Monotone normality and stratifiability from a pointfree point of view[☆,☆☆]

Javier Gutiérrez García[a], Jorge Picado[b,*], María Ángeles de Prada Vicente[a]

[a] *Department of Mathematics, University of the Basque Country UPV/EHU, Apdo. 644, 48080 Bilbao, Spain*
[b] *CMUC, Department of Mathematics, University of Coimbra, 3001-501 Coimbra, Portugal*

Abstract

Monotone normality is usually defined in the class of T_1 spaces. In this paper we study it under the weaker condition of subfitness, a separation condition that originates in pointfree topology. In particular, we extend some well known characterizations of these spaces to the subfit context (notably, their hereditary property and the preservation under surjective continuous closed maps) and present a similar study for stratifiable spaces, an important subclass of monotonically normal spaces. In the second part of the paper, we extend further these ideas to the lattice theoretic setting. In particular, we give the pointfree analogues of the previous results on monotonically normal spaces and introduce and investigate the natural pointfree counterpart of stratifiable spaces.

Keywords: Monotone normality, Borges operator, hereditary monotone normality, monotonically normal operator, stratifiability, subfit space, frame, locale, subfit frame, weakly subfit frame, open sublocale, closed map.
2010 MSC: 54D15, 06D22, 54C20, 54C99.

1. Introduction

Fifty years ago, Borges [4, Lemma 2.1] introduced and Zenor ([34], see [16]) named the notion of monotone normality, a strengthening of normality. Since that pioneering papers, there has been an extensive literature on the topic (see e.g. [5, 7, 25, 32] for references). Every metrizable space and every linearly ordered space is monotonically normal. In fact, it could be argued that whenever a space can be shown "explicitly" to be normal, then it is probably monotonically normal.

[☆] In memory of Sergio Salbany.

[☆☆] Research supported by the Ministry of Economy and Competitiveness of Spain (under grant MTM2012-37894-C02-02), the UPV/EHU (under grants UFI11/52 and GIU12/39) and the Centre for Mathematics of the University of Coimbra (funded by the European Regional Development Fund through the program COMPETE and by the Portuguese Government through the Fundação para a Ciência e a Tecnologia, under the project PEst-C/MAT/UI0324/2011).

[*] Corresponding author

Email addresses: `javier.gutierrezgarcia@ehu.es` (Javier Gutiérrez García), `picado@mat.uc.pt` (Jorge Picado), `mariangeles.deprada@ehu.es` (María Ángeles de Prada Vicente)

URL: `www.ehu.es/javiergutierrezgarcia` (Javier Gutiérrez García), `www.mat.uc.pt/~picado` (Jorge Picado)

Monotone normality is usually treated in the class of T_1 spaces (in this context, a space is monotonically normal iff it is hereditarily monotonically normal [5], i.e., every its subspace is monotonically normal). Apart [21] and, more recently, [12, 13, 15], monotone normality has been considered in the restricted class of T_1 spaces. In [12], Gutiérrez García, Mardones-Pérez and de Prada Vicente undertook the study of monotone normality free of the T_1 property and obtained new characterizations of monotone normality for general spaces. In addition, they showed that monotone normality is not, in general, an hereditary property.

In the present paper, by approaching the problem from a pointfree point of view, we are able at the same time to improve these results and to extend them to the pointfree setting. Our primary motivating question is the following: is there any separation axiom weaker than T_1 under which monotone normality becomes an hereditary condition?

The T_1 axiom for spaces is so heavily dependent on points that one cannot expect an exact pointfree counterpart for it. Subfit frames [19] and, sometimes, unordered (T_U) frames [19, 20] have been considered as candidates but both fail to coincide with the T_1 property in the spatial case. The former form a strictly weaker counterpart of T_1 spaces [24]. They were introduced by Isbell in [19] and independently (as *conjunctivity*, because it is the opposite of the disjunctive property for distributive lattices) by Simmons [30]. A frame L is said to be *subfit* if

$$a \nleq b \quad \Longrightarrow \quad \exists\, c \in L\colon a \vee c = 1 \neq b \vee c. \tag{Sfit}$$

As remarked by Isbell [19] (and also by Simmons [30]), given a space (X, OX), the frame OX of open sets is subfit if and only if the underlying space satisfies the following condition:

$$\forall U \in OX,\ \forall x \in U, \quad \exists\, y \in \overline{\{x\}} \text{ such that } \overline{\{y\}} \subseteq U. \tag{Conj}$$

Simmons (see e.g. [31, Lemma 4.8]) noted that

$$T_1 = (\mathrm{Conj}) + T_D$$

where T_D is the familiar separation axiom between T_0 and T_1 (in fact, much closer to T_0 than to T_1) due to Aull and Thron [1], requiring that each point $x \in X$ has an open neighborhood U such that $U \smallsetminus \{x\}$ is also open.

Our main goal with this paper is to study the role of subfitness within monotone normality, first in spaces and then in the more general pointfree setting. The notion of a stratifiable frame will appear naturally as an interesting subclass of monotonically normal frames. They are the pointfree counterpart of the stratifiable spaces introduced by Ceder [6] and also studied by Borges [4] (to whom the name stratifiable is due). In particular, we will see that monotone normality is hereditary under subfitness while stratifiability is always hereditary. Further, we will study the preservation of both properties under closed maps.

The paper is organized as follows. In Section 2 we study the role of the subfitness axiom on monotonically normal spaces with the aim of extending the results in [5] from the class of T_1 spaces to the broader class of subfit spaces. In Section 3 we address perfectly normal spaces and stratifiable spaces. In Section 4 we show how those classical topological variants of normality can be naturally stated in a general lattice, yielding natural dual concepts closely related to that of extremal disconnectedness. These first sections emphasize the role (and usefulness) of the pointfree point of view in clarifying classical topological concepts and ideas and underlying principles. After recalling, in Section 5, the background on the category of frames and the corresponding pointfree approach to topology needed in the last two sections of the paper, we broaden the extent of the topological ideas of the first sections, by introducing and investigating monotonically normal frames (Section 6) and stratifiable frames (Section 7).

2. Monotonically normal spaces

What is the monotonization process of a topological concept? Quoting [11], take any property of a space, like normality, that can be formulated in terms of a map $\Delta\colon P \to Q$. By partially ordering sets P and Q and imposing Δ to be a monotone (i.e, order-preserving) map one defines the corresponding monotone variant of the given property.

For instance, let X be a topological space with lattice of open sets $\mathcal{O}X$ and let $\mathcal{C}X$ denote the corresponding family of closed subsets. Recall that X is normal if there is a map $\Delta\colon \mathscr{D}_X \to \mathcal{O}X$, where

$$\mathscr{D}_X = \{(F,U) \in \mathcal{C}X \times \mathcal{O}X \mid F \subseteq U\},$$

such that $F \subseteq \Delta(F,U) \subseteq \overline{\Delta(F,U)} \subseteq U$ for every $(F,U) \in \mathscr{D}_X$. Then a space is *monotonically normal* [16] whenever Δ is monotone (where $\mathscr{D}_X$ is assumed to have the componentwise order) and it is *hereditarily monotonically normal* if every its subspace is monotonically normal. The function Δ that witnesses monotone normality is then referred to as a monotone normality operator. Note that contrary to the usual practice in the literature, we do not assume monotonically normal spaces as being T_1.

For more examples of monotonization of topological concepts see [26, 17, 3, 33, 10].

When the space is assumed to be T_1, we have the following:

Proposition 2.1 ([5]). *Let X be a T_1 topological space. The following are equivalent:*

(1) *X is monotonically normal.*

(2) *For each pair (x,U), where U is an open set containing x, there is an open set $G(x,U)$ such that:*
   - (i) *$x \subseteq G(x,U) \subseteq U$;*
   - (ii) *if $x \in U \subseteq V$ then $G(x,U) \subseteq G(x,V)$;*
   - (iii) *if $G(x,U) \cap G(y,V) \neq \varnothing$ then either $x \in V$ or $y \in U$.*

Such an assignment $(x,U) \mapsto G(x,U)$ is sometimes called a *Borges monotone normality operator* or simply a *Borges operator*.

Remarks 2.2. (1) It is worth mention that the T_1 axiom is only needed in the implication (1) $\Rightarrow$ (2) since any Borges operator G induces a monotone normality operator (cf. [5, Th. 1.2]) by defining, for each $(F,U) \in \mathscr{D}_X$,

$$\Delta_G(F,U) = \bigcup_{x \in F} G(x,U).$$

(2) It is really easy to check that the property of having a Borges operator is hereditary. Indeed, if X has a Borges operator G, then for each $\varnothing \neq A \subseteq X$ and each (x,U) such that U is an open in A and $x \in U$, let

$$G_A(x,U) = G(x, X \smallsetminus \overline{A \smallsetminus U}) \cap A. \tag{$*$}$$

Clearly enough, $G_A(x,U)$ is open in A, $x \in G_A(x,U) \subseteq (X \smallsetminus \overline{A \smallsetminus U}) \cap A = U$ and $G_A(x,U) \subseteq G_A(x,V)$ whenever $x \in U \subseteq V$. Also, if $G_A(x,U) \cap G_A(y,V) \neq \varnothing$ then either $x \in X \smallsetminus \overline{A \smallsetminus V}$ or $y \in X \smallsetminus \overline{A \smallsetminus U}$ and so either $x \in V$ or $y \in U$.

In conclusion, we have the following for any space X:

$$\begin{gathered} X \text{ has a Borges operator} \\ \Downarrow \text{(A)} \\ X \text{ is hereditarily monotonically normal} \\ \Downarrow \text{(B)} \\ \text{Every open subspace of } X \text{ is monotonically normal} \\ \Downarrow \text{(C)} \\ X \text{ is monotonically normal.} \end{gathered}$$

Of course, as a consequence of the characterization in Proposition 2.1, for the class of T_1 spaces the converse implications also hold (see [5, Thm. 1.2]) and hence, in particular, monotone normality is an hereditary property. However, as proved in [12], Proposition 2.1 (in particular, the implication (1) $\Rightarrow$ (2)) is no longer valid if the space fails to be T_1, and monotone normality is not hereditary with respect to open subspaces (for example, given a non-normal topological space X and $X^* = X \cup \{\omega\}$ endowed with the topology $\mathcal{O}X^* = \mathcal{O}X \cup \{X^*\}$, then X is a dense open subspace of the monotonically normal space X^*). This means that the converse to the implication (C) above fails to be true in general.

What about the converses to (A) and (B) in the general case?

Even if the converse to (B) is well-known to hold in the case of normal spaces, as we have already mentioned, the usual proof is not “monotonizable”. Concerning the converse of (A), [12] contains the following improvement of Proposition 2.1:

Proposition 2.3 ([12, Prop. 3.2])**.** *Let X be a topological space. The following are equivalent:*

(1) *X is monotonically normal;*
(2) *For each pair (x, U), where U is an open set containing $\overline{\{x\}}$, there is an open set $H(x, U)$ such that:*
   (i) *$\overline{\{x\}} \subseteq H(x, U) \subseteq U$;*
   (ii) *if $\overline{\{x\}} \subseteq U \subseteq V$ then $H(x, U) \subseteq H(x, V)$;*
   (iii) *if $H(x, U) \cap H(y, V) \neq \varnothing$ then either $x \in V$ or $y \in U$.*

Note that the implication (2) $\Rightarrow$ (1) above follows by the same argument used in Remark 2.2 (1), since no separation axiom is required: any such H induces a monotone normality operator by defining $\Delta_H(F, U) = \bigcup_{x\in F} H(x, U)$ for each $(F, U) \in \mathscr{D}_X$. The converse implication merely uses the fact that the subsets of the form $\overline{\{x\}}$ for some $x \in X$ are the minimal closed subsets in X.

Now, recall the subfitness condition (Conj) from the Introduction.

Lemma 2.4. *Let X be a normal space. The following are equivalent:*

(1) *X is subfit.*
(2) *X is weakly regular, i.e. if $U \in \mathcal{O}X$ and $x \in U$ then $\overline{\{x\}} \subseteq U$.*
(3) *X is regular, i.e. if $U \in \mathcal{O}X$ and $x \in U$ then there exists an $V \in \mathcal{O}X$ such that $x \in V \subseteq \overline{V} \subseteq U$.*

Moreover, if X is additionally T_0 then X is subfit iff it is T_1 iff it is T_2.

Proof. (1) $\Longrightarrow$ (2): Let $x \in U$. By subfitness, we may conclude that there exists $y \in \overline{\{x\}}$ such that $\overline{\{y\}} \subseteq U$. Since X is normal, it follows that there exists an open set V such that $y \in \overline{\{y\}} \subseteq V \subseteq \overline{V} \subseteq U$. It follows then that $x \in V$ (since $y \in \overline{\{x\}} \cap V$) and thus $\overline{\{x\}} \subseteq \overline{V} \subseteq U$.

(2) $\Longrightarrow$ (3) follows immediately from normality and (3) $\Longrightarrow$ (1) is trivial.

The last assertion is obvious. □

Remark 2.5. The notion of weak regularity is due to Morita [23] (this is also the R_0 condition of Davis [8]). Contrarily to what is mentioned in [18, p. 72], it is in general stronger than subfitness. Nevertheless, subfitness, weak regularity and regularity do coincide under normality, as shown above.

Now we can go back to Proposition 2.3 and conclude immediately the following:

Proposition 2.6. *Let X be a subfit topological space. The following are equivalent:*

(1) *X is monotonically normal.*
(2) *X has a Borges operator.*

Finally, it follows from Remark 2.2(2) and Proposition 2.3 that:

Corollary 2.7. *Let X be a subfit topological space. The following are equivalent:*

(1) *X has a Borges operator.*
(2) *X is hereditarily monotonically normal.*
(3) *Every open subspace of X is monotonically normal.*
(4) *X is monotonically normal.*

Further, as it is well known, monotone normality is preserved under surjective closed continuous functions in the class of T_1 spaces. Since the proof does not require any separation axiom, it holds in the more general context in which we have defined monotone normality (cf. Remark 6.13 below).

3. Stratifiable spaces

Recall that a space X is said to be *perfectly normal* if it is normal and every closed set is a G_δ set. Here again the T_1 axiom is not assumed as part of the definition of normality. However, it follows immediately from the definition that any perfectly normal space is subfit. Consequently, by Lemma 2.4, any T_0 perfectly normal space is T_2. It is easy to prove the following characterization of perfect normality which merely uses the lattice $\mathcal{O}X$ of open sets of X (cf. Exercise 1.5.K in [9]):

Fact 3.1. *A topological space X is perfectly normal if and only if it is normal and there exist* $\{\alpha_n, \beta_n : \mathcal{O}X \to \mathcal{O}X\}_{n\in\mathbb{N}}$ *satisfying:*

(1) $\bigcup_{n\in\mathbb{N}} \alpha_n(U) = U$ *for every* $U \in \mathcal{O}X$*;*
(2) $U \cup \beta_n(U) = X$ *and* $\alpha_n(U) \cap \beta_n(U) = \varnothing$ *for every* $n \in \mathbb{N}$ *and* $U \in \mathcal{O}X$*;*
(3) $\alpha_n(U) \subseteq \alpha_m(U)$ *and* $\beta_n(U) \supseteq \beta_m(U)$ *for every* $n \leq m$ *in* $\mathbb{N}$ *and* $U \in \mathcal{O}X$.

Remark 3.2. Note that the assumption of normality in Fact 3.1 is redundant since, by the infinite distributive law in the lattice $\mathcal{O}X$, conditions (1) and (2) imply that X is normal. The reason for including it here will be apparent later on when we will treat this characterization in a general lattice.

This of course leads to the monotone variant of perfect normality: a space X is *monotonically perfectly normal* whenever X is monotonically normal, each α_n is order-preserving and each β_n is order-reversing. This is not a new notion; it goes back to Ceder [6] (who called them M_3 spaces) and it was Borges [4] who named them *stratifiable spaces* (always with the additional assumption of the T_1 property) and called each sequence $\{\alpha_n(U)\}_{n\in\mathbb{N}}$ a *stratification* of X. We give here an equivalent formulation of the original definition:

Definition 3.3. A topological space X is *stratifiable* if and only if it is monotonically normal and there exist $\{\alpha_n, \beta_n : \mathcal{O}X \to \mathcal{O}X\}_{n\in\mathbb{N}}$ satisfying:

(1) $\bigcup_{n\in\mathbb{N}} \alpha_n(U) = U$ for every $U \in \mathcal{O}X$;

(2) $U \cup \beta_n(U) = X$ and $\alpha_n(U) \cap \beta_n(U) = \varnothing$ for every $n \in \mathbb{N}$ and $U \in \mathcal{O}X$;

(3′) $\alpha_n(U) \subseteq \alpha_m(V)$ and $\beta_n(U) \supseteq \beta_m(V)$ for every $n \leq m$ in $\mathbb{N}$ and $U \subseteq V$ in $\mathcal{O}X$.

Remark 3.4. Here again the assumption of monotone normality can be avoided, since it is implied by conditions (1), (2) and (3′) and the infinite distributive law (see Definition 4.4 and Proposition 7.1).

It is well known and worth mention here that, in the class of T_1-spaces, stratifiability is an hereditary property and it is preserved by closed continuous functions [4, 6]. The extension of these properties to a larger class of spaces will be obtained as corollaries of our pointfree results (namely, Corollaries 7.7 and 7.11). Note that, in contrast with Corollary 2.7, the subfitness condition is not needed since stratifiable spaces are always subfit.

4. Dual notions: extremally disconnected spaces

Those properties on a space which merely depend on the lattice of the open subsets, as normality or its dual (extremal disconnectedness) [14], can be described in pure lattice theoretical terms: a topological space $(X, \mathcal{O}X)$ is

– normal if

$$a \vee b = 1 \text{ in } \mathcal{O}X \implies \exists u, v \in \mathcal{O}X\colon u \wedge v = 0 \text{ and } a \vee u = 1 = b \vee v. \tag{4.1}$$

– extremally disconnected if

$$a \wedge b = 0 \text{ in } \mathcal{O}X \implies \exists u, v \in \mathcal{O}X\colon u \vee v = 1 \text{ and } a \wedge u = 0 = b \wedge v. \tag{4.2}$$

Conditions (4.1) and (4.2) are dual to each other and are formulable in any lattice and so one may speak more generally about *normal* and *extremally disconnected lattices*. Evidently, a lattice L is normal iff its dual lattice L^{op} is extremally disconnected. Introducing

$$\mathscr{D}_L = \{(a,b) \in L \times L \mid a \vee b = 1\} \quad (\text{hence } \mathscr{D}_{L^{op}} = \{(a,b) \in L \times L \mid a \wedge b = 0\})$$

we have:

– A lattice L is *normal* if there exists a map $\Delta\colon \mathscr{D}_L \to \mathscr{D}_{L^{op}}$, $(a,b) \mapsto (\Delta_1(a,b), \Delta_2(a,b))$, such that

$$(a, \Delta_1(a,b)), (b, \Delta_2(a,b)) \in \mathscr{D}_L \quad \text{for all } (a,b) \in \mathscr{D}_L. \tag{N}$$

– A lattice L is *extremally disconnected* if L^{op} is normal, i.e. if there exists a map $\Delta\colon \mathscr{D}_{L^{op}} \to \mathscr{D}_L$ such that

$$(a, \Delta_1(a,b)), (b, \Delta_2(a,b))) \in \mathscr{D}_{L^{op}} \quad \text{for all } (a,b) \in \mathscr{D}_{L^{op}}. \tag{ED}$$

A. Monotone normality vs monotone extremal disconnectedness

By naturally endowing $\mathscr{D}_L$ with the componentwise order inherited from $L^{op} \times L$, i.e.,

$$(a, b) \leq_{\mathscr{D}_L} (c, d) \equiv a \geq c \text{ and } b \leq d,$$

we may extend the notion of monotone normality from spaces to arbitrary lattices and hence, by applying it to the dual lattice, we may arrive to the natural notion of monotone extremal disconnectedness. Does this give any interesting new concept? Our main goal in this section is precisely to investigate what do the duals of the topological notions studied in the preceding sections yield.

Definition 4.1. Let L be a lattice. We say that:

(1) L is *monotonically normal* if there exists a monotone $\Delta\colon \mathscr{D}_L \to \mathscr{D}_{L^{op}}$ such that

$$(a, \Delta_1(a, b)), (b, \Delta_2(a, b)) \in \mathscr{D}_L \quad \text{for all } (a, b) \in \mathscr{D}_L. \tag{MN}$$

(2) L is *monotonically extremally disconnected* if L^{op} is monotonically normal, i.e. if there exists a monotone $\Delta\colon \mathscr{D}_{L^{op}} \to \mathscr{D}_L$ such that

$$(a, \Delta_1(a, b)), (b, \Delta_2(a, b)) \in \mathscr{D}_{L^{op}} \quad \text{for all } (a, b) \in \mathscr{D}_{L^{op}}. \tag{MED}$$

In particular when $L = \mathcal{O}X$ for some topological space X we obtain a new topological notion, dual to that of monotone normality.

Let L be a complete lattice in which finite joins distribute over arbitrary meets. The *pseudosupplement* of each $a \in L$ (i.e., the pseudocomplement in L^{op}) does exist and is given by $a^\# = \bigwedge\{b \in L \mid a \vee b = 1\}$ (see the following section for related information). It satisfies rules like

$$a \geq a^{\#\#}, \quad a \leq b \implies b^\# \leq a^\#, \quad a^{\#\#\#} = a^\# \quad \text{and} \quad \Big(\bigwedge_{i\in I} a_i\Big)^\# = \bigvee_{i\in I} a_i^\# \tag{4.3}$$

that will be used in the next proposition.

Proposition 4.2. *Let L be a complete lattice in which finite joins distribute over arbitrary meets. Then L is monotonically normal if and only if L is normal.*

Proof. Suppose L is normal and let $\Delta\colon \mathscr{D}_L \to \mathscr{D}_{L^{op}}$ be a a normality operator satisfying (N). For each $(a, b) \in \mathscr{D}_L$ we have

$$\Delta_1(a, b) \wedge \Delta_2(a, b) = 0, \quad a \vee \Delta_1(a, b) = 1 \quad \text{and} \quad b \vee \Delta_2(a, b) = 1.$$

Then, immediately, $\Delta_1(a, b) \geq a^\#$ and $\Delta_2(a, b) \geq b^\#$, and thus $a^\# \wedge b^\# = 0$. This ensures that the map $\Delta'\colon \mathscr{D}_L \to \mathscr{D}_{L^{op}}$ given by $\Delta'(a, b) = (a^\#, b^\#)$ for all $(a, b) \in \mathscr{D}_L$ is well-defined. Moreover:

(1) $a \vee \Delta'_1(a, b) = a \vee a^\# = 1 = b \vee b^\# = b \vee \Delta'_2(a, b)$.
(2) If $(a, b) \leq_{\mathscr{D}_L} (c, d)$ then $a \geq c$ and $b \leq d$. It follows that $a^\# \leq c^\#$ and $b^\# \geq d^\#$ and so $(a^\#, b^\#) \leq_{\mathscr{D}_{L^{op}}} (c^\#, d^\#)$.

Hence, Δ' witnesses the monotone normality of L. □

In particular, for any topological space X the lattice of closed subsets satisfies the conditions of Proposition 4.2 and we have:

Corollary 4.3. *Any extremally disconnected space is monotonically extremally disconnected.*

This shows that monotone extremal disconnectedness is equivalent to extremal disconnectedness and, therefore, in this case the monotonization process does not produce a new notion.

B. Stratifiability vs co-stratifiability

Let us now analyse what happens with the notions of perfect normality and stratifiability when we extend them to arbitrary lattices and dualize.

Definition 4.4. Let L be a complete lattice. We say that:

(a) L is *perfectly normal* if L is normal and there exist $(\alpha_n, \beta_n\colon L \to L)_{n\in\mathbb{N}}$ such that:
   (1) $\bigvee\limits_{n\in\mathbb{N}} \alpha_n(a) = a$ for every $a \in L$;
   (2) $a \vee \beta_n(a) = 1$ and $\alpha_n(a) \wedge \beta_n(a) = 0$ for every $n \in \mathbb{N}$ and $a \in L$;
   (3) $\alpha_n(a) \le \alpha_m(a)$ and $\beta_n(a) \ge \beta_m(a)$ for every $n \le m$ in $\mathbb{N}$ and $a \in L$.

(b) L is *stratifiable* if L is monotonically normal and there exist $(\alpha_n, \beta_n\colon L \to L)_{n\in\mathbb{N}}$ satisfying (1) and (2) above and
   (3′) $\alpha_n(a) \le \alpha_m(b)$ and $\beta_n(a) \ge \beta_m(b)$ for every $n \le m$ in $\mathbb{N}$ and $a \le b$ in L.

(c) L is perfectly extremally disconnected if L^{op} is perfectly normal.

(d) L is *co-stratifiable* in case L^{op} is stratifiable.

In particular, when $L = \mathcal{O}X$ for some space X, we have that X is perfectly normal (resp. stratifiable) if and only if the lattice $\mathcal{O}X$ is perfectly normal (resp. stratifiable).

Proposition 4.5. *Let L be a complete lattice in which finite joins distribute over arbitrary meets. Then the following are equivalent:*

(1) *L is stratifiable;*
(2) *L is perfectly normal;*
(3) *L is a complete Boolean algebra.*

Proof. (1) $\Longrightarrow$ (2) is obvious and for (3) $\Longrightarrow$ (1) just consider $\alpha_n(a) = a$ and $\beta_n(a) = a^{\mathrm{c}}$ for each $n \in \mathbb{N}$ (where a^c stands for the complement of a) for each $n \in \mathbb{N}$ and $a \in L$.

(2) $\Longrightarrow$ (3): Since L is distributive, it is enough to prove that, for any $a \in L$, the pseudosupplement $a^\#$ is in fact a complement. For each $n \in \mathbb{N}$ and $a \in L$, we have $a \vee \beta_n(a) = 1$ which implies $\beta_n(a) \ge a^\#$. Also, since L^{op} is a frame and $\alpha_n(a) \wedge \beta_n(a) = 0$, it follows that $\alpha_n(a) \le \beta_n(a)^\#$. Hence, $\alpha_n(a) \le \beta_n(a)^\# \le a^{\#\#} \le a$ and, therefore, $a = \bigvee_{n\in\mathbb{N}} \alpha_n(a) \le a^{\#\#} \le a$. Finally, since $a \vee a^\# = 1$, there exist (by normality) $u, v \in L$ such that $a \vee u = 1 = a^\# \vee v$ and $u \wedge v = 0$. Hence, $0 = u \wedge v \ge a^\# \wedge a^{\#\#} = a^\# \wedge a$ and a is complemented. □

In particular, for any topological space X the lattice of closed subsets of X satisfies the conditions of Proposition 4.5 and we have:

Corollary 4.6. *The following are equivalent for any topological space X:*

(1) *X is co-stratifiable.*
(2) *X is perfectly extremally disconnected.*
(3) *$\mathcal{C}X = \mathcal{O}X$, i.e. any open set is clopen.*

Proof. (1) $\Longrightarrow$ (2) and (3) $\Longrightarrow$ (1) are clear. For (2) $\Longrightarrow$ (3), by Proposition 4.5 we have that $\mathcal{C}X$ is a complete Boolean algebra. Since finite joins and meets in $\mathcal{C}X$ coincide respectively with unions and intersections in $\mathscr{P}(X)$, it follows that the complement of each $F \in \mathcal{C}X$ is precisely $X \smallsetminus F$. Consequently, for any closed set F its complement $X \smallsetminus F$ is also open, i.e. F is clopen. □

In spite that perfect normality and its monotone version (stratifiability) are different concepts, this shows that their duals do coincide and describe a very special class of extremally disconnected spaces: the ones where every open is clopen.

5. Basics on frames

Recall that a *frame* is a complete lattice satisfying the distributive law

$$(\bigvee S) \wedge a = \bigvee\{s \wedge a \mid s \in S\}$$

for all subsets $S \subseteq L$ and all $a \in L$. Thus, the mapping $x \mapsto x \wedge a$ preserves suprema and has a right Galois adjoint $y \mapsto (a \to y)$ which makes a frame a *complete Heyting algebra*, that is, a lattice with an extra binary operation $\to$ on L satisfying

$$a \wedge b \leq c \quad \text{iff} \quad a \leq b \to c. \tag{H}$$

The Heyting operation satisfies properties like

(H1) $a \leq b$ iff $a \to b = 1$,
(H2) $a \leq b \to a$,
(H3) $a \to (\bigwedge_{i\in I} b_i) = \bigwedge_{i\in I}(a \to b_i)$, and
(H4) $(\bigvee_{i\in I} a_i) \to b = \bigwedge_{i\in I}(a_i \to b)$.

In particular, there are the *pseudocomplements* $a^* = a \to 0 = \bigvee\{b \in L \mid a \wedge b = 0\}$ satisfying $a \wedge a^* = 0$. They satisfy rules dual to (4.3):

$$a \leq a^{**}, \quad a \leq b \Rightarrow b^* \leq a^*, \quad a^{***} = a^* \quad \text{and} \quad \Big(\bigvee_{i\in I} a_i\Big)^* = \bigwedge_{i\in I} a_i^*. \tag{5.1}$$

A frame homomorphism $h\colon L \to M$ preserves all joins (including the bottom element 0) and all *finite* meets (including the top element 1). The resulting category is the category **Frm** of frames. A typical frame is the lattice $\mathcal{O}X$ of all open sets of a topological space X: joins are given by set-theoretical unions and finite meets by intersections, $\varnothing$ is the least element and X the biggest element X. If $f\colon X \to Y$ is a continuous map then $\mathcal{O}(f) = (U \mapsto f^{-1}[U])\colon \mathcal{O}Y \to \mathcal{O}X$ is a frame homomorphism. Thus one has a contravariant functor $\mathcal{O}\colon$ **Top** $\to$ **Frm** (where **Top** is the category of topological spaces).

For more information about frames see, e.g., [20] or [27].

Since the relation between spaces and frames is contravariant, it is the dual category of **Frm**, usually denoted by **Loc**, that should be regarded as the category of pointfree (generalized) spaces. This is the category of *locales* and *localic maps*. Localic maps may be represented by maps $f\colon L \to M$ defined as the right (Galois) adjoints of the frame homomorphisms $h\colon M \to L$, that is, maps f such that $h(x) \leq y$ iff $x \leq f(y)$.

Subobjects in **Loc** are represented by *sublocales*. A sublocale S of a locale L is a subset $S \subseteq L$ such that

(S1) for every $M \subseteq S$, $\bigwedge M \in S$ (thus in particular, the top 1 is in S), and
(S2) for every $s \in S$ and every $x \in L$, $x \to s$ is in S.

Sublocales are precisely such subsets for which the embedding map $j\colon S \to L$ is a (one-one) localic map. The corresponding (left adjoint) frame homomorphism is denoted by $\nu_S : L \to S$ and it is given by $\nu_S(x) = \bigwedge\{s \in S \mid s \geq x\}$. Sublocales of L ordered by reverse inclusion constitute a frame $\mathcal{S}(L)$.

Each sublocale S of L is itself a frame with the same meets as L (and since the Heyting operation depends on the meet structure only, with the same Heyting operation) but, however, the joins in S and L do not necessarily coincide. We shall denote the join of a and b in S by $a \vee_S b$, to distinguish it from the join $a \vee b$ in L.

In **Loc**, the role of open (resp. closed) subspaces is taken by *open* (resp. *closed*) *sublocales* (associated to each $a \in L$)

$$\mathfrak{o}(a) = \{a \to x \mid x \in L\} = \{x \in L \mid a \to x = x\} \quad \text{(resp.} \quad \mathfrak{c}(a) = \{a \vee x \mid x \in L\}). \tag{5.2}$$

They are complemented with each other and one has $\mathfrak{o}(\bigvee a_i) = \bigwedge \mathfrak{o}(a_i)$, $\mathfrak{o}(a \wedge b) = \mathfrak{o}(a) \vee \mathfrak{o}(b)$, $\mathfrak{c}(\bigvee a_i) = \bigvee \mathfrak{c}(a_i)$ and $\mathfrak{c}(a \wedge b) = \mathfrak{c}(a) \wedge \mathfrak{c}(b)$. In particular, $\mathfrak{c}(a) \leq \mathfrak{c}(b)$ if and only if $a \leq b$ and $\mathfrak{o}(b) \leq \mathfrak{o}(a)$ if and only if $a \leq b$.

6. Monotonically normal frames

From now on, we restrict our study from general lattices to frames and locales. In this case, condition (4.1) yields the usual notion of a normal frame and it is clear that one can take $v = u^*$. Thus, L is normal if and only if, whenever $a \vee b = 1$, there exists a $u \in L$ satisfying $a \vee u = 1 = b \vee u^*$. Then, a frame L is normal if and only if there exists a function $\Delta\colon \mathscr{D}_L = \{(a, b) \in L \times L \mid a \vee b = 1\} \to L$ such that

$$a \vee \Delta(a, b) = 1 = b \vee \Delta(a, b)^*$$

for all $(a, b) \in \mathscr{D}_L$. The function Δ is called a *normality operator* [11].

Correspondingly, a frame L is monotonically normal if there exists a monotone function $\Delta\colon \mathscr{D}_L \to L$ such that

$$a \vee \Delta(a, b) = 1 = b \vee \Delta(a, b)^*$$

for all $(a, b) \in \mathscr{D}_L$ (and Δ is called a *monotone normality operator* [11]).

This is a conservative extension of the point-set notion, that is, a topological space X is monotonically normal iff OX is a monotonically normal frame.

Examples 6.1. (1) If $a \vee b = 1$ implies $a = 1$ or $b = 1$, then the frame is trivially monotonically normal. Indeed the function $\Delta\colon \mathscr{D}_L \to L$ given by

$$\Delta(a, b) = \begin{cases} 0, & \text{if } a = 1, \\ 1, & \text{if } a \neq 1, \end{cases}$$

is a monotone normality operator. For example, any chain (completely ordered set) satisfies this condition and hence is trivially monotonically normal.

(2) As proved in [11], any metrizable frame ([28]) is monotonically normal. More generally, any frame that admits a chain of admissible covers is monotonically normal [11]. Note that, in particular, any nearness with a countable basis is of this kind. It may be also worth mentioning that since this general condition is preserved by taking homomorphic images it is automatic that the frames in question are hereditarily monotonically normal.

Given a function $\Delta\colon \mathscr{D}_L \to L$, let $\Delta^{op}(a, b) = \Delta(b, a)$. We say that Δ is *self-disjoint* whenever the pointwise meet $\Delta \wedge \Delta^{op}$ is equal to 0.

Remark 6.2. It is important to recall here the following results from [11]. If $\Delta\colon \mathscr{D}_L \to L$ is a monotone normality operator, then so is $\Delta^{\circledast}$ defined by $\Delta^{\circledast}(a,b) = \Delta(b,a)^*$. On the other hand, if Δ_1 and Δ_2 are monotone normality operators, then so is the pointwise meet $\Delta_1 \wedge \Delta_2$. Consequently, if Δ is a monotone normality operator, then for $\Theta = \Delta \wedge \Delta^{\circledast}$ one has $\Theta(a,b) \wedge \Theta^{op}(a,b) \leq \Delta(a,b) \wedge \Delta(a,b)^* = 0$ and hence Θ is self-disjoint. It follows that each monotonically normal frame L admits a self-disjoint monotone normality operator.

Recall that a frame is said to be *hereditarily monotonically normal* if every its sublocale is monotonically normal. Of course, the following implications are true for any frame L:

$$\begin{gathered} L \text{ is hereditarily monotonically normal} \\ \Downarrow \text{(A)} \\ \text{Every open sublocale of } L \text{ is monotonically normal} \\ \Downarrow \text{(B)} \\ L \text{ is monotonically normal.} \end{gathered}$$

And what can we say about the converse implications of (A) and (B)?

Proposition 6.3. *If every open sublocale of L is monotonically normal, then L is hereditarily monotonically normal.*

Proof. Let S be an arbitrary sublocale of L, given by the surjective homomorphism $\nu_S : L \to S$. In order to prove that S is monotonically normal, let $a, b \in S$ be such that $a \vee_S b = 1$ and consider the open sublocale

$$T = \mathfrak{o}(a \vee b) = \{(a \vee b) \to x \mid x \in L\} = \{\nu_T(x) \mid x \in L\}.$$

By (H4), $\nu_T(a) = (a \vee b) \to a = (a \to a) \wedge (b \to a) = b \to a$. Likewise, $\nu_T(b) = a \to b$. So, we have

$$(b \to a) \vee_T (a \to b) = \nu_T(a) \vee_T \nu_T(b) = \nu_T(a \vee b) = 1.$$

Since T is monotonically normal, there exists by Remark 6.2 a self-disjoint monotone normality operator $\Delta\colon \mathscr{D}_T \to T$, i.e., such that $\Delta \wedge \Delta^{op}$ is equal to $0_T = (a \vee b)^*$. We have

$$\begin{aligned} (a \vee b) \to (a \vee \Delta(b \to a,\, a \to b)) &= \nu_T(a \vee \Delta(b \to a,\, a \to b)) \\ &= \nu_T(a) \vee_T \Delta(b \to a,\, a \to b) \\ &= (b \to a) \vee_T \Delta(b \to a,\, a \to b) = 1, \end{aligned}$$

which, by (H1), yields $a \vee b \leq a \vee \Delta(b \to a,\, a \to b)$. Thus

$$a \vee b = (a \vee b) \wedge (a \vee \Delta(b \to a,\, a \to b)) = a \vee (b \wedge \Delta(b \to a,\, a \to b)). \tag{$*$}$$

We now show that $\Delta_S : \mathscr{D}_S \to S$ defined by $\Delta_S(a,b) = \nu_S(b \wedge \Delta(b \to a,\, a \to b))$ is a monotone normality operator for S. Indeed, we first note that

$$(\Delta_S \wedge \Delta_S^{op})(a,b) = \nu_S(a \wedge b \wedge (\Delta \wedge \Delta^{op})(b \to a,\, a \to b)) = \nu_S((a \wedge b) \wedge (a \vee b)^*) = \nu_S(0) = 0_S$$

and hence $\Delta_S(b,a) \leq \Delta_S(a,b)^*$. Now, by ($*$) we get

$$a \vee_S \Delta_S(a,b) = \nu_S(a \vee (b \wedge \Delta(b \to a,\, a \to b))) = \nu_S(a \vee b) = a \vee_S b = 1,$$

and dually, $b \vee_S \Delta_S(a,b)^* \geq b \vee_S \Delta_S(b,a) = 1$. This shows that S is normal.

Finally, let $(a_1, b_1) \leq (a_2, b_2)$ in $\mathscr{D}_S$, that is, $a_1 \geq a_2$ and $b_1 \leq b_2$. Since the Heyting operator $(\cdot) \to (\cdot)$ is antitone on the left and monotone on the right (recall (H3) and (H4)), then $b_1 \to a_1 \geq b_2 \to a_2$ and $a_1 \to b_1 \leq a_2 \to b_2$, that is, $(b_1 \to a_1, a_1 \to b_1) \leq (b_2 \to a_2, a_2 \to b_2)$ in $\mathscr{D}_T$. Then, since Δ is monotone, we have $\Delta(b_1 \to a_1, a_1 \to b_1) \leq \Delta(b_2 \to a_2, a_2 \to b_2)$. Therefore $b_1 \wedge \Delta(b_1 \to a_1, a_1 \to b_1) \leq b_2 \wedge \Delta(b_2 \to a_2, a_2 \to b_2)$ and hence $\Delta_S(a_1, b_1) \leq \Delta_S(a_2, b_2)$. ◻

In conclusion, the converse of (A) is true.

We can also show the converse of (B) in case the frame is subfit:

Theorem 6.4. *Let L be a subfit and monotonically normal frame. Then every open sublocale of L is monotonically normal.*

Proof. Given L as stated, let $0 \neq a \in L$ (if $a = 0$ then $\mathfrak{o}(a) = \{1\}$ is trivially monotonically normal). For each $(x,y) \in \mathscr{D}_{\mathfrak{o}(a)}$ we have that $x \vee_{\mathfrak{o}(a)} y = a \to (x \vee y) = 1$ (i.e. $a \leq x \vee y$). Since L is subfit and $a \nleq 0$, there exists $c \in L$ such that $a \vee c = 1 \neq c$. Moreover, for each such $c \in L$ we have

$$(x \wedge a) \vee y \vee c = (x \vee y \vee c) \wedge (a \vee y \vee c) \geq a \vee c = 1.$$

On the other hand, by monotone normality, Remark 6.2 ensures the existence of a self-disjoint monotone normality operator $\Delta\colon \mathscr{D}_L \to L$. We now show that $\Delta_{\mathfrak{o}(a)}\colon \mathscr{D}_{\mathfrak{o}(a)} \to \mathfrak{o}(a)$, defined for each $(x,y) \in \mathscr{D}_{\mathfrak{o}(a)}$ by

$$\Delta_{\mathfrak{o}(a)}(x,y) = a \to \bigvee\{\Delta(x \wedge a, y \vee c) \mid a \vee c = 1 \neq c\},$$

is a monotone normality operator for S.

Indeed, we first note that since $\{c \in L \mid a \vee c = 1 \neq c\} \neq \varnothing$ we have

$$\begin{aligned}(\Delta_{\mathfrak{o}(a)} \wedge {\Delta_{\mathfrak{o}(a)}}^{op})(x,y) &= \Delta_{\mathfrak{o}(a)}(x,y) \wedge \Delta_{\mathfrak{o}(a)}(y,x)\\ &= a \to \left(\bigvee\{\Delta(x \wedge a, y \vee c) \mid a \vee c = 1 \neq c\} \wedge \bigvee\{\Delta(y \wedge a, x \vee d) \mid a \vee d = 1 \neq d\}\right)\\ &= a \to \left(\bigvee\{\Delta(x \wedge a, y \vee c) \wedge \Delta(y \wedge a, x \vee d) \mid a \vee c = 1 \neq c \text{ and } a \vee d = 1 \neq d\}\right).\end{aligned}$$

But $y \wedge a \leq (y \wedge a) \vee c = (y \vee c) \wedge (a \vee c) = y \vee c$ and $x \vee d = (x \vee d) \wedge (a \vee d) = (x \wedge a) \vee d \geq x \wedge a$ whenever $a \vee c = 1 \neq c$ and $a \vee d = 1 \neq d$. Consequently, we have $\Delta(y \wedge a, x \vee d) \leq \Delta(y \vee c, x \wedge a)$, and since Δ is self-disjoint, $\Delta(x \wedge a, y \vee c) \wedge \Delta(y \wedge a, x \vee d) = 0$. Hence

$$(\Delta_{\mathfrak{o}(a)} \wedge {\Delta_{\mathfrak{o}(a)}}^{op})(x,y) = a \to 0 = 0_{\mathfrak{o}(a)}$$

and therefore $\Delta_{\mathfrak{o}(a)}(y,x) \leq \Delta_{\mathfrak{o}(a)}(x,y)^*$.

Now,

$$\begin{aligned}x \vee_{\mathfrak{o}(a)} \Delta_{\mathfrak{o}(a)}(x,y) &= a \to (x \vee \Delta_{\mathfrak{o}(a)}(x,y))\\ &= a \to \left((a \to x) \vee (a \to \bigvee\{\Delta(x \wedge a, y \vee c) \mid a \vee c = 1\})\right)\\ &= a \to \left(x \vee \bigvee\{\Delta(x \wedge a, y \vee c) \mid a \vee c = 1\}\right)\\ &= a \to \left(\bigvee\{x \vee \Delta(x \wedge a, y \vee c) \mid a \vee c = 1\}\right)\\ &\geq a \to \left(\bigvee\{(x \wedge a) \vee \Delta(x \wedge a, y \vee c) \mid a \vee c = 1\}\right)\\ &= a \to 1 = 1\end{aligned}$$

and dually, $y \vee_{\mathfrak{o}(a)} \Delta_{\mathfrak{o}(a)}(x,y)^* \geq y \vee_{\mathfrak{o}(a)} \Delta_{\mathfrak{o}(a)}(y,x) = 1$. This shows that $\mathfrak{o}(a)$ is normal.

Finally, let $(x_1, y_1) \leq (x_2, y_2)$ in $\mathscr{D}_{\mathfrak{o}(a)}$, that is, $x_1 \geq x_2$ and $y_1 \leq y_2$. Then

$$\begin{aligned}\Delta_{\mathfrak{o}(a)}(x_1, y_1) &= a \to \bigvee\{\Delta(x_1 \wedge a, y_1 \vee c) \mid a \vee c = 1\} \\ &\leq a \to \bigvee\{\Delta(x_2 \wedge a, y_2 \vee c) \mid a \vee c = 1\} \\ &= \Delta_{\mathfrak{o}(a)}(x_2, y_2). \quad \square\end{aligned}$$

Remark 6.5. In the above proof, subfitness is not used in its full force; the proposition actually holds already in the stronger form which only requires L to satisfy

$$a \nleq 0 \quad \Longrightarrow \quad \exists c \in L\colon a \vee c = 1 \neq c. \tag{WSfit}$$

Interestingly, this is precisely the *weak subfitness* of [18] (also *puny* in [31]). This is indeed a condition weaker than subfitness (see Example 6.8 below).

The following results give us a more precise idea of the relationship between subfitness and weak subfitness in terms of sublocales.

Proposition 6.6. *A frame L is weakly subfit if and only if every open sublocale of L is weakly subfit.*

Proof. Let L be weakly subfit and consider $b \in \mathfrak{o}(a)$ such that $b \neq 0_{\mathfrak{o}(a)} = a^*$. Then $b \wedge a \neq 0$ and by (WSfit) there is a $c \in L$, $c \neq 1$, such that $(b \wedge a) \vee c = 1$. Then $1 = b \vee c \leq b \vee (a \to c)$ and $a \to c \neq 1$ (if, otherwise, $a \to c = 1$ then we would have $a \leq c$, i.e. $1 = a \vee c = c \neq 1$, a contradiction). Hence $c' = a \to c \in \mathfrak{o}(a)$ and $b \vee_{\mathfrak{o}(a)} c' = a \to (b \vee c') = a \to 1 = 1 \neq c'$. $\square$

Proposition 6.7. *A frame L is subfit if and only if every closed sublocale of L is weakly subfit.*

Proof. Let L be subfit and consider $b \in \mathfrak{c}(a)$ such that $b \nleq 0_{\mathfrak{c}(a)} = a$. By (Sfit) there exists $c \in L$ satisfying $b \vee c = 1 \neq a \vee c$. Take $c' = a \vee c \in \mathfrak{c}(a)$. Then $b \vee_{\mathfrak{c}(a)} c' = a \vee b \vee c = 1 \neq c'$.

Conversely, let $b \nleq a$ in L. Since $\mathfrak{c}(a)$ is weakly subfit and $b' = a \vee b \nleq a = 0_{\mathfrak{c}(a)}$, there exists $c \in \mathfrak{c}(a)$ such that $b' \vee_{\mathfrak{c}(a)} c = 1 \neq c$. Hence $b \vee c = a \vee b \vee c = b' \vee c = 1$. $\square$

It may be also worth pointing that, by Lemma 4.2 of [18], the frames in which every sublocale is weakly subfit are the hereditarily subfit ones. The latter are precisely the *fit frames* [27, Prop. V.1.5], that is, the frames that satisfy

$$a \nleq b \quad \Longrightarrow \quad \exists c \in L\colon a \vee c = 1 \text{ and } c \to b \nleq b. \tag{Fit}$$

Thus, the situation is as the following table depicts:

$$\begin{array}{ccc}
\boxed{L \text{ satisfies (WSfit)}} & \Longleftrightarrow & \boxed{\text{every open sublocale of } L \text{ satisfies (WSfit)}} \\
\Uparrow \not\Downarrow & & \\
\boxed{L \text{ satisfies (Sfit)}} & \Longleftrightarrow & \boxed{\text{every closed sublocale of } L \text{ satisfies (WSfit)}} \\
\Uparrow \not\Downarrow & & \\
\boxed{L \text{ satisfies (Fit)}} & \Longleftrightarrow & \boxed{\text{every sublocale of } L \text{ satisfies (WSfit)}}
\end{array}$$

Example 6.8. For a space X, the frame OX of open sets is weakly subfit if and only if the space satisfies the following condition:

$$\forall \varnothing \neq U \in OX, \quad \exists\, x \in X \text{ such that } \overline{\{x\}} \subseteq U \qquad \text{(WConj)}$$

(equivalently, if each nonempty open set contains a nonempty closed set). Let $X = \mathbb{N} \cup \{\omega_1, \omega_2\}$ with

$$\begin{aligned} OX &= \{A \cup B \mid A \subseteq \mathbb{N} \text{ with } \mathbb{N} \smallsetminus A \text{ finite and } B = \varnothing,\ \{\omega_1\} \text{ or } \{\omega_1, \omega_2\}\} \cup \{\varnothing\} \\ &= \{A \mid A \subseteq \mathbb{N} \text{ with } \mathbb{N} \smallsetminus A \text{ finite}\} \cup \{A \cup \{\omega_1\} \mid A \subseteq \mathbb{N} \text{ with } \mathbb{N} \smallsetminus A \text{ finite}\} \cup \\ &\quad \cup \{A \cup \{\omega_1, \omega_2\} \mid A \subseteq \mathbb{N} \text{ with } \mathbb{N} \smallsetminus A \text{ finite}\} \cup \{\varnothing\}. \end{aligned}$$

Then

$$\begin{aligned} CX &= \{F \cup B \mid F \subseteq \mathbb{N} \text{ with } F \text{ finite and } B = \varnothing,\ \{\omega_2\} \text{ or } \{\omega_1, \omega_2\}\} \cup \{\varnothing\} \\ &= \{F \mid F \subseteq \mathbb{N} \text{ with } F \text{ finite}\} \cup \{F \cup \{\omega_2\} \mid F \subseteq \mathbb{N} \text{ with } F \text{ finite}\} \cup \\ &\quad \cup \{F \cup \{\omega_1, \omega_2\} \mid F \subseteq \mathbb{N} \text{ with } F \text{ finite}\} \cup \{\varnothing\} \end{aligned}$$

and therefore

$$\overline{\{n\}} = \{n\} \text{ for all } n \in \mathbb{N}, \quad \overline{\{\omega_1\}} = \{\omega_1, \omega_2\} \quad \text{and} \quad \overline{\{\omega_2\}} = \{\omega_2\}.$$

So, OX is weakly subfit, since each nonempty open set contains a closed set of the form $\{n\}$ with $n \in \mathbb{N}$. However, OX is not subfit since $U = \mathbb{N} \cup \{\omega_1\}$ is open, $\omega_1 \in U$, $\overline{\{\omega_1\}} = \{\omega_1, \omega_2\}$ and $\overline{\{\omega_1\}} = \{\omega_1, \omega_2\} \nsubseteq \mathbb{N}$ and $\overline{\{\omega_2\}} = \{\omega_2\} \nsubseteq \mathbb{N}$. Note that, on the other hand, OX is not normal.

As a consequence of Proposition 6.3 and Theorem 6.4 it follows that:

Corollary 6.9. *Every open subspace of a subfit monotonically normal space is monotonically normal.*

Proof. Let X be a subfit monotonically normal space and $U \subseteq X$ an open subset. By hypothesis the open sublocale $\mathfrak{o}(U)$ is a monotonically normal frame. Since the frames $\mathfrak{o}(U)$ and OU are isomorphic, it follows that the subspace U is monotonically normal. □

Corollary 6.10. *The following are equivalent for a subfit frame L:*

(1) *L is monotonically normal.*
(2) *L is hereditarily monotonically normal.*
(3) *Every open sublocale of L is monotonically normal.*

Remark 6.11. Evidently, by Remark 6.5, Corollaries 6.9 and 6.10 hold more generally for weakly subfit spaces and weakly subfit frames respectively.

We end this section with a result about the behaviour of monotone normality under closed maps. For that, we need to recall that the right adjoint $h_*\colon M \to L$ of a frame homomorphism $h\colon L \to M$ (i.e., the localic map corresponding to h) has the following properties:

(L1) h_* preserves arbitrary meets (in particular, $h_*(1) = 1$).
(L2) If $h_*(b) = 1$ then $b = 1$.
(L3) h_* is surjective iff h is injective iff $h_*(h(a)) = a$ for every $a \in L$.

Further, h is *closed* if

$$h_*(h(a) \vee b) = a \vee h_*(b) \quad \text{for every } a \in L \text{ and } b \in M.$$

Proposition 6.12. *Let $h\colon L \to M$ be an injective closed frame homomorphism. If M is monotonically normal then so is L.*

Proof. Let Δ_M be a monotone normality operator on M, i.e. a monotone $\Delta_M\colon \mathscr{D}_M \to M$ satisfying

$$a \vee \Delta_M(a,b) = 1 = b \vee \Delta_M(a,b)^*$$

for all $(a,b) \in \mathscr{D}_M$. Let us define a further map $\Delta_L\colon \mathscr{D}_L \to L$ as follows:

For each $(c,d) \in \mathscr{D}_L$, since h is a frame homomorphism it follows that $(h(c),h(d)) \in \mathscr{D}_M$. So we define

$$\Delta_L(c,d) = h_*(\Delta_M(h(c),h(d))) \in L.$$

We first show that Δ_L is a normality operator. Since h is closed, then, for each $(c,d) \in \mathscr{D}_L$, we have

$$c \vee \Delta_L(c,d) = c \vee h_*(\Delta_M(h(c),h(d))) = h_*(h(c) \vee \Delta_M(h(c),h(d))) = h_*(1_M) = 1_L.$$

On the other hand, since h_* preserves arbitrary meets and is surjective we have

$$\Delta_L(c,d) \wedge h_*(\Delta_M(h(c),h(d))^*) = h_*(\Delta_M(h(c),h(d)) \wedge \Delta_M(h(c),h(d))^*) = h_*(0_M) = 0_L$$

and thus

$$d \vee \Delta_L(c,d)^* \geq d \vee h_*(\Delta_M(h(c),h(d))^*) = h_*(h(d) \vee \Delta_M(h(c),h(d))^*) = h_*(1_M) = 1_L.$$

Finally, Δ_L is monotone since Δ_M, h and h_* are monotone. □

Remark 6.13. Translated to the category of locales this proposition asserts that any closed surjective localic map preserves monotone normality. This is the pointfree monotone version of the classical Hausdorff mapping invariance theorem that states that the image of any normal space under any closed continuous map is normal.

7. Stratifiable frames

We close with a few results on stratifiable frames. As in the case of monotone normality, the notion of stratifiability introduced in Definition 4.4 for arbitrary lattices L admits a simpler characterization when L is a frame:

Proposition 7.1. *A frame L is stratifiable if and only if there exists $(\alpha_n\colon L \to L)_{n\in\mathbb{N}}$ such that:*

(1) $\bigvee_{n\in\mathbb{N}} \alpha_n(a) = a$ *for every* $a \in L$.
(2) $a \vee \alpha_n(a)^* = 1$ *for every* $n \in \mathbb{N}$ *and* $a \in L$.
(3) $\alpha_n(a) \leq \alpha_m(b)$ *for every* $n \leq m$ *in* $\mathbb{N}$ *and* $a \leq b$ *in* L.

Proof. $\Rightarrow$: This is clear since $a \vee \alpha_n(a)^* \geq a \vee \beta_n(a) = 1$ for each $n \in \mathbb{N}$ and $a \in L$.

$\Leftarrow$: First, let us show that L is monotonically normal. Let $(\alpha_n\colon L \to L)_{n\in\mathbb{N}}$ as above. We define the map $\Delta\colon \mathscr{D}_L \to L$ as follows:

$$\Delta(a,b) = \bigvee_{n\in\mathbb{N}} (\alpha_n(a)^* \wedge \alpha_n(b)).$$

Then we have

$$a \vee \Delta(a,b) = \bigvee_{n\in\mathbb{N}} ((a \vee \alpha_n(a)^*) \wedge (a \vee \alpha_n(b))) = \bigvee_{n\in\mathbb{N}} (a \vee \alpha_n(b)) = a \vee b = 1.$$

On the other hand, for every n, m, the element $c_{n,m} = (\alpha_n(a)^* \wedge \alpha_n(b)) \wedge (\alpha_m(a) \wedge \alpha_m(b)^*)$ is equal to 0. Indeed, if $n \leq m$, then $c_{n,m} \leq \alpha_m(b) \wedge \alpha_m(b)^* = 0$; similarly, if $n > m$, then $c_{n,m} \leq \alpha_n(a)^* \wedge \alpha_n(a) = 0$. Consequently, by the frame distribution law, $\Delta(a,b) \wedge (\bigvee_{m\in\mathbb{N}}(\alpha_m(a) \wedge \alpha_m(b)^*)) = 0$, that is, $\bigvee_{m\in\mathbb{N}}(\alpha_m(a) \wedge \alpha_m(b)^*) \leq \Delta(a,b)^*$, and thus

$$b \vee \Delta(a,b)^* \geq \bigvee_{m\in\mathbb{N}} ((b \vee \alpha_m(a)) \wedge (b \vee \alpha_n(b)^*)) = \bigvee_{m\in\mathbb{N}} (b \vee \alpha_m(a)) = b \vee a = 1.$$

To complete the proof, just notice that α and $\beta = \alpha^*$ satisfy the conditions of Definition 4.4(b). □

It follows that the concept of stratifiability for frames is conservative, that is, a topological space is stratifiable (in the sense of our Definition 3.3) if and only if the frame of its open sets is stratifiable. Moreover:

Corollary 7.2. *Any stratifiable frame is monotonically normal and perfectly normal.*

Concerning examples of stratifiable frames, we will now observe that the monotonically frames of Example 6.1(2) are already stratifiable. For that, we need to recall a few things about frame covers and nearnesses. A subset of a frame $A \subseteq L$ is a *cover* of L if $\bigvee A = 1$. The set of all covers of L, denoted as $\operatorname{Cov} L$, can be preordered as follows: a cover A *refines* a cover B, written $A \preccurlyeq B$, if for each $a \in A$ there is some $b \in B$ with $a \leq b$. For any $A \in \operatorname{Cov} L$ and $b \in L$, the element Ab of L is defined by $Ab = \bigvee\{a \in A \mid a \wedge b \neq 0\}$.

For any $\mathcal{A} \subseteq \operatorname{Cov} L$, the relation $\lhd_{\mathcal{A}}$ (or simply $\lhd$) on L is defined by

$$x \lhd_{\mathcal{A}} y \quad \text{if} \quad Ax \leq y \quad \text{for some} \quad A \in \mathcal{A},$$

and $\mathcal{A}$ is said to be *admissible* if $a = \bigvee\{b \in L \mid b \lhd_{\mathcal{A}} a\}$ for each $a \in L$. A *nearness* on L is an admissible filter $\mathcal{A}$ in $(\operatorname{Cov} L, \preccurlyeq)$ ([2]). A *nearness frame* [2] is a pair $(L, \mathcal{A})$ where $\mathcal{A}$ is a nearness on L. Given a nearness frame $(L, \mathcal{A})$, a system of covers $\mathfrak{B} \subseteq \mathcal{A}$ is said to be a *basis* of $\mathcal{A}$ if for each $A \in \mathcal{A}$ there exists some $B \in \mathfrak{B}$ such that $B \preccurlyeq A$. Note that the relation $\lhd_{\mathcal{B}}$ coincides with $\lhd_{\mathcal{A}}$. Next proposition extends Remark 4.6 and Proposition 4.5 of [11].

Proposition 7.3. *Let L be a frame and $\mathcal{A} = \{A_n \mid n \in \mathbb{N}\}$ a system of admissible covers of L such that $A_m \preccurlyeq A_n$ whenever $n \leq m$ in $\mathbb{N}$. Then L is stratifiable.*

Proof. Let us define $(\alpha_n \colon L \to L)_{n\in\mathbb{N}}$ by

$$\alpha_n(a) = \bigvee\{b \in L \mid A_n b \leq a\}$$

for all $n \in \mathbb{N}$ and $a \in L$. Then we have:

(1) $a = \bigvee\{b \in L \mid b \lhd_{\mathcal{A}} a\} = \bigvee_{n\in\mathbb{N}} \bigvee\{b \in L \mid A_n b \leq a\} = \bigvee_{n\in\mathbb{N}} \alpha_n(a)$ for each $a \in L$.

(2) For each $x \in A_n$ satisfying $x \wedge \alpha_n(a) \neq 0$ there is a $b \in L$ such that $A_n b \leq a$ and $x \wedge b \neq 0$ (hence $x \leq A_n b \leq a$). Therefore

$$1 = \bigvee A_n = (\bigvee\{x \in A_n \mid x \wedge \alpha_n(a) \neq 0\}) \vee (\bigvee\{x \in A_n \mid x \wedge \alpha_n(a) = 0\}) \leq a \vee \alpha_n(a)^*$$

for each $n \leq m$ in $\mathbb{N}$ and $a \leq b$ in L.

(3) $\alpha_n(a) = \bigvee\{x \in L \mid A_n x \leq a\} \leq \bigvee\{x \in L \mid A_m x \leq a\} \leq \bigvee\{x \in L \mid A_m x \leq b\} = \alpha_n(b)$ for each $n \leq m$ in $\mathbb{N}$ and $a \leq b$ in L. □

Corollary 7.4. *Every nearness frame $(L, \mathcal{A})$ with a countable basis is stratifiable.*

In particular, each metrizable frame is stratifiable.

We conclude with two propositions that provide conditions under which frame homomorphisms preserve and reflect stratifiability.

Proposition 7.5. *Let $h\colon L \to M$ be a surjective frame homomorphism. If L is stratifiable then so is M.*

Proof. Let $(\alpha_n\colon L \to L)_{n\in\mathbb{N}}$ be a stratification of L and consider

$$\beta_n = M \xrightarrow{h_*} L \xrightarrow{\alpha_n} L \xrightarrow{h} M$$

where h_* is the right adjoint of h. The fact that $(\beta_n\colon M \to M)_{n\in\mathbb{N}}$ is a stratification for M is easily established by checking conditions (1)-(3) of Proposition 7.1:

(1) For each $b \in M$ there is some $a \in L$ such that $b = h(a)$. Then

$$\bigvee_{n\in\mathbb{N}} \beta_n(b) = \bigvee_{n\in\mathbb{N}} h(\alpha_n(h_*(b))) = h\Big(\bigvee_{n\in\mathbb{N}} \alpha_n(h_*(b))\Big) = h(h_*(b)) = h(h_*(h(a))) = h(a) = b.$$

(2) For each $b \in M$ we have

$$\begin{aligned} b \vee \beta_n(b)^* &= b \vee h(\alpha_n(h_*(b)))^* \geq b \vee h(\alpha_n(h_*(b))^*) \\ &= h(h_*(b)) \vee h(\alpha_n(h_*(b))^*) = h(h_*(b) \vee \alpha_n(h_*(b))^*) = h(1) = 1. \end{aligned}$$

(3) is obvious since both h and h_* are order-preserving. □

We have now the following corollaries:

Corollary 7.6. *Any sublocale of a stratifiable frame is stratifiable.*

Corollary 7.7. *Any subspace of a stratifiable space is stratifiable.*

Proof. Let A be a subspace of a topological space X and $\iota_A\colon A \to X$ the inclusion map. Then ${\iota_A}^{-1}\colon \mathcal{O}X \to \mathcal{O}A$ is a surjective frame homomorphism. Since $\mathcal{O}X$ is stratifiable, so is $\mathcal{O}A$. □

Remark 7.8. We note that given a subset $A \subseteq X$, the right adjoint $({\iota_A}^{-1})_*\colon \mathcal{O}A \to \mathcal{O}X$ is given by the formula

$$({\iota_A}^{-1})_*(U) = X \smallsetminus \overline{A \smallsetminus U} \quad \text{for each } U \in \mathcal{O}A,$$

which was already used in Remark 2.2(1) when proving that the property of having a Borges operator is hereditary.

Finally we establish that stratifiability is preserved under injective closed frame homomorphisms. We first need the following lemma whose proof is closely related with the arguments used in the proof of Proposition 7.1.

Lemma 7.9. *Let L be a stratifiable frame with a stratification $(\alpha_n\colon L \to L)_{n\in\mathbb{N}}$. For each $a, b \in L$ there exists $u(a, b) \leq a$ in L such that:*

(1) *$u(a_1, b_1) \leq u(a_2, b_2)$ whenever $a_1 \leq a_2$ and $b_1 \geq b_2$.*
(2) *$u(a, b) \vee b \geq a$ and $a \vee u(a, b)^* \geq b$.*

(3) *If $a \vee b = 1$ then $a \vee u(a,b)^* = 1 = u(a,b) \vee b$.*

(Note that $u(a,b) \vee b = a \vee b$.)

Proof. It suffices to take
$$u(a,b) = \bigvee_{n\in\mathbb{N}} (\alpha_n(a) \wedge \alpha_n(b)^*)\,.$$
Indeed, $u(a,b) \leq a$ since $\alpha_n(a) \leq a$ for every $n \in \mathbb{N}$. Moreover:
(1) is obvious.
(2) On one hand,
$$u(a,b) \vee b = \bigvee_{n\in\mathbb{N}} ((\alpha_n(a) \vee b) \wedge (\alpha_n(b)^* \vee b)) = \bigvee_{n\in\mathbb{N}} (\alpha_n(a) \vee b) = a \vee b \geq a.$$
On the other hand, for every $m, n \in \mathbb{N}$ we have
$$\begin{aligned} &\alpha_m(b) \wedge \alpha_m(a)^* \wedge \alpha_n(a) \wedge \alpha_n(b)^* \leq \alpha_n(b) \wedge \alpha_n(b)^* = 0 \quad \text{whenever } m \leq n \text{ and} \\ &\alpha_m(b) \wedge \alpha_m(a)^* \wedge \alpha_n(a) \wedge \alpha_n(b)^* \leq \alpha_m(a)^* \wedge \alpha_m(a) = 0 \quad \text{otherwise.} \end{aligned}$$
Consequently, $\alpha_m(b) \wedge \alpha_m(a)^* \leq (\alpha_n(a) \wedge \alpha_n(b)^*)^*$ for every $m, n \in \mathbb{N}$ and therefore
$$\begin{aligned} a \vee u(a,b)^* &= a \vee \bigwedge_{n\in\mathbb{N}} (\alpha_n(a) \wedge \alpha_n(b)^*)^* \geq a \vee \bigvee_{m\in\mathbb{N}} (\alpha_m(b) \wedge \alpha_m(a)^*) \\ &= \bigvee_{m\in\mathbb{N}} ((a \vee \alpha_m(b) \wedge (a \vee \alpha_m(a)^*)) = \bigvee_{m\in\mathbb{N}} (a \vee \alpha_m(b)) = a \vee b \geq b. \end{aligned}$$
(3) is immediate from (2). □

Theorem 7.10. *Let $h\colon L \to M$ be an injective closed frame homomorphism. If M is stratifiable then so is L.*

Proof. Let $(\alpha_n\colon M \to M)_{n\in\mathbb{N}}$ be a stratification of M. We need to present a stratification $(\beta_n\colon L \to L)_{n\in\mathbb{N}}$ of L. So, let $a \in L$ and $b_n = h(h_*(\alpha_n(h(a))^*)) \in M$ for each $n \in \mathbb{N}$. Note that
$$h(a) \vee b_n = 1. \tag{7.10.1}$$
Indeed, since h is closed, we get
$$\begin{aligned} h(a) \vee b_n &= h(a) \vee h(h_*(\alpha_n(h(a))^*)) = h(a \vee h_*(\alpha_n(h(a))^*)) \\ &= h(h_*(h(a) \vee \alpha_n(h(a))^*)) = h(h_*(b \vee \alpha_n(h(a))^*)) = h\,(h_*(1)) = 1. \end{aligned}$$
Moreover, $b_n = h\,(h_*\,(\alpha_n(h(a))^*)) \leq \alpha_n(h(a))^*$ for each $n \in \mathbb{N}$ and therefore
$$\bigvee_{n\in\mathbb{N}} {b_n}^* \geq \bigvee_{n\in\mathbb{N}} \alpha_n(h(a)) = h(a). \tag{7.10.2}$$
Hence, Lemma 7.9 yields a $u(h(a), b_n) \leq h(a)$ satisfying the corresponding conditions (1), (2) and (3). Then,
$$\beta_n(a) = h_*(u(h(a), b_n))$$
defines a stratification of L, since it satisfies conditions (1)-(3) of Proposition 7.1:

(1) By (7.10.1) and Lemma 7.9(3), $u(h(a), b_n) \vee b_n = 1$. Consequently, since h is closed, we get

$$\begin{aligned} 1 = h\,(h_*(1)) &= h(h_*(u(h(a), b_n) \vee b_n)) = h(h_*(u(h(a), b_n) \vee h(h_*(\alpha_n(h(a))^*)))) \\ &= h(h_*(u(h(a), b_n)) \vee h_*(\alpha_n(h(a))^*)) = h(h_*(u(h(a), b_n))) \vee h(h_*(\alpha_n(h(a))^*)) \\ &= h(h_*(u(h(a), b_n))) \vee b_n. \end{aligned}$$

Hence $h(\beta_n(a)) = h(h_*(u(h(a), b_n))) \geq b_n{}^*$. Using (7.10.2) we conclude that

$$h\Big(\bigvee_{n\in\mathbb{N}} \beta_n(a)\Big) = \bigvee_{n\in\mathbb{N}} h(\beta_n(a)) \geq \bigvee_{n\in\mathbb{N}} b_n{}^* \geq h(a).$$

In fact, since h is injective it follows that $\bigvee_{n\in\mathbb{N}} \beta_n(a) \geq a$, and the converse inequality is a consequence of condition (2) proved below.

(2) First, $a \vee \beta_n(a)^* = a \vee h_*(u(h(a), b_n))^* \geq a \vee h_*\,(u(h(a), b_n)^*)$. By the closedness of h, this is equal to $h_*\,(h(a) \vee u(h(a), b_n)^*)$. But by (7.10.1) and Lemma 7.9(3), $h(a) \vee u(h(a), b_n) = 1$, so it is shown that $a \vee \beta_n(a)^* = 1$.

(3) It is a straightforward checking using assertion (1) of the lemma. □

Translated to the category of locales this proposition asserts that any closed surjective localic map preserves stratifiability. Since axiom T_D is precisely equivalent with representability of closed continuous maps as closed frame homomorphisms (see [29] or [27, III.7.3.1]), we have also the following corollary from the preceding result:

Corollary 7.11. *Let X and Y be T_D-spaces and $f\colon X \to Y$ a closed surjective continuous map. If X is stratifiable then so is Y.*

References

[1] C. E. Aull and W. J. Thron, Separation axioms between T_0 and T_1, *Indag. Math.* 24 (1962) 26–37.
[2] B. Banaschewski and A. Pultr, Cauchy points of uniform and nearness frames, *Quaest. Math.* 19 (1996) 101–127.
[3] H. Bennet, D. Lutzer and M. Matveev, The monotone Lindelöf property and separability in ordered spaces, *Topology Appl.* 151 (2005) 180–186.
[4] C. Borges, On stratifiable spaces, *Pacific J. Math.* 17 (1966) 1-16.
[5] C. Borges, A study of monotonically normal spaces, *Proc. Amer. Math. Soc.* 38 (1973) 211–214.
[6] J. G. Ceder, Some generalizations of metric spaces, *Pacific J. Math.* 11 (1961) 105–125.
[7] P. J. Collins, Monotone normality, *Topology Appl.* 74 (1996) 179–198.
[8] A. S. Davis, Indexed systems of neighborhoods for general topological spaces, *Amer. Math. Monthly* 68 (1961) 886–893.
[9] R. Engelking, *General Topology*, Heldermann Verlag, 1989.
[10] C. Good, R. Knight and I. Stares, Monotone countable paracompactness, *Topology Appl.* 101 (2000) 281–298.
[11] J. Gutiérrez García, T. Kubiak and J. Picado, Monotone insertion and monotone extension of frame homomorphisms, *J. Pure Appl. Algebra* 212 (2008) 955–968.
[12] J. Gutiérrez García, I. Mardones Pérez and M. A. de Prada Vicente, Monotone normality free of T_1 axiom, *Acta Math. Hungar.* 122 (2009) 71–80.
[13] J. Gutiérrez García, I. Mardones Pérez and M. A. de Prada Vicente, Insertion and extension theorems for lattice-valued functions on preordered topological spaces, *Topology Appl.* 158 (2011) 60–68.
[14] J. Gutiérrez García and J. Picado, On the parallel between normality and extremal disconnectedness, *J. Pure Appl. Algebra* 218 (2014) 784–803.
[15] J. Gutiérrez García, S. Romaguera and J. M. Sánchez-Álvarez, Quasi-metrics and monotone normality, *Topology Appl.* 158 (2011) 2049–2055.
[16] R. W. Heath, D. J. Lutzer and P. Zenor, Monotonically normal spaces, *Trans. Amer. Math. Soc.* 178 (1973) 481–493.
[17] J. Heikki, K. Junnila and H.-P. Künzi, Ortho-bases and monotonic properties, *Proc. Amer. Math. Soc.* 119 (1993) 1335–1345.

[18] H. Herrlich and A. Pultr, Nearness, subfitness and sequential regularity, *Appl. Categ. Structures* 8 (2000) 67–80.
[19] J. R. Isbell, Atomless parts of spaces, *Math. Scand.* 31 (1972) 5–32.
[20] P. T. Johnstone, *Stone Spaces*, Cambridge University Press, 1982.
[21] T. Kubiak, Monotone insertion of continuous functions, *Questions Answers Gen. Topology* 11 (1993) 51–59.
[22] I. Mardones Pérez and M. A. de Prada Vicente, Monotone insertion of lattice-valued functions, *Acta Math. Hungar.* 117 (2007) 187–200.
[23] K. Morita, On the simple extension of a space with respect to a uniformity I, *Proc. Japan Acad.* 27 (1951) 65–72.
[24] G. S. Murchiston and M. G. Stanley, A T_1 space with no closed points and a T_1 locale which is not T_1, *Math. Proc. Cambridge Phil. Soc.* 95 (1984) 421–422.
[25] D. G. P. Palentz, Monotone normality and paracompactness, *Topology Appl.* 14 (1982) 171–182.
[26] C. Pan, Monotonically CP spaces, *Questions Answers Gen. Topology* 15 (1997) 25–32.
[27] J. Picado and A. Pultr, *Frames and Locales: topology without points*, Frontiers in Mathematics, Vol. 28, Springer, Basel, 2012.
[28] A. Pultr, Pointless uniformities II. (Dia)metrization, *Comment. Math. Univ. Carolinae* 25 (1984) 105–120.
[29] A. Pultr and A. Tozzi, Separation axioms and frame representation of some topological facts, *Appl. Categ. Structures* 2 (1994) 107-118.
[30] H. Simmons, The lattice theoretic part of topological separation properties, *Proc. Edinburgh Math. Soc. (2)* 21 (1978) 41–48.
[31] H. Simmons, Regularity, fitness, and the block structure of frames, *Appl. Categ. Structures* 14 (2006) 1–34.
[32] I. Stares, Monotone normality and extension of functions, *Comment. Math. Univ. Carolin.* 36 (1995) 563–578.
[33] K. Yamazaki, Locally bounded set-valued mappings and monotone countable paracompactness, *Topology Appl.* 154 (2007) 2817–2825.
[34] P. Zenor, Monotonically normal spaces, *Notices Amer. Math. Soc.* 17 (1970) 1034.