THE STRUCTURE OF SPLIT REGULAR BIHOM-LIE ALGEBRAS

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ABSTRACT. We introduce the class of split regular BiHom-Lie algebras as the natural extension of the one of split Hom-Lie algebras and so of split Lie algebras. We show that an arbitrary split regular BiHom-Lie algebra $\mathfrak{L}$ is of the form $\mathfrak{L} = U + \sum_j I_j$ with U a linear subspace of a fixed maximal abelian subalgebra H and any I_j a well described (split) ideal of $\mathfrak{L}$, satisfying $[I_j, I_k] = 0$ if $j \neq k$. Under certain conditions, the simplicity of $\mathfrak{L}$ is characterized and it is shown that $\mathfrak{L}$ is the direct sum of the family of its simple ideals.

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1. INTRODUCTION AND FIRST DEFINITIONS

A BiHom-algebra is an algebra in such a way that the identities defining the structure are twisted by two homomorphisms ϕ, ψ. This class of algebras was introduced from a categorical approach in [5] as an extension of the class of Hom-algebras. The origin of Hom-structures can be found in the physics literature around 1900, appearing in the study of quasi-deformations of Lie algebras of vector fields, in particular q-deformations of Witt and Virasoro algebras, [6]. Since then, many authors have been interested in the study of Hom-algebras but we refer to [7, 8], and the references therein, for a good review of the matter. The reference [5] is also fundamental for getting the basic notions, motivations and results on BiHom-algebras.

In the present paper we introduce the class of split regular BiHom-Lie algebras $\mathfrak{L}$ as the natural extension of the one of split Hom-Lie algebras and so of split Lie algebras, and study its structure. In §2 we develop connections of roots techniques in the framework of BiHom-algebras, which becomes the main tool in our study. In §3 we apply all of these techniques to show that $\mathfrak{L}$ is of the form $\mathfrak{L} = U + \sum I_j$ with U a linear subspace of a fixed maximal abelian subalgebra H and any I_j a well described ideal of $\mathfrak{L}$, satisfying $[I_j, I_k] = 0$ if $j \neq k$. Finally, in §4, and under certain conditions, the simplicity of $\mathfrak{L}$ is characterized and it is shown that $\mathfrak{L}$ is the direct sum of the family of its simple ideals.

Definition 1.1. *A* BiHom-Lie algebra *over a field* $\mathbb{K}$ *is a* 4*-tuple* $(\mathfrak{L}, [\cdot,\cdot], \phi, \psi)$, *where* $\mathfrak{L}$ *is a* $\mathbb{K}$*-linear space,* $[\cdot,\cdot] : \mathfrak{L} \times \mathfrak{L} \to \mathfrak{L}$ *a bilinear map and* $\phi, \psi : \mathfrak{L} \to \mathfrak{L}$ *linear mappings satisfying the following identities:*

1. $\phi \circ \psi = \psi \circ \phi$,
2. $[\psi(x), \phi(y)] = -[\psi(y), \phi(x)]$, *(BiHom-skew-symmetry)*

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3. $[\psi^2(x), [\psi(y), \phi(z)]] + [\psi^2(y), [\psi(z), \phi(x)]] + [\psi^2(z), [\psi(x), \phi(y)]] = 0$, *(BiHom-Jacobi identity),*

for any $x, y, z \in \mathfrak{L}$. When ϕ, ψ furthermore are algebra automorphisms it is said that $\mathfrak{L}$ is a regular *BiHom-Lie algebra.*

Lie algebras are examples of BiHom-Lie algebras by taking $\phi = \psi = Id$. Hom-Lie algebras are also examples of BiHom-Lie algebras by considering $\psi = \phi$.

Example 1.1. *Let $(L, [\cdot, \cdot])$ be a Lie algebra and $\phi, \psi : L \to L$ two automorphisms. If we endow the underlying liner space L with a new product $[\cdot, \cdot]' : L \times L \to L$ defined by $[x, y]' := [\phi(x), \psi(y)]$ for any $x, y \in L$, we have that $(L, [\cdot, \cdot]', \phi, \psi)$ becomes a regular BiHom-Lie algebra.*

Throughout this paper $\mathfrak{L}$ will denote a regular BiHom-Lie algebra. A *subalgebra* A of $\mathfrak{L}$ is a linear subspace such that $[A, A] \subset A$ and $\phi(A) = \psi(A) = A$. A subalgebra I of $\mathfrak{L}$ is called an *ideal* if $[I, \mathfrak{L}] \subset I$, (and so necessarily $[\mathfrak{L}, I] \subset I$). A regular BiHom-Lie algebra $\mathfrak{L}$ is called *simple* if $[\mathfrak{L}, \mathfrak{L}] \neq 0$ and its only ideals are $\{0\}$ and $\mathfrak{L}$.

Finally, we would like to note that $\mathfrak{L}$ is considered of arbitrary dimension and over an arbitrary base field $\mathbb{K}$ and that we will denote by $\mathbb{N}$ the set of all non-negative integers and by $\mathbb{Z}$ the set of all integers.

Let us introduce the class of split algebras in the framework of regular BiHom-Lie algebras $\mathfrak{L}$. First, we recall that a Lie algebra $(L, [\cdot, \cdot])$, over a base field $\mathbb{K}$, is called *split* respect to a maximal abelian subalgebra H of L, if L can be written as the direct sum

$$L = H \oplus (\bigoplus_{\alpha \in \Gamma} L_\alpha)$$

where

$$L_\alpha := \{v_\alpha \in L : [h, v_\alpha] = \alpha(h) v_\alpha \text{ for any } h \in H\}$$

being any $\alpha : H \longrightarrow \mathbb{K}$, $\alpha \in \Gamma$, a non-zero linear functional on H such that $L_\alpha \neq 0$.

Let us return to a regular BiHom-Lie algebra $\mathfrak{L}$. Denote by H a maximal abelian, (in the sense $[H, H] = 0$), subalgebra of $\mathfrak{L}$. For a linear functional

$$\alpha : H \longrightarrow \mathbb{K},$$

we define the *root space* of $\mathfrak{L}$ (respect to H) associated to α as the subspace

$$\mathfrak{L}_\alpha := \{v_\alpha \in \mathfrak{L} : [h, \phi(v_\alpha)] = \alpha(h) \phi\psi(v_\alpha) \text{ for any } h \in H\}.$$

The elements $\alpha : H \longrightarrow \mathbb{K}$ satisfying $\mathfrak{L}_\alpha \neq 0$ are called *roots* of $\mathfrak{L}$ with respect to H and we denote $\Lambda := \{\alpha \in (H)^* \setminus \{0\} : \mathfrak{L}_\alpha \neq 0\}$.

Definition 1.2. *We say that $\mathfrak{L}$ is a* split regular BiHom-Lie algebra*, with respect to H, if*

$$\mathfrak{L} = H \oplus (\bigoplus_{\alpha \in \Lambda} \mathfrak{L}_\alpha).$$

We also say that Λ is the roots system *of $\mathfrak{L}$.*

As examples of split regular BiHom-Lie algebras we have the split Hom-Lie algebras and the split Lie algebras. Hence, the present paper extends the results in [1] and in [2]. Let us see another example.

Example 1.2. *Let* $(L = H \oplus (\bigoplus_{\alpha\in\Gamma} L_\alpha), [\cdot,\cdot])$ *be a split Lie algebra and* $\phi, \psi : L \to L$ *two automorphisms such that* $\phi(H) = \psi(H) = H$. *By Example 1.1, we know that* $(L, [\cdot,\cdot]', \phi, \psi)$, *where* $[x, y]' := [\phi(x), \psi(y)]$ *for any* $x, y \in L$, *is a regular BiHom-Lie algebra. Then it is straightforward to verify that the direct sum*

$$L = H \oplus (\bigoplus_{\alpha\in\Gamma} L_{\alpha\psi^{-1}})$$

makes of the regular BiHom-Lie algebra $(L, [\cdot,\cdot]', \phi, \psi)$ *a split regular BiHom-Lie algebra, being the roots system* $\Lambda = \{\alpha\psi^{-1} : \alpha \in \Gamma\}$.

From now on $\mathfrak{L} = H \oplus (\bigoplus_{\alpha\in\Lambda} \mathfrak{L}_\alpha)$ denotes a split regular BiHom-Lie algebra. Also, and for an easier notation, the mappings $\phi|_H, \psi|_H, \phi|_H^{-1}, \psi|_H^{-1} : H \to H$ will be denoted by $\phi, \psi, \phi^{-1}, \psi^{-1}$ respectively.

Lemma 1.1. *For any* $\alpha \in \Lambda \cup \{0\}$ *the following assertions hold.*

1. $\phi(\mathfrak{L}_\alpha) = \mathfrak{L}_{\alpha\phi^{-1}}$ *and* $\psi(\mathfrak{L}_\alpha) = \mathfrak{L}_{\alpha\psi^{-1}}$.
2. $\phi^{-1}(\mathfrak{L}_\alpha) = \mathfrak{L}_{\alpha\phi}$ *and* $\psi^{-1}(\mathfrak{L}_\alpha) = \mathfrak{L}_{\alpha\psi}$.

Proof. 1. For any $h \in H$ and $v_\alpha \in \mathfrak{L}_\alpha$, since

$$[h, \phi(v_\alpha)] = \alpha(h)\phi\psi(v_\alpha) \tag{1}$$

we have that by writing $h' = \phi(h)$ then

$$[h', \phi^2(v_\alpha)] = \phi([h, \phi(v_\alpha)]) = \alpha(h)\phi^2\psi(v_\alpha) = \alpha\phi^{-1}(h')\phi^2\psi(v_\alpha) =$$
$$= \alpha\phi^{-1}(h')\phi\psi(\phi(v_\alpha)).$$

That is, $\phi(v_\alpha) \in \mathfrak{L}_{\alpha\phi^{-1}}$ and so

$$\phi(\mathfrak{L}_\alpha) \subset \mathfrak{L}_{\alpha\phi^{-1}}. \tag{2}$$

Now, let us show

$$\mathfrak{L}_{\alpha\phi^{-1}} \subset \phi(\mathfrak{L}_\alpha).$$

Indeed, for any $h \in H$ and $v_\alpha \in \mathfrak{L}_\alpha$, Equation (1) shows $[\phi^{-1}(h), v_\alpha] = \alpha(h)\psi(v_\alpha)$. From here we get $[\phi(h), v_\alpha] = \alpha\phi^2(h)\psi(v_\alpha)$ and conclude

$$\phi^{-1}(\mathfrak{L}_\alpha) \subset \mathfrak{L}_{\alpha\phi}. \tag{3}$$

Hence, since for any $x \in \mathfrak{L}_{\alpha\phi^{-1}}$ we can write $x = \phi(\phi^{-1}(x))$ and by Equation (3) we have $\phi^{-1}(x) \in \mathfrak{L}_\alpha$, we conclude $\mathfrak{L}_{\alpha\phi^{-1}} \subset \phi(\mathfrak{L}_\alpha)$. This fact together with Equation (2) show $\phi(\mathfrak{L}_\alpha) = \mathfrak{L}_{\alpha\phi^{-1}}$.

To verify

$$\psi(\mathfrak{L}_\alpha) \subset \mathfrak{L}_{\alpha\psi^{-1}}, \tag{4}$$

observe that Equation (1) gives us $[\psi(h), \psi\phi(v_\alpha)] = \alpha(h)\psi\phi\psi(v_\alpha)$ and so $[\psi(h), \phi\psi(v_\alpha)] = \alpha\psi^{-1}(\psi(h))\phi\psi(\psi(v_\alpha))$. Since Equation (1) and the identity $\psi^{-1}\phi = \phi\psi^{-1}$ also give us

$$\psi^{-1}(\mathfrak{L}_\alpha) \subset \mathfrak{L}_{\alpha\psi}, \tag{5}$$

we conclude as above that $\psi(\mathfrak{L}_\alpha) = \mathfrak{L}_{\alpha\psi^{-1}}$.

2. The fact $\phi^{-1}(\mathfrak{L}_\alpha) \subset \mathfrak{L}_{\alpha\phi}$ is Equation (3), while the fact $\mathfrak{L}_{\alpha\phi} \subset \phi^{-1}(\mathfrak{L}_\alpha)$ is consequence of writing any element $x \in \mathfrak{L}_{\alpha\phi}$ of the form $x = \phi^{-1}(\phi(x))$ and apply Equation (2). We can argue similarly with Equations (5) and (4) to get $\psi^{-1}(\mathfrak{L}_\alpha) = \mathfrak{L}_{\alpha\psi}$. □

Lemma 1.2. *For any* $\alpha, \beta \in \Lambda \cup \{0\}$ *we have* $[\mathfrak{L}_\alpha, \mathfrak{L}_\beta] \subset \mathfrak{L}_{\alpha\phi^{-1}+\beta\psi^{-1}}$.

Proof. For each $h \in H, v_\alpha \in \mathfrak{L}_\alpha$ and $v_\beta \in \mathfrak{L}_\beta$ we can write

$$[h, \phi([v_\alpha, v_\beta])] = [\psi^2\psi^{-2}(h), \phi([v_\alpha, v_\beta])].$$

So, by denoting $h' = \psi^{-2}(h)$, we can apply BiHom-Jacobi identity and BiHom-skew-symmetry to get

$$\begin{aligned}
&[\psi^2(h'), \phi([v_\alpha, v_\beta])] = [\psi^2(h'), [\psi\psi^{-1}\phi(v_\alpha), \phi(v_\beta)]] = \\
&-[\psi\phi(v_\alpha), [\psi(v_\beta), \phi(h')]] - [\psi^2(v_\beta), [\psi(h'), \phi\psi^{-1}\phi(v_\alpha)]] = \\
&[\psi\phi(v_\alpha), [\psi(h'), \phi(v_\beta)]] - [\psi^2(v_\beta), [\phi\phi^{-1}\psi(h'), \phi\psi^{-1}\phi(v_\alpha)]] = \\
&[\psi\phi(v_\alpha), [\psi(h'), \phi(v_\beta)]] - [\psi(\psi(v_\beta)), \phi([\phi^{-1}\psi(h'), \psi^{-1}\phi(v_\alpha)])] = \\
&[\psi\phi(v_\alpha), [\psi(h'), \phi(v_\beta)]] + [[\psi^2\phi^{-1}(h'), \phi(v_\alpha)], \phi\psi(v_\beta)] = \\
&\beta\psi(h')[\psi\phi(v_\alpha), \phi\psi(v_\beta)] + \alpha\psi^2\phi^{-1}(h')[\phi\psi(v_\alpha)], \phi\psi(v_\beta)] = \\
&(\beta\psi + \alpha\psi^2\phi^{-1})(h')[\psi\phi(v_\alpha), \phi\psi(v_\beta)] = \\
&(\beta\psi + \alpha\psi^2\phi^{-1})(h')[\phi\psi(v_\alpha), \phi\psi(v_\beta)] = \\
&(\beta\psi + \alpha\psi^2\phi^{-1})(h')\phi\psi([v_\alpha, v_\beta]).
\end{aligned}$$

Taking now into account $h' = \psi^{-2}(h)$ we have shown

$$[h, \phi([v_\alpha, v_\beta])] = (\beta\psi^{-1} + \alpha\phi^{-1})(h)\phi\psi([v_\alpha, v_\beta]).$$

From here $[\mathfrak{L}_\alpha, \mathfrak{L}_\beta] \subset \mathfrak{L}_{\alpha\phi^{-1}+\beta\psi^{-1}}$. □

Lemma 1.3. *The following assertions hold.*

1. *If $\alpha \in \Lambda$ then $\alpha\phi^{-z_1}\psi^{-z_2} \in \Lambda$ for any $z_1, z_2 \in \mathbb{Z}$.*
2. $\mathfrak{L}_0 = H$.

Proof. 1. Consequence of Lemma 1.1-1,2.

2. The fact $H \subset \mathfrak{L}_0$ is a direct consequence of the character of abelian subalgebra of H. Let us now show $\mathfrak{L}_0 \subset H$. For any $0 \neq x \in \mathfrak{L}_0$ we can express $x = h \oplus (\bigoplus_{i=1}^{m} v_{\alpha_i})$ with $h \in H$, any $v_{\alpha_i} \in \mathfrak{L}_{\alpha_i}$ and with $\alpha_i \neq \alpha_j$ when $i \neq j$. Since for any $h' \in H$ we have $[h', x] = 0$, then Lemma 1.1 allows us to get $0 = [h', x] = [h', h + \bigoplus_{i=1}^{m} \phi\phi^{-1}(v_{\alpha_i})] = \bigoplus_{i=1}^{m} \alpha_i\phi(h')\psi(v_{\alpha_i}) = 0$. From here, Lemma 1.1 together with the fact $\alpha_i \neq 0$ give us that any $v_{\alpha_i} = 0$. Hence $x = h \in H$. □

Maybe the main topic in the theory of Hom-algebras consists in studying if a known result for a class of, non-deformed, algebra still holds true for the corresponding class of Hom-algebras. Following this line, the present paper shows how the structure theorems getting in [2] and in [1] for split Lie algebras and split regular Hom-Lie algebras respectively, also hold for the class of split regular BiHom-Lie algebras. We would like to know that all of the constructions carried out along this paper strongly involve both of the structure mappings ϕ and ψ, which makes the proofs different from the non-bi-deformed cases.

2. Connections of roots techniques

As in the previous section, $\mathfrak{L}$ denotes a split regular BiHom-Lie algebra and

$$\mathfrak{L} = \mathfrak{L}_0 \oplus (\bigoplus_{\alpha\in\Lambda} \mathfrak{L}_\alpha)$$

the corresponding root spaces decomposition. Given a linear functional $\alpha : H \to \mathbb{K}$, we denote by $-\alpha : H \to \mathbb{K}$ the element in H^* defined by $(-\alpha)(h) := -\alpha(h)$ for all $h \in H$. We also denote by

$$-\Lambda := \{-\alpha : \alpha \in \Lambda\} \text{ and } \pm\Lambda := \Lambda\dot{\cup}(-\Lambda).$$

Definition 2.1. *Let $\alpha, \beta \in \Lambda$. We will say that α is* connected *to β if*

- *Either $\beta = \epsilon\alpha\phi^{z_1}\psi^{z_2}$ for some $z_1, z_2 \in \mathbb{Z}$ and $\epsilon \in \{1, -1\}$, or*
- *Either there exists $\{\alpha_1, \alpha_2, ..., \alpha_k\} \subset \pm\Lambda$, with $k \geq 2$, such that*
  1. $\alpha_1 \in \{\alpha\phi^{-n}\psi^{-r} : n, r \in \mathbb{N}\}$.
  2. $\alpha_1\phi^{-1} + \alpha_2\psi^{-1} \in \pm\Lambda$,
     $\alpha_1\phi^{-2} + \alpha_2\phi^{-1}\psi^{-1} + \alpha_3\psi^{-1} \in \pm\Lambda$,
     $\alpha_1\phi^{-3} + \alpha_2\phi^{-2}\psi^{-1} + \alpha_3\phi^{-1}\psi^{-1} + \alpha_4\psi^{-1} \in \pm\Lambda$,
     ....
     $\alpha_1\phi^{-i} + \alpha_2\phi^{-i+1}\psi^{-1} + \alpha_3\phi^{-i+2}\psi^{-1} + \cdots + \alpha_i\phi^{-1}\psi^{-1} + \alpha_{i+1}\psi^{-1} \in \pm\Lambda$,
     ....
     $\alpha_1\phi^{-k+2} + \alpha_2\phi^{-k+3}\psi^{-1} + \alpha_3\phi^{-k+4}\psi^{-1} + \cdots + \alpha_{k-2}\phi^{-1}\psi^{-1} + \alpha_{k-1}\psi^{-1} \in \pm\Lambda$.
  3. $\alpha_1\phi^{-k+1} + \alpha_2\phi^{-k+2}\psi^{-1} + \alpha_3\phi^{-k+3}\psi^{-1} + \cdots + \alpha_i\phi^{-k+i}\psi^{-1} + \cdots + \alpha_{k-1}\phi^{-1}\psi^{-1} + \alpha_k\psi^{-1} \in \{\pm\beta\phi^{-m}\psi^{-s} : m, s \in \mathbb{N}\}$.

  We will also say that $\{\alpha_1, ..., \alpha_k\}$ is a connection *from α to β.*

Observe that for any $\alpha \in \Lambda$, we have that $\alpha\phi^{z_1}\psi^{z_2}$ is connected to $\alpha\phi^{z_3}\psi^{z_4}$ for any $z_1, z_2, z_3, z_4 \in \mathbb{Z}$, and also to $-\alpha\phi^{z_3}\psi^{z_4}$ in case $-\alpha \in \Lambda$.

Lemma 2.1. *The relation $\sim$ in Λ, defined by $\alpha \sim \beta$ if and only if α is connected to β, is symmetric.*

Proof. Suppose $\alpha \sim \beta$. In case $\beta = \epsilon\alpha\phi^{z_1}\psi^{z_2}$ with $z_1, z_2 \in \mathbb{Z}$ and $\epsilon \in \{1, -1\}$ we clearly have $\beta \sim \alpha$. So, let us consider a connection

$$\{\alpha_1, \alpha_2, ..., \alpha_k\} \subset \pm\Lambda, \tag{6}$$

$k \geq 2$, from α to β. Observe that condition 3. in Definition 2.1 allows us to distinguish two possibilities. In the first one

$$\alpha_1\phi^{-k+1} + \alpha_2\phi^{-k+2}\psi^{-1} + \cdots + \alpha_i\phi^{-k+i}\psi^{-1} + \cdots + \alpha_k\psi^{-1} = \beta\phi^{-m}\psi^{-s}, \tag{7}$$

while in the second one

$$\alpha_1\phi^{-k+1} + \alpha_2\phi^{-k+2}\psi^{-1} + \cdots + \alpha_i\phi^{-k+i}\psi^{-1} + \cdots + \alpha_k\psi^{-1} = -\beta\phi^{-m}\psi^{-s} \tag{8}$$

for some $m, s \in \mathbb{N}$.

Suppose we have the first above possibility (7). Lemma 1.3-1 shows that the set

$$\{\beta\phi^{-m}\psi^{-s}, -\alpha_k\phi^{-1}, -\alpha_{k-1}\phi^{-3}, -\alpha_{k-2}\phi^{-5}, ..., -\alpha_{k-i}\phi^{-2i-1}, ..., -\alpha_2\phi^{-2k+3}\} \subset \pm\Lambda.$$

We are going to show that this set is a connection from β to α. It is clear that satisfies condition 1. of Definition 2.1, so let us check that also satisfies condition 2. We have

$$(\beta\phi^{-m}\psi^{-s})\phi^{-1} - (\alpha_k\phi^{-1})\psi^{-1} = (\beta\phi^{-m}\psi^{-s} - \alpha_k\psi^{-1})\phi^{-1} =$$
$$(\alpha_1\phi^{-k+1} + \alpha_2\phi^{-k+2}\psi^{-1} + \cdots + \alpha_{k-1}\phi^{-1}\psi^{-1})\phi^{-1},$$

last equality being consequence of Equation (7), and so

$$(\beta\phi^{-m}\psi^{-s})\phi^{-1} - (\alpha_k\phi^{-1})\psi^{-1} = (\alpha_1\phi^{-k+2} + \alpha_2\phi^{-k+3}\psi^{-1} + \cdots + \alpha_{k-1}\psi^{-1})\phi^{-2}.$$

Taking into account

$$\alpha_1\phi^{-k+2}+\alpha_2\phi^{-k+3}\psi^{-1}+\cdots+\alpha_{k-1}\psi^{-1}\in\pm\Lambda$$

by condition 2. of Definition 2.1 applied to the connection (6), Lemma 1.3-1 allows us to assert $(\beta\phi^{-n}\psi^{-s})\phi^{-1}-(\alpha_k\phi^{-1})\psi^{-1}\in\pm\Lambda$.

For any $1\leq i\leq k-2$ we also have that,

$$(\beta\phi^{-m}\psi^{-s})\phi^{-i}-(\alpha_k\phi^{-1})\phi^{-i+1}\psi^{-1}-(\alpha_{k-1}\phi^{-3})\phi^{-i+2}\psi^{-1}-\cdots-(\alpha_{k-(i-1)}\phi^{-2i+1})\psi^{-1}=$$
$$(\beta\phi^{-m}\psi^{-s}-\alpha_k\psi^{-1}-\alpha_{k-1}\phi^{-1}\psi^{-1}-\cdots-\alpha_{k-(i-1)}\phi^{-i+1}\psi^{-1})\phi^{-i}=$$
$$(\alpha_1\phi^{-k+1}+\alpha_2\phi^{-k+2}\psi^{-1}+\cdots+\alpha_{k-i}\phi^{-i}\psi^{-1})\phi^{-i},$$

last equality being consequence of Equation (7). From here,

$$(\beta\phi^{-m}\psi^{-s})\phi^{-i}-(\alpha_k\phi^{-1})\phi^{-i+1}\psi^{-1}-(\alpha_{k-1}\phi^{-3})\phi^{-i+2}\psi^{-1}-\cdots-(\alpha_{k-(i-1)}\phi^{-2i+1})\psi^{-1}=$$
$$(\alpha_1\phi^{-k+i+1}+\alpha_2\phi^{-k+i+2}\psi^{-1}+\cdots+\alpha_{k-i}\psi^{-1})\phi^{-2i}.$$

Taking now into account that, by condition 2. of Definition 2.1 applied to (6),

$$\alpha_1\phi^{-k+i+1}+\alpha_2\phi^{-k+i+2}\psi^{-1}+\cdots+\alpha_{k-i}\psi^{-1}\in\pm\Lambda,$$

we get as consequence of Lemma 1.3-1 that

$$(\beta\phi^{-m}\psi^{-s})\phi^{-i}-(\alpha_k\phi^{-1})\phi^{-i+1}\psi^{-1}-(\alpha_{k-1}\phi^{-3})\phi^{-i+2}\psi^{-1}-\cdots$$
$$\cdots-(\alpha_{k-(i-1)}\phi^{-2i+1})\psi^{-1}\in\pm\Lambda.$$

Consequently, our set satisfies condition 2. of Definition 2.1. Let us prove that this set also satisfies condition 3. of this definition. We have as above that

$$(\beta\phi^{-m}\psi^{-s})\phi^{-k+1}-(\alpha_k\phi^{-1})\phi^{-k+2}\psi^{-1}-(\alpha_{k-1}\phi^{-3})\phi^{-k+3}\psi^{-1}-\cdots-(\alpha_2\phi^{-2k+3})\psi^{-1}=$$
$$(\beta\phi^{-m}\psi^{-s}-\alpha_k\psi^{-1}-\alpha_{k-1}\phi^{-1}\psi^{-1}-\cdots-\alpha_2\phi^{-k+2}\psi^{-1})\phi^{-k+1}=$$
$$(\alpha_1\phi^{-k+1})\phi^{-k+1}.$$

Condition 1. of Definition 2.1 applied to the connection (6) gives us now that $\alpha_1=\alpha\phi^{-n}\psi^{-r}$ for some $n,r\in\mathbb{N}$ and so

$$(\beta\phi^{-m}\psi^{-s})\phi^{-k+1}-(\alpha_k\phi^{-1})\phi^{-k+2}\psi^{-1}-(\alpha_{k-1}\phi^{-3})\phi^{-k+3}\psi^{-1}-\cdots-(\alpha_2\phi^{-2k+3})\psi^{-1}=$$
$$\alpha\phi^{-(2k-2+n)}\psi^{-r}\in\{\alpha\phi^{-h}\psi^{-r}:h,r\in\mathbb{N}\}.$$

We have showed that our set is actually a connection from β to α.

Suppose now we are in the second possibility given by Equation (8). Then we can prove as in the above first possibility, given by Equation (7), that

$$\{\beta\phi^{-m}\psi^{-s},\alpha_k\phi^{-1},\alpha_{k-1}\phi^{-3},\alpha_{k-2}\phi^{-5},...,\alpha_{k-i}\phi^{-2i-1},...,\alpha_2\phi^{-2k+3}\}$$

is a connection from β to α. We conclude $\beta\sim\alpha$ and so the relation $\sim$ is symmetric. □

Lemma 2.2. *Let $\{\alpha_1,...,\alpha_k\}$, $k\geq 2$, be a connection from α to β with $\alpha_1=\alpha\phi^{-n}\psi^{-r}$, $n,r\in\mathbb{N}$. Then for any $\epsilon\in\{1,-1\}$ and $m,s\in\mathbb{N}$ with $m\geq n$ and $s\geq r$, there exists a connection $\{\bar{\alpha}_1,...,\bar{\alpha}_k\}$ from α to β such that $\bar{\alpha}_1=\alpha\phi^{-m}\psi^{-s}$.*

Proof. By Lemma 1.3-1,2 we have $\{\alpha_1\phi^{n-m}\psi^{r-s},...,\alpha_k\phi^{n-m}\psi^{r-s}\}\subset\pm\Lambda$. Define $\bar{\alpha}_i:=\alpha_i\phi^{n-m}\psi^{r-s}$, $i=1,...,k$, then Lemma 1.3-1 allows us to verify that $\{\bar{\alpha}_1,...,\bar{\alpha}_k\}$ is a connection from α to β which clearly satisfies

$$\bar{\alpha}_1=\alpha_1\phi^{n-m}\psi^{r-s}=(\alpha\phi^{-n}\psi^{-r})\phi^{n-m}\psi^{r-s}=\alpha\phi^{-m}\psi^{-s}.$$

□

Lemma 2.3. *Let $\{\alpha_1, ..., \alpha_k\}$, $k \geq 2$, be a connection from α to β with*

$$\alpha_1\phi^{-k+1}+\alpha_2\phi^{-k+2}\psi^{-1}+\alpha_3\phi^{-k+3}\psi^{-1}+\cdots+\alpha_i\phi^{-k+i}\psi^{-1}+\cdots+\alpha_k\psi^{-1} = \epsilon\beta\phi^{-m}\psi^{-s},$$

being $m, s \in \mathbb{N}$ and $\epsilon \in \{1, -1\}$. Then for any $q, p \in \mathbb{N}$ such that $q \geq m, p \geq s$, there exists a connection $\{\bar{\alpha}_1, ..., \bar{\alpha}_k\}$ from α to β such that

$$\bar{\alpha}_1\phi^{-k+1}+\bar{\alpha}_2\phi^{-k+2}\psi^{-1}+\bar{\alpha}_3\phi^{-k+3}\psi^{-1}+\cdots+\bar{\alpha}_i\phi^{-k+i}\psi^{-1}+\cdots+\bar{\alpha}_k\psi^{-1} = \epsilon\beta\phi^{-q}\psi^{-p}.$$

Proof. Lemma 1.3-1 allows us to assert that $\{\alpha_1\phi^{m-q}\psi^{s-p}, ..., \alpha_k\phi^{m-q}\psi^{s-p}\} \subset \pm\Lambda$. Define now $\bar{\alpha}_i := \alpha_i\phi^{m-q}\psi^{s-p}, i = 1, ..., k$. Then as in the previous item, Lemma 1.3-1 gives us that $\{\bar{\alpha}_1, ..., \bar{\alpha}_k\}$ is a connection from α to β. Finally

$$\begin{aligned}
&\bar{\alpha}_1\phi^{-k+1} + \bar{\alpha}_2\phi^{-k+2}\psi^{-1} + \bar{\alpha}_3\phi^{-k+3}\psi^{-1} + \cdots + \bar{\alpha}_k\psi^{-1} = \\
&\quad = \alpha_1\phi^{m-q}\psi^{s-p}\phi^{-k+1} + \alpha_2\phi^{m-q}\psi^{s-p}\phi^{-k+2}\psi^{-1} + \cdots + \alpha_k\phi^{m-q}\psi^{s-p}\psi^{-1} \\
&\quad = (\alpha_1\phi^{-k+1} + \alpha_2\phi^{-k+2}\psi^{-1} + \cdots + \alpha_k\psi^{-1})\phi^{m-q}\psi^{s-p} \\
&\quad = (\epsilon\beta\phi^{-m}\psi^{-s})\phi^{m-q}\psi^{s-p} \\
&\quad = \epsilon\beta\phi^{-q}\psi^{-p}.
\end{aligned}$$

□

Lemma 2.4. *The relation $\sim$ in Λ, defined by $\alpha \sim \beta$ if and only if α is connected to β, is transitive.*

Proof. Suppose $\alpha \sim \beta$ and $\beta \sim \gamma$.

If $\beta = \epsilon\alpha\phi^{z_1}\psi^{z_2}$ for some $z_1, z_2 \in \mathbb{Z}, \epsilon \in \{1, -1\}$ and $\gamma = \epsilon'\beta\phi^{z_3}\psi^{z_4}$ for some $z_3, z_4 \in \mathbb{Z}$, it is clear that $\alpha \sim \gamma$.

Suppose $\beta = \epsilon\alpha\phi^{z_1}\psi^{z_2}$ for some $z_1, z_2 \in \mathbb{Z}, \epsilon \in \{1, -1\}$ and β is connected to γ through a connection $\{\tau_1, ..., \tau_p\}$, $p \geq 2$, being $\tau_1 = \beta\phi^{-n}\psi^{-r}$, $n, r \in \mathbb{N}$. By choosing $m, s \in \mathbb{N}$ such that $m \geq n$, $s \geq r$ and $z_1 - m \leq 0$ and $z_2 - s \leq 0$, Lemma 2.2 allows us to assert that β is connected to γ through a connection $\{\bar{\tau}_1, \bar{\tau}_2, ..., \bar{\tau}_k\}$ such that $\bar{\tau}_1 = \beta\phi^{-m}\psi^{-s}$. From here, $\{\epsilon\bar{\tau}_1, \epsilon\bar{\tau}_2, ..., \epsilon\bar{\tau}_k\}$ is a connection form α to γ.

Finally, let us write $\{\alpha_1, ..., \alpha_k\}$, $k \geq 2$, for a connection from α to β, which satisfies

$$\alpha_1\phi^{-k+1} + \alpha_2\phi^{-k+2}\psi^{-1} + \cdots + \alpha_k\psi^{-1} = \epsilon\beta\phi^{-m}\psi^{-s}, \tag{9}$$

for some $m, s \in \mathbb{N}, \epsilon \in \{1, -1\}$; and write $\{\tau_1, ..., \tau_p\}$ for a connection from β to γ, being then

$$\tau_1 = \beta\phi^{-q}\psi^{-p} \tag{10}$$

for some $n, q \in \mathbb{N}$. Note that Lemmas 2.2 and 2.3 allows us to suppose $m = q$ and $s = p$.

From here, taking into account Equations (9), and (10); and the fact $m = q$ and $s = p$, we can easily verify that $\{\alpha_1, ..., \alpha_k, \tau_2, ..., \tau_p\}$ is a connection from α to γ if $\epsilon = 1$; and that $\{\alpha_1, ..., \alpha_k, -\tau_2, ..., -\tau_p\}$ it is if $\epsilon = -1$. □

Corollary 2.1. *The relation $\sim$ in Λ, defined by $\alpha \sim \beta$ if and only if α is connected to β, is an equivalence relation.*

Proof. Since clearly the relation $\sim$ is reflexive, the result follows of Lemmas 2.1 and 2.4.

□

3. Decompositions as sum of ideals

By Corollary 2.1 the connection relation is an equivalence relation in Λ. From here, we can consider the quotient set

$$\Lambda/\sim = \{[\alpha] : \alpha \in \Lambda\},$$

becoming $[\alpha]$ the set of nonzero roots $\mathfrak{L}$ which are connected to α.

Our next goal in this section is to associate an (adequate) ideal $I_{[\alpha]}$ to any $[\alpha]$.

Fix $\alpha \in \Lambda$, we start by defining the set $I_{0,[\alpha]} \subset \mathfrak{L}_0$ as follows:

$$I_{0,[\alpha]} := span_{\mathbb{K}}\{[\mathfrak{L}_\beta, \mathfrak{L}_\gamma] : \beta, \gamma \in [\alpha] \cup \{0\}\} \cap \mathfrak{L}_0.$$

By applying Lemma 1.1-2 and 1.2 we get

$$I_{0,[\alpha]} := span_{\mathbb{K}}\{[\mathfrak{L}_{\beta\psi^{-1}}, \mathfrak{L}_{-\beta\phi^{-1}}] : \beta \in [\alpha]\}.$$

Next, we define

$$V_{[\alpha]} := \bigoplus_{\beta \in [\alpha]} \mathfrak{L}_\beta.$$

Finally, we denote by $I_{[\alpha]}$ the direct sum of the two subspaces above, that is,

$$I_{[\alpha]} := I_{0,[\alpha]} \oplus V_{[\alpha]}.$$

Proposition 3.1. *For any $[\alpha] \in \Lambda/\sim$, the following assertions hold.*

1. $[I_{[\alpha]}, I_{[\alpha]}] \subset I_{[\alpha]}$.
2. $\phi(I_{[\alpha]}) = I_{[\alpha]}$ *and* $\psi(I_{[\alpha]}) = I_{[\alpha]}$.

Proof. 1. Since $I_{0,[\alpha]} \subset \mathfrak{L}_0 = H$, then $[I_{0,[\alpha]}, I_{0,[\alpha]}] = 0$ and we have

$$[I_{0,[\alpha]} \oplus V_{[\alpha]}, I_{0,[\alpha]} \oplus V_{[\alpha]}] \subset [I_{0,[\alpha]}, V_{[\alpha]}] + [V_{[\alpha]}, I_{0,[\alpha]}] + [V_{[\alpha]}, V_{[\alpha]}]. \tag{11}$$

Let us consider the first summand in Equation (11). Given $\beta \in [\alpha]$ we have $[I_{0,[\alpha]}, \mathfrak{L}_\beta] \subset \mathfrak{L}_{\beta\psi^{-1}}$, being $\beta\psi^{-1} \in [\alpha]$ by Lemma 1.3-1. Hence $[I_{0,[\alpha]}, \mathfrak{L}_\beta] \subset V_{[\alpha]}$. In a similar way we get $[\mathfrak{L}_\beta, I_{0,[\alpha]}] \subset V_{[\alpha]}$. Consider now the third summand in Equation (11). Given $\beta, \gamma \in [\alpha]$ such that $[\mathfrak{L}_\beta, \mathfrak{L}_\gamma] \neq 0$, then $[\mathfrak{L}_\beta, \mathfrak{L}_\gamma] \subset \mathfrak{L}_{\beta\phi^{-1}+\gamma\psi^{-1}}$. If $\beta\phi^{-1} + \gamma\psi^{-1} = 0$ we have $[\mathfrak{L}_\beta, \mathfrak{L}_{-\gamma}] \subset \mathfrak{L}_0$ and so $[\mathfrak{L}_\beta, \mathfrak{L}_{-\gamma}] \subset I_{0,[\alpha]}$. Suppose then $\beta\phi^{-1} + \gamma\psi^{-1} \in \Lambda$. We have that $\{\beta, \gamma\}$ is a connection from β to $\beta\phi^{-1} + \gamma\psi^{-1}$. The transitivity of $\sim$ gives now that $\beta\phi^{-1} + \gamma\psi^{-1} \in [\alpha]$ and so $[\mathfrak{L}_\beta, \mathfrak{L}_\gamma] \subset \mathfrak{L}_{\beta\phi^{-1}+\gamma\psi^{-1}} \subset V_{[\alpha]}$. Hence $[\bigoplus_{\beta \in [\alpha]} \mathfrak{L}_\beta, \bigoplus_{\beta \in [\alpha]} \mathfrak{L}_\beta] \subset I_{0,[\alpha]} \oplus V_{[\alpha]}$. That is,

$$[V_{[\alpha]}, V_{[\alpha]}] \subset I_{[\alpha]}. \tag{12}$$

From Equations (11) and (12) we get $[I_{[\alpha]}, I_{[\alpha]}] = [I_{0,[\alpha]} \oplus V_{[\alpha]}, I_{0,[\alpha]} \oplus V_{[\alpha]}] \subset I_{[\alpha]}$.

2. The facts $\phi(I_{[\alpha]}) = I_{[\alpha]}$ and $\psi(I_{[\alpha]}) = I_{[\alpha]}$ are direct consequences of Lemma 1.1-1. □

Proposition 3.2. *For any $[\alpha] \neq [\gamma]$ we have $[I_{[\alpha]}, I_{[\gamma]}] = 0$.*

Proof. We have

$$[I_{0,[\alpha]} \oplus V_{[\alpha]}, I_{0,[\gamma]} \oplus V_{[\gamma]}] \subset$$

$$[I_{0,[\alpha]} V_{[\gamma]}] + [V_{[\alpha]}, I_{0,[\gamma]}] + [V_{[\alpha]}, V_{[\gamma]}]. \tag{13}$$

Consider the above third summand $[V_{[\alpha]}, V_{[\gamma]}]$ and suppose there exist $\alpha_1 \in [\alpha]$ and $\gamma_1 \in [\gamma]$ such that $[\mathfrak{L}_{\alpha_1}, \mathfrak{L}_{\gamma_1}] \neq 0$. As necessarily $\alpha_1\phi^{-1} \neq -\gamma_1\psi^{-1}$, then $\alpha_1\phi^{-1} + \gamma_1\psi^{-1} \in$

Λ. So $\{\alpha_1, \gamma_1, -\alpha_1\phi^{-1}\}$ is a connection between α_1 and γ_1. By the transitivity of the connection relation we have $\alpha \in [\gamma]$, a contradiction. Hence $[\mathfrak{L}_{\alpha_1}, \mathfrak{L}_{\gamma_1}] = 0$ and so

$$[V_{[\alpha]}, V_{[\gamma]}] = 0. \tag{14}$$

Consider now the first summand $[I_{0,[\alpha]}, V_{[\gamma]}]$ in Equation (13). Let us take $\alpha_1 \in [\alpha]$ and $\gamma_1 \in [\gamma]$ and show that

$$\gamma_1([\mathfrak{L}_{\alpha_1\psi^{-1}}, \mathfrak{L}_{-\alpha_1\phi^{-1}}]) = 0.$$

Indeed, by BiHom-Jacobi identity we have

$$[\psi^2(\mathfrak{L}_{\gamma_1}), [\psi(\mathfrak{L}_{\alpha_1}), \phi(\mathfrak{L}_{-\alpha_1})]] + [\psi^2(\mathfrak{L}_{\alpha_1}), [\psi(\mathfrak{L}_{-\alpha_1}), \phi(\mathfrak{L}_{\gamma_1})]] + [\psi^2(\mathfrak{L}_{-\alpha_1}), [\psi(\mathfrak{L}_{\gamma_1}), \phi(\mathfrak{L}_{\alpha_1})]] = 0.$$

Now by Equation (14) we get

$$[\psi^2(\mathfrak{L}_{\gamma_1}), [\psi(\mathfrak{L}_{\alpha_1}), \phi(\mathfrak{L}_{-\alpha_1})]] = 0$$

and so

$$0 = [\psi^2(\mathfrak{L}_{\gamma_1}), [\psi(\mathfrak{L}_{\alpha_1}), \phi(\mathfrak{L}_{-\alpha_1})]] = [\psi^2(\mathfrak{L}_{\gamma_1}), \phi\phi^{-1}([\psi(\mathfrak{L}_{\alpha_1}), \phi(\mathfrak{L}_{-\alpha_1})])] = [\psi\phi^{-1}([\psi(\mathfrak{L}_{\alpha_1}), \phi(\mathfrak{L}_{-\alpha_1})]), \phi\psi(\mathfrak{L}_{\gamma_1})].$$

Since $\psi\phi^{-1}([\psi(\mathfrak{L}_{\alpha_1}), \phi(\mathfrak{L}_{-\alpha_1})]) \subset \mathfrak{L}_0 = H$ and $\psi(\mathfrak{L}_{\gamma_1}) \subset \mathfrak{L}_{\gamma_1\psi^{-1}}$ we obtain

$$\gamma_1\phi^{-1}([\psi(\mathfrak{L}_{\alpha_1}), \phi(\mathfrak{L}_{-\alpha_1})])\phi\psi^2(\mathfrak{L}_{\gamma_1}) = 0.$$

From here

$$\gamma_1\phi^{-1}([\mathfrak{L}_{\alpha_1\psi^{-1}}, \mathfrak{L}_{-\alpha_1\phi^{-1}}]) = \gamma_1\phi^{-1}([\psi(\mathfrak{L}_{\alpha_1}), \phi(\mathfrak{L}_{-\alpha_1})]) = 0 \tag{15}$$

for any $\alpha_1 \in [\alpha]$.

Since

$$\phi([\mathfrak{L}_{\alpha_1\psi^{-1}}, \mathfrak{L}_{-\alpha_1\phi^{-1}}]) \subset [\mathfrak{L}_{\alpha_1\psi^{-1}\phi^{-1}}, \mathfrak{L}_{-\alpha_1\phi^{-2}}]),$$

we get

$$[\mathfrak{L}_{\alpha_1\psi^{-1}}, \mathfrak{L}_{-\alpha_1\phi^{-1}}] \subset \phi^{-1}([\mathfrak{L}_{\alpha_1\psi^{-1}\phi^{-1}}, \mathfrak{L}_{-\alpha_1\phi^{-2}}]) = \phi^{-1}([\mathfrak{L}_{\alpha_1\phi^{-1}\psi^{-1}}, \mathfrak{L}_{-\alpha_1\phi^{-2}}]).$$

Taking now into account that Equation (15) and the fact $\alpha_1\phi^{-1} \in [\alpha]$ give us

$$\gamma_1\phi^{-1}([\mathfrak{L}_{\alpha_1\phi^{-1}\psi^{-1}}, \mathfrak{L}_{-\alpha_1\phi^{-2}}]) = 0$$

we conclude

$$\gamma_1([\mathfrak{L}_{\alpha_1\psi^{-1}}, \mathfrak{L}_{-\alpha_1\phi^{-1}}]) = 0.$$

From here $[[\mathfrak{L}_{\alpha_1\psi^{-1}}, \mathfrak{L}_{-\alpha_1\phi^{-1}}], \mathfrak{L}_{\gamma_1}] \subset \gamma_1([\mathfrak{L}_{\alpha_1\psi^{-1}}, \mathfrak{L}_{-\alpha_1\phi^{-1}}])\phi\psi(\mathfrak{L}_{\gamma_1}) = 0$. We have showed $[I_{0,[\alpha]}, V_{[\gamma]}] = 0$. In a similar way we get $[V_{[\alpha]}, I_{0,[\gamma]}] = 0$ and we conclude, together with Equations (13) and (14), that $[I_{[\alpha]}, I_{[\gamma]}] = 0$. □

Theorem 3.1. *The following assertions hold.*

1. *For any $[\alpha] \in \Lambda/\sim$, the linear space*
$$I_{[\alpha]} = I_{0,[\alpha]} \oplus V_{[\alpha]}$$
of $\mathfrak{L}$ associated to $[\alpha]$ is an ideal of $\mathfrak{L}$.
2. *If $\mathfrak{L}$ is simple, then there exists a connection from α to β for any $\alpha, \beta \in \Lambda$; and $H = \sum_{\alpha\in\Lambda} [\mathfrak{L}_{\alpha\psi^{-1}}, \mathfrak{L}_{-\alpha\phi^{-1}}]$.*

Proof. 1. Since $[I_{[\alpha]}, H] \subset I_{[\alpha]}$ we have by Proposition 3.1 and Proposition 3.2 that

$$[I_{[\alpha]}, \mathfrak{L}] = [I_{[\alpha]}, H \oplus (\bigoplus_{\beta \in [\alpha]} \mathfrak{L}_\beta) \oplus (\bigoplus_{\gamma \notin [\alpha]} \mathfrak{L}_\gamma)] \subset I_{[\alpha]}.$$

In a similar way we get $[\mathfrak{L}, I_{[\alpha]}] \subset I_{[\alpha]}$ and, finally, as we also have by Proposition 3.1 that $\phi(I_{[\alpha]}) = \psi(I_{[\alpha]}) = I_{[\alpha]}$ we conclude $I_{[\alpha]}$ is an ideal of $\mathfrak{L}$.

2. The simplicity of $\mathfrak{L}$ implies $I_{[\alpha]} = \mathfrak{L}$. From here, it is clear that $[\alpha] = \Lambda$ and $H = \sum_{\alpha \in \Lambda} [\mathfrak{L}_{\alpha\psi^{-1}}, \mathfrak{L}_{-\alpha\phi^{-1}}]$. □

Theorem 3.2. *We have*

$$\mathfrak{L} = U + \sum_{[\alpha] \in \Lambda/\sim} I_{[\alpha]},$$

where U is a linear complement in H of $\sum_{\alpha \in \Lambda} [\mathfrak{L}_{\alpha\psi^{-1}}, \mathfrak{L}_{-\alpha\phi^{-1}}]$ and any $I_{[\alpha]}$ is one of the ideals of $\mathfrak{L}$ described in Theorem 3.1-1. Furthermore $[I_{[\alpha]}, I_{[\gamma]}] = 0$ when $[\alpha] \neq [\gamma]$.

Proof. We have $I_{[\alpha]}$ is well defined and, by Theorem 3.1-1, an ideal of $\mathfrak{L}$, being clear that

$$\mathfrak{L} = H \oplus (\bigoplus_{\alpha \in \Lambda} \mathfrak{L}_\alpha) = U + \sum_{[\alpha] \in \Lambda/\sim} I_{[\alpha]}.$$

Finally, Proposition 3.2 gives us $[I_{[\alpha]}, I_{[\gamma]}] = 0$ if $[\alpha] \neq [\gamma]$. □

Let us denote by $\mathcal{Z}(\mathfrak{L}) := \{v \in \mathfrak{L} : [v, \mathfrak{L}] + [\mathfrak{L}, v] = 0\}$ the *center* of $\mathfrak{L}$.

Corollary 3.1. *If $\mathcal{Z}(\mathfrak{L}) = 0$ and $H = \sum_{\alpha \in \Lambda} [\mathfrak{L}_{\alpha\psi^{-1}}, \mathfrak{L}_{-\alpha\phi^{-1}}]$. Then $\mathfrak{L}$ is the direct sum of the ideals given in Theorem 3.1,*

$$\mathfrak{L} = \bigoplus_{[\alpha] \in \Lambda/\sim} I_{[\alpha]}.$$

Furthermore $[I_{[\alpha]}, I_{[\gamma]}] = 0$ when $[\alpha] \neq [\gamma]$.

Proof. Since $H = \sum_{\alpha \in \Lambda} [\mathfrak{L}_{\alpha\psi^{-1}}, \mathfrak{L}_{-\alpha\phi^{-1}}]$ we get $\mathfrak{L} = \sum_{[\alpha] \in \Lambda/\sim} I_{[\alpha]}$. Finally, to verify the direct character of the sum, take some $v \in I_{[\alpha]} \cap (\sum_{[\beta] \in \Lambda/\sim, [\beta] \neq [\alpha]} I_{[\beta]})$. Since $v \in I_{[\alpha]}$, the fact $[I_{[\alpha]}, I_{[\beta]}] = 0$ when $[\alpha] \neq [\beta]$ gives us

$$[v, \sum_{[\beta] \in \Lambda/\sim, [\beta] \neq [\alpha]} I_{[\beta]}] + [\sum_{[\beta] \in \Lambda/\sim, [\beta] \neq [\alpha]} I_{[\beta]}, v] = 0.$$

In a similar way, since $v \in \sum_{[\beta] \in \Lambda/\sim, [\beta] \neq [\alpha]} I_{[\beta]}$ we get $[v, I_{[\alpha]}] + [I_{[\alpha]}, v] = 0$. That is, $v \in \mathcal{Z}(\mathfrak{L})$ and so $v = 0$. □

4. THE SIMPLE COMPONENTS

In this section we are interested in studying under which conditions $\mathfrak{L}$ decomposes as the direct sum of the family of its simple ideals, obtaining so a second Wedderburn-type theorem for a class of BiHom-Lie algebras. We recall that a roots system Λ of a split regular BiHom-Lie algebra $\mathfrak{L}$ is called *symmetric* if it satisfies that $\alpha \in \Lambda$ implies $-\alpha \in \Lambda$. From now on we will suppose Λ is symmetric.

Lemma 4.1. *If I is an ideal of $\mathfrak{L}$ such that $I \subset H$, then $I \subset \mathcal{Z}(\mathfrak{L})$.*

Proof. Consequence of $[I,H]+[H,I]\subset[H,H]=0$ and $[I,\bigoplus_{\alpha\in\Lambda}\mathfrak{L}_\alpha]+[\bigoplus_{\alpha\in\Lambda}\mathfrak{L}_\alpha,I]\subset(\bigoplus_{\alpha\in\Lambda}\mathfrak{L}_\alpha)\cap H=0$. □

Lemma 4.2. *For any $\alpha,\beta\in\Lambda$ with $\alpha\neq\beta$ there exists $h_0\in H$ such that $\alpha(h_0)\neq 0$ and $\alpha(h_0)\neq\beta(h_0)$.*

Proof. As $\alpha\neq\beta$, there exists $h\in H$ such that $\alpha(h)\neq\beta(h)$. If $\alpha(h)\neq 0$ we have finished, so let us suppose $\alpha(h)=0$ what implies $\beta(h)\neq 0$. Since $\alpha\neq 0$, we can fix some $h'\in H$ such that $\alpha(h')\neq 0$. We can distinguish two cases, in the first one $\alpha(h')\neq\beta(h')$ and in the second one $\alpha(h')=\beta(h')$. Then we have that by taking $h_0:=h'$ in the first case and $h_0:=h+h'$ in the second one we complete the proof. □

Lemma 4.3. *If I is an ideal of $\mathfrak{L}$ and $x=h+\sum_{j=1}^{n}v_{\alpha_j}\in I$, with $h\in H, v_{\alpha_j}\in\mathfrak{L}_{\alpha_j}$ and $\alpha_j\neq\alpha_k$ if $j\neq k$. Then any $v_{\alpha_j}\in I$.*

Proof. If $n=1$ we have $x=h+v_{\alpha_1}\in I$. By taking $h'\in H$ such that $\alpha_1(h')\neq 0$ we get $[h',x]=[h',\phi\phi^{-1}(h)]+[h',\phi\phi^{-1}(v_{\alpha_1})]=\alpha_1\phi(h')\psi(v_{\alpha_1})\in I$ and so $\psi(v_{\alpha_1})\in I$. From here $\psi^{-1}(\psi(v_{\alpha_1}))=v_{\alpha_1}\in I$.

Suppose now $n>1$ and consider α_1 and α_2. By Lemma 4.2 there exists $h_0\in H$ such that $\alpha_1(h_0)\neq 0$ and $\alpha_1(h_0)\neq\alpha_2(h_0)$. Then we have

$$[h_0,x]=[h_0,\phi\phi^{-1}(h)]+[h_0,\phi\phi^{-1}(v_{\alpha_1})]+[h_0,\phi\phi^{-1}(v_{\alpha_2})]+\cdots+[h_0,\phi\phi^{-1}(v_{\alpha_n})]=$$

$$\alpha_1\phi(h_0)\psi(v_{\alpha_1})+\alpha_2\phi(h_0)\psi(v_{\alpha_2})+\cdots+\alpha_n\phi(h_0)\psi(v_{\alpha_n})\in I \tag{16}$$

and

$$\psi(x)=$$

$$\psi(h)+\psi(v_{\alpha_1})+\psi(v_{\alpha_2})+\cdots+\psi(v_{\alpha_n})\in I. \tag{17}$$

By multiplying Equation (17) by $\alpha_2\phi(h_0)$ and subtracting Equation (16) we get

$$\alpha_2\phi(h_0)\psi(h)+(\alpha_2\phi(h_0)-\alpha_1\phi(h_0))\psi(v_{\alpha_1})+$$

$$(\alpha_2\phi(h_0)-\alpha_3\phi(h_0))\psi(v_{\alpha_3})+\cdots+(\alpha_2\phi(h_0)-\alpha_n\phi(h_0))\psi(v_{\alpha_n})\in I.$$

By denoting $\tilde{h}:=\alpha_2\phi(h_0)\psi(h)\in H$ and $v_{\alpha_i\psi^{-1}}:=(\alpha_2\phi(h_0)-\alpha_i\phi(h_0))\psi(v_{\alpha_i})\in\mathfrak{L}_{\alpha_i\psi^{-1}}$ we can write

$$\tilde{h}+v_{\alpha_1\psi^{-1}}+v_{\alpha_3\psi^{-1}}+\cdots+v_{\alpha_n\psi^{-1}}\in I. \tag{18}$$

Now we can argue as above with Equation (18) to get

$$\tilde{\tilde{h}}+v_{\alpha_1\psi^{-2}}+v_{\alpha_4\psi^{-2}}+\cdots+v_{\alpha_n\psi^{-2}}\in I$$

for $\tilde{\tilde{h}}\in H$ and any $v_{\alpha_i\psi^{-2}}\in\mathfrak{L}_{\alpha_i\psi^{-2}}$. By iterating this process we obtain

$$\bar{h}+v_{\alpha_1\psi^{-n+1}}\in I$$

with $\bar{h}\in H$ and $v_{\alpha_1\psi^{-n+1}}\in\mathfrak{L}_{\alpha_1\psi^{-n+1}}$. As in the above case $n=1$, we get $v_{\alpha_1\psi^{-n+1}}\in I$ and consequently $v_{\alpha_1}\in\mathbb{K}\psi^{-n+1}(v_{\alpha_1\psi^{-n+1}})\in I$.

In a similar way we can prove any $v_{\alpha_i}\in I$ for $i\in\{2,...,n\}$ and the proof is complete. □

Let us introduce the concepts of root-multiplicativity and maximal length in the framework of split BiHom-Lie algebras, in a similar way to the ones for split Hom-Lie algebras, split Lie algebras, split triple systems, split Leibniz structures and so on (see [1, 2, 3, 4] for these notions and examples).

Definition 4.1. *We say that a split regular BiHom-Lie algebra $\mathfrak{L}$ is* root-multiplicative *if given $\alpha, \beta \in \Lambda$ such that $\alpha\phi^{-1} + \beta\psi^{-1} \in \Lambda$, then $[\mathfrak{L}_\alpha, \mathfrak{L}_\beta] \neq 0$.*

Definition 4.2. *It is said that a split regular BiHom-Lie algebra $\mathfrak{L}$ is of* maximal length *if $\dim \mathfrak{L}_\alpha = 1$ for any $\alpha \in \Lambda$.*

Theorem 4.1. *Let $\mathfrak{L}$ be a split regular BiHom-Lie algebra of maximal length and root-multiplicative. Then $\mathfrak{L}$ is simple if and only if $\mathcal{Z}(\mathfrak{L}) = 0, H = \sum\limits_{\alpha\in\Lambda}[\mathfrak{L}_{\alpha\psi^{-1}}, \mathfrak{L}_{-\alpha\phi^{-1}}]$ and Λ has all of its elements connected.*

Proof. Suppose $\mathfrak{L}$ is simple. Since $\mathcal{Z}(\mathfrak{L})$ is an ideal of $\mathfrak{L}$ then $\mathcal{Z}(\mathfrak{L}) = 0$. From here, Theorem 3.1-2 completes the proof of the first implication. To prove the converse, consider I a nonzero ideal of $\mathfrak{L}$. By Lemma 4.3 we can write $I = (I \cap H) \oplus (\bigoplus\limits_{\alpha\in\Lambda} I_\alpha)$, where $I_\alpha := I \cap \mathfrak{L}_\alpha$. By the maximal length of $\mathfrak{L}$, if we denote by $\Lambda_I := \{\alpha \in \Lambda : I_\alpha \neq 0\}$, we can write $I = (I \cap H) \oplus (\bigoplus\limits_{\alpha\in\Lambda_I} \mathfrak{L}_\alpha)$, being also $\Lambda_I \neq \emptyset$ as consequence of Lemma 4.1. Let us fix some $\alpha_0 \in \Lambda_I$ being then $0 \neq \mathfrak{L}_{\alpha_0} \subset I$. Since $\phi(I) = I$ and $\psi(I) = I$ and by making use of Lemma 1.1-1 we can assert that

$$\text{if } \alpha \in \Lambda_I \text{ then } \{\alpha\phi^{z_1}\psi^{z_2} : z_1, z_2 \in \mathbb{Z}\} \subset \Lambda_I. \tag{19}$$

In particular

$$\{\mathfrak{L}_{\alpha_0\phi^{z_1}\psi^{z_2}} : z_1, z_2 \in \mathbb{Z}\} \subset I. \tag{20}$$

Now, let us take any $\beta \in \Lambda$ satisfying $\beta \notin \{\pm\alpha_0\phi^{z_1}\psi^{z_2} : z_1, z_2 \in \mathbb{Z}\}$. Since α_0 and β are connected, we have a connection $\{\alpha_1, ..., \alpha_k\}, k \geq 2$, from α_0 to β satisfying:

$\alpha_1 = \alpha_0\phi^{-n}\psi^{-r}$ for some $n, r \in \mathbb{N}$,
$\alpha_1\phi^{-1} + \alpha_2\psi^{-1} \in \Lambda$,
$\alpha_1\phi^{-2} + \alpha_2\phi^{-1}\psi^{-1} + \alpha_3\psi^{-1} \in \Lambda$,
....
$\alpha_1\phi^{-i+1} + \alpha_2\phi^{-i+2} + \alpha_3\phi^{-i+3} + \cdots + \alpha_i\psi^{-1} \in \Lambda$,
....
$\alpha_1\phi^{-k+2} + \alpha_2\phi^{-k+3}\psi^{-1} + \alpha_3\phi^{-k+4}\psi^{-1} + \cdots + \alpha_{k-2}\phi^{-1}\psi^{-1} + \alpha_{k-1}\psi^{-1} \in \Lambda$,
$\alpha_1\phi^{-k+1} + \alpha_2\phi^{-k+2}\psi^{-1} + \alpha_3\phi^{-k+3}\psi^{-1} + \cdots + \alpha_i\phi^{-k+i}\psi^{-1} + \cdots + \alpha_{k-}\phi^{-1}\psi^{-1} + \alpha_k\psi^{-1} = \epsilon\beta\phi^{-m}\psi^{-s}$ for some $m, s \in \mathbb{N}$ and $\epsilon \in \{1, -1\}$.

Taking into account $\alpha_1, \alpha_2 \in \Lambda$ and $\alpha_1\phi^{-1} + \alpha_2\psi^{-1} \in \Lambda$, the root-multiplicativity and maximal length of $\mathfrak{L}$ allow us to assert $0 \neq [\mathfrak{L}_{\alpha_1}, \mathfrak{L}_{\alpha_2}] = \mathfrak{L}_{\alpha_1\phi^{-1}+\alpha_2\psi^{-1}}$. Since $0 \neq \mathfrak{L}_{\alpha_1} \subset I$ as consequence of Equation (20) we get

$$0 \neq \mathfrak{L}_{\alpha_1\phi^{-1}+\alpha_2\psi^{-1}} \subset I.$$

A similar argument applied to $\alpha_1\phi^{-1} + \alpha_2\psi^{-1}, \alpha_3$ and

$$(\alpha_1\phi^{-1} + \alpha_2\psi^{-1})\phi^{-1} + \alpha_3\psi^{-1} = \alpha_1\phi^{-2} + \alpha_2\phi^{-1}\psi^{-1} + \alpha_3\psi^{-1}$$

gives us $0 \neq \mathfrak{L}_{\alpha_1\phi^{-2}+\alpha_2\phi^{-1}\psi^{-1}+\alpha_3\psi^{-1}} \subset I$. We can follow this process with the connection $\{\alpha_1, ..., \alpha_k\}$ to get

$$0 \neq \mathfrak{L}_{\alpha_1\phi^{-k+1}+\alpha_2\phi^{-k+2}\psi^{-1}+\cdots+\alpha_k\psi^{-1}} \subset I$$

and then

$$\text{either } \mathfrak{L}_{\beta\phi^{-m}\psi^{-s}} \subset I \text{ or } \mathfrak{L}_{-\beta\phi^{-m}\psi^{-s}} \subset I.$$

From Equations (19) and (20), we now get

(21)

$$\text{either } \{\mathfrak{L}_{\alpha\phi^{-z_1}\psi^{-z_2}} : z_1, z_2 \in \mathbb{Z}\} \subset I \text{ or } \{\mathfrak{L}_{-\alpha\phi^{-z_1}\psi^{-z_2}} : z_1, z_2 \in \mathbb{Z}\} \subset I \text{ for any } \alpha \in \Lambda.$$

Equation (21) can be reformulated by asserting that given any $\alpha \in \Lambda$ either $\{\alpha\phi^{-z_1}\psi^{-z_2} : z_1, z_2 \in \mathbb{Z}\}$ or $\{-\alpha\phi^{-z_1}\psi^{-z_2} : z_1, z_2 \in \mathbb{Z}\}$ is contained in Λ_I. Taking now into account $H = \sum_{\alpha\in\Lambda} [\mathfrak{L}_{\alpha\psi^{-1}}, \mathfrak{L}_{-\alpha\phi^{-1}}]$ we have

$$H \subset I. \tag{22}$$

If we consider now any $\alpha \in \Lambda$, since $\mathfrak{L}_\alpha = [H, \mathfrak{L}_{\alpha\psi}]$ by the maximal length of $\mathfrak{L}$, Equation (22) gives us $\mathfrak{L}_\alpha \subset I$ and so $I = \mathfrak{L}$. That is, $\mathfrak{L}$ is simple. □

Theorem 4.2. *Let $\mathfrak{L}$ be a split regular BiHom-Lie algebra of maximal length, root multiplicative, with $\mathcal{Z}(\mathfrak{L}) = 0$ and satisfying $H = \sum_{\alpha\in\Lambda} [\mathfrak{L}_{\alpha\psi^{-1}}, \mathfrak{L}_{-\alpha\phi^{-1}}]$. Then*

$$\mathfrak{L} = \bigoplus_{[\alpha]\in\Lambda/\sim} I_{[\alpha]},$$

where any $I_{[\alpha]}$ is a simple (split) ideal having its roots system, $\Lambda_{I_{[\alpha]}}$, with all of its elements $\Lambda_{I_{[\alpha]}}$-connected.

Proof. Taking into account Corollary 3.1 we can write $\mathfrak{L} = \bigoplus_{[\alpha]\in\Lambda/\sim} I_{[\alpha]}$ as the direct sum of the family of ideals

$$I_{[\alpha]} = I_{0,[\alpha]} \oplus V_{[\alpha]} = (\sum_{\alpha\in\Lambda} [\mathfrak{L}_{\alpha\psi^{-1}}, \mathfrak{L}_{-\alpha\phi^{-1}}]) \oplus \bigoplus_{\beta\in[\alpha]} \mathfrak{L}_\beta,$$

being each $I_{[\alpha]}$ a split regular BiHom-Lie algebra having as roots system $\Lambda_{I_{[\alpha]}} := [\alpha]$. To make use of Theorem 4.1 in each $I_{[\alpha]}$, we have to observe that the root-multiplicativity of $\mathfrak{L}$ and Proposition 3.2 show that $\Lambda_{I_{[\alpha]}}$ has all of its elements $\Lambda_{I_{[\alpha]}}$-connected, that is, connected through connections contained in $\Lambda_{I_{[\alpha]}}$. We also get that any of the $I_{[\alpha]}$ is root-multiplicative as consequence of the root-multiplicativity of $\mathfrak{L}$. Clearly $I_{[\alpha]}$ is of maximal length, and finally its center $\mathcal{Z}_{I_{[\alpha]}}(I_{[\alpha]}) := \{x \in I_{[\alpha]} : [x, I_{[\alpha]} = 0]\} = 0$ as consequence of $[I_{[\alpha]}, I_{[\gamma]}] = 0$ if $[\alpha] \neq [\gamma]$ (see Theorem 3.2) and $\mathcal{Z}(\mathfrak{L}) = 0$. We can apply Theorem 4.1 to any $I_{[\alpha]}$ so as to conclude $I_{[\alpha]}$ is simple. It is clear that the decomposition $\mathfrak{L} = \bigoplus_{[\alpha]\in\Lambda/\sim} I_{[\alpha]}$ satisfies the assertions of the theorem. □

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