A NEW MULTICOMPONENT POINCARÉ-BECKNER INEQUALITY

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Abstract. We prove a new vectorial functional inequality of Poincaré-Beckner type. The inequality may be interpreted as an entropy-entropy production one for a gradient flow in the metric space of Radon measures. The proof uses subtle analysis of combinations of related super- and sub-level sets employing the coarea formula and the relative isoperimetric inequality.

Keywords: Poincaré inequalities, coarea formula, optimal transport, gradient flow

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1. Introduction

Let $\Omega \subset \mathbb{R}^d$ be a bounded, connected, open domain. Fix a vector function $\mathbf{m} \in C^1(\overline{\Omega}; \mathbb{R}^N)$ and a matrix function $A \in C^1(\overline{\Omega}; M_N(\mathbb{R}))$. In this paper, we contemplate the inequality

$$\int_\Omega \sum_{i=1}^N |f_i|^p \,\mathrm{d}x \le C \int_\Omega \sum_{i=1}^N u_i(|f_i|^p + |\nabla f_i|^p)\,\mathrm{d}x, \tag{1}$$

where $\mathbf{u} \in W^1_p(\Omega; \mathbb{R}^N)$ is a vector function with non-negative components $u_i \ge 0$, and $\mathbf{f} = \mathbf{f}(\mathbf{u}) = \mathbf{m} - A\mathbf{u}$.

A quick glimpse suggests that (1) is trivial when all components of $\mathbf{u}$ are bounded away from zero. On the other hand, given an index set $I \subset \{1, \dots, N\}$ and a solution $\mathbf{u}$ to the linear system

$$\begin{cases} u_i = 0 & (i \in I) \\ f_j = 0 & (j \notin I), \end{cases} \tag{2}$$

inequality (1) is clearly violated unless $I = \varnothing$. Under suitable structural assumptions on A and $\mathbf{m}$ (roughly speaking, we need that (2) has a unique non-negative solution $\mathbf{u}_I$ for any I), we will show that it is enough for a function $\mathbf{u}$ to stay away from the solutions to (2) with $I \ne \varnothing$ in order to comply with inequality (1). The only solution of (2) compatible with the inequality is thus

$$\mathbf{u}^\infty := \mathbf{u}_\varnothing = A^{-1}\mathbf{m}.$$

In the case $N = 1$ the only solution of (2) with $I \ne \varnothing$ is $\mathbf{u} \equiv 0$, thus C in (1) is expected to blow up only as $\mathbf{u} = u_1$ approaches zero in some sense. Indeed, we have recently proved in [8] that C can be chosen in the form $1/\Phi(\int_\Omega u_1)$, where Φ is a strictly increasing continuous function with $\Phi(0) = 0$ (provided $N = 1$, $\mathbf{m} = m_1(x) > 0$ and $A \equiv 1$). The proof in [8] uses a generalized Beckner inequality [1, Lemma 4], that is why we refer to (1) as Poincaré-Beckner inequality. However, that proof completely fails in the multicomponent case due to implicit cross-diffusion effects.

Our interest to (1) comes from the fact that in the case of symmetric positive-definite matrix $A(x)$ and $p = 2$ inequality (1) is equivalent to an entropy-entropy production inequality corresponding to the gradient flow of the geodesically non-convex entropy functional

$$\mathcal{E}(\mathbf{u}) = \frac{1}{2}\int_\Omega A(\mathbf{u} - \mathbf{u}^\infty) \cdot (\mathbf{u} - \mathbf{u}^\infty) \tag{3}$$

on the space of N-dimensional non-negative Radon measures on Ω equipped with the unbalanced optimal transport distance and induced Riemannian structure as recently introduced in [8] (see also [9, 3, 10, 2, 6]). This gradient flow coincides with a fitness-driven PDE system of population dynamics involving degenerate cross-diffusion. Inequality (1) implies exponential convergence of the trajectories of this gradient flow to the coexistence steady-state $\mathbf{u}^\infty$ which corresponds to the so-called ideal free distribution [5, 4] of the populations. We refer to our companion paper [7] for the details of this interpretation of (1) and its implications.

The proof of (1) which we carry out in this paper is non-standard, being based on a subtle analysis of suitable unions of super-level sets of the components of $\mathbf{f}$ employing the coarea formula and the relative isoperimetric inequality. Assuming that there exists a sequence violating the inequality, either we conclude that it converges one of the degenerate states $\mathbf{u}_I$, or we can detect a drop of f_i that can be exploited to estimate the total variation of f_i by means of the coarea formula. To apply this consideration to the term

$$\int_\Omega \sum_{i=1}^N u_i |\nabla f_i|^p \, \mathrm{d}x,$$

we must consider the variation of f_i over the region where u_i is not small. However, due to the hidden cross-diffusion nature of the problem, this produces "holes" in the level sets of f_i, and we cannot use the relative isoperimetric inequality to estimate the perimeter of the super-level sets. We patch the holes by merging certain super- and sub-level sets of different f_i. Since we argue by contradiction, we are not able to quantify the constant C in (1).

The paper is organized as follows. In Section 2, we give our structural conditions on A and $\mathbf{m}$, and present the main results. In Section 3, we state some algebraic and analytical properties of $\mathbf{f}(\mathbf{u})$ and related nonlinear functions whose proofs may be found in the Appendix. In Section 4.1, we derive the main estimates for the sequences allegedly violating (1). In Section 4.2, we identify three possible scenarios which are determined by behavior of suitable combinations of super- and sub-level sets of f_i. The first alternative leads to the convergence to $\mathbf{u}_I$ (Section 4.3). The second and the third are the most involved ones, and employ the geometric ideas described above, see Sections 4.4 and 4.5.

2. The main results

Let $\Omega \subset \mathbb{R}^d$ be a bounded, connected, open domain. We assume that it admits the relative isoperimetric inequality, cf. [11, Remark 12.39]:

$$P(A;\Omega) \geq c_\Omega |A|^{\frac{d-1}{d}}, \quad A \subset \Omega,\ |A| \leq \frac{1}{2}|\Omega|. \tag{4}$$

Here $P(A;\Omega)$ denotes the relative perimeter of a Lebesgue measurable A of locally finite perimeter with respect to Ω.

Suppose we are given a vector function $\mathbf{m} = (m_1, \dots, m_N) \in C^1(\overline{\Omega}; \mathbb{R}^N)$ and a matrix function $A = (a_{ij}) \in C^1(\overline{\Omega}; M_N(\mathbb{R}))$. We assume that there exists $\kappa > 0$ independent of $x \in \overline{\Omega}$ such that

Assumption 1. *We have pointwise*

$$|a_{ij}| \leq \frac{1}{\kappa} \quad (i,j = 1,\dots,N;\ x \in \overline{\Omega}), \tag{5}$$

$$m_i \leq \frac{1}{\kappa} \quad (i = 1,\dots,N;\ x \in \overline{\Omega}). \tag{6}$$

Assumption 2. *For any $I = \{i_1,\dots,i_r\} \subset \{1,\dots,N\}$, $i_1 < \dots < i_r$, we have*

$$\begin{vmatrix} a_{i_1 i_1} & \cdots & a_{i_1 i_r} \\ \vdots & \ddots & \vdots \\ a_{i_r i_1} & \cdots & a_{i_r i_r} \end{vmatrix} \geq \kappa. \tag{7}$$

Assumption 3. *For any $I = \{i_1,\dots,i_r\} \subset \{1,\dots,N\}$, $i_1 < \dots < i_r$, and $j \notin I$ we have*

$$\begin{vmatrix} a_{i_1 i_1} & \cdots & a_{i_1 i_r} & m_{i_1} \\ \vdots & \ddots & \vdots & \vdots \\ a_{i_r i_1} & \cdots & a_{i_r i_r} & m_{i_r} \\ a_{j i_1} & \cdots & a_{j i_r} & m_j \end{vmatrix} \geq \kappa. \tag{8}$$

Remark 1. Letting $I = \varnothing$ in (8), we see that all the functions m_j are necessarily positive.

Remark 2. Assumptions 2 and 3 allow for a geometrical interpretation, see Section 3.

Remark 3. For a symmetric matrix A, Assumption 2 is equivalent to uniform positive definiteness. However, we do not assume A to be necessarily symmetric.

Given a vector function $\mathbf{u} = (u_1,\dots,u_N)\colon \overline{\Omega} \to \mathbb{R}^N$, set

$$f_i = m_i - \sum_{i=1}^{N} a_{ij} u_j \colon \overline{\Omega} \to \mathbb{R}.$$

Theorem 1. *Suppose that A and $\mathbf{m}$ satisfy Assumptions 1–3 and let $p \geq 1$ and $\mathbf{U} \subset W_p^1(\Omega;\mathbb{R}^N)$ be a set of functions such that*

(i) $\mathbf{u} \geq 0$ for any $\mathbf{u} \in \mathbf{U}$;

(ii) no bounded with respect to the L^p norm sequence $\{\mathbf{u}_n = (u_{1n}, \dots, u_{Nn})\} \subset \mathbf{U}$ admits a nonempty index set $I \subset \{1, \dots, N\}$ so that

$$u_{in} \xrightarrow[n\to\infty]{} 0 \quad (i \in I) \quad \textit{in measure} \tag{9}$$

$$f_{kn} \xrightarrow[n\to\infty]{} 0 \quad (k \notin I) \quad \textit{in measure} \tag{10}$$

Then there exists $C > 0$ such that

$$\int_\Omega \sum_{i=1}^N |f_i|^p \,\mathrm{d}x \le C \int_\Omega \sum_{i=1}^N u_i(|f_i|^p + |\nabla f_i|^p)\,\mathrm{d}x \quad (\mathbf{u} = (u_1, \dots, u_N) \in \mathbf{U}). \tag{11}$$

Remark 4. The integrand in the right-hand side of (11) is nonnegative, and the integral may be infinite.

Remark 5. For $p > 1$, by Vitali's theorem, the convergence in measure in (9), (10) can be replaced by the convergence in L^q, $1 \le q < p$.

Condition (ii) of Theorem 1 means that the set $\mathbf{U}$ must be separated from a finite number of specific points in the topology of convergence in measure. Specifically, it follows from Assumption 2 that given $I \subset \{1, \dots, N\}$, the linear system

$$\begin{cases} u_i = 0 & (i \in I) \\ f_j = 0 & (j \notin I), \end{cases} \tag{12}$$

has a unique solution $\mathbf{u}_I \in C^1(\overline{\Omega}; \mathbb{R}^N)$. It is easy to see that (9) and (10) are equivalent to

$$\mathbf{u}_n \to \mathbf{u}_I \quad (n \to \infty) \quad \text{in measure.} \tag{13}$$

Theorem 1 admits the following stronger formulation.

Solving (12) by Cramer's rule and recalling the assumptions, we see that all the functions $\mathbf{u}_I = (u_{I1}, \dots, u_{In})$ are bounded by a constant depending only on κ. Let $M = M(\kappa)$ be an arbitrary number such that

$$M > \sup\{u_{Ii} \colon I \subset \{1, \dots, N\}; i \in \{1, \dots, N\}\}. \tag{14}$$

Theorem 2. *Let $p \ge 1$, $\mathcal{A} \subset C^1(\overline{\Omega}; M_N(\mathbb{R}) \times \mathbb{R}^N)$, and $\mathbf{U} \subset W^1_p(\Omega; \mathbf{R}^N)$ be such that*

(i) any $(A, \mathbf{m}) \in \mathcal{A}$ satisfies Assumptions 1–3 with a constant $\kappa = \kappa(\mathcal{A})$;

(ii) $\mathbf{u} \ge 0$ for any $\mathbf{u} \in \mathbf{U}$;

(iii) one cannot choose sequences $\{\mathbf{u}_n\} \subset \mathbf{U}$ and $\{(A_n, \mathbf{m}_n)\} \subset \mathcal{A}$ such that $\{\mathbf{u}_n\}$ is bounded in L^p, and (9) and (10) hold for an $I \ne \varnothing$.

Then there exists $C(\Omega, p, \kappa, \mathcal{A}, \mathbf{U}) > 0$ such that

$$\int_\Omega \sum_{i=1}^N |f_i|^p \,\mathrm{d}x \le C(\Omega, p, \kappa, \mathcal{A}, \mathbf{U}) \int_\Omega \sum_{i=1}^N \tilde{u}_i(|f_i|^p + |\nabla f_i|^p)\,\mathrm{d}x \tag{15}$$

for all $\mathbf{u} = (u_1, \dots, u_N) \in \mathbf{U}$ and $(A, \mathbf{m}) \in \mathcal{A}$, where

$$\tilde{u}_{in}(x) = \min(u_{in}(x), M). \tag{16}$$

3. Auxiliary functions

In this section we collect a few auxiliary results concerning systems of affine functions on $\mathbb{R}^N$

$$f_i(u_1,\dots,u_N) = m_i - \sum_{j=1}^{N} a_{ij}u_j \quad (i=1,\dots,N) \tag{17}$$

with scalar coefficients. The proofs of the statements can be found in the Appendix.

We say that the coefficients in (17) are *admissible*, if they satisfy Assumptions 1, 2, and 3 with a fixed $\kappa > 0$.

Given $I = \{i_1,\dots,i_r\} \subset \{1,\dots,N\}$, $i_1 < \cdots < i_r$, and $j \in \{1,\dots,N\}$, denote the determinants in the right-hand sides of (7) and (8) by

$$\Delta_I = \begin{vmatrix} a_{i_1i_1} & \cdots & a_{i_1i_r} \\ \vdots & \ddots & \vdots \\ a_{i_ri_1} & \cdots & a_{i_ri_r} \end{vmatrix}, \quad \Delta_{I,j} = \begin{vmatrix} a_{i_1i_1} & \cdots & a_{i_1i_r} & m_{i_1} \\ \vdots & \ddots & \vdots & \vdots \\ a_{i_ri_1} & \cdots & a_{i_ri_r} & m_{i_r} \\ a_{ji_1} & \cdots & a_{ji_r} & m_j \end{vmatrix}.$$

Remark 6. If the determinants Δ_I are nonzero, the systems (12) are exactly determined. Denoting the solution of (12) by $\mathbf{u}_I = (u_{I1},\dots,u_{IN})$ as before, for any I we have

$$u_{\complement I i} = \frac{\Delta_{I\setminus\{i\},i}}{\Delta_I}, \quad f_j(\mathbf{u}_{\complement I}) = \frac{\Delta_{I,j}}{\Delta_I}, \tag{18}$$

where $\complement I = \{1,\dots,N\}\setminus I$. Thus, in the case of admissible coefficients the values of $\mathbf{u}_I$ and of $f_j(\mathbf{u}_I)$ are nonnegative and bounded by a constant depending only on κ, but not on the particular choice of coefficients. Moreover, if $j \notin I$, then u_{Ij} are bounded away from zero by a constant depending on κ, but not on the coefficients. If $i \in I$, the same is true for $f_i(\mathbf{u}_I)$.

We want to geometrically interprete the positivity of Δ_I and $\Delta_{I,j}$, involved in Assumptions 2 and 3. To this end, consider the system of linear inequalities in $\mathbb{R}^N$:

$$\begin{cases} f_i \geq 0 & (i=1,\dots,N), \\ u_i \geq 0 & (i=1,\dots,N). \end{cases} \tag{19}$$

Proposition 1. *Suppose that $\Delta_I \neq 0$ for any $I \subset \{1,\dots,N\}$. Then $\Delta_I > 0$ and $\Delta_{I,j} > 0$ for any $I \subset \{1,\dots,N\}$ and $j \notin I$ if and only if the solution set of (19) is a polytope with vertices $\{\mathbf{u}_I : I \subset \{1,\dots,N\}\}$ combinatorially isomorphic to a cube.*

One corollary of Proposition 1 is that in the case of admissible coefficients no vertex (and hence, no point whatsoever) of the polytope (19) satisfies any of the the equations $u_i = 0 = f_i$. A strengthened version of this observation stated in the following lemma plays a crucial role in our proof.

Lemma 1. *There exists $\sigma = \sigma(\kappa)$ such that if for some admissible coefficients, some index j, and some $\mathbf{u} = (u_1,\dots,u_N) \geq 0$ we have $u_j \leq \sigma$ and $f_j(\mathbf{u}) \leq \sigma$, then*

$$\min_i f_i(\mathbf{u}) \leq -\sigma. \tag{20}$$

Now we introduce a few auxiliary functions. Fix $p \geq 1$ and set

$$g = \sum_{i=1}^{N} |f_i|^p,$$

$$v = \begin{cases} \frac{1}{g} \sum_{i=1}^{N} u_i |f_i|^p & \text{if } g \neq 0 \\ \text{whatever between } \min_i u_i \text{ and } \max_i u_i & \text{if } g = 0. \end{cases}$$

Observe that g and v are nonnegative on $\mathbb{R}_+^N$ and

$$g = 0 \Leftrightarrow f_i = 0 \ (i = 1, ..., N) \Leftrightarrow \mathbf{u} = \mathbf{u}_\varnothing, \tag{21}$$

$$v = 0 \Leftrightarrow \mathbf{u} = \mathbf{u}_I \quad (I \neq \varnothing). \tag{22}$$

Also note the identity

$$vg = \sum_{i=1}^{N} u_i |f_i|^p \tag{23}$$

and the inequality

$$\min_i u_i \leq v \leq \max_i u_i. \tag{24}$$

Developing observation (22), the following lemma and its corollaries state that $v(\mathbf{u})$ is small only in the neighbourhood of the set $\{\mathbf{u}_I : I \neq \varnothing\}$. This allows to use the function v to prove convergence of the form (13) and (9)–(10).

Lemma 2. *We have*

$$\lim_{\substack{v(\mathbf{u}) \to 0 \\ \mathbf{u} \in \mathbb{R}_+^N}} \min_{I \neq \varnothing} |\mathbf{u} - \mathbf{u}_I| = 0, \tag{25}$$

where the limit is uniform with respect to admissible coefficients.

Corollary 1. *There exists $\sigma > 0$ such that for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, \kappa) > 0$ such that if $\mathbf{u} \geq 0$ and $v(\mathbf{u}) < \delta$ for some admissible coefficients, then there exists $I \neq \varnothing$ such that*

$$\sum_{i \in I} u_i + \sum_{j \notin I} |f_j(\mathbf{u})| < \varepsilon, \tag{26}$$

$$f_i(\mathbf{u}) > \sigma \quad (i \in I). \tag{27}$$

Corollary 2. *We have*

$$\lim_{\substack{v(\mathbf{u}) \to 0 \\ \mathbf{u} \in \mathbb{R}_+^N}} \min_{I \neq \varnothing} \left(\sum_{i \in I} u_i + \sum_{j \notin I} |f_j| \right) = 0, \tag{28}$$

where the limit is uniform with respect to admissible coefficients.

As v vanishes only at the points $\mathbf{u}_I$ $(I \neq \varnothing)$, it follows from Remark 6 that $v = 0$ implies $f_i > 0$ for some i. The following corollary of Lemma 2 extends this observation to the case of small v.

Corollary 3. *There exist* $\varepsilon_0 > 0$ *and* $\sigma > 0$ *depending on* κ*, but not on admissible coefficients, such that if* $v(\mathbf{u}) < \varepsilon_0$*, there exists* i *such that* $f_i(\mathbf{u}) > \sigma$*.*

4. Proof of the theorems

4.1. **Preliminaries.** We prove Theorem 2, and Theorem 1 follows.

Assume that the theorem is not true and there exist a sequence $\{\varepsilon_n\}$, sequences of coefficients $\{A_n\} \subset C^1(\overline{\Omega}; M_N(\mathbb{R}))$ and $\{\mathbf{m}_n\} \subset C^1(\overline{\Omega}; \mathbb{R}^N)$ satisfying Assumptions 1–3 with some $\kappa > 0$ and a sequence $\{(u_{1n}, \dots, u_{Nn})\} \subset \mathbf{U}$ such that $\varepsilon_n \to 0$ and

$$\int_\Omega \sum_{i=1}^N \tilde{u}_{in}(|f_{in}|^p + |\nabla f_{in}|^p)\,\mathrm{d}x \le \varepsilon_n^2 \int_\Omega \sum_{i=1}^N |f_{in}|^p\,\mathrm{d}x, \tag{29}$$

where f_{in} corresponds to A_n and $\mathbf{m}_n$.

We claim that without loss of generality the functions $\{\mathbf{u}_n\}$ can be assumed to be smooth. Indeed, we always can assume that $\mathbf{U}$ is open in the relative topology of the cone of nonnegative functions in W_p^1, otherwise we can replace it by a small enlargement of $\mathbf{U}$ without affecting the hypothesis of the theorem. Then by Meyers-Serrin theorem we can approximate $\mathbf{u}_n$ by smooth functions from $\mathbf{U}$ such that (29) holds with ε_n^2 replaced by $2\varepsilon_n^2$.

Denote

$$\tilde{v}_n(x) = \begin{cases} \frac{1}{g_n(x)} \sum_{i=1}^N \tilde{u}_{in}(x)|f_{in}(x)|^p & \text{if } g_n(x) \neq 0 \\ v_n(x) & \text{if } g_n(x) = 0. \end{cases}$$

It is obvious that

$$u_{in}(x) \le \varepsilon_n \quad \text{if and only if} \quad \tilde{u}_{in}(x) \le \varepsilon_n.$$

It follows from Lemma 2 and (14) that there exists $\delta > 0$ independent of n and x such that if $v_n(x) < \delta$, then for any i we have $u_{in}(x) < M$ and, consequently, $u_{in}(x) = \tilde{u}_{in}(x)$ and $v_n(x) = \tilde{v}(x)$. In particular, there is no loss of generality in assuming that

$$v_n(x) \le \varepsilon_n \quad \text{if and only if} \quad \tilde{v}_n(x) \le \varepsilon_n.$$

Write (29) in the form

$$\int_\Omega \tilde{v}_n g_n\,\mathrm{d}x + \int_\Omega \sum_{i=1}^N \tilde{u}_{in}|\nabla f_{in}|^p\,\mathrm{d}x \le \varepsilon_n^2 \int_\Omega g_n\,\mathrm{d}x,$$

whence

$$\int_{[v_n \le \varepsilon_n]} \tilde{v}_n g_n\,\mathrm{d}x + \varepsilon_n \int_{[v_n > \varepsilon_n]} g_n\,\mathrm{d}x + \int_\Omega \sum_{i=1}^N \tilde{u}_{in}|\nabla f_{in}|^p\,\mathrm{d}x \le \varepsilon_n^2 \left(\int_{[v_n \le \varepsilon_n]} g_n\,\mathrm{d}x + \int_{[v_n > \varepsilon_n]} g_n\,\mathrm{d}x \right).$$

Dropping a nonnegative term on the left-hand side and dividing both sides by ε_n, we obtain

$$\frac{1}{\varepsilon_n}\int_\Omega \sum_{i=1}^N \tilde{u}_{in}|\nabla f_{in}|^p \,\mathrm{d}x \le -(1-\varepsilon_n)\int_{[v_n>\varepsilon_n]} g_n \,\mathrm{d}x + \varepsilon_n \int_{[v_n\le\varepsilon_n]} g_n \,\mathrm{d}x. \tag{30}$$

Lemma 2 implies that if v is bounded, so is $\mathbf{u}$, so there exists $M > 0$ such that $g \le M$ whenever $v < 1$. Without loss of generality, $\varepsilon_n < 1$, so from (30) we conclude

$$\frac{1}{\varepsilon_n}\int_\Omega \sum_{i=1}^N \tilde{u}_{in}|\nabla f_{in}|^p \,\mathrm{d}x \le M\varepsilon_n |[v_n \le \varepsilon_n]|. \tag{31}$$

Moreover, it follows from (30) that the integral

$$\int_{[v_n>\varepsilon_n]} g_n \,\mathrm{d}x$$

is bounded uniformly with respect to n. Hence the sequence $\{g_n\}$ is bounded in L^1 and $\{\mathbf{u}_n\}$ is bounded in L^p. It remains to show that $\{\mathbf{u}_n\}$ satisfies (9) and (10) for some nonempty I in order to obtain a contradiction.

Lemma 3. *Given $a > 0$,*

$$\lim_{n\to\infty} |[v_n > \varepsilon_n] \cap [g_n > a]| = 0. \tag{32}$$

Proof. We have:

$$|[v_n > \varepsilon_n] \cap [g_n > a]| \le \frac{1}{a}\int_{[v_n>\varepsilon_n]\cap[g_n>a]} g_n \,\mathrm{d}x \le \frac{1}{a}\int_{[v_n>\varepsilon_n]} g_n \,\mathrm{d}x. \tag{33}$$

Inequality (30) implies

$$-(1-\varepsilon_n)\int_{[v_n>\varepsilon_n]} g_n \,\mathrm{d}x + \varepsilon_n \int_{[v_n\le\varepsilon_n]} g_n \,\mathrm{d}x \ge 0,$$

so we can estimate the last integral in (33) and obtain

$$|[v_n > \varepsilon_n] \cap [g_n > a]| \le \frac{\varepsilon_n}{a(1-\varepsilon_n)}\int_{[v_n\le\varepsilon_n]} g_n \,\mathrm{d}x \le \frac{M\varepsilon_n|[v_n\le\varepsilon_n]|}{a(1-\varepsilon_n)} \to 0 \quad (n\to\infty)$$

and (32) is proved. □

Lemma 4. *Given a, there exists C_a such that for large n,*

$$|[g_n > a]| \le C_a |[v_n \le \varepsilon_n]|. \tag{34}$$

Proof. Using the estimate

$$|[v_n > \varepsilon_n] \cap [g_n > a]| \le \frac{M\varepsilon_n|[v_n\le\varepsilon_n]|}{a(1-\varepsilon_n)}$$

obtained in the proof of Lemma 3, we get

$$|[g_n > a]| \le |[v_n \le \varepsilon_n]| + |[v_n > \varepsilon_n] \cap [g_n > a]| \le \left(1 + \frac{M\varepsilon_n}{a(1-\varepsilon_n)}\right)|[v_n \le \varepsilon_n]|,$$

and the lemma follows. □

4.2. **Limit behaviour of the sequences.** Now we are ready to consider the dynamics of $\mathbf{u}_n$ in detail.

We choose and fix $\varepsilon_0 \in (0,1)$ and $\sigma > 0$ satisfying Corollary 3 and Lemma 1. As those numbers do not depend on admissible coefficients, they satisfy

Condition 1. *If $v_n(x) < \varepsilon_0$, there exists i such that $|f_{in}(x)| > \sigma$.*

Condition 2. *If $f_{in}(x) \le \sigma$ and $u_{in}(x) \le \sigma$, then there exists $j \ne i$ such that $f_j \le -\sigma$.*

Given $I \subset \{1, \dots, N\}$, define

$$A_n(I) = \left(\bigcap_{i\in I} [f_{in} > \sigma]\right) \cap \left(\bigcap_{j\notin I} \left[|f_{jn}| < \frac{\sigma}{2}\right]\right).$$

Lemma 5. *We have*

$$\lim_{n\to\infty} \sum_I |A_n(I)| = |\Omega|. \tag{35}$$

Proof. We prove the lemma by showing the inclusion

$$[v_n \le \varepsilon_n] \cup [g_n \le a] \subset \bigcup_I A_n(I) \tag{36}$$

with suitable a and evoking Lemma 3.

Take $a = (\sigma/2)^p$, then the inequality $g_n \le a$ clearly implies $|f_i| \le \sigma/2$ for any i, so

$$[g_n \le a] \subset \bigcap_{i=1}^{N} \left[|f_{in}| \le \frac{\sigma}{2}\right] = A_n(\varnothing). \tag{37}$$

Now suppose that $v_n(x) \le \varepsilon_n$ for some x. Applying Corollary 1 with $\varepsilon = \sigma/2$, we find $I \ne \varnothing$ such that

$$f_{in}(x) > \sigma \quad (i \in I)$$

and

$$\sum_{j\notin I} |f_{jn}(x)| < \frac{\sigma}{2}$$

whenever $\varepsilon_n < \delta$ for some $\delta > 0$ independent of n and x. Consequently,

$$x \in A_n(I)$$

and we have the inclusion

$$[v_n \le \varepsilon_n] \subset \bigcup_{I\ne\varnothing} A_n(I). \tag{38}$$

Combining (37) and (38), we obtain (36).

By Lemma 3, the measure of the left-hand side of inclusion (36) converges to $|\Omega|$, and (35) follows. □

We can assume that the limits

$$\lim_{k\to\infty} A_n(I)$$

exist. In view of Lemma 5 we face three logical possibilities:
(i) $\lim_{n\to\infty} A_n(I) = |\Omega|$ for some $I \neq \varnothing$;
(ii) $\lim_{n\to\infty} A_n(\varnothing) = |\Omega|$;
(iii) $\lim_{n\to\infty} A_n(I_s) > 0$ $(s = 1, 2)$ with $I_1 \neq I_2$.

We conclude the proof of Theorem 2 by examining the alternatives (i)–(iii). It is fairly straightforward to demonstrate that (i) implies (9) and (10). A more subtle analysis based on the coarea formula and the relative isoperimetric inequality shows that (ii) and (iii) are in fact impossible.

Recall that the relative perimeter of a Lebesgue measurable set A of (locally) finite perimeter is defined as

$$P(A; \Omega) = \mu_A(\Omega),$$

where μ_A is the total variation of the Gauss–Green measure of A (see [11]). We need the following properties of the perimeters:

Lemma 6 ([11], Proposition 12.19 and Lemma 12.22)**.** *If A is a set of locally finite perimeter in $\mathbb{R}^d$, then*

$$\operatorname{supp} \mu_A \subset \partial A;$$

if A and B are sets of (locally) finite perimeter in $\mathbb{R}^d$, then $A \cup B$ is a set of (locally) finite perimeter in $\mathbb{R}^d$, and, for $\Omega \subset \mathbb{R}^d$ open,

$$P(A \cup B; \Omega) \leq P(A; \Omega) + P(B; \Omega).$$

4.3. **Convergence in case (i).** Assume that (i) holds. We claim that (9) and (10) are valid, i. e. for any $\varepsilon > 0$,

$$\lim_{k\to\infty} \left| \left[\sum_{j\in I} u_{in} + \sum_{i\notin I} |f_{in}| \geq \varepsilon \right] \right| = 0. \tag{39}$$

By assumption, $I \neq \varnothing$, so for any $x \in A_n(I)$ we have $f_{in}(x) > \sigma$ for at least one i and thus $g_n(x) > \sigma^p$. It follows from Lemma 3 that

$$\lim_{k\to\infty} |A_n(I) \cap [v_n > \varepsilon_n]| = 0$$

and consequently

$$\lim_{k\to\infty} |A_n(I) \cap [v_n \leq \varepsilon_n]| = |\Omega|. \tag{40}$$

Take $x \in A_n(I) \cap [v_n \leq \varepsilon_n]$. By Corollary 1, there exists $\delta > 0$ independent of k and x such that for some $I_{k,x}$ we have

$$\sum_{j\in I_{k,x}} u_{in}(x) + \sum_{i\notin I_{k,x}} |f_{in}(x)| < \varepsilon_n, \tag{41}$$

$$f_{jn}(x) > \sigma \quad (j \in I_{k,x}), \tag{42}$$

provided that $v_n(x) < \delta$, which holds without loss of generality. However, (41) and (42) are only compatible with the definition of $A_n(I)$ in the case $I_{k,x} = I$. Thus,

$$\sum_{j\in I} u_{in}(x) + \sum_{i\notin I} |f_{in}(x)| < \varepsilon \quad (x \in A_n(I) \cap [v_n \le \varepsilon_n]),$$

or, equivalently,

$$A_n(I) \cap [v_n \le \varepsilon_n] \subset \left[\sum_{j\in I} u_{in} + \sum_{i\notin I} |f_{in}| < \varepsilon\right].$$

Consequently,

$$\left|\left[\sum_{j\in I} u_{in} + \sum_{i\notin I} |f_{in}| \ge \varepsilon\right]\right| \le |\Omega| - |A_n(I) \cap [v_n \le \varepsilon_n]|$$

and by (40), the limit (39) holds.

4.4. **Impossibility of case (ii).** We argue by contradiction that case (ii) is impossible. Thus, we assume that

$$\lim_{n\to\infty} \left|\bigcap_{i=1}^{N} \left[|f_{in}| \le \frac{\sigma}{2}\right]\right| = |\Omega|. \tag{43}$$

Given $t \in (\sigma/2, \sigma)$ define the set

$$A_n(t) = \bigcup_{i=1}^{N} [|f_{in}| > t].$$

For fixed n, $A_n(t)$ decreases with respect to t. We need to establish several properties of these sets.

Lemma 7. *For fixed t,*

$$\lim_{n\to\infty} |A_n(t)| = 0.$$

Proof. We have:

$$A_n(t) \subset \bigcup_{i=1}^{N} \left[|f_{in}| > \frac{\sigma}{2}\right] = \Omega \setminus \bigcap_{i=1}^{N} \left[|f_{in}| \le \frac{\sigma}{2}\right],$$

so

$$|A_n(t)| \le |\Omega| - \left|\bigcap_{i=1}^{N} [|f_{in}| \le \sigma]\right| \to 0 \quad (n \to \infty)$$

according to (43). □

Lemma 8. *The following inclusion holds:*

$$\partial_\Omega A_n(t) \subset \bigcap_{i=1}^{N} [u_{in} > \varepsilon_n]. \tag{44}$$

Proof. We take an $x \in \Omega$ such that $u_{jn}(x) \leq \varepsilon_n$ for some j and show that x is an interior point of $A_n(t)$. There are two possibilities: either $f_{jn}(x) > t$ or $f_{jn}(x) \leq t$. In the former case we see immediately that x is an interior point of $A_n(t)$. In the latter case we have $f_{jn}(x) \leq \sigma$, and applying Condition 2 we find f_{in} such that $f_{in}(x) \leq -\sigma < -t$. But then $|f_{in}(x)| > t$, and again x is an interior point of $A_n(t)$. □

Lemma 9. *If n is sufficiently large, the following inclusion holds:*

$$A_n(t) \supset [v_n \leq \varepsilon_n]. \tag{45}$$

Proof. By Condition 1, for large n we have

$$[v_n \leq \varepsilon_n] \subset \bigcup_{i=1}^{N} [|f_{in}| > \sigma] \subset A_n(t).$$

□

It follows from Lemma 7 that we can write the isoperimetric inequality

$$P(A_n(t);\Omega) \geq c_\Omega |A_n(t)|^{\frac{d-1}{d}}. \tag{46}$$

Estimate the left-hand side of (31):

$$\begin{aligned}
\frac{1}{\varepsilon_n}\int_\Omega \sum_{i=1}^{N} \tilde{u}_{in}|\nabla f_{in}|^p \,\mathrm{d}x &\geq \frac{1}{\varepsilon_n}\sum_{i=1}^{N}\int_{[g_n > (\frac{\sigma}{2})^p]} \tilde{u}_{in}|\nabla f_{in}|^p \,\mathrm{d}x \\
&\geq \frac{1}{\varepsilon_n \left|\left[g_n > \left(\frac{\sigma}{2}\right)^p\right]\right|^{p-1}} \sum_{i=1}^{N}\left(\int_{[g_n > (\frac{\sigma}{2})^p]} \tilde{u}_{in}^{1/p}|\nabla f_{in}| \,\mathrm{d}x\right)^p \\
&\geq \frac{1}{\varepsilon_n N^{p-1}\left|\left[g_n > \left(\frac{\sigma}{2}\right)^p\right]\right|^{p-1}} \left(\sum_{i=1}^{N}\int_{[g_n > (\frac{\sigma}{2})^p]} \tilde{u}_{in}^{1/p}|\nabla f_{in}| \,\mathrm{d}x\right)^p \\
&\geq \frac{1}{N^{p-1}\left|\left[g_n > \left(\frac{\sigma}{2}\right)^p\right]\right|^{p-1}} \left(\sum_{i=1}^{N}\int_{[g_n > (\frac{\sigma}{2})^p]\cap[u_{in} > \varepsilon_n]} |\nabla f_{in}| \,\mathrm{d}x\right)^p.
\end{aligned}$$

Apply the coarea formula [11, Theorem 13.1, formula (13.10)] and Lemma 6:

$$\begin{aligned}
&\frac{1}{\varepsilon_n}\int_\Omega \sum_{i=1}^N \tilde{u}_{in}|\nabla f_{in}|^p \,\mathrm{d}x \\
&\quad \geq \frac{1}{N^{p-1}\left|\left[g_n > \left(\frac{\sigma}{2}\right)^p\right]\right|^{p-1}} \left(\sum_{i=1}^N \int_0^\infty P\left([f_{in} > t]; \left[g_n > \left(\frac{\sigma}{2}\right)^p\right] \cap [u_{in} > \varepsilon_n]\right) \mathrm{d}t \right. \\
&\qquad\qquad \left. + \int_{-\infty}^0 P\left([f_{in} < t]; \left[g_n > \left(\frac{\sigma}{2}\right)^p\right] \cap [u_{in} > \varepsilon_n]\right) \mathrm{d}t \right)^p \\
&\quad \geq \frac{1}{N^{p-1}\left|\left[g_n > \left(\frac{\sigma}{2}\right)^p\right]\right|^{p-1}} \left(\sum_{i=1}^N \int_{\sigma/2}^\sigma \left(P\left([f_{in} > t]; \left[g_n > \left(\frac{\sigma}{2}\right)^p\right] \cap [u_{in} > \varepsilon_n]\right) \right.\right. \\
&\qquad\qquad \left.\left. + P\left([f_{in} < -t]; \left[g_n > \left(\frac{\sigma}{2}\right)^p\right] \cap [u_{in} > \varepsilon_n]\right) \right) \mathrm{d}t \right)^p \\
&\quad \geq \frac{1}{N^{p-1}\left|\left[g_n > \left(\frac{\sigma}{2}\right)^p\right]\right|^{p-1}} \left(\sum_{i=1}^N \int_{\sigma/2}^\sigma P\left([|f_{in}| > t]; \left[g_n > \left(\frac{\sigma}{2}\right)^p\right] \cap [u_{in} > \varepsilon_n]\right) \mathrm{d}t \right)^p \\
&\quad = \frac{1}{N^{p-1}\left|\left[g_n > \left(\frac{\sigma}{2}\right)^p\right]\right|^{p-1}} \left(\sum_{i=1}^N \int_{\sigma/2}^\sigma \mu_{[|f_{in}|>t]}\left(\left[g_n > \left(\frac{\sigma}{2}\right)^p\right] \cap [u_{in} > \varepsilon_n]\right) \mathrm{d}t \right)^p
\end{aligned}$$

Observe that for $t \in (\sigma/2, \sigma)$ by Lemma 6 we have

$$\operatorname{supp} \mu_{[|f_{in}|>t]} \subset \partial[|f_{in}| > t] \subset [|f_{in}| = t] \subset \left[g_n > \left(\frac{\sigma}{2}\right)^p\right],$$

so we can proceed as follows:

$$\begin{aligned}
&\frac{1}{\varepsilon_n}\int_\Omega \sum_{i=1}^N \tilde{u}_{in}|\nabla f_{in}|^p \,\mathrm{d}x \\
&\quad \geq \frac{1}{N^{p-1}\left|\left[g_n > \left(\frac{\sigma}{2}\right)^p\right]\right|^{p-1}} \left(\sum_{i=1}^N \int_{\sigma/2}^\sigma \mu_{[|f_{in}|>t]}([u_{in} > \varepsilon_n]) \,\mathrm{d}t \right)^p \\
&\quad \geq \frac{1}{N^{p-1}\left|\left[g_n > \left(\frac{\sigma}{2}\right)^p\right]\right|^{p-1}} \left(\sum_{i=1}^N \int_{\sigma/2}^\sigma \mu_{[|f_{in}|>t]}\left(\bigcap_{i=1}^N [u_{in} > \varepsilon_n]\right) \mathrm{d}t \right)^p \\
&\quad = \frac{1}{N^{p-1}\left|\left[g_n > \left(\frac{\sigma}{2}\right)^p\right]\right|^{p-1}} \left(\sum_{i=1}^N \int_{\sigma/2}^\sigma P\left([|f_{in}| > t]; \left(\bigcap_{i=1}^N [u_{in} > \varepsilon_n]\right)\right) \mathrm{d}t \right)^p \\
&\quad \geq \frac{1}{N^{p-1}\left|\left[g_n > \left(\frac{\sigma}{2}\right)^p\right]\right|^{p-1}} \left(\int_{\sigma/2}^\sigma P\left(A_n(t); \left(\bigcap_{i=1}^N [u_{in} > \varepsilon_n]\right)\right) \mathrm{d}t \right)^p.
\end{aligned}$$

Now (44) implies that

$$\operatorname{supp} \mu_{A_n(t)} \cap \Omega \subset \bigcap_{i=1}^{N} [u_{in} > \varepsilon_n],$$

so we can write

$$\frac{1}{\varepsilon_n} \int_\Omega \sum_{i=1}^{N} \tilde{u}_{in} |\nabla f_{in}|^p \,\mathrm{d}x \geq \frac{1}{N^{p-1} \left| \left[g_n > \left(\frac{\sigma}{2}\right)^p \right] \right|^{p-1}} \left(\int_{\sigma/2}^{\sigma} P(A_n(t); \Omega) \,\mathrm{d}t \right)^p.$$

Employing the isoperimetric inequality (46) to get

$$\frac{1}{\varepsilon_n} \int_\Omega \sum_{i=1}^{N} \tilde{u}_{in} |\nabla f_{in}|^p \,\mathrm{d}x \geq \frac{c_\Omega^p}{N^{p-1} \left| \left[g_n > \left(\frac{\sigma}{2}\right)^p \right] \right|^{p-1}} \left(\int_{\sigma/2}^{\sigma} |A_n(t)|^{\frac{d-1}{d}} \,\mathrm{d}t \right)^p.$$

Estimate $|A_n(t)|$ using the inclusion (45) and Lemma 4:

$$\begin{aligned} \frac{1}{\varepsilon_n} \int_\Omega \sum_{i=1}^{N} \tilde{u}_{in} |\nabla f_{in}|^p \,\mathrm{d}x &\geq \frac{c_\Omega^p (\sigma/2)^p |[v_n \leq \varepsilon_n]|^{p(1-1/d)}}{C_{(\sigma/2)^p} N^{p-1} |[v_n \leq \varepsilon_n]|^{p-1}} \\ &= \frac{c_\Omega^p (\sigma/2)^p}{C_{(\sigma/2)^p} N^{p-1}} |[v_n \leq \varepsilon_n]|^{1-\frac{p}{d}}. \end{aligned}$$

Combining obtained estimate with (31), we get:

$$\frac{c_\Omega^p (\sigma/2)^p}{C_{(\sigma/2)^p} N^{p-1}} |[v_n \leq \varepsilon_n]|^{1-\frac{p}{d}} \leq M \varepsilon_n |[v_n \leq \varepsilon_n]|,$$

whence

$$\frac{c_\Omega^p (\sigma/2)^p}{C_{(\sigma/2)^p} N^{p-1}} \leq M \varepsilon_n |[v_n \leq \varepsilon_n]|^{\frac{p}{d}} \to 0 \quad (n \to \infty),$$

contrary to the fact that the left-hand side is a positive constant independent of n.

This contradiction means that at least assumption (43) is impossible.

4.5. **Impossibility of case (iii).** We complete the proof of Theorem 1 by demonstrating that the case (iii) is also impossible. We argue by contradiction. We assume that there exist $I_1 \neq I_2$ such that

$$|A_n(I_s)| \geq \mu_s > 0 \quad (s = 1, 2). \tag{47}$$

Without loss of generality, $1 \in I_1 \setminus I_2$.

Given $t \in (\sigma/2, \sigma)$, define the set

$$A_n(t) = [f_{1n} > t] \cup \left(\bigcup_{i=1}^{N} [f_{in} < -t] \right). \tag{48}$$

If n is fixed, the sets $A_n(t)$ decrease with respect to t.

Lemma 10. *The relative perimeter of $A_n(t)$ can be estimated as*

$$P(A_n(t);\Omega) \geq p_0, \tag{49}$$

where $p_0 > 0$ is independent of t and n.

Proof. First of all, observe the inclusions

$$A_n(I_1) \subset A_n(t) \subset \Omega \setminus A_n(I_2). \tag{50}$$

Indeed, if $x \in A_n(I_1)$, then $f_{1n}(x) > \sigma$, so x belongs to the first set in the right-hand side of (48), and the first inclusion in (50) holds. On the other hand, if $x \in A_n(t)$, then either $f_{1n}(x) > t > \sigma/2$ or $f_{in}(x) < -t < -\sigma/2$ for some i. As $1 \notin I_2$, it is clear that in both cases $x \notin A_n(I_2)$, so the second inclusion in (50) is also valid.

The isoperimetric inequality for $A_n(t)$ reads

$$P(A_n(t);\Omega) \geq c_\Omega \Big(\min(|A_n(t)|, |\Omega \setminus A_n(t)|) \Big)^{\frac{d-1}{d}}$$

Estimating by means of (50), we have:

$$P(A_n(t);\Omega) \geq c_\Omega (\min(\mu_1, 1-\mu_2))^{\frac{d-1}{d}}$$

and (49) follows. □

Lemma 11. *The following inclusions hold:*

$$\partial_\Omega A_n(t) \cap \partial[f_{1n} > t] \subset [u_{1n} > \varepsilon_n] \tag{51}$$

$$\partial_\Omega A_n(t) \cap \partial[f_{in} < -t] \subset [u_{in} > \varepsilon_n] \tag{52}$$

Proof. If $u_{1n}(x) \leq \varepsilon_n$ and $x \in \partial[f_{1n} > t]$, then $f_{1n}(x) = t \leq \sigma$. If $u_{in}(x) \leq \varepsilon_n$ and $x \in \partial[f_{in} < -t]$, then $f_{in}(x) = -t \leq \sigma$. In both cases by Condition 2 there exists j such that $f_{jn}(x) \leq -\sigma < -t$, so x belongs to the interior of $A_n(t)$ and the lemma follows. □

Estimate the left-hand side of (31):

$$\begin{aligned}
\frac{1}{\varepsilon_n} \int_\Omega \sum_{i=1}^N \tilde{u}_{in} |\nabla f_{in}|^p \,\mathrm{d}x &\geq \sum_{i=1}^N \int_{[u_{in} > \varepsilon_n]} |\nabla f_{in}|^p \,\mathrm{d}x \\
&\geq \sum_{i=1}^N \frac{1}{|[u_{in} > \varepsilon_n]|^{p-1}} \left(\int_{[u_{in} > \varepsilon_n]} |\nabla f_{in}| \,\mathrm{d}x \right)^p \\
&\geq \frac{1}{N^{p-1} |\Omega|^{p-1}} \left(\sum_{i=1}^N \int_{[u_{in} > \varepsilon_n]} |\nabla f_{in}| \,\mathrm{d}x \right)^p .
\end{aligned}$$

Now we apply the coarea formula:

$$\frac{1}{\varepsilon_n}\int_\Omega \sum_{i=1}^N \tilde{u}_{in}|\nabla f_{in}|^p \,\mathrm{d}x$$
$$\geq \frac{1}{N^{p-1}|\Omega|^{p-1}}\left(\sum_{i=1}^N\int_0^\infty P([f_{in}>t];[u_{in}>\varepsilon_n])\,\mathrm{d}t+\sum_{i=1}^N\int_{-\infty}^0 P([f_{in}<t];[u_{in}>\varepsilon_n])\,\mathrm{d}t\right)^p$$
$$\geq \frac{1}{N^{p-1}|\Omega|^{p-1}}\left(\int_{\frac{\sigma}{2}}^\sigma\left(P([f_{1n}>t];[u_{1n}>\varepsilon_n])+\sum_{i=1}^N P([f_{in}<-t];[u_{in}>\varepsilon_n])\right)\mathrm{d}t\right)^p$$
$$=\frac{1}{N^{p-1}|\Omega|^{p-1}}\left(\int_{\frac{\sigma}{2}}^\sigma\left(\mu_{[f_{1n}>t]}([u_{1n}>\varepsilon_n])+\sum_{i=1}^N \mu_{[f_{in}<-t]}([u_{in}>\varepsilon_n])\right)\mathrm{d}t\right)^p.$$

Using (51) and (52), we obtain

$$\frac{1}{\varepsilon_n}\int_\Omega \sum_{i=1}^N \tilde{u}_{in}|\nabla f_{in}|^p \,\mathrm{d}x \geq \frac{1}{N^{p-1}|\Omega|^{p-1}}$$
$$\times\left(\int_{\frac{\sigma}{2}}^\sigma\left(\mu_{[f_{1n}>t]}(\partial_\Omega A_n(t)\cap\partial[f_{1n}>t])+\sum_{i=1}^N \mu_{[f_{in}<-t]}(\partial_\Omega A_n(t)\cap\partial[f_{in}<-t])\right)\mathrm{d}t\right)^p.$$

Using the inclusions

$$\operatorname{supp}\mu_{[f_{1n}>t]}\subset\partial[f_{1n}>t],$$
$$\operatorname{supp}\mu_{[f_{in}<t]}\subset\partial[f_{in}<-t],$$
$$\operatorname{supp}\mu_{A_n(t)}\cap\Omega\subset\partial_\Omega A_n(t),$$

we get

$$\frac{1}{\varepsilon_n}\int_\Omega \sum_{i=1}^N \tilde{u}_{in}|\nabla f_{in}|^p \,\mathrm{d}x$$
$$\geq \frac{1}{N^{p-1}|\Omega|^{p-1}}\left(\int_{\frac{\sigma}{2}}^\sigma\left(\mu_{[f_{1n}>t]}(\partial_\Omega A_n(t))+\sum_{i=1}^N \mu_{[f_{in}<-t]}(\partial_\Omega A_n(t))\right)\mathrm{d}t\right)^p$$
$$=\frac{1}{N^{p-1}|\Omega|^{p-1}}\left(\int_{\frac{\sigma}{2}}^\sigma\left(P([f_{1n}>t];\partial_\Omega A_n(t))+\sum_{i=1}^N P([f_{in}<-t];\partial_\Omega A_n(t))\right)\mathrm{d}t\right)^p$$
$$\geq\frac{1}{N^{p-1}|\Omega|^{p-1}}\left(\int_{\frac{\sigma}{2}}^\sigma P(A_n(t);\partial_\Omega A_n(t))\mathrm{d}t\right)^p$$
$$=\frac{1}{N^{p-1}|\Omega|^{p-1}}\left(\int_{\frac{\sigma}{2}}^\sigma P(A_n(t);\Omega)\mathrm{d}t\right)^p.$$

Estimating the relative perimeter by Lemma 10, we conclude that

$$\frac{1}{\varepsilon_n}\int_\Omega \sum_{i=1}^N \tilde{u}_{in}|\nabla f_{in}|^p \,\mathrm{d}x \geq \frac{1}{N^{p-1}|\Omega|^{p-1}}\left(\frac{\sigma p_0}{2}\right)^p.$$

Now from (31) we get

$$\frac{1}{N^{p-1}|\Omega|^{p-1}}\left(\frac{\sigma p_0}{2}\right)^p \leq M\varepsilon_n|[v_n \leq \varepsilon_n]|,$$

where the left-hand side is a positive constant, and the right-hand side goes to 0 as $n \to \infty$, a contradiction.

Appendix

Proof of Proposition 1. Denote the solution set of (19) by P.

Step 1. Basis of the induction. We use induction over N. A direct verification shows that the statement is true for $N = 1$.

Step 2. Positivity of the determinants Δ_I implies that P is bounded. It is well-known that the polyhedron P is bounded if and only if the homogeneous system

$$\left\{\begin{aligned} a_{11}u_1 + \cdots + a_{1N}u_N &\leq 0,\\ &\dots\\ a_{N1}u_1 + \cdots + a_{NN}u_N &\leq 0,\\ u_1 &\geq 0,\\ &\dots\\ u_N &\geq 0, \end{aligned}\right. \tag{53}$$

admits only the trivial solution. Assume that contrary to the claim, system (53) has a nontrivial solution. As the system has rank N, it has a nontrivial solution $\mathbf{b} = (b_1, \dots, b_N)$ satisfying exactly $N-1$ independent inequalities with equality. If

$$b_N = 0 = a_{N1}b_1 + \cdots + a_{NN}b_N,$$

then $(b_1, \dots, b_{N-1})$ is a nontrivial solution of

$$\left\{\begin{aligned} a_{11}u_1 + \cdots + a_{1,N-1}u_{N-1} &\leq 0,\\ &\dots\\ a_{N-1,1}u_1 + \cdots + a_{N-1,N-1}u_{N-1} &\leq 0,\\ u_1 &\geq 0,\\ &\dots\\ u_{N-1} &\geq 0, \end{aligned}\right. ,$$

which contradicts the induction assumption. More generally, for no i can we have

$$b_i = 0 = a_{i1}b_1 + \cdots + a_{iN}b_N.$$

Thus, without loss of generality we can assume that $b_{r+1} = \dots = b_N = 0$, $b_r = 1$, while b_1, $\dots$, b_{r-1} are positive, solve

$$\begin{cases} a_{11}u_1 + \dots + a_{1,r-1}u_{r-1} = -a_{1r}, \\ \dots \\ a_{r-1,1}u_1 + \dots + a_{r-1,r-1}u_{N-1} = -a_{r-1,r}, \end{cases} \tag{54}$$

and satisfy

$$a_{r1}b_1 + \dots + a_{r,r-1}b_{N-1} + a_{rr} < 0. \tag{55}$$

Using Cramer's rule to solve (54), we see that

$$b_r = \frac{\Delta_i}{\Delta_r} \quad (i = 1, \dots, r-1), \tag{56}$$

where Δ_i is the r, i-cofactor of the matrix

$$\begin{bmatrix} a_{11} & \dots & a_{1r} \\ \vdots & \ddots & \vdots \\ a_{r1} & \dots & a_{rr} \end{bmatrix}.$$

for $i = 1, \dots, r$. Plugging the representation (56) into (55) and applying the Laplace formula, we obtain

$$\frac{\begin{vmatrix} a_{11} & \dots & a_{1r} \\ \vdots & \ddots & \vdots \\ a_{r1} & \dots & a_{rr} \end{vmatrix}}{\begin{vmatrix} a_{11} & \dots & a_{1,r-1} \\ \vdots & \ddots & \vdots \\ a_{r-1,1} & \dots & a_{r-1,r-1} \end{vmatrix}} < 0,$$

which contradicts the positivity of all Δ_I. Thus, P is bounded.

Step 3. Positivity of the determinants implies that the set of vertices of P is $\{\mathbf{u}_I\colon I \subset \{1, \dots, N\}\}$. According to Remark 6, each $\mathbf{u}_I$ solves (19). Moreover, $\mathbf{u}_I$ satisfies with equality the subsystem

$$\begin{cases} f_j \geq 0 & (j \notin I), \\ u_i \geq 0 & (i \in I) \end{cases}$$

of (19) of rank N. Consequently, each $\mathbf{u}_I$ is a vertex of P. We must prove that P has no other vertices.

In the hyperplane $u_N = 0$ consider the polyhedron P' being the solution set of

$$\begin{cases} \tilde{f}_i \geq 0 & (i = 1, \dots, N-1), \\ u_i \geq 0 & (i = 1, \dots, N-1), \end{cases} \tag{57}$$

where $\tilde{f}_i$ is the restriction of f_i to the hyperplane. By the induction assumption, P' is an $(N-1)$-dimensional polytope with vertices $\{\mathbf{u}_I\colon I \ni N\}$. We claim that it is a facet of P.

Indeed, let $\widetilde{P}$ be the face of P on the hyperplane $\{u_N = 0\}$. On this hyperplane $\widetilde{P}$ is given by

$$\begin{cases} \tilde{f}_i \geq 0 & (i = 1, \dots, N), \\ u_i \geq 0 & (i = 1, \dots, N-1), \end{cases} , \tag{58}$$

and the inclusion $\widetilde{P} \subset P$ is immediate. On the other hand, by the induction assumption, any vertex of P' is one of the points $\mathbf{u}_I$ lying in the hyperplane, so it is a vertex of P and also of $\widetilde{P}$. Consequently, $P' \subset \widetilde{P}$. Thus, we have $P' = \widetilde{P}$, and P' is an $(N-1)$-dimensional face of P. The vertex $\mathbf{u}_\varnothing$ of P does not belong to the hyperplane $\{u_N = 0\}$, so P has dimension N, and P' is its facet.

Likewise, the interection of P with any hyperplane $u_i = 0$ is a facet of P having the vertices $\{\mathbf{u}_I \colon I \ni i\}$.

Let $\mathbf{v} = (v_1, \dots, v_N)$ be a vertex of P. Then $\mathbf{v}$ satisfies with equalities a subsystem of (19) of rank N consisting of N inequalities. If this subsystem is $f_i \geq 0$ $(i = 1, \dots, N)$, then $\mathbf{v} = \mathbf{u}_\varnothing$. Otherwise, we have $v_i = 0$ for some i, so $\mathbf{v}$ lies in the hyperplane $u_i = 0$ and by the above, coincides with one of $\mathbf{u}_I$. Thus, the set of vertices of P is exactly $\{\mathbf{u}_I \colon I \subset \{1, \dots, N\}\}$.

Step 4. Positivity of the determinants implies that the facets of P are the intersections of P with the hyperplanes $u_i = 0$ and $f_i = 0$, $i = 1, \dots, N$. As P is given by (19), each facet of P is the intersection of P with a hyperplane of the form $u_i = 0$ or $f_i = 0$. Conversely, each intersection of this form is a facet of P. Indeed, we have already checked this in the case of the hyperplanes $u_i = 0$. Now let $P' = P \cap \{f_1 = 0\}$. The face P' contains, among others, the vertices $\mathbf{u}_I$, where $I \not\ni 1$ and $I \ni N$. By the induction assumption, these are precisely the vertices of an $(N-2)$-dimensional facet of $P'' = P \cap \{u_N = 0\}$. Thus, $\dim P' \geq N - 2$. On the other hand, P' contains $\mathbf{u}_\varnothing$, which is affinely independent of $\{\mathbf{u}_I \colon I \ni N\}$, so actually $\dim P' = N - 1$, as claimed.

Step 5. Positivity of the determinants implies that P is combinatorially isomorphic to a cube. Indeed, considering the cube as the solution set of

$$\begin{cases} 1 - u_i \geq 0 & (i = 1, \dots, N), \\ u_i \geq 0 & (i = 1, \dots, N), \end{cases} ,$$

we see that the mapping $\mathbf{u}_I \mapsto (\alpha_1, \dots, \alpha_N)$, where

$$\alpha_i = \begin{cases} 0, & \text{if } i \in I, \\ 1 & \text{otherwise}, \end{cases}$$

preserves facets.

Step 6. The geometric properties of P imply the positivity of the determinants. Conversely, assume that solution set of (19) is a polytope with vertices $\{\mathbf{u}_I \colon I \subset \{1, \dots, N\}\}$ combinatorially isomorphic to a cube. Observe that given i, the set of vertices of the facet $P \cap \{u_i = 0\}$ is $\{\mathbf{u}_I \colon I \ni i\}$. Consequently, if $i \in I$, the vertex $\mathbf{u}_{\complement I}$ does not belong to this facet, so $u_{\complement I i} > 0$ whenever $i \in I$. Likewise, $f_j(\mathbf{u}_{\complement I}) > 0$ whenever $j \notin I$. Now it follows from (18) that all the determinants Δ_I and $\Delta_{I,j}$ have the same sign. As the facet

$P' = P \cap \{u_N = 0\}$ enjoys analogous geometric properties, by induction we see that the determinants are actually positive. □

Proof of Lemma 1. We fix an admissible set of coefficients and j and prove that

$$\sup\{\min_i f_i(\mathbf{u})\colon \mathbf{u} = (u_1, \dots, u_N) \geq 0, u_j \leq \sigma, f_j(\mathbf{u}) \leq \sigma\} \leq -\sigma \tag{59}$$

with some σ independent of the coefficients. For definiteness, assume that $j = N$.

As above, denote the polytope given by (19) by P.

It follows from Remark 6 that there exists $c > 0$ independent of the coefficients such that P has no vertices in the open slab $\{0 < u_N < 2c\}$. In other words, all the vertices of the polytope $P_{2c} := P \cap \{0 \leq u_N \leq 2c\}$ lie on the hyperplanes $u_N = 0$ and $u_N = 2c$. It is easy to check that any point of P_{2c} belonging to the hyperplane $u_N = c$ is the midpoint of a line segment with the endpoints on the facets $P'_{2c} = P_{2c} \cap \{u_N = 0\}$ and $P''_{2c} = P_{2c} \cap \{u_N = 2c\}$ of P, the former being also a facet of P.

By Remark 6, there exists $c' > 0$ independent of the coefficients such that $f_N \geq 2c'$ on each vertex of the facet $P \cap \{u_N = 0\} = P'_{2c}$, so $f_N \geq 2c'$ on P'_{2c}. Also $f_N \geq 0$ on $P''_{2c} \subset P$. Consequently, $f_N \geq c'$ on $P_{2c} \cap \{u_N = c\} = P \cap \{u_N = c\}$.

By the above, all the vertices of the polytope $P_c = P \cap \{0 \leq u_N \leq c\}$ lie in the halfspace $f_N \geq c'$, therefore so does the polytope itself. In other words, the polytope P_c is the solution set of

$$\begin{cases} f_i \geq 0 \quad (i = 1, \dots, N-1), \\ u_i \geq 0 \quad (i = 1, \dots, N), \\ u_N \leq c, \end{cases}$$

and this system implies the inequality $f_N - c' \geq 0$. By the Minkowski–Farkas theorem, there exist nonnegative α_i, β_i, γ, and δ such that

$$f_N - c' = \sum_{i=1}^{N-1} \alpha_i f_i + \sum_{i=1}^{N} \beta_i u_i + \gamma(c - u_N) + \delta. \tag{60}$$

Generally speaking, representation (60) is not unique, but we claim that possible values of α_i are uniformly bounded with respect to admissible coefficients. Indeed, plugging the zero vertex into (60), we get

$$f_N(0) - c' = \sum_{i=1}^{N-1} \alpha_i \tilde{f}_i(0) + \gamma c + \delta,$$

whence

$$\sum_{i=1}^{N-1} \alpha_i f_i(0) \leq f_N(0) - c'.$$

By Remark 6, the right-hand side is bounded, and the values of $f_i(0)$ in the left-hand side are bounded away from 0. Consequently, there exists $C > 0$ independent of admissible coefficients such that

$$\alpha_i \leq C$$

for any possible choice of α_i, as claimed.

Now write (60) in the form

$$\sum_{i=1}^{N-1} \alpha_i f_i = -\frac{c'}{2} + \left(f_N - \frac{c'}{2}\right) - \sum_{i=1}^{N} \beta_i u_i - \gamma(c - u_n) - \delta$$

and observe that whenever $u_N \le c$ and $f_N \le c'/2$, we have

$$\sum_{i=1}^{N-1} \alpha_i f_i \le -\frac{c'}{2},$$

whence

$$\sum_{f_i<0} f_i \le -\frac{c'}{2C}.$$

There are at most N summands in the left-hand side, so for any $\mathbf{u}$ satisfying said requirements there exists f_i such that $f_i(\mathbf{u}) \le -c'/(2CN)$. Thus, (59) is valid with $\sigma = \min\{c, c'/2, c'/(2CN)\}$, which is clearly independent of particular choice of admissible coefficients. □

The following lemma is the first step towards proving Lemma 2. It ensures that v does not vanish at infinity.

Lemma 12.

$$\lim_{\substack{|\mathbf{u}|\to\infty \\ \mathbf{u}\in\mathbb{R}_+^N}} v = \infty \tag{61}$$

uniformly with respect to admissible coefficients.

Proof. Assume the contrary: there exist $C > 0$, a sequence of admissible sets of coefficients $\{(A_n, \mathbf{m}_n)\}$ and a sequence $\{\mathbf{u}_n\} \subset \mathbb{R}^N$ such that $\mathbf{u}_n \ge 0$, $\mathbf{u}_n \to \infty$, and $0 \le v_n(\mathbf{u}_n) \le C$, where v_n is corresponding to the coefficients $(A_n, \mathbf{m}_n)$. By f_{in} denote the affine functions corresponding to $(A_n, \mathbf{m}_n)$, and by f'_{in}, associated linear functionals.

By Assumption 1, the sequence $\{(A_n, \mathbf{m}_n)\}$ is bounded. Without loss of generality, $(A_n, \mathbf{m}_n) \to (A_*, \mathbf{m}_*)$, the limiting coefficients also being admissible. Let f_{i*} be the corresponding affine functions, and f'_{i*} be the associated linear functions.

Without loss of generality,

$$u_{in} = b_i \tau_n + o(\tau_n), \tag{62}$$

where $b_i \ge 0$ is finite, $\mathbf{b} = (b_1, \dots, b_N) \ne 0$, $\tau_n \to \infty$. We have:

$$f_{in}(\mathbf{u}_n) = m_{in} + f'_{in}(\mathbf{u}_n) = m_{in} + f'_{in}(\mathbf{b})\tau_n + o(\tau_n) = f'_{in}(\mathbf{b})\tau_n + o(\tau_n)$$

As $f'_{in} \to f'_{i*}$ and the sequences $\{m_{in}\}$ are bounded, we obtain

$$f_{in}(\mathbf{u}_n) = f'_{i*}(\mathbf{b})\tau_n + o(\tau_n). \tag{63}$$

Plugging representations (62) and (63) into the right-hand side of (23), we obtain:

$$\sum_{i=1}^{N} u_{in} |f_{in}(\mathbf{u}_n)|^p = \left(\sum_{i=1}^{N} b_i |f'_{i*}(\mathbf{b})|^p\right) \tau_n^{p+1} + o(\tau_n^{p+1}).$$

Here the leading coefficient does not vanish. If it did, we would have $b_i|f'_{i*}| = 0$ for any i, so $\mathbf{b}$ would solve the linear system

$$\begin{cases} f'_{i*}(\mathbf{b}) = 0 \quad (i \in I), \\ b_i = 0 \quad (i \notin I) \end{cases}$$

for some $I \subset \{1, \dots, N\}$. But due to Assumption 2 this system only has the trivial solution.

Thus, the right-hand side of (23) grows as τ_n^{p+1}. On the other hand, a trivial verification shows that the left-hand side of (23) is $O(\tau_n^p)$, a contradiction. □

Proof of Lemma 2. Suppose, contrary to our claim, that there exist $\varepsilon > 0$, a sequence $\{(A_n, \mathbf{m}_n)\}$ of admissible coefficients, and a sequence $\{\mathbf{u}_n = (u_{1n}, \dots, u_{Nn})\} \subset \mathbb{R}_+$ such that

$$v_n(\mathbf{u}_n) \to 0$$

and

$$|\mathbf{u}_n - \mathbf{u}_{In}| \geq \varepsilon \quad (I \neq \varnothing), \tag{64}$$

where v_n and $\mathbf{u}_{In}$ corresponds to the coefficients $(A_n, \mathbf{m})$. Due to Assumptions 1–3 (see also Remark 6) there is no loss of generality in assuming that $(A_n, \mathbf{m}) \to (A_*, \mathbf{m}_*)$, where the limit is also admissible, and $\mathbf{u}_{I_n} \to \mathbf{u}_{I*}$ for each $I \neq \varnothing$, where the limit satisfies (12). Thus, if v_* corresponds to the limiting coefficients, we have $v_*(\mathbf{u}) = 0$ if and only if $\mathbf{u} = \mathbf{u}_{I*}$ for some $I \neq \varnothing$.

Due to Lemma 12, $\{\mathbf{u}_n\}$ is bounded, and we can assume that $\mathbf{u}_n \to \mathbf{u}_* = (u_{1*}, \dots, u_{N*}) \in \mathbb{R}_+^N$. Passing to the limit in (64), we get

$$|\mathbf{u}_* - \mathbf{u}_{I*}| \geq \varepsilon \quad (I \neq \varnothing). \tag{65}$$

By (24),

$$\min_i u_{i*} = \lim_{n\to\infty} \min_i u_{in} \leq \lim_{n\to\infty} v_n(\mathbf{u}_n) = 0,$$

so $\mathbf{u}_* \neq \mathbf{u}_\varnothing$ (Remark 6). Thus, $g_*(\mathbf{u}_*) \neq 0$ and we can pass to the limit:

$$v_*(\mathbf{u}_*) = \lim_{n\to\infty} v_n(\mathbf{u}_n) = 0.$$

This and the fact that $\mathbf{u}_*$ is nonnegative implies $\mathbf{u}_* = \mathbf{u}_{I*}$ for some I, which contradicts (65). □

Proof of Corollary 1. Given $I \neq \varnothing$, consider the norm

$$|\mathbf{u}|_I = \sum_{i\in I} |u_i| + \sum_{j\notin I} |f'_j(\mathbf{u})|,$$

which implicitly depends on the choice admissible coefficients. Observe that

$$C_1|\mathbf{u}| \leq |\mathbf{u}|_I \leq C_2|\mathbf{u}|, \tag{66}$$

where C_1 and C_2 depend on κ but not on admissible coefficients. Indeed, letting for simplicity $I = \{1, \dots, r\}$, we have $|\mathbf{u}|_I = |A_I \mathbf{u}|$, where

$$A_I = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & \mathbf{0} & \\ & & 1 & & & \\ & & & -a_{r+1,r+1} & \dots & -a_{rN} \\ & \mathbf{0} & & \vdots & \ddots & \vdots \\ & & & -a_{N,r+1} & \dots & -a_{NN} \end{bmatrix}$$

and it follows from Assumptions 1 and 2 that the norms $\|A_I\|$ and $\|A_I^{-1}\|$ are bounded uniformly with respect to admissible coefficients.

By Assumption 1, there exits C depending on κ such that for any admissible coefficients $\|f_i'\| \le C$ for any i, where $\|\cdot\|$ is the norm of a linear functional on $\mathbb{R}^N$.

Take $\varepsilon > 0$. By Lemma 2 there exists $\delta > 0$ independent of admissible coefficients such that whenever $v(\mathbf{u}) < \delta$, we have

$$\min_{I \neq \varnothing} |\mathbf{u} - \mathbf{u}_I| < \frac{\varepsilon}{C + C_2}.$$

Take $I \neq \varnothing$ such that

$$|\mathbf{u} - \mathbf{u}_I| < \frac{\varepsilon}{C + C_2},$$

then (26) holds.

By Remark 6, for any $i \in I$ we have $f_i(\mathbf{u}_I) \geq c$ with c independent of admissible coefficients, so

$$f_i(\mathbf{u}) = f_i(\mathbf{u}_I) + f_i'(\mathbf{u} - \mathbf{u}_I) \geq c - \varepsilon.$$

Without loss of generality, $\varepsilon < c/2$, so (27) holds with $\sigma = c/2$. □

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