GRADINGS AND SYMMETRIES ON HEISENBERG TYPE ALGEBRAS

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ABSTRACT. We describe the fine (group) gradings on the Heisenberg algebras, on the Heisenberg superalgebras and on the generalized Heisenberg algebras. We compute the Weyl groups of these gradings. Also the results obtained respect to Heisenberg superalgebras are applied to the study of Heisenberg Lie color algebras.

2010 MSC: 17B70, 17B75, 17B40, 16W50.
Key words and phrases: Heisenberg algebra, graded algebra, Weyl group.

1. INTRODUCTION

In the last years there has been an increasing interest in the study of the group gradings on Lie theoretic structures. In the case of Lie algebras, this study has been focussed on the simple ones. In fact the (complex) finite-dimensional simple case has been studied, among other authors, by Bahturin, Elduque, Havlíček, Kochetov, Patera, Pelantová, Shestakov, Zaicev and Zassenhaus [5, 6, 8, 19, 24, 30] in the classical case ([19] encloses $\mathfrak{d}_4$), while the exceptional cases $\mathfrak{g}_2, \mathfrak{f}_4$ and $\mathfrak{d}_4$ have been studied by Bahturin, Draper, Elduque, Kochetov, Martín, Tvalavadze and Viruel [7, 15, 16, 17, 20]. The fine group gradings on the real forms of $\mathfrak{g}_2$ and $\mathfrak{f}_4$, also simple algebras, have been classified by the three first authors [11]. With respect to the group gradings on $\mathbb{Z}_2$-graded algebras, these have been considered by the three first authors in the case of the Jordan superalgebra K_{10} [12], and by the second and third authors together with Elduque in the case of exceptional Lie superalgebras [18]. In relation with other Lie structures, the Lie triple systems of exceptional type have also been considered from the viewpoint of gradings (see [13]).

In the present paper we are interested in studying the gradings on a family of non-simple Lie algebras, superalgebras, color algebras and extended algebras, the Heisenberg algebras (resp. superalgebras, Lie color algebras and extended ones). Since Heisenberg (super) algebras are nilpotent and extended Heisenberg algebras are solvable we also have to mention the recent references [3] and [4].

This family of algebraic structures was introduced by A. Kaplain in [27] and has played an important role in Quantum Mechanics, where for instance extended Heisenberg algebras appear by a quantizing process from the classical Heisenberg algebra $H(4)$ [1], where coherent states for power-law potentials are constructed by using generalized Heisenberg algebras, being also shown that these coherent states are useful for describing the states of real and ideal lasers [9] or where a deformation of a Heisenberg algebra it is used to describe the solutions of the N-particle rational Calogero model and to solve the problem of proving the existence of supertraces [29]; and also in Geometry, where for instance the set

The authors are partially supported by the MCYT grant MTM2010-15223 and by the Junta de Andalucía grants FQM-336, FQM-2467, FQM-3737. The first and fourth authors are also supported by the PCI of the UCA 'Teoría de Lie y Teoría de Espacios de Banach' and by the PAI with project number FQM-298.

of superderivations of a Heisenberg superalgebra is applied to the theory of cohomology, [14], among other works [26, 28, 39].

The study of Weyl groups of Lie gradings was inaugurated by Patera and Zassenhaus in [30]. Some concrete examples were developed, for instance, in [25]. Recently, Elduque and Kochetov have determined the Weyl groups of the fine gradings on matrix, octonions, Albert and simple Lie algebras of types A, B, C and D (see [21, 22]). The (extended) Weyl group of a simple Lie algebra is the Weyl group of the Cartan grading on that algebra (which is of course fine). Thus the notion of Weyl group of a grading encompasses that of the usual Weyl group with its countless applications. This is one of the reasons motivating the study of Weyl groups on fine Lie gradings. On the other hand, if we consider the category of graded Lie algebras (not in the sense of Lie superalgebras), then the automorphism group G of such an object is defined as the group of automorphisms of the algebra which preserve the grading, and the Weyl group of the grading is an epimorphic image of G. Thus the symmetries of a graded Lie algebra are present in the Weyl group of the grading. Our work includes the description of the Weyl group of the fine gradings on Heisenberg algebras, on Heisenberg superalgebras and on extended Heisenberg algebras.

We would like to mention the work [34] which contains a detailed study of the automorphisms on Heisenberg-type algebras. The study of gradings is always strongly related to the notion of diagonalizable group of automorphisms. Thus the mentioned work has been illuminating though the study of gradings goes a step further. We have also computed the group of automorphisms in the case of Heisenberg superalgebras which is not present in [34].

The definition of extended Heisenberg algebra is motivated by the following result, which allows to compute the groups which can act by isometries in a Lorentzian manifold.

Theorem. [2, Theorem 11.7.3] *Let M be a compact connected Lorentzian manifold and G a connected Lie group acting isometrically and locally faithfully on M. Then its Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{s}$ is a direct sum of a compact semisimple Lie algebra $\mathfrak{k}$, an abelian algebra $\mathfrak{a}$ and a Lie algebra $\mathfrak{s}$, which is either trivial, or isomorphic to $\mathfrak{aff}(\mathbb{R})$, to a Heisenberg algebra H_n, to a generalized Heisenberg algebra $H_n(\lambda)$ with $\lambda \in \mathbb{Q}_+^{(n-1)/2}$, or to $\mathfrak{sl}_2(\mathbb{R})$.*

Moreover, according to [2, Chapter 8], the converse of this result is also true: if G is a connected simply connected Lie group whose Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{s}$ is as above, then there is a locally faithful isometric action of G on a compact connected Lorentz manifold. The physical meaning of this result claims that if you want a Lie group to act on a compact connected Lorentzian manifold, then you must be ready to admit that the group may have a Heisenberg section. More results about the relationship between extended Heisenberg algebras and Lorentzian manifolds can be found in [23].

Finally we devote some words to the distribution of results in our work. After a background on gradings in Section 2, we study the group gradings on Heisenberg algebras in Section 3, by showing that all of them are toral and by computing the Weyl group of the only (up to equivalence) fine one. In Section 4 we study the fine group gradings on Heisenberg superalgebras and calculate the Weyl group of the fine ones. In Section 5 we discuss on the concept of Heisenberg Lie color algebra, give a description of the same and show how the results in the previous section can be applied to classify a certain family of Heisenberg Lie color algebras. Finally, in Section 6, we devote some attention to the concept of extended Heisenberg algebras and also compute their fine group gradings and their symmetries, which turn out to be very abundant.

2. PRELIMINARIES

Throughout this work the base field will be denoted by $\mathbb{F}$. Let A be an algebra over $\mathbb{F}$, a *grading* on A is a decomposition

$$\Gamma : A = \bigoplus_{s\in S} A_s$$

of A into direct sum of nonzero subspaces such that for any $s_1, s_2 \in S$ there exists $s_3 \in S$ such that $A_{s_1}A_{s_2} \subset A_{s_3}$. The grading Γ is said to be a *group grading* if there is a group G containing S such that $A_{s_1}A_{s_2} \subset A_{s_1s_2}$ (multiplication of indices in the group G) for any $s_1, s_2 \in S$. Then we can write

$$\Gamma : A = \bigoplus_{g\in G} A_g,$$

by setting $A_g = 0$ if $g \in G \setminus S$. In this paper all the gradings we consider will be group gradings where G is a finitely generated abelian group and G is generated by the set of all the elements $g \in G$ such that $A_g \neq 0$, usually called the *support* of the grading (the above S).

Given two gradings $A = \oplus_{g\in G}U_g$ and $A = \oplus_{h\in H}V_h$, we shall say that they are *isomorphic* if there is a group isomorphism $\sigma\colon G \to H$ and an (algebra) automorphism $\varphi\colon A \to A$ such that $\varphi(U_g) = V_{\sigma(g)}$ for all $g \in G$. The above two gradings are said to be *equivalent* if there are a bijection $\sigma\colon S \to S'$ between the supports of the first and second gradings respectively and an algebra automorphism φ of A such that $\varphi(U_g) = V_{\sigma(g)}$ for any $g \in S$.

Let Γ and Γ' be two gradings on A. The grading Γ is said to be a *refinement* of Γ' (or Γ' a *coarsening* of Γ) if each homogeneous component of Γ' is a (direct) sum of some homogeneous components of Γ. A grading is called *fine* if it admits no proper refinements. A fundamental concept to obtain the coarsenings of a given grading is the one of universal grading group. Given a grading $\Gamma : A = \oplus_{g\in G}A_g$, one may consider the abelian group $\tilde{G}$ generated by the support of Γ subject only to the relations $g_1g_2 = g_3$ if $0 \neq [A_{g_1}, A_{g_2}] \subset A_{g_3}$. Then A is graded over $\tilde{G}$; that is $\tilde{\Gamma} : A = \oplus_{\tilde{g}\in\tilde{G}}A_{\tilde{g}}$, where $A_{\tilde{g}}$ is the sum of the homogeneous components A_g of Γ such that the class of g in $\tilde{G}$ is $\tilde{g}$. Note that there is at most one such homogeneous component and that this $\tilde{G}$-grading $\tilde{\Gamma}$ is equivalent to Γ, since $G \hookrightarrow \tilde{G}$, $g \mapsto \tilde{g}$ is an injective map (not homomorphism). This group $\tilde{G}$ has the following universal property: given any coarsening $A = \oplus_{h\in H}A'{}_h$ of $\tilde{\Gamma}$, there exists a unique group epimorphism $\alpha\colon \tilde{G} \to H$ such that

$$A'_h = \bigoplus_{\tilde{g}\in\alpha^{-1}(h)} A_{\tilde{g}}.$$

The group $\tilde{G}$ is called the *universal grading group* of Γ. Since we always consider abelian group gradings, the *general principle for nontorality* of gradings can be recalled in the following terms: if G is the universal grading group of a fine grading and G is not torsion-free, then the grading is nontoral. Throughout this paper, the gradings will be considered over their universal grading groups.

It is also well-known that any grading on A is induced by a finitely generated abelian subgroup of diagonalizable automorphisms of Aut(A), the automorphism group of the algebra. A special kind of gradings arises when we consider the inducing automorphisms in a torus. Indeed, a grading of an algebra A is said to be *toral* if it is produced by automorphisms within a torus of the automorphism group of the algebra.

For a grading $\Gamma : A = \oplus_{g\in G} A_g$, the *automorphism group* of Γ, denoted Aut(Γ), consists of all self-equivalences of Γ, i.e., automorphisms of A that permute the components of Γ. The *stabilizer* of Γ, denoted Stab(Γ), consists of all automorphisms of the graded algebra A, i.e., automorphisms of A that leave each component of Γ invariant. The quotient group Aut(Γ)/ Stab(Γ) will be called the *Weyl group* of Γ and denoted by $\mathcal{W}(\Gamma)$.

Next, we recall that if $L = L_0 \oplus L_1$ is a Lie superalgebra over $\mathbb{F}$ and G a finitely generated abelian group. A G-grading on L is a decomposition $\Gamma : L = \bigoplus_{g\in G} \big((L_0)_g \oplus (L_1)_g\big)$ where any $(L_i)_g$ is a linear subspace of L_i and where $[(L_i)_{g_1}, (L_j)_{g_2}] \subset (L_{i+j})_{g_1 g_2}$ holds for any $g_1, g_2 \in G$ and any $i, j \in \{0, 1\}$ (sum modulo 2). Here the support of Γ is $\{g \in G\colon (L_i)_g \neq 0 \text{ for some } i\}$ and everything works analogously to the case of a Lie algebra with a grading. Note only a subtle difference: assuming that L is non-abelian, the trivial grading, $L = L_0 \oplus L_1$, has as universal grading group $\mathbb{Z}_2$, while the trivial grading on L as a Lie algebra has the trivial group as the universal grading group.

We will have the occasion to use basic terminology of finite groups: $\mathbb{Z}_n$ for the cyclic group of order n, S_n for the permutation group of n elements and D_n for the dihedral group of order $2n$.

Finally we give two fundamental lemmas of purely geometrical nature that will be applied in future sections. Recall that a symplectic space V is a linear space provided with an alternative nondegenerate bilinear form $\langle\cdot,\cdot\rangle$, and that in the finite-dimensional case a standard result states the existence of a "symplectic basis", that is, a basis: $\{u_1, u_1', \dots, u_n, u_n'\}$ such that $\langle u_i, u_i'\rangle = 1$ while any other inner product is zero.

Lemma 1. *Let $(V, \langle\cdot,\cdot\rangle)$ be a finite-dimensional symplectic space and assume that V is the direct sum of linear subspaces $V = \oplus_{i\in I} V_i$ where for any $i \in I$ there is a unique $j \in I$ such that $\langle V_i, V_j\rangle \neq 0$. Then there is a basis $\{u_1, u_1', \dots, u_n, u_n'\}$ of V such that:*

- *The basis is contained in $\cup_i V_i$.*
- *For any i, j we have $\langle u_i, u_j\rangle = \langle u_i', u_j'\rangle = 0$.*
- *For each i and j we have $\langle u_i, u_j'\rangle = \delta_{i,j}$ (Kronecker's delta).*

Proof. First we split I into a disjoint union $I = I_1 \cup I_2$ such that I_1 is the set of all $i \in I$ such that $\langle V_i, V_i\rangle \neq 0$ and in I_2 we have all the indices i such that there is a $j \neq i$ with $\langle V_i, V_j\rangle \neq 0$. Now for each $i \in I_1$ the space V_i is symplectic with relation to the restriction of $\langle\cdot,\cdot\rangle$ to V_i. So we fix in such V_i a symplectic basis. Take now $i \in B$ and let $j \in I$ be the unique index such that $\langle V_i, V_j\rangle \neq 0$. Consider now the restriction $\langle\cdot,\cdot\rangle\colon V_i \times V_j \to \mathbb{F}$. This map is nondegenerate in the obvious sense (which implies $\dim(V_i) = \dim(V_j)$). If we fix a basis $\{e_1, \dots, e_q\}$ of V_i, then by standard linear algebra arguments we get that there is basis $\{f_1, \dots f_q\}$ in V_j such that $\langle e_i, f_i\rangle = 1$ being the remaining inner products among basic elements zero. Thus, putting together these basis suitable reordered we get the symplectic basis whose existence is claimed in the Lemma. □

Since all the elements in the basis constructed above are in some component V_i we will refer to this basis as a "homogeneous basis" of V.

Lemma 2. *Let $(V, \langle\cdot,\cdot\rangle)$ be a finite-dimensional linear space V with a symmetric nondegenerate bilinear form $\langle\cdot,\cdot\rangle\colon V \times V \to \mathbb{F}$ where $\mathbb{F}$ is of characteristic other than 2. Assume that $V = \oplus_{i\in I} V_i$ is the direct sum of linear subspaces in such a way that for each $i \in I$ there is a unique $j \in I$ such that $\langle V_i, V_j\rangle \neq 0$. Then there is a basis $B = \{u_1, v_1, \dots, u_r, v_r, z_1, \dots z_q\}$ of V such that*

- *$B \subset \cup_i V_i$.*
- *$\langle z_i, z_i\rangle \neq 0$, $\langle u_i, v_i\rangle = 1$.*

- *Any other inner product of elements in B is zero.*

Proof. Let I_1 be the subset of I such that for any $i \in I_1$ we have $\langle V_i, V_i\rangle \neq 0$ and I_2 the complementary $I_2 := I \setminus I_1$. Thus, for any $i \in I_1$ there is unique $j \neq i$ such that $\langle V_i, V_j\rangle \neq 0$. Now each V_i (with $i \in I_1$) has an orthogonal basis and for any $i \in I_2$ consider the unique $j \neq i$ such that $\langle V_i, V_j\rangle \neq 0$. The couple (V_i, V_j) gives a dual pair $\langle\cdot,\cdot\rangle\colon V_i \times V_j \to \mathbb{F}$ (implying $\dim(V_i) = \dim(V_j)$) and for any basis $\{e_k\}$ of V_i, there is a dual basis $\{f_h\}$ in V_j such that $\langle e_k, f_h\rangle = \delta_{k,h}$ (Kronecker's delta). So putting together all these bases (suitably ordered) the required base on V is getting. □

Observe that if $\mathbb{F}$ is algebraically closed the inner products $\langle z_i, z_i\rangle$ in Lemma 2 may be chosen to be 1. The basis constructed in Lemma 2 will be also termed an "homogeneous basis".

3. GRADINGS ON HEISENBERG ALGEBRAS

A Lie algebra H is called a *Heisenberg algebra* if it is nilpotent in two steps (that is, not abelian and $[[H,H],H] = 0$) with one-dimensional center $\mathcal{Z}(H) := \{x \in H : [x,H] = 0\}$. If $z \neq 0$ is a fixed element in $\mathcal{Z}(H)$ and we take P any complementary subspace of $\mathbb{F}z$, then the map $\langle\cdot,\cdot\rangle\colon P \times P \to \mathbb{F}$ given by $\langle v, v'\rangle z = [v, v']$ is a nondegenerate skewsymmetric bilinear form, or, in other words, $(P, \langle\cdot,\cdot\rangle)$ is a symplectic space. Of course, any Lie algebra constructed from a symplectic space $(P, \langle\cdot,\cdot\rangle)$ as $H = P \oplus \mathbb{F}z$ with $z \in \mathcal{Z}(H)$ and $[v, v'] = \langle v, v'\rangle z$ for all $v, v' \in P$, is a Heisenberg algebra. Recall that the dimension of P is necessarily even.

In particular, there is one Heisenberg algebra up to isomorphism for each odd dimension $n = 2k+1$, which we will denote H_n, characterized by the existence of a basis

$$B = \{e_1, \hat{e}_1, \dots, e_k, \hat{e}_k, z\} \tag{1}$$

in which the nonzero products are $[e_i, \hat{e}_i] = -[\hat{e}_i, e_i] = z$ for $1 \le i \le k$.

A fine grading on H_n is obviously provided by this basis as:

$$H_n = \langle e_1\rangle \oplus \cdots \oplus \langle e_k\rangle \oplus \langle \hat{e}_1\rangle \oplus \cdots \oplus \langle \hat{e}_k\rangle \oplus \langle z\rangle. \tag{2}$$

This grading is also a group grading. For instance, it can be considered as a $\mathbb{Z}$-grading by letting $H_n = \bigoplus_{i=-k}^{k} (H_n)_i$ for

$$(H_n)_{-i} = \langle \hat{e}_i\rangle,\ (H_n)_0 = \langle z\rangle,\ (H_n)_i = \langle e_i\rangle,\quad i = 1, \dots, k.$$

Moreover, this grading on H_n is toral. It is enough to observe that the group of automorphisms $\mathcal{T}$ which are represented by scalar matrices relative to the basis B is a torus. In fact, it is a maximal torus of dimension $k+1$. Indeed, an element $f \in \mathcal{T}$ will be determined by the nonzero scalars $(\lambda_1, \dots, \lambda_k, \lambda) \in \mathbb{F}^{k+1}$ such that $f(z) = \lambda z$ and $f(e_i) = \lambda_i e_i$, being then $f(\hat{e}_i) = \frac{\lambda}{\lambda_i}\hat{e}_i$. If we denote such an automorphism by $t_{(\lambda_1,\dots,\lambda_k;\lambda)}$, it is straightforward that $t_{(\lambda_1,\dots,\lambda_k;\lambda)} t_{(\lambda'_1,\dots,\lambda'_k;\lambda')} = t_{(\lambda_1\lambda'_1,\dots,\lambda_k\lambda'_k;\lambda\lambda')}$ and that any automorphism commuting with every element in $\mathcal{T}$ preserves the common diagonalization produced by $\mathcal{T}$, which is precisely the one given by (2). All this in particular implies that this fine toral grading can be naturally considered as a grading over the group $\mathbb{Z}^{k+1}$, which is its universal grading group:

$$\Gamma:\quad \begin{aligned} &(H_n)_{(0,\dots,0;2)} = \langle z\rangle,\\ &(H_n)_{(0,\dots,1,\dots,0;1)} = \langle e_i\rangle,\ \text{(1 in the i-th slot)}\\ &(H_n)_{(0,\dots,-1,\dots,0;1)} = \langle \hat{e}_i\rangle, \end{aligned} \tag{3}$$

if $i = 1, \ldots, k$.

Our first aim is to prove that, essentially, this is the unique fine grading.

Theorem 1. *For any group grading of H_n there is a basis $B = \{z, u_1, u'_1, \ldots, u_n, u'_n\}$ of homogeneous elements of H_n such that $[u_i, u'_i] = z$ and the remaining possible brackets among elements of B are zero. In particular, any group grading on H_n is toral.*

Proof. As before we will consider the bilinear alternate form $\langle \cdot, \cdot \rangle : H_n \times H_n \to \mathbb{F}$. In any (group) graded Lie algebra the center admits a basis of homogeneous elements so if H_n is graded by a group G, there is some $g_0 \in G$ such that $z \in (H_n)_{g_0}$. Thus, denoting by $\mathcal{Z}(H_n) = \mathbb{F}z$ the center of H_n, the quotient Lie algebra $P := H_n/\mathcal{Z}(H_n)$ is a symplectic space relative to $\langle x+\mathcal{Z}(H_n), y+\mathcal{Z}(H_n)\rangle := \langle x, y\rangle$. Denote by $\pi : H_n \to P$ the canonical projection. By defining for each $g \in G$ the subspaces $P_g := \pi((H_n)_g)$, it is easy to check that $P = \oplus_{g\in G} P_g$ and that for any $g \in G$ there is a unique h, ($h = -g + g_0$), such that $\langle P_g, P_h\rangle \neq 0$. Then Lemma 1 provides a symplectic basis of P of homogeneous elements. Let $\{u_1 + \mathcal{Z}(H_n), u'_1 + \mathcal{Z}(H_n), \ldots, u_n + \mathcal{Z}(H_n), u'_n + \mathcal{Z}(H_n)\}$ be such basis (observe that each u_i and u'_j may be chosen is some homogeneous component of H_n being $u_i, u'_j \notin \mathbb{F}z$). Then $B := \{z, u_1, u'_1, \ldots, u_n, u'_n\}$ is a basis of homogeneous elements of H_n such that $[u_i, u'_i] = z$ and the remaining possible brackets among basis elements being zero. To finish the proof we can consider the maximal torus of $\mathrm{Aut}(H_n)$ diagonalizing the basis B. Up to conjugations, this torus is formed by all the automorphisms $t_{(\lambda_1,\ldots,\lambda_k;\lambda)}$ constructed above. Since B is diagonalized by any of these elements, the grading is toral and in fact a coarsening of the fine toral grading described above. $\square$

Corollary 1. *Up to equivalence, the unique fine grading of H_n is the $\mathbb{Z}^{k+1}$-grading given by (3).*

In order to work on the group of symmetries of this grading, the Weyl group, we compute first the automorphism group of the Heisenberg algebra, $\mathrm{Aut}(H_n)$. For any $f \in \mathrm{Aut}(H_n)$ we have $f(z) \in \mathcal{Z}(H_n)$ and so $f(z) = \lambda_f z$ for some $\lambda_f \in \mathbb{F}^\times$. If we denote by $i : P \to H_n$ the inclusion map, by $\pi : H_n \to P$ the projection map and define $\bar{f} := \pi \circ f \circ i$, we easily get that $\bar{f}$ is a linear automorphism of P satisfying $\langle \bar{f}(x_P), \bar{f}(y_P)\rangle = \lambda_f \langle x_P, y_P\rangle$ for any $x_P, y_P \in P$. Indeed, given any $x = x_P + \lambda z$, and $y = y_P + \mu z$ in H_n, $\lambda, \mu \in \mathbb{F}$, we have, taking into account $[z, H_n] = 0$, that

$$f([x, y]) = f([x_P, y_P]) = f(\langle x_P, y_P\rangle z) = \langle x_P, y_P\rangle f(z) = \langle x_P, y_P\rangle \lambda_f z$$

and

$$[f(x), f(y)] = [\bar{f}(x_P), \bar{f}(y_P)] = \langle \bar{f}(x_P), \bar{f}(y_P)\rangle z.$$

Hence $\bar{f}$ belongs to $\mathrm{GSp}(P) := \{g \in \mathrm{End}(P) :$ there is $\lambda_g \in \mathbb{F}^\times$ with $\langle g(x), g(y)\rangle = \lambda_g \langle x, y\rangle\ \forall x, y \in P\}$, the *similitude group* of $(P, \langle \cdot, \cdot \rangle)$.

As a consequence, an arbitrary $f \in \mathrm{Aut}(H_n)$ has as associated matrix relative to the basis B in Equation (1),

$$M_B(f) = \left(\begin{array}{c|c} M_{B_P}(\bar{f}) & 0 \\ \hline \underline{\lambda} & \lambda_f \end{array}\right)$$

for $\bar{f} = \pi \circ f \circ i \in \mathrm{GSp}(P)$ and for the vector $\underline{\lambda} = (\lambda_1, ..., \lambda_{2k}) \in \mathbb{F}^{2k}$ given by $\lambda_{2i-1} z = (f - \bar{f})(e_i)$ and $\lambda_{2i} z = (f - \bar{f})(\hat{e}_i)$ if $i \leq k$. Moreover, if we define in the set $\mathrm{GSp}(P) \times \mathbb{F}^{2n}$ the (semidirect) product

$$(\bar{f}, \underline{\lambda})(\bar{g}, \underline{\eta}) := (\overline{fg}, \underline{\lambda} M_{B_P}(\bar{g}) + \lambda_f\, \underline{\eta}),$$

it is straightforward to verify that the mapping

$$\Omega\colon \operatorname{Aut}(H_n) \to \operatorname{GSp}(P) \ltimes \mathbb{F}^{2k}$$

given by $\Omega(f) = (\bar{f}, \underline{\lambda})$ is a group isomorphism.

Take, for each permutation $\sigma \in S_k$, the map $\tilde{\sigma}\colon H_n \to H_n$ given by $\tilde{\sigma}(e_i) = e_{\sigma(i)}$, $\tilde{\sigma}(\hat{e}_i) = \hat{e}_{\sigma(i)}$ and $\tilde{\sigma}(z) = z$. It is clear that $\tilde{\sigma}$ is an automorphism permuting the homogeneous component of Γ, the grading in (3), that is, $\tilde{\sigma} \in \operatorname{Aut}(\Gamma)$.

Other remarkable elements in the automorphism group of the grading are the following ones: for each index $i \leq k$, take $\mu_i\colon H_n \to H_n$ given by $\mu_i(e_i) = \hat{e}_i, \mu_i(\hat{e}_i) = -e_i, \mu_i(e_j) = e_j, \mu_i(\hat{e}_j) = \hat{e}_j$ (for $j \neq i$) and $\mu_i(z) = z$.

Denote by $[f]$ the class of an automorphism $f \in \operatorname{Aut}(\Gamma)$ in the quotient $\mathcal{W}(\Gamma) = \operatorname{Aut}(\Gamma)/\operatorname{Stab}(\Gamma)$.

Proposition 1. *The Weyl group $\mathcal{W}(\Gamma)$ is generated by $\{[\tilde{\sigma}] : \sigma \in S_k\}$ and $[\mu_1]$.*

Proof. Let f be an arbitrary element in $\operatorname{Aut}(\Gamma)$. The elements in $\operatorname{Aut}(\Gamma)$ permute the homogeneous components of the grading Γ, but $\mathbb{F}z$ remains always invariant. Thus $f(e_1)$ belongs to some homogeneous component different from $\mathbb{F}z$, and there is an index $i \leq k$ such that either $f(e_1) \in \mathbb{F}^\times e_i$ or $f(e_1) \in \mathbb{F}^\times \hat{e}_i$. We can assume that $f(e_1) \in \mathbb{F}^\times e_i$ by replacing, if necessary, f with $\mu_i f$. Now, take the permutation $\sigma = (1, i)$ which interchanges 1 and i, so that $f' = \tilde{\sigma} f$ maps e_1 into αe_1 for some $\alpha \in \mathbb{F}^\times$. Note that $\alpha[e_1, f'(\hat{e}_1)] = [f'(e_1), f'(\hat{e}_1)] = f'([e_1, \hat{e}_1]) = f'(z)$ is a nonzero multiple of z, hence $f'(\hat{e}_1) \notin \{x \in H_n : [x, e_1] = 0\} = \langle z, e_1, e_i, \hat{e}_i \mid 2 \leq i \leq k\rangle$. But $f' \in \operatorname{Aut}(\Gamma)$, so $f'(\hat{e}_1) \in \langle \hat{e}_1 \rangle$. In a similar manner the automorphism f' sends e_2 to some $\langle e_j \rangle$ or some $\langle \hat{e}_j \rangle$ for $j \neq 1$, hence we can replace f' by $f'' \in \{\widetilde{(2,j)} f', \widetilde{(2,j)} \mu_j f'\}$ such that f'' preserves the homogeneous components $\langle e_1 \rangle$, $\langle \hat{e}_1 \rangle$, $\langle e_2 \rangle$, $\langle \hat{e}_2 \rangle$ and $\langle z \rangle$. By arguing as above, we can multiply f by an element in the subgroup generated by μ_j and $\tilde{\sigma}$ for $1 \leq j \leq k$ and $\sigma \in S_k$ such that the product stabilizes all the components, so that it belongs to $\operatorname{Stab}(\Gamma)$. The proof finishes if we observe that $\tilde{\sigma}\mu_i = \mu_{\sigma(i)}\tilde{\sigma}$ for all $i \leq k$, so that all μ_i's belong to the noncommutative group generated by $\{[\tilde{\sigma}] : \sigma \in S_k\}$ and $[\mu_1]$. □

Hence $\mathcal{W}(\Gamma) = \{[\mu_{i_1} \dots \mu_{i_s} \tilde{\sigma}] : \sigma \in S_k,\ 1 \leq i_1 \leq \cdots \leq i_s \leq k\}$ has $2^k k!$ elements. Observe that, although any μ_i has order 4, its class $[\mu_i]$ has order 2. Besides μ_i and μ_j commute, so we can identify $\mathcal{W}(\Gamma)$ with the group $\mathcal{P}(K) \rtimes S_k$ with the product given by $(A, \sigma)(B, \eta) = (A \bigtriangleup \sigma(B), \sigma\eta)$ if $A, B \subset K = \{1, \dots, k\}$, $\sigma, \eta \in S_k$, and where the elements in $\mathcal{P}(K)$ are the subsets of $\{1, \dots, k\}$ and $\bigtriangleup$ denotes the symmetric difference. Thus

$$\mathcal{W}(\Gamma) \cong \mathbb{Z}_2^k \rtimes S_k.$$

4. Gradings on Heisenberg superalgebras

In this section we will assume the ground field $\mathbb{F}$ to be algebraically closed and of characteristic other than 2. A Heisenberg superalgebra $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ is a nilpotent in two steps Lie superalgebra with one-dimensional center such that $[\mathcal{H}_0, \mathcal{H}_1] = 0$. In particular this implies that the even part is a Heisenberg algebra, so that it is determined up to isomorphism by its dimension. Note that, if $x, y \in \mathcal{H}_1$, then $[x, y] = [y, x] \in \mathbb{F}z = \mathcal{Z}(\mathcal{H})$, so there is a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle\colon \mathcal{H}_1 \times \mathcal{H}_1 \to \mathbb{F}$ such that $[x, y] = \langle x, y \rangle z$ for all $x, y \in \mathcal{H}_1$. Hence there is a basis of $\mathcal{H}_1$ in which the matrix of $\langle \cdot, \cdot \rangle$ is the identity matrix. If $B_1 = \{w_1, ..., w_m\}$ denotes such a basis, and

$B_0 = \{z, e_1, \hat{e}_1, ..., e_k, \hat{e}_k\}$ denotes the basis of $\mathcal{H}_0$ as in Section 3, then the product is given by

$$\begin{aligned} &[e_i, \hat{e}_i] = -[\hat{e}_i, e_i] = z, \quad 1 \le i \le k, \\ &[w_j, w_j] = z, \quad 1 \le j \le m, \end{aligned}$$

with all other products being zero. As this algebra is completely determined by $n = 2k+1$ and m, the dimensions of the even and odd part respectively, we denote it by $H_{n,m}$.

If we denote by

$$\circ\colon (H_{n,m})_1 \times (H_{n,m})_1 \to (H_{n,m})_0$$

the bilinear mapping $x_1 \circ y_1 := [x_1, y_1]$, we get that the product in $H_{n,m}$ can be expressed by

$$[(x_0, x_1), (y_0, y_1)] = [x_0, y_0] + x_1 \circ y_1,$$

for all $x_i, y_i \in (H_{n,m})_i$.

Let us compute the group $\mathrm{Aut}(H_{n,m})$ of automorphisms of $H_{n,m}$. Recall that $\mathrm{Aut}(H_{n,m})$ is formed by the linear automorphisms $f\colon H_{n,m} \to H_{n,m}$ such that $f([x,y]) = [f(x), f(y)]$ for all $x, y \in H_{n,m}$ and $f((H_{n,m})_i) = (H_{n,m})_i$ for $i \in \{0, 1\}$.

If we identify $(H_{n,m})_1$ with the underlying vector space endowed with the inner product $\langle\, , \,\rangle$ then we can consider the group $\mathrm{GO}((H_{n,m})_1) = \{g \in \mathrm{End}((H_{n,m})_1) : \langle g(x), g(y)\rangle = \lambda\langle x, y\rangle\, \forall x, y \in (H_{n,m})_1$ for some $\lambda \in \mathbb{F}^\times\}$, and the group homomorphism $\pi\colon\ \mathrm{Aut}(H_{n,m}) \to \mathrm{GO}((H_{n,m})_1)$ such that $\pi(f) = f|_{(H_{n,m})_1}$. We prove next that π is an epimorphism. Given $f \in \mathrm{GO}((H_{n,m})_1)$ we know that there is $\lambda_f \in \mathbb{F}^\times$ such that $\langle f(x), f(y)\rangle = \lambda_f\langle x, y\rangle$. Then we can define the linear map $\hat{f}\colon H_{n,m} \to H_{n,m}$ such that $\hat{f}$ restricted to $(H_{n,m})_1$ is f and $\hat{f}(z) := \lambda_f z$, $\hat{f}(e_i) := \lambda_f e_i$ and $\hat{f}(\hat{e}_i) := \hat{e}_i$ for any i. Then $\hat{f} \in \mathrm{aut}(H_{n,m})$ and $\pi(\hat{f}) = f$. Furthermore $\ker(\pi) \cong \mathrm{Sp}(P) \times \mathbb{F}^{2k}$, taking into account the results in Section 3 and that any $f \in \mathrm{aut}(H_{n,m})$ such that $\pi(f) = f|_{(H_{n,m})_1} = 1$ satisfies that $f(z) = z$. Therefore we have a short exact sequence

$$1 \to \mathrm{Sp}(P) \times \mathbb{F}^{2k} \xrightarrow{i} \mathrm{Aut}(H_{n,m}) \xrightarrow{\pi} \mathrm{GO}((H_{n,m})_1) \to 1$$

which is split since $j\colon \mathrm{GO}((H_{n,m})_1) \to \mathrm{Aut}(H_{n,m})$ defined by $j(f) := \hat{f}$ satisfies $\pi j = 1$. Then $\mathrm{Aut}(H_{n,m})$ is the semidirect product

$$\mathrm{Aut}(H_{n,m}) \cong (\mathrm{Sp}(P) \times \mathbb{F}^{2k}) \rtimes \mathrm{GO}((H_{n,m})_1).$$

Assume now that $H_{n,m}$ is a graded superalgebra and G is the grading group. Then the even part $(H_{n,m})_0$ admits a basis $\{z, e_1, \hat{e}_1, \ldots e_k, \hat{e}_k\}$ of homogeneous elements as has been proved in the previous section. On the other hand the product in $(H_{n,m})_1$ is of the form $x \circ y = \langle x, y\rangle z$ for any $x, y \in (H_{n,m})_1$ and where $\langle\cdot,\cdot\rangle\colon (H_{n,m})_1 \times (H_{n,m})_1 \to \mathbb{F}$ is a symmetric nondegenerate bilinear form. Furthermore, for any $g \in G$, denote by $\mathcal{L}_g$ the subspace $\mathcal{L}_g := (H_{n,m})_1 \cap (H_{n,m})_g$ (that is the odd part of the homogeneous component of degree g of $H_{n,m}$). Then $(H_{n,m})_1 = \oplus_{g\in G}\mathcal{L}_g$ is a decomposition on linear subspaces and for any $g \in G$ there is a unique $h \in G$ such that $\langle \mathcal{L}_g, \mathcal{L}_h\rangle \neq 0$: indeed, assume $\langle \mathcal{L}_g, \mathcal{L}_h\rangle \neq 0$. Then $0 \neq \mathcal{L}_g \circ \mathcal{L}_h \in \mathbb{F}z$ and this implies that $g + h = g_0$ where g_0 is the degree of z. Thus $h = g_0 - g$ is unique. Next we apply Lemma 2 to get a basis $\{z, e_1, \hat{e}_1, \ldots e_k, \hat{e}_k, u_1, v_1, \ldots, u_r, v_r, z_1, \ldots, z_q\}$ of $H_{n,m}$ (of homogeneous elements) such that $z, e_i, \hat{e}_i$ generate the even part of $H_{n,m}$ while $u_1, v_1, \ldots, u_r, v_r, z_1, \ldots, z_q$ generate the odd part and the nonzero products are:

$$[e_i, \hat{e}_i] = [u_j, v_j] = [z_l, z_l] = z. \tag{4}$$

for $i \in \{1, \ldots, k\}$, $j \in \{1, \ldots, r\}$ and $l \in \{1, \ldots q\}$.

This basis provides a $\mathbb{Z}^{1+k+r}\times\mathbb{Z}_2^{m-2r}$-grading on $H_{n,m}$ given by

$$
\Gamma^r: \quad \begin{array}{l}
(H_{n,m})_{(2;0,\dots,0;0,\dots,0;\bar 0,\dots,\bar 0)} = \langle z\rangle,\\
(H_{n,m})_{(1;0,\dots,1,\dots,0;0,\dots,0;\bar 0,\dots,\bar 0)} = \langle e_i\rangle,\\
(H_{n,m})_{(1;0,\dots,-1,\dots,0;0,\dots,0;\bar 0,\dots,\bar 0)} = \langle \hat e_i\rangle,\\
(H_{n,m})_{(1;0,\dots,0;0,\dots,1,\dots,0;\bar 0,\dots,\bar 0)} = \langle u_j\rangle,\\
(H_{n,m})_{(1;0,\dots,0;0,\dots,-1,\dots,0;\bar 0,\dots,\bar 0)} = \langle v_j\rangle,\\
(H_{n,m})_{(1;0,\dots,0;0,\dots,0;\bar 0,\dots,\bar 1,\dots,\bar 0)} = \langle z_l\rangle,
\end{array} \tag{5}
$$

if $i\le k, j\le r, l\le q$. This grading is a refinement of the original G-grading of the algebra.

Observe that for each r such that $0\le 2r\le m$, there exists a basis of $H_{n,m}$ satisfying the relations (4) by taking for $j\le r, l\le m-2r$,

$$
\begin{array}{l}
u_j := \frac{1}{\sqrt 2}(w_{2j-1}+\mathbf{i}w_{2j}),\\
v_j := \frac{1}{\sqrt 2}(w_{2j-1}-\mathbf{i}w_{2j}),\\
z_l := w_{l+2r},
\end{array}
$$

if $\mathbf{i}\in\mathbb{F}$ is a primitive square root of the unit. If the starting G-grading is fine, then it is equivalent to the $\mathbb{Z}^{1+k+r}\times\mathbb{Z}_2^{m-2r}$-grading Γ^r provided by the above basis. Therefore we have proved the following result.

Theorem 2. *Up to equivalence, there are $\frac{m}{2}+1$ fine gradings on $H_{n,m}$ if m is even and $\frac{m+1}{2}$ in case m is odd, namely, $\{\Gamma^r : 2r\le m\}$. All of these are inequivalent, and only one is toral, $\Gamma^{m/2}$ when m is even.*

In order to compute the Weyl groups of these fine gradings, consider, as in Section 3, the maps $\tilde\sigma,\mu_i\in\mathrm{Aut}((H_{n,m})_0)$ if $\sigma\in S_k$, $i\le k$, and extend to automorphisms of $H_{n,m}$ by setting $\tilde\sigma|_{(H_{n,m})_1}=\mu_i|_{(H_{n,m})_1}=\mathrm{id}$. Thus, $\tilde\sigma,\mu_i\in\mathrm{Aut}(\Gamma^r)$, for Γ^r the grading in (5). Take too, for each permutation $\sigma\in S_r$, the map $\bar\sigma\colon H_{n,m}\to H_{n,m}$ given by $\bar\sigma|_{(H_{n,m})_0}=\mathrm{id}$, $\bar\sigma(u_i)=u_{\sigma(i)}$, $\bar\sigma(v_i)=v_{\sigma(i)}$ and $\bar\sigma(z_l)=z_l$. Also consider for each permutation $\rho\in S_q$, the map $\hat\rho\colon H_{n,m}\to H_{n,m}$ given by $\hat\rho|_{(H_{n,m})_0}=\mathrm{id}$, $\hat\rho(u_i)=u_i$, $\hat\rho(v_i)=v_i$ and $\hat\rho(z_l)=z_{\rho(l)}$. Finally consider for each index $i\le r$, the map $\mu_i'\colon H_{n,m}\to H_{n,m}$ given by $\mu_i'|_{(H_{n,m})_0}=\mathrm{id}$, $\mu_i'(u_i)=v_i$, $\mu_i'(v_i)=-u_i$, $\mu_i'(u_k)=u_k$, $\mu_i'(v_k)=v_k$ for $k\ne i$ and $\mu_i'(z_l)=z_l$. It is clear that $\bar\sigma,\hat\rho,\mu_i'\in\mathrm{Aut}(\Gamma^r)$ in any case.

Proposition 2. *The Weyl group $\mathcal{W}(\Gamma^r)$ is generated by $[\mu_1]$, $[\mu_1']$, $\{[\tilde\sigma]:\sigma\in S_k\}$, $\{[\bar\sigma]:\sigma\in S_r\}$ and $\{[\hat\sigma]:\sigma\in S_q\}$, with $k=(n-1)/2, q=m-2r$.*

Proof. We have by Section 3 that the subgroup W of $\mathcal{W}(\Gamma^r)$ generated by the classes of the elements fixing all the homogeneous components of $(H_{(n,m)})_1$ is $\{[\mu_{i_1}\dots\mu_{i_s}\tilde\sigma]:\sigma\in S_k,\ 1\le i_1\le\dots\le i_s\le k\}$.

Let f be an arbitrary element in $\mathrm{Aut}(\Gamma^r)$. As $f|_{(H_{(n,m)})_0}$ preserves the grading Γ in Equation (3), then we can compose f with an element in W to assume that f preserves all the homogeneous components of $(H_{(n,m)})_0$.

Then the element $f(u_1)$ belongs to some homogeneous component of $(H_{(n,m)})_1$, but it does not happen that there is $j\le q$ such that $f(u_1)\in\mathbb{F}z_j$, since then $0=f([u_1,u_1])=z_j\circ z_j=z$. So there is an index $i\le r$ such that either $f(u_1)\in\mathbb{F}u_i$ or $f(u_1)\in\mathbb{F}v_i$. The same arguments as in the proof of Proposition 1 show that we can replace f by $\mu_{j_1}'\dots\mu_{j_s}'\bar\sigma f$ for $1\le j_1\le\dots\le j_s\le r$ and $\sigma\in S_r$ to get that $f(u_i)\in\mathbb{F}u_i$ and $f(v_i)\in\mathbb{F}v_i$ for all $i\le r$.

Now it is clear that $f(z_1)\in\mathbb{F}z_l$ for some $1\le l\le q$. If $l\ne 1$, we can replace f with $\hat\rho f$, for $\rho=(1,l)$, so that we can assume $f(z_1)\in\mathbb{F}z_1$. And, in the same way, we can assume that $f(z_l)\in\mathbb{F}z_l$ for $1\le l\le q$. Our new f belongs to $\mathrm{Stab}(\Gamma^r)$.

The proof finishes if we observe that $\bar{\sigma}\mu'_i = \mu'_{\sigma(i)}\bar{\sigma}$ for all $i \leq r$ and $\sigma \in S_r$, and that $\bar{\sigma}$ as well as μ'_i commute with $\hat{\rho}$ for all $\rho \in S_q$. $\square$

Hence, an arbitrary element in $\mathcal{W}(\Gamma^r)$ is

$$[\mu_{i_1} \dots \mu_{i_s} \tilde{\sigma} \mu'_{j_1} \dots \mu'_{j_t} \bar{\eta} \hat{\rho}]$$

for $1 \leq i_1 \leq \cdots \leq i_s \leq k$, $1 \leq j_1 \leq \cdots \leq j_t \leq r$, $\tilde{\sigma} \in S_k$, $\bar{\eta} \in S_r$, $\hat{\rho} \in S_q$, so that $\mathcal{W}(\Gamma^r)$ is isomorphic to

$$(\mathcal{P}(\{1, \dots, k\}) \rtimes S_k) \times (\mathcal{P}(\{1, \dots, r\}) \rtimes S_r) \times S_q$$

with the product as in Section 3, and in a more concise form,

$$\mathcal{W}(\Gamma^r) \cong \mathbb{Z}_2^{r+k} \rtimes (S_k \times S_r \times S_q).$$

5. AN APPLICATION TO HEISENBERG LIE COLOR ALGEBRAS

As in §4, the base field $\mathbb{F}$ will be supposed throughout this section algebraically closed and of characteristic other that 2. Lie color algebras were introduced in [33] as a generalization of Lie superalgebras and hence of Lie algebras. This kind of algebras has attracted the interest of several authors in the last years, (see [10, 31, 32, 37, 38]), being also remarkable the important role they play in theoretical physic, specially in conformal field theory and supersymmetries ([35, 36]).

Definition 1. *Let G be an abelian group. A* skew-symmetric bicharacter *of G is a map $\epsilon\colon G \times G \longrightarrow \mathbb{F}^\times$ satisfying*

$$\epsilon(g_1, g_2) = \epsilon(g_2, g_1)^{-1},$$
$$\epsilon(g_1, g_2 + g_3) = \epsilon(g_1, g_2)\epsilon(g_1, g_3),$$

for any $g_1, g_2, g_3 \in G$.

Observe that $\epsilon(g, 0) = 1$ for any $g \in G$, where 0 denotes the identity element of G.

Definition 2. *Let $L = \bigoplus_{g \in G} L_g$ be a G-graded $\mathbb{F}$-vector space. For a nonzero homogeneous element $v \in L$, denote by $\deg v$ the unique element in G such that $v \in L_{\deg v}$. We shall say that L is a* Lie color algebra *if it is endowed with a $\mathbb{F}$-bilinear map (the Lie color bracket)*

$$[\cdot, \cdot]\colon L \times L \longrightarrow L$$

satisfying $[L_g, L_h] \subset L_{g+h}$ for all $g, h \in G$ and

$$[v, w] = -\epsilon(\deg v, \deg w)[w, v], \quad \text{(color skew-symmetry)}$$
$$[v, [w, t]] = [[v, w], t] + \epsilon(\deg v, \deg w)[w, [v, t]], \quad \text{(Jacobi color identity)}$$

for all homogeneous elements $v, w, t \in L$ and for some skew-symmetric bicharacter ϵ.

Two Lie color algebras are *isomorphic* if they are isomorphic as graded algebras.

Clearly any Lie algebra is a Lie color algebra and also Lie superalgebras are examples of Lie color algebras, (take $G = \mathbb{Z}_2$ and $\epsilon(i, j) = (-1)^{ij}$, for any $i, j \in \mathbb{Z}_2$).

Heisenberg Lie color algebras have been previously considered in the literature (see [26]). Fixed a skew-symmetric bicharacter $\epsilon\colon G \times G \longrightarrow \mathbb{F}^\times$, a Heisenberg Lie color algebra H is defined in [26] as a G-graded vector space, where G is a torsion-free abelian group with a basis $\{\epsilon_1, \epsilon_2, ..., \epsilon_n\}$, in the form

$$H = \bigoplus_{i=1}^{n} \mathbb{F}p_i \oplus \bigoplus_{j=1}^{n} \mathbb{F}q_j \oplus \mathbb{F}c$$

where $p_i \in H_{\epsilon_i}$, $q_j \in H_{-\epsilon_j}$, and $c \in H_0$; and where the Lie color bracket is given by $[p_i, q_i] = \delta_{ij} c$ and $[p_i, p_j] = [q_i, q_j] = [p_i, c] = [c, p_i] = [q_j, c] = [c, q_j] = 0$.

Observe that, following this definition, the class of Heisenberg superalgebras is not contained in the one of Heisenberg Lie color algebras. Hence, this definition seems to us very restrictive. So let us briefly discuss about the concept of Heisenberg Lie color algebras. Any Heisenberg algebra (respectively Heisenberg superalgebra) H is characterized among the Lie algebras (respectively among the Lie superalgebras) for satisfying $[H, H] = \mathcal{Z}(H)$ and $\dim(\mathcal{Z}(H)) = 1$. Hence, it is natural to introduce the following definition.

Definition 3. *A Heisenberg Lie color algebra is a Lie color algebra L such that $[L, L] = \mathcal{Z}(L)$ and $\dim(\mathcal{Z}(L)) = 1$.*

Examples.

1. The above so called Heisenberg Lie color algebras, given in [26], satisfy these conditions and so can be seen as a particular case of the ones given by Definition 3. As examples of Heisenberg Lie color algebras we also have the Heisenberg algebras ($G = \{0\}$) and the Heisenberg superalgebras ($G = \{\mathbb{Z}_2\}$). We can also consider any Heisenberg G-graded algebra as a Heisenberg Lie color algebra for the group G and the trivial bicharacter given by $\epsilon(g, h) = 1$ for all $g, h \in G$.

2. Any G-graded $L = \oplus_{g \in G} M_g$ Heisenberg superalgebra $L = L_0 \oplus L_1$ gives rise to a Heisenberg Lie color algebra relative to the group $G \times \mathbb{Z}_2$ and an adequate skew-symmetric bicharacter ϵ. In fact, we just have to $(G \times \mathbb{Z}_2)$-graduate L as $L = \oplus_{(g,i) \in G \times \mathbb{Z}_2} M_{(g,i)}$ where $M_{(g,i)} = M_g$ with $M_{(g,i)} \subset L_i$, and define $\epsilon : (G \times \mathbb{Z}_2) \times (G \times \mathbb{Z}_2) \to \mathbb{F}^{\times}$ as $\epsilon((g, i), (h, j)) := (-1)^{ij}$. Observe that both gradings on L are equivalent.

3. Consider some $g_0 \in G$, a graded vector space $V = \bigoplus_{g \in G} V_g$ such that $\dim(V_g) = \dim(V_{-g+g_0})$ for any $g \notin \{g_0, 0\}$ and $\dim(V_{g_0}) = \dim(V_0) + 1$ in case $g_0 \neq 0$; and any skew-symmetric bicharacter ϵ: $G \times G \longrightarrow \mathbb{F}^{\times}$ satisfying $\epsilon(g, g) = -1$ for any $g \in G$ such that $2g = g_0$ and $V_g \neq 0$. Fix bases $\{z, u_{g_0,1}, ..., u_{g_0,n_{g_0}}\}$ and $\{\hat{u}_{g_0,1}, ..., \hat{u}_{g_0,n_{g_0}}\}$ of V_{g_0} and V_0 respectively when $g_0 \neq 0$ or $\{z, u_{g_0,1}, ..., u_{g_0,n_{g_0}}, \hat{u}_{g_0,1}, ..., \hat{u}_{g_0,n_{g_0}}\}$ when $g_0 = 0$. For any subset $\{g, -g + g_0\} \neq \{g_0, 0\}$ of G, fix also basis $\{u_{g,1}, ..., u_{g,n_g}\}$ and $\{\hat{u}_{g,1}, ..., \hat{u}_{g,n_g}\}$ of V_g and V_{-g+g_0} in case $2g \neq g_0$ and $\{u_{g,1}, ..., u_{g,n_g}\}$ basis of V_g in case $2g = g_0$. Then by defining a product on V given by $[u_{g,i}, \hat{u}_{g,i}] = z$, $[\hat{u}_{g,i}, u_{g,i}] = -\epsilon(-g + g_0, g)z$ in the cases $g = 0$ or $2g \neq g_0$, $[u_{g,i}, u_{g,i}] = z$ in the cases $2g = g_0$ with $g \neq 0$, and the remaining brackets zero, for any subset $\{g, -g + g_0\}$ of G, we get that V becomes a Heisenberg Lie color algebra that we call of *type* (G, g_0, ϵ). We note that for an easier notation we allow empty basis in the above definition which correspond to trivial subspaces $\{0\}$.

Lemma 3. *A Heisenberg Lie color algebra L of type (G, g_0, ϵ) is a grading of a Heisenberg superalgebra if and only if $\epsilon(g, -g + g_0) \in \{\pm 1\}$ for any $g \in G$ such that $L_g \neq 0$.*

Proof. Suppose L is a grading of a Heisenberg superalgebra $L = L_0 \oplus L_1$ and there exists $g \in G$ with $L_g \neq 0$ and such that $\epsilon(g, -g + g_0) \notin \{\pm 1\}$. Since $[L_0, L_1] = 0$,

$L_g + L_{-g+g_0} \subset L_i$ for some $i \in \mathbb{Z}_2$. By taking either $u_g \in L_g$ and $\hat{u}_g \in L_{-g+g_0}$ in the cases $g = 0$ or $2g \neq g_0$; or $u_g \in L_g$ in the case $2g = g_0$ with $g \neq 0$, elements of the standard basis of (G, g_0, ϵ) described in Example 3, we have either $0 \neq [u_g, \hat{u}_g] = -\epsilon(g, -g + g_0)[\hat{u}_g, u_g]$ if $2g \neq g_0$ or $0 \neq [u_g, u_g] = -\epsilon(g, -g + g_0)[u_g, u_g]$ if $2g = g_0$, being $\epsilon(g, -g + g_0) \neq \pm 1$, which contradicts the identities of a Lie superalgebra.

Conversely, if $\epsilon(g, -g + g_0) \in \{\pm 1\}$ for any $g \in G$ with $L_g \neq 0$, we can $\mathbb{Z}_2$-graduate L as

$$L = \Big(\bigoplus_{\{g \in \mathrm{supp}(G) : \epsilon(g, -g+g_0) = 1\}} L_g\Big) \oplus \Big(\bigoplus_{\{h \in \mathrm{supp}(G) : \epsilon(h, -h+g_0) = -1\}} L_h\Big),$$

this one becoming a Heisenberg Lie superalgebra, graded by the group generated by the support $\mathrm{supp}(G) := \{g \in G : L_g \neq 0\}$. □

Proposition 3. *Any Heisenberg Lie color algebra is isomorphic to a Heisenberg Lie color algebra of type* (G, g_0, ϵ).

Proof. Consider a Heisenberg Lie color algebra $L = \bigoplus_{g \in G} L_g$. Since $\dim(\mathcal{Z}(L)) = 1$, we can write $\mathcal{Z}(L) = \langle z \rangle$, being $z = \sum_{i=1}^n x_{g_i}$, with any $0 \neq x_{g_i} \in L_{g_i}$ and $g_i \neq g_j$ if $i \neq j$. If $n \neq 1$, then $[x_{g_i}, L_h] \subset L_{g_i+h} \cap \langle z \rangle = 0$ for any $h \in G$ and $i \in \{1, .., n\}$. Hence any $x_{g_i} \in \mathcal{Z}(L) = \langle z \rangle$, a contradiction. Thus $z \in L_{g_0}$ for some $g_0 \in G$. Now the fact $[L, L] \subset \mathcal{Z}(L)$ gives us that for any $g \in G$, $[L_g, L_h] = 0$ if $g + h \neq g_0$, and consequently $[L_g, L_{-g+g_0}] \neq 0$ if $g \neq g_0$.

Since for any skew-symmetric bicharacter $\epsilon\colon G \times G \longrightarrow \mathbb{F}^\times$ and $g \in G$ we have $\epsilon(g, g) \in \{\pm 1\}$, we can $\mathbb{Z}_2$-graduate L as

$$L = \Big(\bigoplus_{\{g \in G : \epsilon(g,g) = 1\}} L_g\Big) \oplus \Big(\bigoplus_{\{h \in G : \epsilon(h,h) = -1\}} L_h\Big).$$

This is a grading of the algebra L since $\epsilon(g + h, g + h) = \epsilon(g, g)\epsilon(g, h)\epsilon(h, g)\epsilon(h, h) = \epsilon(g, g)\epsilon(h, h)$.

Let us distinguish two cases, according to g_0 is the identity element or not.

First, assume $g_0 = 0$. For any $g \in G$ it is easy to check that $\epsilon(g, g) = \epsilon(-g, -g) = \epsilon(g, -g) = \epsilon(-g, g) \in \{\pm 1\}$. This fact together with the observations in the previous paragraph tell us that $L = (\bigoplus_{\{g \in G : \epsilon(g,g) = 1\}} L_g) \oplus (\bigoplus_{\{h \in G : \epsilon(h,h) = -1\}} L_h)$ is actually a Lie superalgebra, satisfying $[L, L] = \mathcal{Z}(L)$ and $\dim(\mathcal{Z}(L)) = 1$, and being the initial Lie color grading a refinement of the $\mathbb{Z}_2$-grading as superalgebra. By arguing as in [14] we easily get that L is of type $(G, 0, \epsilon)$, being also a grading of a Heisenberg superalgebra.

Second, assume that $g_0 \neq 0$. Since $[L, L] \subset \langle z \rangle \subset L_{g_0}$, then $[L_{g_0}, L_{g_0}] \subset L_{2g_0} \cap L_{g_0} = 0$, so that $L' := L_{g_0} \oplus L_0$ is a Lie algebra (take into consideration $\epsilon(g_0, 0) = \epsilon(0, 0) = 1$). If $L_0 = 0$, then $L' = \langle z \rangle$ and otherwise $[L', L'] = \mathcal{Z}(L')$ with $\dim(\mathcal{Z}(L')) = 1$. In the second situation, we have that $L' = L_{g_0} \oplus L_0$ is a Heisenberg algebra, so that taking into account Section 3, the grading is toral and there exist basis $\{z, u_{g_0,1}, ..., u_{g_0,n_{g_0}}\}$ and $\{\hat{u}_{g_0,1}, ..., \hat{u}_{g_0,n_{g_0}}\}$ of L_{g_0} and L_0 respectively such that $[u_{g_0,i}, \hat{u}_{g_0,i}] = z$, $[\hat{u}_{g_0,i}, u_{g_0,i}] = -z$ and such that the remaining products in L' are zero. Consider now any subset $\{g, -g + g_0\} \neq \{g_0, 0\}$ of G. In case $L_g \neq 0$, then necessarily $L_{-g+g_0} \neq 0$ and we can distinguish two possibilities. First, if $2g \neq g_0$, the facts $[L_g, L_h] = 0$ if $g + h \neq g_0$ and $[L_g, L_{-g+g_0}] = \langle z \rangle$ allow us to apply standard linear algebra arguments to obtain basis $\{u_{g,1}, ..., u_{g,n_g}\}$ and $\{\hat{u}_{g,1}, ..., \hat{u}_{g,n_g}\}$ of L_g and L_{-g+g_0} such that $[u_{g,i}, \hat{u}_{g,i}] = z$, $[\hat{u}_{g,i}, u_{g,i}] = -\epsilon(-g + g_0, g)z$ and being null the rest of the products among the elements of the basis. Second, in the case $2g = g_0$, a similar argument gives us $\{u_{g,1}, ..., u_{g,n_g}\}$ a basis of L_g such that

$[u_{g,i}, u_{g,i}] = z$ with the remaining brackets zero. Also observe that necessarily $\epsilon(g, g) = -1$ for any $g \in G$ such that $2g = g_0$ and $L_g \neq 0$ because in the opposite case $\epsilon(g, g) = 1$ and then $0 \neq L_g \subset \mathcal{Z}(L) \subset L_{g_0}$, a contradiction. Summarizing, we have showed that L is isomorphic to a Heisenberg Lie color algebra of type (G, g_0, ϵ) with $g_0 \neq 0$. □

We finish this section by showing how the results in Section 4 can be applied to the classification of Heisenberg Lie color algebras. Following Lemma 3 and the arguments in the proof of Proposition 3, any Heisenberg Lie color algebra $L = \oplus_{g\in G} L_g$ is isomorphic to a grading of a Heisenberg superalgebra if and only if either $\mathcal{Z}(L) \subset L_0$ or L is of the type (G, g_0, ϵ) with $\epsilon(g, -g + g_0) \in \{\pm 1\}$ for any $g \in G$ such that $L_g \neq 0$. In particular, this is the case of the Heisenberg Lie color algebras considered in [26]. Hence, L is isomorphic to a coarsening of a fine grading $H_{n,m} = \oplus_{k\in K}(H_{n,m})_k$ of a Heisenberg superalgebra. Since it is known the procedure to compute all of the coarsenigs of a given grading when this is given by its universal group grading (see Section 2 and [17]), we can apply Theorem 2 to get the list of all of these Heisenberg Lie color algebras L, in the moment G and K are generated by their supports and K is the universal grading group. Of course this procedure does not hold for an Heisenberg Lie color algebra which is not a grading of a Heisenberg superalgebra.

6. Gradings on extended Heisenberg algebras

In this final section we consider the so called extended Heisenberg algebras. As mentioned in the introduction, these algebras appear naturally as some of the direct summands of the Lie algebras of connected Lie groups acting isometrically and locally faithfully on compact connected Lorentzian manifolds.

6.1. **Definition of extended Heisenberg algebra.** Consider the Heisenberg algebra H_n over the field $\mathbb{F}$ and take d to be any derivation of H_n. Then one can define in $\mathbb{F} \times H_n$ the product

$$[(\lambda, a), (\mu, b)] := (0, \lambda d(b) - \mu d(a) + [a, b]).$$

This defines a Lie algebra structure in $\mathbb{F} \times H_n$ and we will denote this Lie algebra by H_n^d. There is a particular derivation of H_n given in terms of the basis $\{z, e_1, \hat{e}_1, ..., e_k, \hat{e}_k\}$ of H_n ($n = 2k + 1$) by $d(z) = 0$, $d(e_i) = \lambda_i \hat{e}_i$ and $d(\hat{e}_i) = -\lambda_i e_i$ for a fixed k-tuple $\lambda := (\lambda_1, \ldots, \lambda_k) \in (\mathbb{F}^\times)^k$. This Lie algebra will be denoted $H_n(\lambda)$. Returning to the general case H_n^d, if we define $u = (1, 0)$, then $H_n^d = \mathbb{F}u \oplus H_n$ and its product can be rewritten as $[\lambda u + a, \mu u + b] = \lambda[u, b] - \mu[u, a] + [a, b]$. In this case $d = \mathrm{ad}(u)|_{H_n}$. Thus for instance in H_n^λ we have $[u, e_i] = \lambda_i \hat{e}_i$ while $[u, \hat{e}_i] = -\lambda_i e_i$. We will take as our "official"definition of extended Heisenberg Lie algebra the following:

Definition 4. *Let $\lambda = (\lambda_1, \ldots, \lambda_k) \in (\mathbb{F}^\times)^k$, $k > 0$. The corresponding extended Heisenberg algebra $H_n(\lambda)$ of dimension $n = 2k + 2$ is the Lie algebra spanned by the elements*

$$\{z, u, e_1, \hat{e}_1, ..., e_k, \hat{e}_k\}, \tag{6}$$

and the nonvanishing Lie brackets are given by

$$[e_i, \hat{e}_i] = \lambda_i z\,, \quad [u, e_i] = \lambda_i \hat{e}_i \text{ and } [u, \hat{e}_i] = -\lambda_i e_i,$$

for any $i = 1, \ldots, k$.

As before, $H_n(\lambda) = \mathbb{F}u \oplus H_n$ where H_n can be identified with the subalgebra spanned by $\{z, e_1, \hat{e}_1, ..., e_k, \hat{e}_k\}$ (by scaling the basis). This algebra is not nilpotent, but it is solvable, since $[H_n(\lambda), H_n(\lambda)] = H_n$.

The above definition of the extended Heisenberg algebra depends heavily on the basis $\{z, u, e_1, \hat{e}_1, ..., e_k, \hat{e}_k\}$. On the other hand the definition of H_n^d, though being less basis-dependent, still relies on the derivation $d\colon H_n \to H_n$. There is a more intrinsic definition of this kind of algebras which is equivalent to that of H_n^d. On the one hand, the Lie algebra H_n^d fits in a split exact sequence

$$0 \to H_n \xrightarrow{i} H_n^d \xrightarrow{p} \mathbb{F} \to 0$$

where i is the inclusion map and $p(\lambda, a) = \lambda$ for any $a \in H_n$. The sequence is split because we can define $j\colon \mathbb{F} \to H_n^d$ by $j(1) = (1, 0)$ and then $pj = 1_{\mathbb{F}}$. On the other hand if we consider any algebra A which is a split extension of the type

$$0 \to H_n \to A \xrightarrow{p} \mathbb{F} \to 0,$$

then A is isomorphic to some H_n^d for a suitable derivation d of H_n. Indeed: since the extension is split there is a monomorphism $j\colon \mathbb{F} \to A$ such that $pj = 1_{\mathbb{F}}$. If we take $u := j(1)$, then $A = \mathrm{Im}(j) \oplus H_n$ (we identify H_n with its image under the monomorphism $H_n \to A$). Since $\mathrm{Im}(j) = \mathbb{F}u$ we already have a decomposition $A = \mathbb{F}u \oplus H_n$. So the product in A is given by $[\lambda u + a, \mu u + b] = \lambda[u, a] - \mu[u, b] + [a, b]$. Next we prove that $\mathrm{ad}(u)(H_n) \subset H_n$. It suffices to prove that $p([u, a]) = 0$ for any $a \in H_n$. But $p([u, a]) = [p(u), p(a)] = [1, p(a)] = 0$ since $\mathbb{F}$ is abelian. Thus we can take $d := \mathrm{ad}(u)|_{H_n}$ and $A \cong H_n^d$.

In spite of the possibility of considering extended Heisenberg algebras as split extensions as previously mentioned, we adhere to the official definition $H_n(\lambda)$ because of our necessity of making explicit computations when dealing with fine gradings. Since we are working under the hypothesis that the ground field is algebraically closed (and of characteristic zero) we can modify slightly the definition for our convenience. We call a extended Heisenberg algebra H_n^λ over a field $\mathbb{F}$, a Lie algebra such that there is a basis

$$\{z, u, e_1, \hat{e}_1, ..., e_k, \hat{e}_k\}, \tag{7}$$

and the nonvanishing Lie brackets are given by

$$[e_i, \hat{e}_i] = \lambda_i z\,, \quad [u, e_i] = \lambda_i \hat{e}_i \text{ and } [u, \hat{e}_i] = \lambda_i e_i,$$

for any $i = 1, \dots, k$. Under our assumptions on the field, we can replace $e_1, \dots, e_k$ in Equation (6) by $\mathbf{i}e_1, \dots, \mathbf{i}e_k$ (where $\mathbf{i} = \sqrt{-1}$) to check that $H_n(\lambda) = H_n^{\mathbf{i}\lambda}$.

6.2. **Torality and basic examples.** We are now dealing with two fine gradings on H_n^λ which will be relevant to our work. One of them is toral while the other is not.

A fine grading on H_n^λ is obviously provided by our basis in Equation (7):

$$H_n^\lambda = \langle z \rangle \oplus \langle u \rangle \oplus (\oplus_{i=1}^k \langle e_i \rangle) \oplus (\oplus_{i=1}^k \langle \hat{e}_i \rangle).$$

Again it is also a group grading. In order to find G_0 the universal grading group, note that necessarily (we denote $\deg x = g$ when $x \in (H_n^\lambda)_g$) the following assertions about the degrees are verified:

$$\begin{aligned} &\deg e_i + \deg \hat{e}_i = \deg z, \\ &\deg e_i + \deg u = \deg \hat{e}_i, \\ &\deg \hat{e}_i + \deg u = \deg e_i. \end{aligned}$$

Hence $\deg u \in G_0$ has order 1 or 2 and $2 \deg e_i = \deg z + \deg u$, so that $\deg z$ can be chosen with freedom and $2(\deg e_i - \deg \hat{e}_i) = 0$ (providing $k - 1$ more order two elements). The universal grading group G_0 is the free abelian group with generators $\deg u$,

$\deg z$, $\deg e_i$, $\deg \hat{e}_i$ and relations above. It can be computed to be $G_0 = \mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_2^{k-1}$ and the grading is given as follows:

$$
\Gamma_1 : \quad \begin{array}{l}
(H_n^\lambda)_{(2;\bar{1};\bar{0},\dots,\bar{0},\dots,\bar{0})} = \langle z \rangle, \\
(H_n^\lambda)_{(0;\bar{1};\bar{0},\dots,\bar{0},\dots,\bar{0})} = \langle u \rangle, \\
(H_n^\lambda)_{(1;\bar{1};\bar{0},\dots,\bar{1},\dots,\bar{0})} = \langle e_i \rangle \ \ (\bar{1} \text{ in the } i\text{-th slot when } i \neq k), \\
(H_n^\lambda)_{(1;\bar{0};\bar{0},\dots,\bar{1},\dots,\bar{0})} = \langle \hat{e}_i \rangle, \\
(H_n^\lambda)_{(1;\bar{1};\bar{0},\dots,\bar{0},\dots,\bar{0})} = \langle e_k \rangle, \\
(H_n^\lambda)_{(1;\bar{0};\bar{0},\dots,\bar{0},\dots,\bar{0})} = \langle \hat{e}_k \rangle.
\end{array}
$$

The general principle of nontorality, (see §2), implies that this fine grading is nontoral.

Now we will find a toral fine grading. First note that, for

$$
\begin{aligned}
u_i &:= e_i + \hat{e}_i, \\
v_i &:= e_i - \hat{e}_i,
\end{aligned} \tag{8}
$$

the following relations are satisfied:

$$
\begin{aligned}
[u, u_i] &= \lambda_i u_i, \\
[u, v_i] &= -\lambda_i v_i, \\
[u_i, v_i] &= -2\lambda_i z.
\end{aligned}
$$

Proposition 4.

a) *The group of automorphisms of H_n^λ which diagonalizes the basis $B := \{u, z\} \cup \{u_i, v_i \mid i = 1, \dots, k\}$ is a maximal torus. It is given by the linear group whose elements are the matrices* $\mathrm{diag}(1, \gamma, \dots, \alpha_i, \gamma\alpha_i^{-1}, \dots)$.

b) *In any toral grading Γ of H_n^λ the identity element of the grading group is in the support of* Γ.

Proof. Denote by T the group of automorphisms of H_n^λ diagonalizing the above mentioned basis. Let $f \in T$ and write $f(u) = \eta u$, $f(z) = \gamma z$, $f(u_i) = \alpha_i u_i$ and $f(v_i) = \beta_i v_i$ with $\eta, \gamma, \alpha_i, \beta_i \in \mathbb{F}^\times$. Applying f to $[u, u_i] = \lambda_i u_i$ we get $\eta\alpha_i = \alpha_i$, hence $\eta = 1$. Moreover, since $[u_i, v_i] = -2\lambda_i z$, again applying f we get $\alpha_i\beta_i = \gamma$, hence $\beta_i = \gamma\alpha_i^{-1}$. We observe that $T \cong (\mathbb{F}^\times)^{k+1}$ is a torus. To prove its maximality take an automorphism g commuting with each element in T. Then it must preserve the simultaneous eigenspaces relative to the elements in T. Given that $\mathbb{F}$ is algebraically closed, these simultaneous eigenspaces are $\mathbb{F}u$, $\mathbb{F}z$, and all the spaces $\mathbb{F}u_i$ and $\mathbb{F}v_i$. Thus g must diagonalize B and so $g \in T$.

Finally take a toral grading Γ of H_n^λ with grading group G. Consider the associated action $\rho\colon \chi(G) \to \mathrm{Aut}(H_n^\lambda)$ where $\chi(G) := \hom(G, \mathbb{F}^\times)$ is the character group. The torality of Γ implies that $\mathrm{Im}(\rho)$ is contained in a maximal torus of $\mathrm{Aut}(H_n^\lambda)$. Since any two maximal tori are conjugated, we may assume that $\mathrm{Im}(\rho) \subset T$. Hence u is fixed by any element in $\mathrm{Im}(\rho)$ and so it is in the zero homogeneous component $(H_n^\lambda)_0$. □

Thus we obtain a toral fine $\mathbb{Z}^{1+k}$-grading given by

$$
\Gamma_2 : \quad \begin{array}{l}
(H_n^\lambda)_{(0;0,\dots,0)} = \langle u \rangle, \\
(H_n^\lambda)_{(2;0,\dots,0)} = \langle z \rangle, \\
(H_n^\lambda)_{(1;0,\dots,1,\dots,0)} = \langle u_i \rangle, \\
(H_n^\lambda)_{(1;0,\dots,-1,\dots,0)} = \langle v_i \rangle,
\end{array} \tag{9}
$$

if $i = 1, \dots, k$.

Observe that the zero homogeneous component of the grading Γ_1 is zero, so applying the last assertion in the previous proposition we recover the result that this grading is nontoral.

Our first aim is to prove that under suitable conditions on the vector λ, Γ_1 and Γ_2 are the unique fine gradings up to equivalence. For that, it is useful to note a more general fact: u (or at least an element whose behavior is that one of u) is homogeneous in any grading. Let us prove it.

Let $\Gamma\colon L = \oplus_{g\in G} L_g$ be any group grading on $L = H_n^\lambda$. As any automorphism leaves invariant $[L, L] = H_n$ and $\mathcal{Z}(L) = \langle z\rangle$, this implies that z is homogeneous in any grading, and H_n is graded (the homogeneous components of any element in H_n are again elements in H_n). Arguing as in Section 3, there is a basis of homogeneous elements $\{x_1, y_1, \dots, x_k, y_k\}$ of $[u, L]$ such that $[x_i, y_i] = z$ and the remaining brackets are zero. We know (scaling if necessary) that there is a homogeneous element u' such that $u' - u \in [L, L]$. Thus $u' = u + \alpha z + \sum \alpha_i u_i + \sum \beta_i v_i$ for some choice of scalars $\alpha_i, \beta_i \in \mathbb{F}$. If we take $u_i' = u_i + 2\beta_i z$ and $v_i' = v_i + 2\alpha_i z$, then the map $f\colon L \to L$ such that $f(u') = u, f(z) = z, f(u_i') = u_i$ and $f(v_i') = v_i$ is a Lie algebra isomorphism. Consequently, there is not loss of generality in supposing that u is homogeneous.

Let us denote by $h \in G$ the degree of u in Γ. Our next aim is to show that h is necessarily of finite order. From now on we are going to denote by φ be the inner derivation

$$\varphi := \mathrm{ad}(u) : H_n^\lambda \to H_n^\lambda, \qquad x \mapsto [u, x],$$

which is going to be a key tool in the study of the group gradings of H_n^λ. If $0 \neq x \in H_n$ is a homogeneous element, then there is $g \in G$ such that $x = \sum_i (c_i u_i + d_i v_i) \in L_g$ for some scalars $c_i, d_i \in \mathbb{F}$, so that $\varphi^t(x) = \sum_i (c_i u_i + (-1)^t d_i v_i)\lambda_i^t \in L_{g+th}$ is not zero for all $t \in \mathbb{N}$, but at most there are $2k$ independent elements in the set

$$\{\sum_{i=1}^{k} (c_i u_i + (-1)^t d_i v_i)\lambda_i^t : t = 0, 1, 2, \dots\} \subset \langle u_1, v_1, \dots, u_k, v_k\rangle,$$

so that there is $r \leq 2k$ with $\varphi^r(x) \in \langle \varphi^t(x) \mid t < r\rangle$. Let r be the minimum positive integer satisfying such condition. So $\varphi^r(x) \in L_{g+rh} \cap (\sum_{t<r} L_{g+th})$ and necessarily $g + rh = g + th$ for some $t < r$, so that $(r - t)h = 0$, as we wanted to show.

Let us denote by l the order of h in G. Recall that the set of eigenvalues of φ is $\{\lambda_1, -\lambda_1, \dots, \lambda_k, -\lambda_k, 0, 0\}$ with respective eigenvectors $\{u_1, v_1, \dots, u_k, v_k, u, z\}$ given by (8), so that the set of eigenvalues of $\varphi|_{[u,L]}$ is

$$\{\lambda_1, -\lambda_1, \dots, \lambda_k, -\lambda_k\} = \mathrm{Spec}(\mathrm{ad}(u)|_{[u,L]}) =: \mathrm{Spec}(u).$$

Fix some $\lambda_i \in \mathrm{Spec}(u)$, consider the eigenspace of φ given by $V_{\lambda_i} := \{x \in L : \varphi(x) = \lambda_i x\}$ and denote by

$$V_{\lambda_i}^l := \{x \in L : \varphi^l(x) = \lambda_i^l x\}.$$

It is obviously nonzero, because $u_i \in V_{\lambda_i} \subset V_{\lambda_i}^l$. Moreover, as $V_{\lambda_i}^l$ is invariant under φ, we have that $\varphi|_{V_{\lambda_i}^l}$ is diagonalizable and

$$V_{\lambda_i}^l = \bigoplus_{j=0}^{l-1} V_{\xi^j \lambda_i} \tag{10}$$

for ξ any primitive lth root of the unit. Note that if $x \in V_{\lambda_i}^l$, then $\sum_{k=0}^{l-1} (\xi^{-j}\lambda_i^{-1})^k \varphi^k(x) \in V_{\xi^j\lambda_i}$.

Recall that if $f \in \mathrm{End}(L)$ satisfies $f(L_g) \subset L_g$ for all $g \in G$, then for each $\alpha \in \mathbb{F}$, the eigenspace $\{x \in L \mid f(x) = \alpha x\}$ is graded. This can be applied to $V_{\lambda_i}^l$, since $\varphi^l(L_g) \subset L_{g+lh} = L_g$, so that $V_{\lambda_i}^l$ is a graded subspace of L. Thus we can take $0 \neq x \in V_{\lambda_i}^l \cap L_g$ for some $g \in G$. Then $0 \neq \sum_{k=0}^{l-1} (\xi^{l-j}\lambda_i^{-1})^k \varphi^k(x) \in \sum_{k=0}^{l-1} L_{g+kh}$, where the involved homogeneous pieces are different, since $g + kh = g + ph$ implies $(k - p)h = 0$. In particular $V_{\xi^j\lambda_i} \neq 0$ for all j, so that

$$\{\xi^j\lambda_i : j = 0, 1, ..., l-1\} \subset \mathrm{Spec}(u) \tag{11}$$

for any $\lambda_i \in \mathrm{Spec}(u)$, and hence $\mathrm{Spec}(u) = \{\pm\xi^j\lambda_i : j < l, i \leq k\}$.

Proposition 5. *Assume that λ_i/λ_j is not a lth root of unit for any $l \geq 1$, if $i \neq j$. Then the unique fine (group) gradings on H_n^λ (up to equivalence) are Γ_1 and Γ_2. Moreover, the Weyl groups of these fine gradings are $\mathcal{W}(\Gamma_1) \cong \mathbb{Z}_2^k$ and $\mathcal{W}(\Gamma_2) \cong \mathbb{Z}_2$.*

Proof. Let $\Gamma\colon L = \oplus_{g\in G}L_g$ be a grading on $L = H_n^\lambda$. By the above discussion we can suppose u is homogeneous with degree $h \in G$ of finite order l. Let us show that either $l = 1$ or $l = 2$. Otherwise, take ξ a primitive lth root of unit. As $\xi\lambda_1 \in \mathrm{Spec}(u)$ by Equation (11), then there is $1 \neq i \leq k$ such that $\xi\lambda_1 \in \{\pm\lambda_i\}$ and hence either $\left(\frac{\lambda_i}{\lambda_1}\right)^l = 1$ or $\left(\frac{\lambda_i}{\lambda_1}\right)^{2l} = 1$, what is a contradiction. Hence, we can distinguish two cases.

First consider $h = e \in G$. Thus $\varphi(L_g) \subset L_g$ for any g. Restrict $\varphi\colon [u, L] \to [u, L]$. We can take a basis of homogeneous elements which are eigenvectors for φ. Recall that the spectrum of $\varphi|_{[u,L]}$ consists of $\{\pm\lambda_1, \dots, \pm\lambda_k\}$. Take $x_1 \neq 0$ some homogeneous element in V_{λ_1}. As $[x_1, V_{-\lambda_1}] \neq 0$, there is some element $y_1 \in V_{-\lambda_1}$ in the above basis such that $[x_1, y_1] = -2\lambda_1 z$. Now $[u, L] = W \oplus \mathcal{Z}_{[u,L]}(W)$ for $W := \langle x_1, y_1\rangle$, where W as well as its centralizer $\mathcal{Z}_{[u,L]}(W)$ are graded and φ-invariant. We continue by induction until finding a basis of homogeneous elements $\{x_1, y_1, \dots, x_k, y_k\}$ of $[u, L]$ such that $[x_i, y_i] = -2\lambda_i z, [u, x_i] = \varphi(x_i) = \lambda_i x_i$ and $[u, y_i] = \varphi(y_i) = -\lambda_i y_i$. Since

$$L = \langle z\rangle \oplus \langle u\rangle \oplus [u, L],$$

we have that the map $u \mapsto u, z \mapsto z, x_i \mapsto u_i$ and $y_i \mapsto v_i$ is a Lie algebra isomorphism which applies Γ into a coarsening of Γ_2.

Second consider the case when $2h = e$ but $h \neq e$. Thus φ^2 preserves the grading Γ and it is diagonalizable with eigenvalues $\{0, 0, \lambda_1^2, \lambda_1^2, \dots, \lambda_k^2, \lambda_k^2\}$ (counting each with multiplicity 1). Observe that φ applies $\{x \in L : \varphi^2(x) = \lambda_i^2 x\} = V_{\lambda_i}^2$ in itself. Moreover these sets are graded, because φ^2 preserves the grading. For any $0 \neq x_1 \in V_{\lambda_1}^2 \cap L_g$ a homogeneous element of the grading, $\varphi(x_1)$ is independent with x_1 (otherwise $\varphi(x_1) \in L_g \cap L_{g+h}$ but $h \neq e$ and $\varphi(x_1) \neq 0$). Take $y_1 = \frac{1}{\lambda_1}\varphi(x_1)$, which verifies $\varphi(y_1) = \lambda_1 x_1$. Since our ground field is algebraically closed, if $[x_1, y_1] \neq 0$, we can scale to get $[x_1, y_1] = \lambda_1 z$, and now we can continue because, as before, $[u, L] = W \oplus \mathcal{Z}_{[u,L]}(W)$ for $W := \langle x_1, y_1\rangle$, where both W and its centralizer are graded and φ-invariant. The case $[x_1, y_1] = 0$ does not occur under the hypothesis of the theorem, since $\lambda_i^2 \neq \lambda_j^2$ if $i \neq j$, so that $\dim V_{\lambda_i}^2 = 2$ for all i. As $[V_\alpha, V_\beta] = 0$ if $\alpha + \beta \neq 0$, this implies that there is $y \in V_{\lambda_1}^2$ with $[x_1, y] \neq 0$ but $V_{\lambda_1}^2 = \langle x_1, y_1\rangle$. To summarize, if h has order 2 and $\lambda_i/\lambda_j \notin \{\pm 1\}$, we find a basis of homogeneous elements $\{x_1, y_1, \dots, x_k, y_k\}$ of $[u, L]$ such that $[x_i, y_i] = \lambda_i z, [u, x_i] = \lambda_i y_i$ and $[u, y_i] = \lambda_i x_i$, so that the map $u \mapsto u, z \mapsto z, x_i \mapsto e_i$ and $y_i \mapsto \hat{e}_i$ is a Lie algebra isomorphism which applies Γ into a coarsening of Γ_1.

In order to compute the Weyl groups of these fine gradings, recall that any $f \in \mathrm{Aut}(L)$ verifies $0 \neq f(z) \in \langle z\rangle$. If besides $f \in \mathrm{Aut}(\Gamma_1)$, then $f(u) \in \langle u\rangle$. Otherwise, there would exist some $i \leq k$ such that either $f(e_i) \in \langle u\rangle$ or $f(\hat{e}_i) \in \langle u\rangle$, so that $0 \neq f(\lambda_i z) = [f(e_i), f(\hat{e}_i)] \in [u, L] \cap \langle z\rangle = 0$. Consider for each index $i \leq k$, the element in $\mathrm{Aut}(\Gamma_1)$ defined by $\mu_i(e_i) = \mathbf{i}\hat{e}_i, \mu_i(\hat{e}_i) = \mathbf{i}e_i, \mu_i(e_j) = e_j, \mu_i(\hat{e}_j) = \hat{e}_j$ for each $j \neq i$, $\mu_i(z) = z$

and $\mu_i(u) = u$. Note that if $r = 1, ..., k$, there are not any $i, j \leq k$ such that $f(e_i) \in \langle e_r \rangle$ and $f(e_j) \in \langle \hat{e}_r \rangle$. Hence we can compose f with some μ_i's if necessary to obtain that $f' := \mu_{i_1} \cdots \mu_{i_s} f$ satisfies

$$f'(e_i) \in \langle e_1 \rangle \cup \langle e_2 \rangle \cup \cdots \cup \langle e_k \rangle$$

for each $i = 1, ..., k$. Thus, there is $\sigma \in S_k$ such that $f'(z) = \mu z$, $f'(u) = \beta u$, $f'(e_i) = \gamma_i e_{\sigma(i)}$ and $f'(\hat{e}_i) = \gamma'_i \hat{e}_{\sigma(i)}$ for any $i = 1, ..., k$, with $\mu, \beta, \gamma_i, \gamma'_i \in \mathbb{F}^\times$. From here, the equality $f'([u, e_i]) = [f'(u), f'(e_i)]$ implies

$$\gamma'_i \lambda_i = \beta \gamma_i \lambda_{\sigma(i)}, \tag{12}$$

and finally the condition $f'([u, \hat{e}_i]) = [f'(u), f'(\hat{e}_i)]$ allows us to assert

$$\gamma_i \lambda_i = \beta \gamma'_i \lambda_{\sigma(i)}. \tag{13}$$

From Equations (12) and (13) we easily get $\lambda_{\sigma(i)} \in \pm\beta^{-1}\lambda_i$ for any $i = 1, ..., k$. By multiplying, $\Pi_{i=1}^k \lambda_{\sigma(i)} \in \pm\beta^{-k}\Pi_{i=1}^k \lambda_i$ so that $\beta^{2k} = 1$. As $\lambda_{\sigma(i)}/\lambda_i$ is not a root of unit if $\sigma(i) \neq i$, we conclude that $\sigma = \mathrm{id}$, so that $f' \in \mathrm{Stab}(\Gamma_1)$. In other words,

$$\mathcal{W}(\Gamma_1) = \{[\mu_{i_1} \dots \mu_{i_s}] : 1 \leq i_1 \leq \cdots \leq i_s \leq k\} \cong \mathbb{Z}_2^k,$$

since $\mu_i \mu_j = \mu_j \mu_i$.

For the other case, define the automorphism $\mu \in \mathrm{Aut}(\Gamma_2)$ by means of $\mu(u_i) = \mathbf{i} v_i$ and $\mu(v_i) = \mathbf{i} u_i$ for all i, $\mu(z) = z$ and $\mu(u) = -u$. Consider $f \in \mathrm{Aut}(\Gamma_2)$, and note that again there is $\beta \in \mathbb{F}^\times$ such that $f(u) = \beta u$. If $f(u_i)$ is a multiple of either u_j or v_j for some j, this clearly implies that $f(v_i)$ also is, so that there is $\sigma \in S_k$ such that $f(u_i) \in \langle u_{\sigma(i)} \rangle \cup \langle v_{\sigma(i)} \rangle$ for all $i \leq k$. As $\beta[u, f(u_i)] = \lambda_i f(u_i)$, then $\lambda_i \in \{\pm\beta\lambda_{\sigma(i)}\}$, and, as before, β is a root of unit, and, by hypothesis, $\sigma = \mathrm{id}$. By composing with μ if necessary, we can assume that $f(u_1) \in \langle u_1 \rangle$, which implies $\beta = 1$. If $f(u_i) \in \langle v_i \rangle$ for some i, then $\beta = -1$, which is a contradiction, so that $f(u_i) \in \langle u_i \rangle$ for all i and f belongs to $\mathrm{Stab}(\Gamma_2)$. We have then proved that

$$\mathcal{W}(\Gamma_2) = \langle \mu \rangle \cong \mathbb{Z}_2.$$

□

6.3. **Fine gradings on extended Heisenberg algebras.** In the general case (possible roots of the unit among the fractions of λ_i's), the situation is much more involved. On one hand, a lot of different fine gradings arise, and on the other hand even the Weyl groups of Γ_1 and Γ_2 change. There is a lot of symmetry in the related extended Lie algebra, and their fine gradings are also symmetric. In order to figure out what is happening, we previously need to show a couple of key examples.

First, for ξ a primitive lth root of the unit and α a nonzero scalar we consider the extended Heisenberg algebra H_{2l+2}^λ corresponding to

$$\lambda = (\lambda_1, \dots, \lambda_l) = (\xi\alpha, \xi^2\alpha, \dots, \xi^{l-1}\alpha, \alpha).$$

Thus $[u, u_i] = \xi^i \alpha u_i, [u, v_i] = -\xi^i \alpha v_i$ and $[u_i, v_i] = -2\xi^i \alpha z$ for $i = 1, \dots, l$, with the definition of u_i's and v_i's as in Equation (8). Take now

$$\begin{array}{l} x_j = \sum_{i=1}^l \xi^{ji} u_i, \\ y_j = -\frac{1}{2l} \sum_{i=1}^l (-1)^j \xi^{(j-1)i} v_i, \end{array} \tag{14}$$

if $j = 1, ..., l$. These vectors verify $[u, x_j] = \alpha x_{j+1}$ and $[u, y_j] = \alpha y_{j+1}$ for all $j \leq l-1$. Besides $[x_i, y_j] = (-1)^j \frac{\alpha}{l} (\sum_{r=1}^l \xi^{r(i+j)}) z$ is not zero if and only if $i + j = l, 2l$, and in such a case $[x_l, y_l] = (-1)^l \alpha z$ and $[x_i, y_{l-i}] = (-1)^{l-i} \alpha z$ for $i = 1, ..., l-1$. Note that obviously $\{x_1, y_1 \dots, x_l, y_l\}$ is a family of independent vectors such that $[x_i, x_j] = 0 = [y_i, y_j]$ for all i, j.

Therefore we have a fine grading on $L = H_n^\lambda$ over the group

$$G = \mathbb{Z}^2 \times \mathbb{Z}_l,$$

given by

$$\begin{aligned} L_{(0,0,\bar{1})} &= \langle u \rangle, \\ L_{(1,1,\bar{0})} &= \langle z \rangle, \\ L_{(1,0,\bar{i})} &= \langle x_i \rangle, \\ L_{(0,1,\bar{i})} &= \langle y_i \rangle, \end{aligned} \tag{15}$$

for all $i = 1, ..., l$.

Take $\gamma \in \mathbb{F}$ such that $\gamma^l = (-1)^l$, and consider $\theta, \vartheta \in \mathrm{Aut}(L)$ defined by

$$\begin{array}{llll} \theta(x_i) = x_{i+1}, & \theta(y_i) = y_{i-1}, & \theta(z) = -z, & \theta(u) = u; \\ \vartheta(x_i) = \gamma^i y_i, & \vartheta(y_i) = -\gamma^i x_i, & \vartheta(z) = z, & \vartheta(u) = \gamma u; \end{array}$$

where the indices are taken modulo $2l$. It is not difficult to check that the Weyl group of the grading described in Equation (15) is generated by the classes $[\theta]$ and $[\vartheta]$, elements of order l and 2 respectively which do not commute, so that the Weyl group is the Dihedral group D_l.

This example motivates the following definition.

Definition 5. *Let L be any Lie algebra, $z \in L$ a fixed element, u an arbitrary element and $\alpha \in \mathbb{F}^\times$. A set $B_l^I(u, \alpha)$, which will be referred as a block of type I, is given by a family of $2l$ independent elements in L,*

$$B_l^I(u, \alpha) = \{x_1, y_1, \dots, x_l, y_l\},$$

satisfying that the only non-vanishing products among them are the following:

$$\begin{array}{ll} [u, x_i] = \alpha x_{i+1} & \forall i = 1, \dots, l-1, \\ [u, x_l] = \alpha x_1, & \\ [u, y_i] = \alpha y_{i+1} & \forall i = 1, \dots, l-1, \\ [u, y_l] = (-1)^l \alpha y_1, & \\ [x_i, y_{l-i}] = (-1)^{l-i} \alpha z & \forall i = 1, \dots, l-1, \\ [x_l, y_l] = (-1)^l \alpha z. & \end{array}$$

As a second example, fix ξ a primitive $2l$th root of the unit and α a nonzero scalar. Consider now the extended Heisenberg algebra H_{2l+2}^λ corresponding to

$$\lambda = (\lambda_1, \dots, \lambda_l) = (\xi\alpha, \xi^2\alpha, \dots, \xi^{l-1}\alpha, -\alpha).$$

Again $[u, u_i] = \xi^i \alpha u_i$, $[u, v_i] = -\xi^i \alpha v_i$ and $[u_i, v_i] = -2\xi^i \alpha z$ for $i = 1, \dots, l$. Take now

$$x_j = \frac{\mathbf{i}}{2\sqrt{l}} \sum_{i=1}^{l} (u_i + (-1)^{j-1} v_i) \xi^{(j-1)i} \tag{16}$$

for each integer j. Observe that $\{x_1, \dots, x_{2l}\}$ is a family of independent vectors satisfying $[u, x_j] = \alpha x_{j+1}$ for any $j = 1, ..., 2l-1$ and $[u, x_{2l}] = \alpha x_1$. A direct computation gives

$$[x_i, x_j] = \frac{1}{2l} \alpha ((-1)^i + (-1)^{j-1}) (\sum_{k=1}^{l} \xi^{(i+j-1)k}) z \tag{17}$$

for any i and j. If $i+j-1=2l$, then i and $j-1$ are either both odd or both even and $[x_i, x_{2l+1-i}] = (-1)^i \alpha z \neq 0$. Hence,

$$[x_1, x_{2l}] = -[x_2, x_{2l-1}] = \cdots = (-1)^{l-1}[x_l, x_{l+1}].$$

Again Equation (17) tells us that the remaining brackets are zero: if $r = i+j-1$ is odd, then $(-1)^i + (-1)^{j-1} = 0$, and, if r is even (different from 0 and $2l$), then $\sum_{k=1}^{l} \xi^{rk} = 0$ since $(\xi^2)^l = 1$.

We note that this provides a fine grading on H^{λ}_{2l+2} over the group

$$G = \mathbb{Z} \times \mathbb{Z}_{2l},$$

given by:

$$\begin{aligned} L_{(0,\bar{1})} &= \langle u \rangle, \\ L_{(2,\bar{1})} &= \langle z \rangle, \\ L_{(1,\bar{i})} &= \langle x_i \rangle, \end{aligned}$$

for $i = 1, ..., 2l$.

Take $\rho \in \mathrm{Aut}(L)$ defined by

$$\rho(x_i) = x_{l+i}, \quad \rho(z) = (-1)^l z, \quad \rho(u) = u,$$

for all $i = 1, \ldots, 2l$ (mod $2l$). This time the Weyl group of the grading is isomorphic to $\mathbb{Z}_2$, since it is easily proved to be generated by the class $[\rho]$.

This example gives rise to the next concept.

Definition 6. *Let L be any Lie algebra, $z \in L$ a fixed element, $u \in L$ an arbitrary element and $\alpha \in \mathbb{F}^{\times}$. A set $B^{II}_l(u, \alpha)$, which will be referred as a block of type II, is given by a family of $2l$ independent elements in L,*

$$B^{II}_l(u, \alpha) = \{x_1, \ldots, x_{2l}\},$$

satisfying that the only non-vanishing products among them are the following:

$$\begin{aligned} &[u, x_i] = \alpha x_{i+1} && \forall i = 1, \ldots, 2l \,(\mathrm{mod}\, 2l), \\ &[x_i, x_{2l-i+1}] = (-1)^i \alpha z && \forall i = 1, \ldots, 2l. \end{aligned}$$

In fact, all of the fine gradings of a extended Heisenberg algebra can be described with blocks of types I and II, according to the following theorem.

Theorem 3. *Let Γ be a fine G-grading on $L = H^{\lambda}_n$. Let $\langle z \rangle$ be the center of L. Then there exist $u \in L$, positive integers l, r, s such that $l(r+2s) = 2k = n-2$ ($r = 0$ when l is odd) and scalars $\beta_1, \ldots, \beta_s, \alpha_1, \ldots, \alpha_r \in \{\pm\lambda_1, \ldots, \pm\lambda_k\}$ such that*

$$\{z, u\} \cup (\bigcup_{j=1}^{s} B^{I}_l(u, \beta_j)) \cup (\bigcup_{i=1}^{r} B^{II}_{\frac{l}{2}}(u, \alpha_i))$$

is a basis of homogeneous elements of Γ, being zero the bracket of any two elements belonging to different blocks.

Proof. Recall that z is always a homogeneous element and that we can assume that u is also homogeneous of degree $h \in G$, necessarily of finite order. Let $l \in \mathbb{Z}_{\geq 0}$ be the order of h. Take $\varphi = \mathrm{ad}(u)$ and consider again V_{λ_i} and $V^l_{\lambda_i}$. Recall that $V^l_{\lambda_i}$ is a φ-invariant graded subspace for all $\lambda_i \in \mathrm{Spec}(u)$.

Let us discuss first the case that l is odd. Fix any $0 \neq x \in V^l_{\lambda_1} \cap L_g$ for some $g \in G$. Since each $\varphi^i(x) \in L_{g+ih}$ we have that

$$\{x, \varphi(x), ..., \varphi^{l-1}(x)\}$$

is a family of linearly independent elements of L. Now observe that Equation (10), together with the fact l is odd, say that $[V^l_{\lambda_1}, V^l_{\lambda_1}] = 0$ and $[V^l_{\lambda_1}, V^l_{-\lambda_1}] \neq 0$. From here,

$$[\varphi^i(x), \varphi^j(x)] = 0 \text{ for any } i, j = 0, 1, ..., l-1, \tag{18}$$

and we can take a nonzero homogeneous element $0 \neq y \in V^l_{-\lambda_1} \cap L_p$ such that $[x, y] \neq 0$. By scaling if necessary we can suppose $[x, y] = \lambda_1 z$, being then $\deg z = g + p$. As above, we also have that

$$\{y, \varphi(y), ..., \varphi^{l-1}(y)\}$$

is a family of linearly independent elements of L satisfying

$$[\varphi^i(y), \varphi^j(y)] = 0 \text{ for any } i, j = 0, 1, ..., l-1. \tag{19}$$

Taking into account $\varphi^i(x) \in L_{g+ih}$ and $\varphi^j(y) \in L_{p+jh}$, we get that in case $[\varphi^i(x), \varphi^j(y)] \neq 0$, then $g + k + (i+j)h = \deg z = g + k$, which is only possible if $i + j$ is a multiple of l. That is, for each $0 \leq i, j < l$,

$$[\varphi^i(x), \varphi^j(y)] = 0 \text{ if } i + j \neq 0, l. \tag{20}$$

Also note that

$$[\varphi^i(x), \varphi^{l-i}(y)] = (-1)^{l-i}\lambda_1^l[x, y] \neq 0. \tag{21}$$

Indeed, take ξ a primitive lth root of the unit and write $x = \sum_{j=0}^{l-1} a_j$ and $y = \sum_{j=0}^{l-1} b_j$ for $a_j \in V_{\xi^j\lambda_1}$ and $b_j \in V_{-\xi^j\lambda_1}$, taking into consideration Equation (10). Then

$$\begin{aligned}[\varphi^i(x), \varphi^{l-i}(y)] &= \textstyle\sum_{j,k}[(\xi)^{ji}\lambda_1^i a_j, (-\xi^k)^{l-i}\lambda_1^{l-i}b_k] = \\ &= \lambda_1^l(-1)^{l-i}\textstyle\sum_j(\xi)^{ji}(\xi)^{j(l-i)}[a_j, b_j] = (-1)^{l-i}\lambda_1^l[x, y].\end{aligned}$$

Finally note that the family $\{x, \varphi(x), ..., \varphi^{l-1}(x), y, \varphi(y), ..., \varphi^{l-1}(y)\}$ is linearly independent. Indeed, in the opposite case some $\varphi^i(y) = \beta\varphi^j(x)$, $\beta \in \mathbb{F}^\times$, because we are dealing with a family of homogeneous elements, and then $[\varphi^i(y), \varphi^{l-j}(y)] = \beta[\varphi^j(x), \varphi^{l-j}(y)] \neq 0$, what contradicts Equation (19).

Taking into account Equations (18), (19), (20) and (21), we have that

$$\{\frac{\varphi(x)}{\lambda_1}, \frac{\varphi(y)}{\lambda_1}, ..., \frac{\varphi^i(x)}{\lambda_1^i}, \frac{\varphi^i(y)}{\lambda_1^i}, ..., \frac{\varphi^l(x)}{\lambda_1^l} = x, \frac{\varphi^l(y)}{\lambda_1^l} = (-1)^l y\}$$

is a block $B^{\mathrm{I}}_l(u, \lambda_1)$ of type I.

Now $[u, L] = W \oplus \mathcal{Z}_{[u,L]}(W)$ for $W := \langle B^{\mathrm{I}}_l(u, \lambda_1)\rangle$, where W as well as its centralizer are graded and φ-invariant. We continue by iterating this process on $\mathcal{Z}_{[u,L]}(W)$ until finding a basis of $[u, L]$ formed by $s = \frac{k}{l}$ blocks of type I of homogeneous elements.

Now consider the case with l even. If we fix as above the linear subspace $0 \neq V^l_{\lambda_1}$, we have two different cases to distinguish.

Assume first that for any $g \in G$ and any $x \in V^l_{\lambda_1} \cap L_g$ we have $[x, \varphi(x)] = 0$. Fix $0 \neq x \in V^l_{\lambda_1} \cap L_g$ for some $g \in G$, being then $\{x, \varphi(x), ..., \varphi^{l-1}(x)\}$ a family of linearly independent elements of L. By induction on n it is easy to verify, taking into account that φ is a derivation, that for any $i = 0, ..., l-1$, we have $[\varphi^i(x), \varphi^{i+n}(x)] = 0$ for any $n = 1, ..., l$. That is, $[\varphi^i(x), \varphi^j(x)] = 0$ for any $i, j = 0, ..., l-1$.

Since the fact that l is even implies $V^l_{\lambda_1} = V^l_{-\lambda_1}$, we can choose a homogeneous element $0 \neq y \in V^l_{\lambda_1} \cap L_p$, for some $p \in G$, such that $0 \neq [x, y] = \lambda_1 z$. The same arguments that in the odd case say that again

$$\{\frac{\varphi(x)}{\lambda_1}, \frac{\varphi(y)}{\lambda_1}, ..., \frac{\varphi^i(x)}{\lambda_1^i}, \frac{\varphi^i(y)}{\lambda_1^i}, ..., \frac{\varphi^l(x)}{\lambda_1^l}, \frac{\varphi^l(y)}{\lambda_1^l}\}$$

is a block $B_l^{\mathrm{I}}(u,\lambda_1)$ of type I. Now we can write

$$[u,L] = W \oplus \mathcal{Z}_{[u,L]}(W) \tag{22}$$

for $W := \langle B_l^{\mathrm{I}}(u,\lambda_1)\rangle$, where W as well as its centralizer are graded and φ-invariant.

Second, assume that there exist $g \in G$ and $0 \neq x \in V_{\lambda_1}^l \cap L_g$ such that $[x,\varphi(x)] \neq 0$. By scaling if necessary we can assume $[x,\varphi(x)] = \lambda_1^2 z$. We have as above that $\{x,\varphi(x),...,\varphi^{l-1}(x)\}$ is a family of homogeneous linearly independent elements of L satisfying $[\varphi^i(x),\varphi^j(x)] = 0$ if $i+j \neq 1, l+1$ (take into account that $\deg z = 2g+h$). Besides Equation (10) allows us to get in a straightforward way that

$$[\varphi^i(x),\varphi^{l-i+1}(x)] = (-1)^i\lambda_1^l[x,\varphi(x)] \neq 0$$

for any $i = 1,\dots,l$. Thus the set

$$\{\frac{\varphi(x)}{\lambda_1},...,\frac{\varphi^i(x)}{\lambda_1^i},...,\frac{\varphi^l(x)}{\lambda_1^l} = x\}$$

is a block $B_{\frac{l}{2}}^{\mathrm{II}}(u,\lambda_1)$ of type II. Now we can write

$$[u,L] = W \oplus \mathcal{Z}_{[u,L]}(W) \tag{23}$$

for W the vector space spanned by the above block $B_{\frac{l}{2}}^{\mathrm{II}}(u,\lambda_1)$, where W as well as its centralizer $\mathcal{Z}_{[u,L]}(W)$ are graded and φ-invariant.

Taking into account Equations (22) and (23), we can iterate this process on $\mathcal{Z}_{[u,L]}(W)$ until finding the required basis of $[u,L]$ formed by s blocks of type I and r blocks of type II. $\square$

Remark. Note that our gradings Γ_1 and Γ_2 should be particular cases of the above situation. Indeed, Γ_1 corresponds to the situation $l=2$, $s=0$ and $r=k$, and Γ_2 corresponds to the situation $l=1$, $s=k$ and $r=0$. Take into consideration that $\{u_i,v_i\} = B_1^{\mathrm{I}}(u,\lambda_i)$ and that $\{e_i,\hat{e}_i\} = B_1^{\mathrm{II}}(u,\lambda_i)$.

Note that this implies that, for ξ a primitive lth root of the unit,

$$\mathrm{Spec}(u) = \tag{24}$$
$$\{\xi^t\alpha_i : i=1,\dots,r, t=0,\dots,l-1\} \cup \{\xi^t\beta_j, -\xi^t\beta_j : j=1,\dots,s, t=0,\dots,l-1\}.$$

Conversely we have:

Theorem 4. *Assume there are $l,r,s \in \mathbb{Z}_{\geq 0}$ ($l>0$) such that $l(r+2s) = 2k = n-2$ ($\neq 0$) and scalars $\alpha_1,\dots,\alpha_r,\beta_1,\dots,\beta_s \in \{\pm\lambda_1,\dots,\pm\lambda_k\}$ such that (24) holds, with l even if $r \neq 0$. Then there exists a (nontoral) grading on $H_n^{(\lambda_1,\dots,\lambda_k)}$ over the group*

$$\begin{array}{ll} \mathbb{Z}^{s+1} \times \mathbb{Z}_2^{r-1} \times \mathbb{Z}_l & \text{if } r \neq 0, \\ \mathbb{Z}^{s+1} \times \mathbb{Z}_l & \text{if } r = 0. \end{array}$$

Proof. Note that $H_n^{(\lambda_1,\dots,\lambda_k)}$ is isomorphic to $H_n^{(-\lambda_1,\lambda_2,\dots,\lambda_k)}$ by means of the automorphism which interchanges u_1 with v_1. Of course it is also isomorphic to $H_n^{(\lambda_{\sigma(1)},\dots,\lambda_{\sigma(k)})}$ if σ is a permutation of $\{1,\dots,k\}$. Now, by considering blocks as in Equations (14) and (16), the fine gradings determined by Theorem 3 exist and can be combined to get the ones of the theorem.

If we take $\{x_1^j, y_1^j, \ldots, x_l^j, y_l^j\}$ a block $B_l^{\mathrm{I}}(u,\beta_j)$ if $j \le s$, and $\{a_1^t, \ldots, a_l^t\}$ a block $B_{\frac{l}{2}}^{\mathrm{II}}(u,\alpha_t)$ if $t \le r$, then the grading is given by

$$\begin{aligned}
&\deg(x_i^j) = (\overline{i+1};\, 0,\ldots,1,\ldots,0;\, 1;\, \bar 0,\ldots,\bar 0) \in \mathbb{Z}_l \times \mathbb{Z}^s \times \mathbb{Z} \times \mathbb{Z}_2^{r-1} \text{ (1 in the } j\text{th slot)},\\
&\deg(y_i^j) = (\bar i;\, 0,\ldots,-1,\ldots,0;\, 1;\, \bar 0,\ldots,\bar 0),\\
&\deg(z) = (\bar 1;\, 0,\ldots,0;\, 2;\, \bar 0,\ldots,\bar 0),\\
&\deg(u) = (\bar 1;\, 0,\ldots,0;\, 0;\, \bar 0,\ldots,\bar 0),\\
&\deg(a_i^t) = (\bar i;\, 0,\ldots,0;\, 1;\, \bar 0,\ldots,\bar 1,\ldots,\bar 0) \qquad (\bar 1 \text{ in the } t\text{th slot if } t \ne r),\\
&\deg(a_i^r) = (\bar i;\, 0,\ldots,0;\, 1;\, \bar 0,\ldots,\bar 0).
\end{aligned}$$

□

In practice, when one wants to know how many gradings are in a particular extended Heisenberg algebra H_n^λ, it is enough to see how many ways are of splitting $\{\pm\lambda_1,\ldots,\pm\lambda_k\}$ in the way described in Equation (24).

Example 1. Let us compute how many fine gradings are there in $L = H_{10}^{(1,1,\mathbf{i},\mathbf{i})}$. As l must divide 8, the possibilities are $l = 1$, $l = 2$ with $(r,s) = (4,0),(2,1),(0,2)$ and $l = 4$ with $(r,s) = (2,0),(0,1)$. Thus we have seven fine gradings over the groups $\mathbb{Z}^5$, $\mathbb{Z}\times\mathbb{Z}_2^4$, $\mathbb{Z}^2\times\mathbb{Z}_2^2$ (two of these ones, with bases of homogeneous elements $\{z,u\} \cup B_1^{\mathrm{II}}(u,1) \cup B_1^{\mathrm{II}}(u,1) \cup B_2^{\mathrm{I}}(u,\mathbf{i})$ and $\{z,u\}\cup B_1^{\mathrm{II}}(u,\mathbf{i}) \cup B_1^{\mathrm{II}}(u,\mathbf{i}) \cup B_2^{\mathrm{I}}(u,1)$ respectively), $\mathbb{Z}^3\times\mathbb{Z}_2$, $\mathbb{Z}\times\mathbb{Z}_4\times\mathbb{Z}_2$ and $\mathbb{Z}^2\times\mathbb{Z}_4$, all of them inequivalent with all the components of dimension 1. The possibility $l = 8$ does not happen taking into account that $\pm\frac{\lambda_i}{\lambda_j}$ is never a primitive eighth root of the unit when $\lambda_i,\lambda_j \in \mathrm{Spec}(u)$.

For computing the Weyl groups of the fine gradings described in the above theorems, we need to make some considerations.

Lemma 4. *Let L be a Lie algebra, $z \in \mathcal{Z}(L)$, $u \in L$, $\alpha,\beta \in \mathbb{F}^\times$ such that L contains blocks of types $B_l^\nu(u,\alpha)$ and $B_l^\nu(u,\beta)$ for some $\nu \in \{I,II\}$. Then*

i) $\langle B_l^I(u,\alpha)\rangle = \langle B_l^I(u,\beta)\rangle$ *if and only if* $\left(\frac{\alpha}{\beta}\right)^l = 1$ *if l is even and* $\left(\frac{\alpha}{\beta}\right)^{2l} = 1$ *if l is odd.*

ii) $\langle B_l^{II}(u,\alpha)\rangle = \langle B_l^{II}(u,\beta)\rangle$ *if and only if* $\left(\frac{\alpha}{\beta}\right)^l = 1$.

Proof. As usual, denote $\varphi = \mathrm{ad}(u)$ and ξ a primitive lth root of the unit. For i), take $B_l^{\mathrm{I}}(u,\alpha) = \{x_1,y_1,\ldots,x_l,y_l\}$ and V the vector space spanned by these elements. Note that φ^l diagonalizes V with eigenvalues α^l and $(-1)^l\alpha^l$ (eigenvectors x_i's and y_i's respectively). Thus $\alpha^l = \beta^l$ or $\alpha^l = (-1)^l\beta^l$.

Conversely, let us see that there is a block of type $B_l^{\mathrm{I}}(u,\xi\alpha)$ contained in $\langle B_l^{\mathrm{I}}(u,\alpha)\rangle$. Indeed, take γ such that $\gamma^2 = \xi$. The elements $x_i' := \gamma^{1-2i}x_i$ and $y_i' := \gamma^{1-2i}y_i$ constitute the required block. Moreover, if we take δ such that $\delta^4 = \xi$, then the elements $x_i' := (-1)^i\delta^{1-2i}y_i$ and $y_i' := (-1)^i\delta^{1-2i}x_i$ constitute a block of type $B_l^{\mathrm{I}}(u,\delta^2\alpha)$ if l is odd.

The case ii) is proved with similar arguments. □

These arguments make convenient to consider the equivalence relation in $\mathbb{F}^\times$ given by: α is related to β if and only if $(\frac{\alpha}{\beta})^l = 1$. The equivalence class of the element α will be denoted by $\bar\alpha := \{\alpha\xi^t : t = 0,\ldots,l-1\}$.

Assume we have fixed a $\mathbb{Z}^{s+1}\times\mathbb{Z}_2^{r-1}\times\mathbb{Z}_l$-grading Γ on $H_n^{(\lambda_1,\ldots,\lambda_k)}$ given by blocks $B_l^{\mathrm{I}}(u,\beta_j) = \{x_1^j,y_1^j,\ldots,x_l^j,y_l^j\}$ if $j \le s$, and blocks $B_{\frac{l}{2}}^{\mathrm{II}}(u,\alpha_i) = \{a_1^i,\ldots,a_l^i\}$ if $i \le r$ ($r \ne 0$), for some $\beta_1,\ldots,\beta_s,\alpha_1,\ldots,\alpha_r \in \{\pm\lambda_1,\ldots,\pm\lambda_k\}$. We have some remarkable elements in the group of automorphisms of the grading, fixing all the spaces spanned by the blocks. For each $j \leq s$, consider $\theta_j \in \mathrm{Aut}(\Gamma)$ leaving invariant $\langle\{z, u, x_i^t, y_i^t, a_i^p : i \leq l, p \leq r, t \leq s, t \neq j\}\rangle$ and such that

$$\theta_j(x_i^j) = \mathbf{i}x_{i+1}^j, \qquad \theta_j(y_i^j) = \mathbf{i}y_{i-1}^j,$$

(indices taken modulo l). As l is necessarily even, we can also consider $\vartheta_j \in \mathrm{Aut}(\Gamma)$ leaving invariant $\langle\{z, u, x_i^t, y_i^t, a_i^p : i \leq l, p \leq r, t \leq s, t \neq j\}\rangle$ and such that

$$\vartheta_j(x_i^j) = y_i^j, \qquad \vartheta_j(y_i^j) = -x_i^j.$$

Finally consider for each $t \leq r$ the automorphism $\varrho_t \in \mathrm{Aut}(\Gamma)$ leaving invariant the subspace $\langle\{z, u, x_i^j, y_i^j, a_i^p : i \leq l, p \leq r, j \leq s, p \neq t\}\rangle$ and such that

$$\varrho_t(a_i^t) = a_{\frac{l}{2}+i}^t$$

for all $i \leq l$ (sum modulo l).

We observe too that the following automorphisms stabilize the grading: for each $\alpha \in \mathbb{F}^\times$, the automorphism given by

$$z \mapsto \alpha^2 z,\ u \mapsto u,\ x_i^j \mapsto \alpha x_i^j,\ y_i^j \mapsto \alpha y_i^j,\ a_i^p \mapsto \alpha a_i^p$$

and for each $t = 0, \dots, l-1$, the automorphism given by

$$z \mapsto \xi^t z,\ u \mapsto \xi^t u,\ x_i^j \mapsto (\xi^t)^i x_i^j,\ y_i^j \mapsto (\xi^t)^{i+1} y_i^j,\ a_i^p \mapsto (\xi^t)^i a_i^p.$$

Thus any $f \in \mathrm{Aut}(\Gamma)$ can be assumed to satisfy $f(z) = z$ and $f(u) = \varepsilon u$ for some $\varepsilon \in \mathbb{F}^\times$, which can be replaced by any $\varepsilon\xi^t$. Notice that $f(B_l^{\mathrm{I}}(u, \beta)) = B_l^{\mathrm{I}}(f(u), \beta)$ for any automorphism, so, in this case $f(B_l^{\mathrm{I}}(u, \beta_j)) = B_l^{\mathrm{I}}(u, \beta_j/\varepsilon)$. This implies that β_j/ε is related with some element in $\{\beta_1, \dots, \beta_s\}$. The same happens with the blocks of type II, so that there are $\sigma \in S_r$ and $\eta \in S_s$ such that

$$\varepsilon\bar{\alpha}_i = \bar{\alpha}_{\sigma(i)},\ \varepsilon\bar{\beta}_j = \bar{\beta}_{\eta(j)}$$

for all $i \leq r$ and $j \leq s$. In particular $\bar{\varepsilon}^r = \bar{1} = \bar{\varepsilon}^s$, so ε is a root of unit. This root is a pth primitive root for some p. This also implies that there are sets X of Y of classes such that $\{\bar{\alpha}_1, \dots, \bar{\alpha}_r\} = \cup_{t=0}^{p-1}\varepsilon^t X$ (disjoint union) and $\{\bar{\beta}_1, \dots, \bar{\beta}_s\} = \cup_{t=0}^{p-1}\varepsilon^t Y$, chosen in such a way that $X \cap \varepsilon^j X$ is either $\emptyset$ or X for any j, and in the same way $Y \cap \varepsilon^j Y$ is either $\emptyset$ or Y. Now reorder the scalars of the blocks such that $\bar{\alpha}_{t\frac{r}{p}+i} = \varepsilon^t\bar{\alpha}_i$ and $\bar{\beta}_{t\frac{s}{p}+j} = \varepsilon^t\bar{\beta}_j$ for all $i = 1, \dots, \frac{r}{p}$, $t = 0, \dots, p-1$ and $j = 1, \dots, \frac{s}{p}$. Moreover we can suppose, by Lemma 4, that just $\alpha_{t\frac{r}{p}+i} = \varepsilon^t\alpha_i$ and $\beta_{t\frac{s}{p}+j} = \varepsilon^t\beta_j$, by replacing the chosen vectors in the blocks (without changing the grading, since changes were produced only when replacing β with $-\beta$ in the case l odd). Now the map g_p given by

$$z \mapsto z,\ u \mapsto \varepsilon^{-1}u,\ x_i^j \mapsto x_i^{j+\frac{s}{p}},\ y_i^j \mapsto y_i^{j+\frac{s}{p}},\ a_i^t \mapsto a_i^{t+\frac{r}{p}},$$

if $t \leq r$, $i \leq l$, $j \leq s$, is an order p automorphism of the grading such that $g_p f$ applies u into itself. Applying besides Lemma 4, we can suppose that our $\sigma \in S_r$ and $\eta \in S_s$ verify that $\alpha_i = \alpha_{\sigma(i)}$ and $\beta_j = \beta_{\eta(j)}$ for all $i \leq r$ and $j \leq s$. If we reorder the set $\{\alpha_1, \dots, \alpha_r\} = \{\gamma_1, \dots, \gamma_1, \dots, \gamma_{r'}, \dots, \gamma_{r'}\}$ with $\bar{\gamma}_i \neq \bar{\gamma}_j$ if $i \neq j$, each γ_i repeated n_i times ($n_1 + \cdots + n_{r'} = r$), and the set $\{\beta_1, \dots, \beta_s\} = \{\delta_1, \dots, \delta_1, \dots, \delta_{s'}, \dots, \delta_{s'}\}$ with $\bar{\delta}_i \neq \bar{\delta}_j$ if $i \neq j$, each δ_i repeated m_i times, then σ leaves invariant the subsets $\{1, \dots, n_1\}$, $\{n_1+1, \dots, n_1+n_2\}$ and so on, thus it determines $\sigma_1 \in S_{n_1}, \dots, \sigma_{r'} \in S_{n_{r'}}$ and also η determines an element in $S_{m_1} \times \cdots \times S_{m_{s'}}$. Conversely each $\sigma = (\sigma_1, \dots, \sigma_{r'}) \in$

$S_{n_1}\times\cdots\times S_{n_{r'}}$ (that is, $\sigma(n_1+\cdots+n_j+t)=n_1+\cdots+n_j+\sigma_{j+1}(t)$ if $1\le t\le n_{j+1}$) and $\eta\in S_{m_1}\times\cdots\times S_{m_{s'}}$ allow to define the automorphism $\Upsilon_{(\eta,\sigma)}$ by

$$z\mapsto z,\ u\mapsto u,\ x_i^j\mapsto x_i^{\eta(j)},\ y_i^j\mapsto y_i^{\eta(j)},\ a_i^t\mapsto a_i^{\sigma(t)},$$

and if we compose it with f, the new automorphism fixes all the blocks globally. It is not hard to prove that then $[f]$ belongs to the group generated by

$$\{[\theta_j],[\vartheta_j]:j\le s\}\cup\{[\varrho_t]:t\le r\},$$

which is isomorphic to $D_l^s\times\mathbb{Z}_2^r$.

To arrive at a conclusion, note that there are always $p\in\mathbb{N}$ and sets X and Y of classes such that $\{\bar\alpha_1,\dots,\bar\alpha_r\}=\cup_{t=0}^{p-1}\varepsilon^tX$ and $\{\bar\beta_1,\dots,\bar\beta_s\}=\cup_{t=0}^{p-1}\varepsilon^tY$, for ε a primitive pth root of 1, and such that $X\cap\varepsilon^jX$ is either the empty set or X and the same happens to Y (for instance take $p=1$). We choose the maximum p with this property (the existence of X and Y). Let t be the minimum positive integer such that $\bar\varepsilon^t=\bar 1$. Clearly t is a divisor of p. If we reorder the set $\{\bar\alpha_1,\dots,\bar\alpha_r\}=\{\bar\gamma_1,\dots,\bar\gamma_1,\dots,\bar\gamma_{r'},\dots,\bar\gamma_{r'}\}$ with $\bar\gamma_i\neq\bar\gamma_j$ if $i\neq j$, each γ_i repeated n_i times ($n_1+\cdots+n_{r'}=r$), and the set $\{\bar\beta_1,\dots,\bar\beta_s\}=\{\bar\delta_1,\dots,\bar\delta_1,\dots,\bar\delta_{s'},\dots,\bar\delta_{s'}\}$ with $\bar\delta_i\neq\bar\delta_j$ if $i\neq j$, each δ_i repeated m_i times, we obtain that the Weyl group of Γ is isomorphic to

$$\frac{\big(S_{n_1}\times\cdots\times S_{n_{r'}}\times S_{m_1}\times\cdots\times S_{m_{s'}}\times\mathbb{Z}_2^r\ltimes D_l^s\big)\rtimes\mathbb{Z}_p}{\mathbb{Z}_{p/t}}. \tag{25}$$

The arguments are the following. First, if $f\in\mathrm{Aut}(\Gamma)$, then $f(u)=\delta u$ for some δ primitive p'th root, and applying the above arguments to fg_p^{-1} (which applies u into $\varepsilon\delta u$), by maximality of p we get $\mathrm{mcm}(p,p')=p$ and δ is a power of ε, so that we can compose f with a power of g_p to get $f(u)=u$. Second, if $f\in\mathrm{Aut}(\Gamma)$ with $f(u)=u$, then $[f]$ belongs to the group generated by

$$\{[\Upsilon_{(\eta,\sigma)}],[\theta_j],[\vartheta_j],[\varrho_t]:j\le s,t\le r\} \tag{26}$$

with η and σ permutations as above. Third, note that $[g_p]$ is an order p element in the Weyl group such that the order $\frac{p}{t}$ element $[g_p]^t$ belongs to the group generated by the set described in Equation (26), since $g_p^t(u)=\varepsilon^tu$ so that the composition of g_p^t with certain element in the stabilizer applies u into itself (ε^t is a power of ξ).

The formula (25) and the related arguments work if l is even, still in case $r=0$.

If l is odd, there are no blocks of type II. In this case it is more convenient to consider α related to β when $\left(\frac{\alpha}{\beta}\right)^{2l}=1$, and the classes of this relation as $\tilde\alpha=\{\alpha\zeta^t:t=0,\dots,2l-1\}$ for ζ primitive $2l$th root. As above, if $f\in\mathrm{Aut}(\Gamma)$ verifies $f(u)=\varepsilon u$, then $f(B_l^{\mathrm{I}}(u,\beta_j))=B_l^{\mathrm{I}}(u,\beta_j/\varepsilon)$. Thus there is $\eta\in S_s$ such that

$$\varepsilon\tilde\beta_j=\tilde\beta_{\eta(j)}$$

for all $j=1,\dots,s$, and ε is a pth root of the unit. So we can divide $\{\tilde\beta_1,\dots,\tilde\beta_s\}=\cup_{t=0}^{p-1}\varepsilon^tY$, with p maximum verifying this property, and take g_p as in the previous case. The maps ϑ_j are not longer automorphisms, but we can consider $\vartheta'\in\mathrm{Aut}(\Gamma)$ given by

$$\vartheta'(z)=z,\ \vartheta'(u)=-u,\ \vartheta'(x_i^j)=(-1)^iy_i^j,\ \vartheta'(y_i^j)=(-1)^{i+1}x_i^j.$$

It is straightforward to get the conclusion that, if $f\in\mathrm{Aut}(\Gamma)$ is an automorphism fixing the subspaces spanned by the blocks, then f belongs to the subgroup generated by

$$\{[\theta_j]:j\le s\}\cup\{[\vartheta']\},$$

which is isomorphic to $\mathbb{Z}_l^s \rtimes \mathbb{Z}_2$, and hence

$$\mathcal{W}(\Gamma) \cong \frac{\left(S_{m_1} \times \cdots \times S_{m_{s'}} \times \mathbb{Z}_2 \ltimes \mathbb{Z}_l^s\right) \rtimes \mathbb{Z}_p}{\mathbb{Z}_{p/t}}. \tag{27}$$

with $(m_1 + \cdots + m_{s'})p = s = \frac{n-2}{2l}$, for m_j's defined as above, and t minimum such that $\tilde{\varepsilon}^t = \tilde{1}$.

Proposition 6. *The Weyl groups of the fine gradings described in Theorem 4 are those in Equations (25) for l even and (27) for l odd.*

Example 2. We now describe the Weyl groups of the fine gradings on $L = H_{10}^{(1,1,\mathbf{i},\mathbf{i})}$ computed in Example 1. For $l = 1$, our grading is Γ_2, and the Weyl group is $\frac{\mathbb{Z}_2^3 \rtimes \mathbb{Z}_4}{\mathbb{Z}_2 = \langle(\bar{1},\bar{1},\bar{1},\bar{2})\rangle}$. Indeed, $\mathcal{W}(\Gamma_2)$ has 4 generators: $[g] \equiv [g_4]$ (the only element with order 4), $[\vartheta']$ and the classes of the two automorphisms f_1 and f_2 coming from permutations, such that the three latter ones commute, $[g]$ commute with $[\vartheta']$, $[gf_1g^{-1}] = [f_2]$ and $[g]^2 = [\vartheta' f_1 f_2]$.

For $(l, r, s) = (2, 4, 0)$, the grading is Γ_1 and now the generators of the Weyl group are $\{[\varrho_i] : i = 1, \ldots, 4\}$, $[g] \equiv [g_4]$, the automorphism interchanging e_1 with e_2 and $\hat{e}_1$ with $\hat{e}_2$ and the one interchanging e_3 with e_4 and $\hat{e}_3$ with $\hat{e}_4$. Hence $\mathcal{W}(\Gamma_1) \cong \frac{\mathbb{Z}_2^6 \rtimes \mathbb{Z}_4}{\mathbb{Z}_2}$. Observe that the results for $\mathcal{W}(\Gamma_1)$ and $\mathcal{W}(\Gamma_2)$ are quite different than those in Proposition 5.

The remaining cases of Example 1 correspond, respectively, to Weyl groups isomorphic to $\mathbb{Z}_2^3 \ltimes \mathbb{Z}_2$, $\mathbb{Z}_2^4$, $\mathbb{Z}_2^2$ and D_4.

Acknowledgment. The authors would like to thank the referee for his exhaustive and deep review of the paper as well as for many suggestions which have helped to improve the work.

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