Y = X$. Consequently, by replacing, in diagram (3.7), X with Y, and ϕ with f_Y, we conclude that the outside and downside triangles of the following diagram are commutative:

(3.9) 

Since the outside triangle is of the type of (3.8), we are assuming the equality $(U'M\bar{a})(u) = (U'M\bar{a})(v)$. In particular, $f_Y(UFj(u)) = f_Y(UFj(v))$. By definition of c_Y, this implies that $c_Y(UFj(u)) = c_Y(UFj(v))$. Hence, $c(UFj(u)) = c(UFj(v))$, and, thus, $U'M(\bar{c}\cdot Fj)(u) = U'M(\bar{c}\cdot Fj)(v)$.

Therefore, since U' locally detects $\mathcal{B}$-morphisms, we conclude that the morphism $U'M\bar{c}\cdot\eta_{U'B} : U'B \to U'M\bar{C}$ is of the form

$$U\bar{c}\cdot\eta_{U'B} = U'\hat{c} \qquad \text{for some} \qquad \hat{c} : B \to M\bar{C}.$$

4. We prove that $U'\hat{c}$ is a monomorphism. Since U' is faithful, it follows that $\hat{c}$ is also a monomorphism. Let $u,\, v \in U'B$, put $X = \{u, v\}$. The set X is isomorphic to $Y = \eta_{U'B}[X]$, which is a subset of $UFU'B$. By (c) of Lemma 3.2, we have that $E_Y = X$. We know that $c\cdot\eta_{U'B}(u) = c\cdot\eta_{U'B}(v)$ if and only if $f_Y\cdot\eta_{U'B}(u) = f_Y\cdot\eta_{U'B}(v)$. But, since $X = E_Y$, we have that the morphism f_Y, obtained as ϕ in (3.7), is isomorphic to $U\bar{a}\cdot\eta_X = U'a\cdot e_Y$, with $U'a$ and e_Y monomorphisms. Thus, f_Y is indeed a monomorphism. Consequently, $c\cdot\eta_{U'B}(u) = c\cdot\eta_{U'B}(v)$ if and only if $u = v$.

5. *Applying (H3).* Finally, we prove that $\bar{C}$ belongs to $\mathcal{A}$, which completes the proof of the theorem. For that, taking into account hypothesis (H3), it suffices to show that $\bar{C}$ has a local $\mathcal{A}$-behaviour.

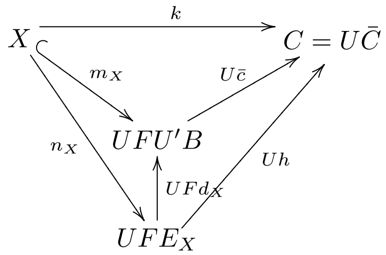
Given a finite subset X of $C = U\bar{C}$, form the pullback of the inclusion map $k : X \hookrightarrow C$ and $c : UFU'B \to C = U\bar{C}$:

$$\begin{array}{ccc} \bar{X} & \xrightarrow{\bar{k}} & UFU'B \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle c} \\ X & \xhookrightarrow{k} & C \end{array}$$

Since we are in **Set**, c is a split epimorphism, and so is r. Hence, there is some $s : X \to \bar{X}$ such that $rs = \mathrm{id}_X$. Then the map

$$(m_X : X \to UFU'B) = (\ X \xrightarrow{s} \bar{X} \xrightarrow{\bar{k}} UFU'B\ )$$

is a monomorphism in **Set**, and, without loss of generality, we assume that it is an inclusion map of X into $UFU'B$. Moreover, $k = krs = c\bar{k}s = cm_X$. Hence, for E_X defined as in (3.6), we have that, putting $h = \bar{c}Fd_X$, the diagram



is commutative. The outside triangle plays the role of triangle (2.5) of (i) of Definition 2.3. In order to show that condition (ii) also holds, let us take a finite set Z such that n_X can be presented as a composition of monomorphisms through Z, say $X \hookrightarrow Z \xrightarrow{s} UFE_X$. Without loss of generality, we may assume the first monomorphism to be an inclusion of X into Z, and assume Z to be a subset of $UFU'B$ via $UFd_X \cdot s$. Then, clearly, we have $E_Z = E_X$ and $n_Z = s$. Let $f_Z = U\bar{a} \cdot n_Z$ be as in (3.7), with $\phi = f_Z$, and $\bar{a} : FE_X = FE_Z \to A_Z$. It follows that $\mathrm{Ker}(U\bar{a} \cdot n_Z) = \mathrm{Ker}(Uh \cdot n_Z)$. Indeed, given $u, v \in Z$, $U\bar{a} \cdot n_Z(u) = U\bar{a} \cdot n_Z(v)$ if and only if $f_Z(u) = f_Z(v)$, which is equivalent to the equality $c_Z(u) = c_Z(v)$, and, then, equivalent to $U\bar{c} \cdot UFd \cdot n_Z(u) = U\bar{c} \cdot UFd \cdot n_Z(v)$, that is, to $U\bar{c}(u) = U\bar{c}(v)$. And, finally, since $m_Z = UFd_X \cdot n_Z$ is the inclusion map of Z into $UFU'B$, the last equality is equivalent to $(Uh \cdot n_Z)(u) = (Uh \cdot n_Z)(v)$. This proves that $\mathrm{Ker}(U\bar{a} \cdot n_Z) = \mathrm{Ker}(Uh \cdot n_Z)$.

Therefore, $\bar{C}$ has a local $\mathcal{A}$-behaviour, and so, by hypothesis (H3), $\bar{C} \in \mathcal{A}$. Then $M\bar{C} = ML\bar{C} = N\bar{C}$, and $\hat{c} : B \to N\bar{C}$ is a monomorphism from B to $N\bar{C}$ with $\bar{C} \in \mathcal{A}$. □

Remark 3.1. *Let $N' : \mathcal{A} \to \mathcal{B}'$ be a faithful functor. Let $\mathcal{B}'$ be a full subcategory of $\mathcal{B}$, closed under subobjects, and whose inclusion functor $I : \mathcal{B}' \hookrightarrow \mathcal{B}$ preserves monomorphisms. It is clear that, if the functor $N = IN'$ is under the hypotheses of Theorem 3.3, then the functor N' also has property (E).*

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