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A GALOIS THEORY FOR MONOIDS

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Dedicated to Manuela Sobral on the occasion of her seventieth birthday

Abstract: We show that the adjunction between monoids and groups obtained via the Grothendieck group construction is admissible, relatively to surjective homomorphisms, in the sense of categorical Galois theory. The central extensions with respect to this Galois structure turn out to be the so-called *special homogeneous surjections*.

Keywords: categorical Galois theory; homogeneous split epimorphism; special homogeneous surjection; central extension; group completion; Grothendieck group.
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Introduction

An **action** of a monoid B on a monoid X can be defined as a monoid homomorphism $B \to \mathrm{End}(X)$, where $\mathrm{End}(X)$ is the monoid of endomorphisms of X. These actions were studied in [15], where it is shown that they are equivalent to a certain class of split epimorphisms, called *Schreier split epimorphisms* in the recent paper [13]. Some properties of Schreier split epimorphisms, as well as the closely related notions of *special Schreier surjection* and *Schreier reflexive relation*, were then studied in [2] and [3], where the foundations for a cohomology theory of monoids are laid. Many typical properties of the category of groups, such as the *Split Short Five Lemma* or the fact that any internal reflexive relation is transitive, remain valid in the category of monoids when, in the spirit of relative homological algebra, those properties are restricted to Schreier split epimorphisms and Schreier reflexive relations. When an action $B \to \mathrm{End}(X)$ factors through the group $\mathrm{Aut}(X)$ of automorphisms of X, the corresponding split epimorphism is called *homogeneous* [2]. Some properties

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of homogeneous split epimorphisms and of the related notions of *special homogeneous surjection* and *homogeneous reflexive relation* were also studied in [2] and [3].

The aim of the present paper is to approach the concept of homogeneous split epimorphism from the point of view of categorical Galois theory [6, 7]. Recall that the classical *Grothendieck group* or *group completion* construction [10, 11, 12] gives an adjunction between the categories Mon of monoids and Gp of groups, which is relevant for instance in K-theory, where it is used in the definition of K_0. We prove that this adjunction is admissible in the sense of categorical Galois theory, when it is considered with respect to the class of surjective homomorphisms both in Mon and in Gp. We further show that the central extensions with respect to this adjunction are the special homogeneous surjections. This gives a positive answer to the question whether homogeneous split epimorphisms can be characterised in a way which does not refer to the underlying split epimorphism of sets.

The paper is organised as follows. In Section 1 we recall some basic notions of categorical Galois theory. In Section 2 we prove that the Grothendieck group adjunction is part of an admissible Galois structure (Theorem 2.2). In Section 3 we recall the definitions of Schreier split epimorphism and homogeneous split epimorphism, special Schreier surjection and special homogeneous surjection together with some of their properties. In Section 4 we show that the central extensions with respect to the Galois structure under consideration are exactly the special homogeneous surjections (Theorem 4.3).

1. Galois structures

We recall the definition of *Galois structure* and the concepts of *trivial*, *normal* and *central extension* arising from it, as introduced in [6, 7, 8]. For the sake of simplicity we restrict ourselves to the context of Barr-exact categories [1].

Definition 1.1. A **Galois structure** $\Gamma = (\mathscr{C}, \mathscr{X}, H, I, \eta, \epsilon, \mathscr{E}, \mathscr{F})$ consists of an adjunction

$$\begin{tikzcd} \mathscr{C} \arrow[r, shift left, "I"] \arrow[r, phantom, "\perp"] & \mathscr{X} \arrow[l, shift left, "H"] \end{tikzcd}$$

with unit $\eta\colon 1_{\mathscr{C}} \Rightarrow HI$ and counit $\epsilon\colon IH \Rightarrow 1_{\mathscr{X}}$ between Barr-exact categories $\mathscr{C}$ and $\mathscr{X}$, as well as classes of morphisms $\mathscr{E}$ in $\mathscr{C}$ and $\mathscr{F}$ in $\mathscr{X}$ such that:

(1) $\mathscr{E}$ and $\mathscr{F}$ contain all isomorphisms;
(2) $\mathscr{E}$ and $\mathscr{F}$ are pullback-stable;

(3) $\mathscr{E}$ and $\mathscr{F}$ are closed under composition;
(4) $H(\mathscr{F}) \subseteq \mathscr{E}$;
(5) $I(\mathscr{E}) \subseteq \mathscr{F}$.

We will follow [7] and call the morphisms in $\mathscr{E}$ and $\mathscr{F}$ **fibrations**.

Definition 1.2. A **trivial extension** is a fibration $f\colon A \to B$ in $\mathscr{C}$ such that the square

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & HI(A) \arrow[d, "HI(f)"] \\ B \arrow[r, "\eta_B"'] & HI(B) \end{tikzcd}$$

is a pullback. A **central extension** is a fibration f whose pullback $p^*(f)$ along *some* fibration p is a trivial extension. A **normal extension** is a fibration such that its kernel pair projections are trivial extensions.

It is well known and easy to see that trivial extensions are always central extensions and that any normal extension is automatically central.

Given an object B in $\mathscr{C}$ we consider the induced adjunction

$$(\mathscr{E} \downarrow B) \underset{H^B}{\overset{I^B}{\underset{\perp}{\rightleftarrows}}} (\mathscr{F} \downarrow I(B)),$$

where we write $(\mathscr{E} \downarrow B)$ for the full subcategory of the slice category $(\mathscr{C} \downarrow B)$ determined by morphisms in $\mathscr{E}$; similarly for $(\mathscr{F} \downarrow I(B))$. Here I^B is the restriction of I, and H^B sends a fibration $g\colon X \to I(B)$ to the pullback

$$\begin{tikzcd} A \arrow[r] \arrow[d, "H^B(g)"'] \arrow[dr, phantom, "\lrcorner", very near start] & H(X) \arrow[d, "H(g)"] \\ B \arrow[r, "\eta_B"'] & HI(B) \end{tikzcd}$$

of $H(g)$ along η_B.

Definition 1.3. A Galois structure $\Gamma = (\mathscr{C}, \mathscr{X}, H, I, \eta, \epsilon, \mathscr{E}, \mathscr{F})$ is said to be **admissible** when all functors H^B are full and faithful.

Proposition 1.4. [9, Proposition 2.4] *If Γ is admissible, then $I\colon \mathscr{C} \to \mathscr{X}$ preserves pullbacks along trivial extensions. In particular, the trivial extensions are pullback-stable, so that every trivial extension is a normal extension.*

2. The Grothendieck group of a monoid

The **Grothendieck group** (or **group completion**) of a monoid $(M, \cdot, 1)$ is given by a group $\mathrm{Gp}(M)$ together with a monoid homomorphism $M \to \mathrm{Gp}(M)$ which is universal with respect to monoid homomorphisms from M to groups [10, 11, 12]. More precisely, we have

$$\mathrm{Gp}(M) = \frac{\mathrm{GpF}(M)}{\mathrm{N}(M)},$$

where $\mathrm{GpF}(M)$ denotes the free group on M and $\mathrm{N}(M)$ is the normal subgroup generated by elements of the form $[m_1][m_2][m_1 \cdot m_2]^{-1}$ (from now on, we simply write m_1m_2 instead of $m_1 \cdot m_2$). This gives us an equivalence relation $\equiv$ on $\mathrm{GpF}(M)$ generated by $[m_1][m_2] \equiv [m_1m_2]$ with equivalence classes $\overline{[m_1][m_2]} = \overline{[m_1m_2]}$. Thus, an arbitrary element in $\mathrm{Gp}(M)$—an equivalence class of words—can be represented by a word of the form

$$[m_1][m_2]^{-1}[m_3][m_4]^{-1}\cdots[m_n]^{\iota(n)} \quad \text{or} \quad [m_1]^{-1}[m_2][m_3]^{-1}[m_4]\cdots[m_n]^{\iota(n)},$$

where $\iota(n) = \pm 1$, $n \in \mathbb{N}$, $m_1, \ldots, m_n \in M$ and no further cancellation is possible.

Let Mon and Gp represent the categories of monoids and of groups, respectively. The group completion of a monoid determines an adjunction

$$\begin{tikzcd} \mathsf{Mon} \arrow[r, shift left, "\mathrm{Gp}"] \arrow[r, phantom, "\perp"] & \mathsf{Gp}, \arrow[l, shift left, "\mathrm{Mon}"] \end{tikzcd} \tag{A}$$

where Mon is the forgetful functor. To simplify notation, we write $\mathrm{Gp}(M)$ instead of $\mathrm{Mon}\mathrm{Gp}(M)$ when referring to the monoid structure of $\mathrm{Gp}(M)$. The counit is $\epsilon = 1_{\mathsf{Gp}}$ and the unit is defined, for any monoid M, by

$$\eta_M \colon M \to \mathrm{Gp}(M) \colon m \mapsto \overline{[m]}.$$

Remark 2.1. It is well known that in general η_M is neither surjective nor injective. For example:

- The additive monoid of natural numbers is such that $\eta_{\mathbb{N}} \colon \mathbb{N} \to \mathbb{Z}$ is an injection. In fact, η_M is injective whenever M is a monoid with cancellation.
- The monoid $M = (\{0, 1\}, \cdot, 1)$ has a trivial Grothendieck group and therefore η_M is surjective.

· The product $\mathbb{N} \times M$, for M as above, is such that $\mathrm{Gp}(\mathbb{N} \times M) = \mathbb{Z}$ (in fact, it is not difficult to see that the group completion functor preserves products) and $\eta_{\mathbb{N}\times M} \colon \mathbb{N} \times M \to \mathbb{Z}$ is neither surjective nor injective.

By choosing the classes of morphisms $\mathscr{E}$ and $\mathscr{F}$ to be the surjections in Mon and Gp, respectively, we obtain a Galois structure

$$\Gamma_{\mathrm{Mon}} = (\mathsf{Mon}, \mathsf{Gp}, \mathrm{Mon}, \mathrm{Gp}, \eta, \epsilon, \mathscr{E}, \mathscr{F}).$$

Since this is the only Galois structure we shall consider in detail, without further mention we take all normal, central and trivial extensions in this paper with respect to Γ_{Mon}.

Theorem 2.2. *The Galois structure* Γ_{Mon} *is admissible.*

Proof: For any monoid M, we must prove that the functor

$$\mathrm{Mon}^M \colon (\mathscr{F} \downarrow \mathrm{Gp}(M)) \to (\mathscr{E} \downarrow M)$$

is fully faithful. Given a morphism $\alpha \colon (A, f) \to (B, g)$ in $(\mathscr{F} \downarrow \mathrm{Gp}(M))$, its image through Mon^M is defined by the universal property of the front pullback below:

$$\begin{tikzcd}
M \times_{\mathrm{Gp}(M)} A \arrow[rr, "\pi_A"] \arrow[dr, dotted, "\mathrm{Mon}^M(\alpha)"] \arrow[dd, twoheadrightarrow, "\mathrm{Mon}^M(f)"'] & & A \arrow[dr, "\alpha"] \arrow[dd, twoheadrightarrow, "f" near end] & \\
& M \times_{\mathrm{Gp}(M)} B \arrow[rr, "\pi_B" near start] \arrow[dl, twoheadrightarrow, "\mathrm{Mon}^M(g)"] & & B \arrow[dl, twoheadrightarrow, "g"] \\
M \arrow[rr, "\eta_M"'] & & \mathrm{Gp}(M). &
\end{tikzcd} \qquad \textbf{(B)}$$

First we prove that Mon^M is faithful. Consider α, $\beta \colon (A, f) \to (B, g)$ such that $\mathrm{Mon}^M(\alpha) = \mathrm{Mon}^M(\beta)$. For any $a \in A$, we prove that $\alpha(a) = \beta(a)$ by induction on the length n (supposing that no cancellations are possible) of the word that represents the class $f(a)$.

If $f(a) = \overline{[m]}$, then $(m, a) \in M \times_{\mathrm{Gp}(M)} A$ and

$$\mathrm{Mon}^M(\alpha)(m, a) = \mathrm{Mon}^M(\beta)(m, a)$$

implies that $\alpha(a) = \beta(a)$. If $f(a) = \overline{[m]^{-1}}$, then $f(a^{-1}) = \overline{[m]}$ and we find $\alpha(a^{-1}) = \beta(a^{-1})$ as in the previous case; hence $\alpha(a) = \beta(a)$.

Suppose that $\alpha(a') = \beta(a')$ for those $a' \in A$ which have $f(a')$ represented by a word of length $n-1$ or smaller. Suppose that $f(a)$ is represented by a word of length n, $n \geqslant 2$. It can be written as the product (= concatenation) of a word of length one and a word of length $n-1$. By the surjectivity of f, their corresponding classes can be written as $f(a_1)$ and $f(a_1^{-1}a)$, for some $a_1 \in A$. Then

$$\alpha(a) = \alpha(a_1)\alpha(a_1^{-1}a) = \beta(a_1)\beta(a_1^{-1}a) = \beta(a)$$

by the induction hypothesis.

Now we have to show that Mon^M is full. The proof goes in two steps: first a proof by induction in the case when M is a free monoid (Lemma 2.3 below), then an extension from the free case to the general case (the subsequent Lemma 2.4). ■

Lemma 2.3. *The functor* Mon^M *is full for all free monoids* M.

Proof: Let M be a free monoid. To simplify notation, we identify the classes in $\mathrm{Gp}(M)$ with their representatives. Consider group surjections f and g as in Diagram (**B**) and a monoid homomorphism

$$\gamma\colon (M \times_{\mathrm{Gp}(M)} A, \mathrm{Mon}^M(f)) \to (M \times_{\mathrm{Gp}(M)} B, \mathrm{Mon}^M(g)).$$

We define a group homomorphism $\alpha\colon (A, f) \to (B, g)$ as follows. For any $a \in A$, we define $\alpha(a)$ by decomposing a into a product of elements in the image of π_A. The main difficulty lies in proving that the result is independent of the chosen decomposition.

If $f(a) = [m]$ for some $m \in M$, then $(m, a) \in M \times_{\mathrm{Gp}(M)} A$ and we define

$$\alpha(a) \coloneqq \pi_B(\gamma(m, a)).$$

If $f(a) = [m]^{-1}$, then $f(a^{-1}) = [m]$ and we define

$$\alpha(a) \coloneqq \pi_B(\gamma(m, a^{-1}))^{-1}.$$

Suppose that $a = a_1 a_2^{-1} \cdots a_n^{\iota(n)}$ such that $f(a_i) = [m_i]$, with $m_i \in M$, and n is the smallest number for which such a decomposition in $\mathrm{Gp}(M)$ exists. Then we must put

$$\alpha(a) = \pi_B(\gamma(m_1, a_1))\pi_B(\gamma(m_2, a_2))^{-1} \cdots \pi_B(\gamma(m_n, a_n))^{\iota(n)}; \qquad (\mathbf{C})$$

the case $a = a_1^{-1} a_2 \cdots a_n^{\iota(n)}$ can be treated similarly.

To prove that α is a homomorphism, it now suffices to show that it is well defined. That is to say, if $a = x_1 x_2^{-1} \cdots x_k^{\iota(k)}$ such that $f(x_i) = [l_i]$, with $l_i \in M$, then (**C**) must agree with

$$\pi_B(\gamma(l_1, x_1))\pi_B(\gamma(l_2, x_2))^{-1} \cdots \pi_B(\gamma(l_k, x_k))^{\iota(k)}.$$

Since M is free, and hence the group $\mathrm{Gp}(M)$ is free, if the words

$$[m_1][m_2]^{-1} \cdots [m_n]^{\iota(n)} \qquad \text{and} \qquad [l_1][l_2]^{-1} \cdots [l_k]^{\iota(k)}$$

are both of minimal length, then $k = n$ and $l_i = m_i$. Thus we only have to prove the result for decompositions of equal length mapping down to the same word in $\mathrm{Gp}(M)$. We do this by induction on n.

Case $n = 1$. Suppose that $a_1 = a = x_1$ and $f(a_1) = [m_1] = f(x_1)$ for some $m_1 \in M$. Then obviously $\alpha(a_1) = \alpha(x_1)$. The same happens if $a_1 = a = x_1$ and $f(a_1) = [m_1]^{-1} = f(x_1)$ for some $m_1 \in M$.

More generally, let a, $x \in A$ be such that $f(a) = [m] = f(x)$ for some $m \in M$. Then $f(x^{-1}a) = [1]$, so

$$\alpha(a) = \pi_B(\gamma(m, a)) = \pi_B(\gamma(m, x))\pi_B(\gamma(1, x^{-1}a)) = \alpha(x)\alpha(x^{-1}a)$$

which implies that

$$\alpha(x^{-1}a) = \alpha(x)^{-1}\alpha(a).$$

This formula will be useful in the sequel of the proof.

Case $n = 2$. Now consider $a \in A$ such that $a_1 a_2^{-1} = a = x_1 x_2^{-1}$ and $f(a_i) = [m_i] = f(x_i)$ with $m_i \in M$. Then $\alpha(x_i^{-1}a_i) = \alpha(x_i)^{-1}\alpha(a_i)$ by the formula above. Hence $x_1^{-1}a_1 = x_2^{-1}a_2$ implies $\alpha(x_1)^{-1}\alpha(a_1) = \alpha(x_2)^{-1}\alpha(a_2)$, so that

$$\alpha(a_1)\alpha(a_2)^{-1} = \alpha(x_1)\alpha(x_2)^{-1}.$$

The case in which $a_1^{-1}a_2 = a = x_1^{-1}x_2$ and $f(a_i) = [m_i] = f(x_i)$ is similar.

Case $n = 3$. Suppose $a_1 a_2^{-1} a_3 = a = x_1 x_2^{-1} x_3$ such that $f(a_i) = [m_i] = f(x_i)$, with $m_i \in M$. Then

$$x_1^{-1}a_1 a_2^{-1} a_3 = x_2^{-1} x_3$$

gives

$$\alpha(a_2 a_1^{-1} x_1)^{-1}\alpha(a_3) = \alpha(x_2)^{-1}\alpha(x_3)$$

because they both map to the same word $[m_2]^{-1}[m_3]$. Similarly,

$$a_1 a_2^{-1} = x_1 x_2^{-1} x_3 a_3^{-1}$$

gives

$$\alpha(a_1)\alpha(a_2)^{-1} = \alpha(x_1)\alpha(a_3 x_3^{-1} x_2)^{-1}$$

because they both map to $[m_1][m_2]^{-1}$. As a consequence, the equality

$$a_3x_3^{-1}x_2 = a_2a_1^{-1}x_1$$

above the word $[m_2]$ of length one gives

$$\alpha(a_3x_3^{-1}x_2) = \alpha(a_2a_1^{-1}x_1)$$

so that $\alpha(x_1)^{-1}\alpha(a_1)\alpha(a_2)^{-1} = \alpha(x_2)^{-1}\alpha(x_3)\alpha(a_3)^{-1}$ and thus

$$\alpha(a_1)\alpha(a_2)^{-1}\alpha(a_3) = \alpha(x_1)\alpha(x_2)^{-1}\alpha(x_3).$$

Again, the case $a_1^{-1}a_2a_3^{-1} = a = x_1^{-1}x_2x_3^{-1}$ can be treated analogously.

Case $n \geqslant 4$. Suppose that the result holds for all decompositions which map down to words of minimal length $n-1$ or shorter in $\mathrm{Gp}(M)$. Suppose that $a_1a_2^{-1}a_3\cdots a_n^{\iota(n)} = a = x_1x_2^{-1}x_3\cdots x_n^{\iota(n)}$ such that $f(a_i) = [m_i] = f(x_i)$, with $m_i \in M$. Then

$$(x_1^{-1}a_1a_2^{-1})a_3\cdots a_n^{\iota(n)} = x_2^{-1}x_3\cdots x_n^{\iota(n)}$$

both map to $[m_2]^{-1}\cdots[m_n]^{\iota(n)}$, so by the induction hypothesis we find

$$\alpha(a_2a_1^{-1}x_1)^{-1}\alpha(a_3)\cdots\alpha(a_n)^{\iota(n)} = \alpha(x_2)^{-1}\alpha(x_3)\cdots\alpha(x_n)^{\iota(n)}.$$

Furthermore, $\alpha(a_2a_1^{-1}x_1) = \alpha(a_2)\alpha(a_1)^{-1}\alpha(x_1)$ as shown above (case $n=3$) so that

$$\alpha(a_1)\alpha(a_2)^{-1}\alpha(a_3)\cdots\alpha(a_n)^{\iota(n)} = \alpha(x_1)\alpha(x_2)^{-1}\alpha(x_3)\cdots\alpha(x_n)^{\iota(n)}.$$

The case $a_1^{-1}a_2a_3^{-1}\cdots a_n^{\iota(n)} = a = x_1^{-1}x_2x_3^{-1}\cdots x_n^{\iota(n)}$ being similar, this concludes the proof. ■

Lemma 2.4. *The functor* Mon^M *is full for all monoids* M*.*

Proof: As in the previous lemma, we simplify notation by identifying the classes in $\mathrm{Gp}(M)$ with their representatives.

Consider group surjections f and g as in Diagram (**B**) as well as a group homomorphism

$$\gamma\colon (M \times_{\mathrm{Gp}(M)} A, \mathrm{Mon}^M(f)) \to (M \times_{\mathrm{Gp}(M)} B, \mathrm{Mon}^M(g)).$$

We cover the monoid M with the free monoid $\mathrm{F}(M)$ on M, then apply the Grothendieck group functor to obtain the following commutative diagram with exact columns:

$$\begin{tikzcd}
\mathrm{Ker}(r_M) \arrow[r] \arrow[d, tail] & \mathrm{N}(M) \arrow[d, tail] \\
\mathrm{F}(M) \arrow[r, tail, "\eta_{\mathrm{F}(M)}"] \arrow[d, two heads, "r_M"'] & \mathrm{GpF}(M) \arrow[d, "q_M"] \\
M \arrow[r, "\eta_M"'] & \mathrm{Gp}(M).
\end{tikzcd}$$

We pull back $\mathrm{Mon}^M(f)$, $\mathrm{Mon}^M(g)$ and the morphism γ between them along the surjection r_M. We thus obtain a diagram

$$\begin{tikzcd}[column sep=small]
\mathrm{F}(M) \times_{\mathrm{Gp}(M)} A \arrow[rr, two heads, "r_M \times_{\mathrm{Gp}(M)} 1_A"] \arrow[dr, "r_M^*(\gamma)"] \arrow[dd, two heads, "r_M^*(\mathrm{Mon}^M(f))"'] & & M \times_{\mathrm{Gp}(M)} A \arrow[rr, "\pi_A"] \arrow[dr, "\gamma"] \arrow[dd, two heads, "\mathrm{Mon}^M(f)"' near start] & & A \arrow[dd, two heads, "f" near start] & \\
& \mathrm{F}(M) \times_{\mathrm{Gp}(M)} B \arrow[rr, two heads, "r_M \times_{\mathrm{Gp}(M)} 1_B" near start] \arrow[dl, two heads, "r_M^*(\mathrm{Mon}^M(g))"] & & M \times_{\mathrm{Gp}(M)} B \arrow[rr, "\pi_B" near start] \arrow[dl, two heads, "\mathrm{Mon}^M(g)"] & & B \arrow[dl, two heads, "g"] \\
\mathrm{F}(M) \arrow[rr, two heads, "r_M"'] & & M \arrow[rr, "\eta_M"'] & & \mathrm{Gp}(M). &
\end{tikzcd}$$

Since $\eta_M r_M = q_M \eta_{F(M)}$, the left hand side triangle of this diagram can also be obtained by taking the pullbacks of f and g along $q_M \eta_{F(M)}$:

$$\begin{tikzcd}[column sep=small]
\mathrm{F}(M) \times_{\mathrm{Gp}(M)} A \arrow[rr, tail, "\eta_{\mathrm{F}(M)} \times_{\mathrm{Gp}(M)} 1_A"] \arrow[dr, "r_M^*(\gamma)"] \arrow[dd, two heads, "r_M^*(\mathrm{Mon}^M(f))"'] & & \mathrm{GpF}(M) \times_{\mathrm{Gp}(M)} A \arrow[rr, "p_A"] \arrow[dr, dotted, "\beta"] \arrow[dd, two heads, "q_M^*(f)"' near start] & & A \arrow[dd, two heads, "f" near start] & \\
& \mathrm{F}(M) \times_{\mathrm{Gp}(M)} B \arrow[rr, tail, "\eta_{\mathrm{F}(M)} \times_{\mathrm{Gp}(M)} 1_B" near start] \arrow[dl, two heads, "r_M^*(\mathrm{Mon}^M(g))"] & & \mathrm{GpF}(M) \times_{\mathrm{Gp}(M)} B \arrow[rr, "p_B" near start] \arrow[dl, two heads, "q_M^*(g)"] & & B \arrow[dl, two heads, "g"] \\
\mathrm{F}(M) \arrow[rr, tail, "\eta_{\mathrm{F}(M)}"'] & & \mathrm{GpF}(M) \arrow[rr, "q_M"'] & & \mathrm{Gp}(M). &
\end{tikzcd}$$

Since the functor $\mathrm{Mon}^{\mathrm{F}(M)}$ is full by Lemma 2.4, we find a morphism β given by the dotted arrow above. It suffices to show that β keeps the elements of the kernel $\mathrm{N}(M)$ (of q_M, thus also) of p_A and p_B fixed, because then it induces the needed $\alpha\colon (A, f) \to (B, g)$ by the universal property of p_A as a cokernel of its kernel.

The group $\mathrm{N}(M)$ is generated by words $[m_1][m_2][m_1m_2]^{-1}$ as a normal subgroup of $\mathrm{GpF}(M)$. Hence it suffices to prove for elements of the type

$$([m_1][m_2][m_1m_2]^{-1}, 1)$$

in $\mathrm{GpF}(M) \times_{\mathrm{Gp}(M)} A$ that

$$\beta([m_1][m_2][m_1m_2]^{-1}, 1) = ([m_1][m_2][m_1m_2]^{-1}, 1) \in \mathrm{GpF}(M) \times_{\mathrm{Gp}(M)} B.$$

Since f is a surjection, there exists an element $a \in A$ such that

$$(r_M \times_{\mathrm{Gp}(M)} 1_A)([m_1][m_2], a) = (m_1m_2, a) = (r_M \times_{\mathrm{Gp}(M)} 1_A)([m_1m_2], a).$$

For some $b \in B$ we have $\gamma(m_1m_2, a) = (m_1m_2, b)$, so using the commutativity of the second diagram, we see that

$$r_M^*(\gamma)([m_1][m_2], a) = ([m_1][m_2], b)$$

and

$$r_M^*(\gamma)([m_1m_2], a) = ([m_1m_2], b).$$

On the other hand, using the commutativity of the third diagram we find

$$\begin{aligned}\beta([m_1][m_2], a) &= \beta(\eta_{\mathrm{F}(M)} \times_{\mathrm{Gp}(M)} 1_A)([m_1][m_2], a)\\ &= (\eta_{\mathrm{F}(M)} \times_{\mathrm{Gp}(M)} 1_B)(r_M^*(\gamma)([m_1][m_2], a))\\ &= ([m_1][m_2], b)\end{aligned}$$

and, similarly, $\beta([m_1m_2], a) = ([m_1m_2], b)$, for some $b \in B$ as above. Since β is a group homomorphism, we obtain

$$\begin{aligned}\beta([m_1][m_2][m_1m_2]^{-1}, 1) &= \beta([m_1][m_2], a)\beta([m_1m_2], a)^{-1}\\ &= ([m_1][m_2], b)([m_1m_2]^{-1}, b^{-1})\\ &= ([m_1][m_2][m_1m_2]^{-1}, 1)\end{aligned}$$

which concludes the proof. ■

Remark 2.5. We can restrict the group completion to commutative monoids: it is easily seen that then Γ_{Mon} restricts to an admissible Galois structure

$$\Gamma_{\mathrm{CMon}} = (\mathsf{CMon}, \mathsf{Ab}, \mathrm{CMon}, \mathrm{Gp}|_{\mathsf{CMon}}, \eta', \epsilon', \mathscr{E}', \mathscr{F}')$$

induced by the (co)restriction

$$\begin{tikzcd} \mathsf{CMon} \arrow[r, shift left, "\mathrm{Gp}|_{\mathsf{CMon}}"] \arrow[r, phantom, "\perp"] & \mathsf{Ab}, \arrow[l, shift left, "\mathrm{CMon}"] \end{tikzcd}$$

of the adjunction (**A**) to commutative monoids and abelian groups.

We end this section with an example showing that the adjunction (**A**) is not **semi-left-exact** [5]: it is not admissible with respect to all morphisms, instead of just the surjections [4].

Example 2.6. Consider $\eta_{\mathbb{N}^2}\colon \mathbb{N}^2 \to \mathbb{Z}^2$ with morphisms f and g as in Diagram (**B**), where A is the subgroup of $\mathbb{Z}^2$ generated by $(1,-1)$, f is determined by $f(1,-1) = (1,-1)$ and

$$g\colon \mathbb{Z}^3 \to \mathbb{Z}^2\colon (k,l,m) \mapsto (k,l).$$

Then $\mathbb{N}^2 \times_{\mathbb{Z}^2} A = 0$ while $\mathbb{N}^2 \times_{\mathbb{Z}^2} \mathbb{Z}^3 = \mathbb{N}^2 \times \mathbb{Z}$, so that the functor Mon^M is not faithful: it maps, for instance, both $\alpha\colon A \to B\colon (1,-1) \mapsto (1,-1,0)$ and $\beta\colon A \to B\colon (1,-1) \mapsto (1,-1,1)$ to the zero morphism $0 \to \mathbb{N}^2 \times \mathbb{Z}$.

3. Schreier split epimorphisms and homogeneous split epimorphisms

In this section we recall some definitions and results from [2] and [3]. We work in the category Mon of monoids.

Definition 3.1. Consider a split epimorphism (f,s) with its kernel:

$$\begin{tikzcd} N \arrow[r, tail, "k"'] & X \arrow[r, shift left, two heads, "f"] & Y. \arrow[l, shift left, "s"] \end{tikzcd} \qquad (\mathbf{D})$$

It is called a **Schreier split epimorphism** when, for any $x \in X$, there exists a unique $n \in N$ such that $x = n\, sf(x)$.

Note that when we say "split epimorphism" we consider the chosen splitting as part of the structure; and for the sake of simplicity, we take canonical kernels—so N is a subset of X.

Definition 3.2. The split epimorphism (**D**) is said to be **right homogeneous** when, for every element $y \in Y$, the function $\mu_y\colon N \to f^{-1}(y)$ defined through multiplication on the right by $s(y)$, so $\mu_y(n) = n\, s(y)$, is bijective. Similarly, by duality, we can define a **left homogeneous** split epimorphism: now the function $N \to f^{-1}(y)\colon n \mapsto s(y)\, n$ must be a bijection for all $y \in Y$. A split epimorphism is said to be **homogeneous** when it is both right and left homogeneous.

Proposition 3.3. [2, Propositions 2.3 and 2.4] *Consider a split epimorphism* (f,s) *as in* $(\mathbf{D})$. *The following statements are equivalent:*

(i) (f,s) *is a Schreier split epimorphism;*
(ii) *there exists a unique function* $q\colon X \to N$ *such that* $q(x)sf(x) = x$, *for all* $x \in X$;
(iii) *there exists a function* $q\colon X \to N$ *such that* $q(x)sf(x) = x$ *and* $q(n\, s(y)) = n$, *for all* $n \in N$, $x \in X$ *and* $y \in Y$;
(iv) (f,s) *is right homogeneous.*

Definition 3.4. Given monoids Y and N, an **action** of Y on N is a monoid homomorphism $\varphi\colon Y \to \mathrm{End}(N)$, where $\mathrm{End}(N)$ is the monoid of endomorphisms of N.

Actions correspond to Schreier split epimorphisms via a semidirect product construction:

3.5. Semidirect products. It is shown in [13] that any Schreier split epimorphism $(\mathbf{D})$ corresponds to an action φ of Y on N defined by

$$\varphi(y)(n) = {}^{y}n = q(s(y)\, n)$$

for $y \in Y$ and $n \in N$. Thus (f,s) is isomorphic, as a split epimorphism, to

$$\begin{tikzcd} N \arrow[r, "\langle 1,0\rangle"'] & N \rtimes_{\varphi} Y \arrow[r, shift left, "\pi_Y"] & Y, \arrow[l, shift left, "\langle 0,1\rangle"] \end{tikzcd}$$

where $N \rtimes_{\varphi} Y$ is the semidirect product of N and Y with respect to φ: the cartesian product of sets $N \times Y$ equipped with the operation

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1\, \varphi_{y_1}(x_2), y_1 y_2),$$

where $\varphi_{y_1} = \varphi(y_1) \in \mathrm{Aut}(N)$. See [13], [2] or Chapter 5 in [3] for more details.

Proposition 3.6. [2, Proposition 3.8] *A Schreier split epimorphism* $(\mathbf{D})$ *is homogeneous if and only if the corresponding action* $\varphi\colon Y \to \mathrm{End}(N)$ *factors through the group* $\mathrm{Aut}(N)$ *of automorphisms of* N.

Lemma 3.7. [2, Lemma 4.1] *Consider the morphism of Schreier split epimorphisms*

$$\begin{tikzcd} N \arrow[r, tail, shift right, "k"'] \arrow[d, "\tilde{u}"'] & X \arrow[l, dashed, shift right, "q"'] \arrow[r, two heads, shift left, "f"] \arrow[d, "u"] & Y \arrow[l, shift left, "s"] \arrow[d, "v"] \\ N' \arrow[r, tail, shift right, "k'"'] & X' \arrow[l, dashed, shift right, "q'"'] \arrow[r, two heads, shift left, "f'"] & Y' \arrow[l, shift left, "s'"] \end{tikzcd}$$

and their kernels, and the restriction $\widetilde{u}$ of u to N. Then the left hand side square consisting of the functions q and q' also commutes: $q'u = \widetilde{u}q$.

This lemma has the following useful consequence.

Corollary 3.8. *Given a morphism between Schreier split epimorphisms as in Lemma 3.7, the homomorphism $\widetilde{u}$ preserves the action of the object Y on N: for all $y \in Y$ and $n \in N$,*

$$\widetilde{u}({}^{y}n) = {}^{v(y)}\widetilde{u}(n).$$

Proof: We have

$$\begin{aligned} \widetilde{u}({}^{y}n) &= \widetilde{u}q(s(y)\, n) = q'u(s(y)\, n) = q'(us(y)\, u(n)) \\ &= q'(s'v(y)\, \widetilde{u}(n)) = {}^{v(y)}\widetilde{u}(n). \end{aligned}$$

■

We now extend these concepts to surjections which are not necessarily split.

Definition 3.9. Given a surjective homomorphism g of monoids and its kernel pair

$$\mathrm{Eq}(g) \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} X \xrightarrow{g} Y, \qquad (\mathbf{E})$$

g is called a **special Schreier surjection** when (π_1, Δ) is a Schreier split epimorphism. It is called a **special homogeneous surjection** when (π_1, Δ) is a homogeneous split epimorphism.

As a consequence of Theorem 5.5 in [2], if g is a special Schreier surjection, then its kernel is necessarily a group.

Remark 3.10. The name *Schreier extension* was used in [16, 14] to describe a different, but closely related concept.

Remark 3.11. A special Schreier (resp. homogeneous) surjection which is a split epimorphism is always a Schreier (resp. homogeneous) split epimorphism. However, a Schreier (resp. homogeneous) split epimorphism is not necessarily a special Schreier (resp. homogeneous) surjection. Indeed, according to Proposition 3.1.12 in [3], a Schreier (resp. homogeneous) split epimorphism is a special Schreier (resp. homogeneous) surjection if and only if its kernel is a group. In fact, by Proposition 2.3.4 in [3], taking the kernel pair of a Schreier split epimorphism (f, s) as in $(\mathbf{D})$, we do obtain a Schreier split epimorphism $(\pi_1, \langle sf, 1_X\rangle)$. Nevertheless, the split epimorphism (π_1, Δ) need not be Schreier.

As a consequence of Theorem 5.5 in [2] and of the remark above we have:

Corollary 3.12. *A surjective homomorphism g as in* (**E**) *is a special Schreier (resp. special homogeneous) surjection if and only if the kernel pair projection π_1 is a special Schreier (resp. special homogeneous) surjection.*

Proposition 3.13. [3, Proposition 7.1.4] *Special Schreier and special homogeneous surjections are stable under products and pullbacks.*

Proposition 3.14. [3, Proposition 7.1.5] *Given any pullback*

$$\begin{tikzcd} X \arrow[r, two heads, "g"] \arrow[d, two heads, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[d, two heads, "f'"] \\ Y \arrow[r, two heads, "h"'] & Y' \end{tikzcd}$$

with g and h surjective homomorphisms, if f is a special Schreier (resp. special homogeneous) surjection, then so is f'.

Proposition 3.15. [2, Proposition 3.4] *Any split epimorphism* (**D**) *such that Y is a group is a homogeneous split epimorphism.*

Remark 3.16. According to the proposition above and to Remark 3.11, a split epimorphism (**D**) such that Y is a group is a special homogeneous surjection if and only if its kernel N is a group. Moreover, every surjective homomorphism between groups is a special homogeneous surjection.

4. Normal extensions and central extensions

In this section we characterise the trivial split extensions, the central and the normal extensions in the Galois structure Γ_{Mon}. The central extensions turn out to be precisely the special homogeneous surjections, while a split epimorphism of monoids is a trivial extension if and only if it is a special homogeneous surjection. This gives a characterisation which does not refer to the underlying split epimorphism of sets: Definition 3.2 in terms of elements, Proposition 3.3 where the splitting q is a *function* rather than a morphism of monoids.

Lemma 4.1. *Any morphism of homogeneous split epimorphisms and their kernels*

$$\begin{tikzcd}
N \arrow[r, tail, "k"] \arrow[d, "\tilde{u}"'] & X \arrow[r, shift left, two heads, "f"] \arrow[d, "u"'] & Y \arrow[l, shift left, "s"] \arrow[d, "\eta_Y"] \\
N' \arrow[r, tail, "k'"'] & X' \arrow[r, shift left, two heads, "f'"] & \mathrm{Gp}(Y) \arrow[l, shift left, "s'"]
\end{tikzcd}$$

factors into the composite

$$\begin{tikzcd}
N \arrow[r, tail, "k"] \arrow[d, equal] & X \arrow[r, shift left, two heads, "f"] \arrow[d, "\overline{\eta_Y}"'] & Y \arrow[l, shift left, "s"] \arrow[d, "\eta_Y"] \\
N \arrow[r, tail, "k''"'] \arrow[d, "\tilde{u}"'] & X'' \arrow[r, shift left, two heads, "f''"] \arrow[d, dotted, "\overline{u}"'] & \mathrm{Gp}(Y) \arrow[l, shift left, "s''"] \arrow[d, equal] \\
N' \arrow[r, tail, "k'"'] & X' \arrow[r, shift left, two heads, "f'"] & \mathrm{Gp}(Y) \arrow[l, shift left, "s'"]
\end{tikzcd}$$

of morphisms of homogeneous split epimorphisms and their kernels, where φ is as in Proposition 3.6 and $X'' = N \rtimes_{\overline{\varphi}} \mathrm{Gp}(Y)$ for $\overline{\varphi}\colon \mathrm{Gp}(Y) \to \mathrm{Aut}(N)$, the unique group homomorphism satisfying $\varphi = \overline{\varphi}\eta_Y$.

Proof: As mentioned above, we have $X \cong N \rtimes_\varphi Y$ for $\varphi\colon Y \to \mathrm{Aut}(N)$. By adjointness, this monoid morphism φ gives rise to a unique group homomorphism $\overline{\varphi}\colon \mathrm{Gp}(Y) \to \mathrm{Aut}(N)$ for which $\overline{\varphi}\eta_Y = \varphi$. Note that $\overline{\varphi}$ is necessarily given by

$$\overline{\varphi}(\overline{[y_1][y_2]^{-1}\cdots[y_n]^{\iota(n)}}) = \varphi_{y_1}\varphi_{y_2}^{-1}\cdots\varphi_{y_n}^{\iota(n)} \in \mathrm{Aut}(N) \tag{F}$$

and

$$\overline{\varphi}(\overline{[y_1]^{-1}[y_2]\cdots[y_n]^{\iota(n)}}) = \varphi_{y_1}^{-1}\varphi_{y_2}\cdots\varphi_{y_n}^{\iota(n)} \in \mathrm{Aut}(N). \tag{G}$$

Via the functoriality of the semidirect product construction this already yields the upper part of the diagram, where $\overline{\eta_Y} = 1_N \rtimes \eta_Y$. This leaves us with finding $\overline{u}\colon X'' \to X'$.

The needed morphism $\overline{u}\colon N \rtimes_{\overline{\varphi}} \mathrm{Gp}(Y) \to N' \rtimes_\psi \mathrm{Gp}(Y)$, where ψ is the action for which $X' \cong N' \rtimes_\psi \mathrm{Gp}(Y)$, is induced once we prove that $\tilde{u}$ is a morphism of $\mathrm{Gp}(Y)$-actions. More precisely, we have to show that

$$\tilde{u}(\overline{\varphi}_z(n)) = \psi_z(\tilde{u}(n))$$

for all $z \in \mathrm{Gp}(Y)$ and $n \in N$. Corollary 3.8 and the fact that $\varphi = \overline{\varphi}\eta_Y$ tell us precisely that this equality holds for generators $z = \eta_Y(y)$ of $\mathrm{Gp}(Y)$, so it suffices to check that it extends to all elements of $\mathrm{Gp}(Y)$. This needs a straightforward verification based on (**F**) and (**G**). ■

Proposition 4.2. *Consider a split epimorphism* (f, s) *as in* (**D**)*. The following statements are equivalent:*

(i) f *is a trivial extension;*
(ii) f *is a special homogeneous surjection.*

Proof: (i) ⇒ (ii) If f is a trivial extension, then by definition the diagram

$$\begin{tikzcd}
X \arrow[r, two heads, shift left, "f"] \arrow[d, "\eta_X"'] & Y \arrow[l, shift left, "s"] \arrow[d, "\eta_Y"] \\
\mathrm{Gp}(X) \arrow[r, two heads, shift left, "\mathrm{Gp}(f)"] & \mathrm{Gp}(Y) \arrow[l, shift left, "\mathrm{Gp}(s)"]
\end{tikzcd} \tag{H}$$

is a pullback. By Remark 3.16, the group homomorphism $\mathrm{Gp}(f)$ is a special homogeneous surjection; hence so is f by Proposition 3.13.

(ii) ⇒ (i) Given a split epimorphism (f, s) which is a special homogeneous surjection, we have to show that the square (**H**) is a pullback. Taking kernels we obtain the morphism of special homogeneous surjections and their kernels

$$\begin{tikzcd}
N \arrow[r, tail, "k"] \arrow[d, "\widetilde{\eta_X}"'] & X \arrow[r, two heads, shift left, "f"] \arrow[d, "\eta_X"'] & Y \arrow[l, shift left, "s"] \arrow[d, "\eta_Y"] \\
K \arrow[r, tail, "k'"'] & \mathrm{Gp}(X) \arrow[r, two heads, shift left, "\mathrm{Gp}(f)"] & \mathrm{Gp}(Y) \arrow[l, shift left, "\mathrm{Gp}(s)"]
\end{tikzcd}$$

where, in particular, the kernel N of f is a group. By Theorem 2.3.7 in [3], the square (**H**) is a pullback precisely when $\widetilde{\eta_X}$ is an isomorphism.

Lemma 4.1 gives us the diagram of solid arrows

$$\begin{tikzcd}
N \arrow[r, tail, "k"] \arrow[d, equal] & X \arrow[r, two heads, shift left, "f"] \arrow[d, "\overline{\eta_Y}"'] & Y \arrow[l, shift left, "s"] \arrow[d, "\eta_Y"] \\
N \arrow[r, tail, "k''"] \arrow[d, "\widetilde{\eta_X}"'] & X'' \arrow[r, two heads, shift left, "f''"] \arrow[d, shift right, "\overline{\eta_X}"'] & \mathrm{Gp}(Y) \arrow[l, shift left, "s''"] \arrow[d, equal] \\
K \arrow[r, tail, "k'"'] & \mathrm{Gp}(X) \arrow[u, dotted, shift right, "g"'] \arrow[r, two heads, shift left, "\mathrm{Gp}(f)"] & \mathrm{Gp}(Y). \arrow[l, shift left, "\mathrm{Gp}(s)"]
\end{tikzcd}$$

On the other hand, since X'' is a group (thanks to Remark 3.16), the universal property of $\mathrm{Gp}(X)$ makes $\overline{\eta_Y}$ induce a unique group homomorphism $g\colon \mathrm{Gp}(X) \to X''$ such that $g\eta_X = \overline{\eta_Y}$. Note that this g is actually a morphism of split epimorphisms:

$$f''g\eta_X = f''\overline{\eta_Y} = \eta_Y f = \mathrm{Gp}(f)\eta_X$$

so that $f''g = \mathrm{Gp}(f)$ by the universal property of η_X, while

$$g\mathrm{Gp}(s)\eta_Y = g\eta_X s = \overline{\eta_Y}s = s''\eta_Y$$

and thus $g\mathrm{Gp}(s) = s''$.

Finally, we have $\overline{\eta_X}g = 1_{\mathrm{Gp}(X)}$ since $\overline{\eta_X}g\eta_X = \overline{\eta_X}\overline{\eta_Y} = \eta_X$. On the other hand, using Lemma 2.1.6 in [3]—which says that Schreier split epimorphisms are strongly split epimorphisms, that is, the kernel and the section are jointly strongly epimorphic—from

$$g\overline{\eta_X}k'' = g\overline{\eta_X}\overline{\eta_Y}k = g\eta_X k = \overline{\eta_Y}k = k'' \qquad \text{and} \qquad g\overline{\eta_X}s'' = g\mathrm{Gp}(s) = s''$$

we conclude that $g\overline{\eta_X} = 1_{X''}$. In particular, the arrow $\widetilde{\eta_X}$ is an isomorphism, hence the square (**H**) is a pullback. ∎

Theorem 4.3. *For a surjective homomorphism of monoids g, the following statements are equivalent:*

(i) *g is a central extension;*
(ii) *g is a normal extension;*
(iii) *g is a special homogeneous surjection.*

Proof: Consider a surjective homomorphism and its kernel pair (**E**). Then g is a normal extension

$$\overset{(1.2)}{\Leftrightarrow} \quad \pi_1 \text{ is a trivial extension}$$

$$\overset{(4.2)}{\Leftrightarrow} \quad \pi_1 \text{ is a special homogeneous surjection}$$

$$\overset{(3.12)}{\Leftrightarrow} \quad g \text{ is a special homogeneous surjection.}$$

A normal extension is always a central extension by definition. To prove that (i) implies (iii), let us suppose that g is a central extension. Then there exists a fibration p such that $p^*(g)$ is a trivial extension, which makes it a normal extension by Proposition 1.4, hence a special homogeneous surjection by (ii) $\Rightarrow$ (iii). Since p is a surjective homomorphism, we can apply Proposition 3.14 to conclude that g is a special homogeneous surjection, too. ∎

4.4. What about special Schreier surjections? A natural question that arises is, whether the special Schreier surjections admit a similar characterisation. More precisely, does the reflection (**A**) factorise in such a way that the special Schreier surjections become the central extensions with respect to this new adjunction? As explained in the proof of Proposition 4.2, any split epimorphism of groups is necessarily special homogeneous, which implies that so are the central extensions. Hence we would need a reflective subcategory $\mathscr{X}$ of Mon in which all spit epimorphisms are Schreier split epimorphisms. By Corollary 3.1.7 in [3] though, this would mean that $\mathscr{X}$ is contained in the category of groups, which defeats the purpose.

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