THE STRUCTURE OF LEIBNIZ SUPERALGEBRAS ADMITTING A MULTIPlicative BASIS

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ABSTRACT. In the literature, most of the descriptions of different classes of Leibniz superalgebras \( \mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1 \), have been made by giving the multiplication table on the elements of a graded basis \( \mathcal{B} = \{ v_k \}_{k \in K} \) of \( \mathcal{L} \), in such a way that for any \( i, j \in K \) we have \( [v_i, v_j] = \lambda_{i,j} [v_j, v_i] \in F v_k \) for some \( k \in K \), where \( F \) denotes the base field and \( \lambda_{i,j} \in F \). In order to give a unifying viewpoint of all these classes of algebras we introduce the category of Leibniz superalgebras admitting a multiplicative basis and study its structure. We show that if a Leibniz superalgebra \( \mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1 \) admits a multiplicative basis then it is the direct sum \( \mathcal{L} = \bigoplus I_\alpha \) with any \( I_\alpha = I_\alpha,0 \oplus I_\alpha,1 \) a well described ideal of \( \mathcal{L} \) admitting a multiplicative basis inherited from \( \mathcal{B} \). Also the \( \mathcal{B} \)-simplicity of \( \mathcal{L} \) is characterized in terms of \( \mathcal{J} \)-connections.

Keywords: Leibniz superalgebra, multiplicative basis, infinite dimension, structure theory.

1. INTRODUCTION AND PREVIOUS DEFINITIONS

Leibniz superalgebras appear as an extension of Leibniz algebras (see [4, 5, 10, 13, 14, 15, 16, 17]), in a similar way than Lie superalgebras generalize Lie algebras, motivated in part for its applications in Physics. The present paper is devoted to the study of the structure of Leibniz superalgebras \( \mathcal{L} \) admitting a multiplicative basis over a field \( \mathbb{F} \). Since a Leibniz algebra is a particular case of a Leibniz superalgebra (with \( \mathcal{L}_1 = \{ 0 \} \)), this work extends the results exhibited in [6]. We would like to remark that the techniques used in this paper also hold in the infinite-dimensional case over arbitrary fields, being adequate enough to provide us a second Wedderburn-type theorem in this general framework (Theorems 2.1 and 3.1). Moreover, although we make use of the ideal \( \mathcal{J} \) which is deeply inherent to Leibniz theory, we believe that our approach can be useful for the knowledge of the structure of wider classes of algebras.

Definition 1.1. A Leibniz superalgebra \( \mathcal{L} \) is a \( \mathbb{Z}_2 \)-graded algebra \( \mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1 \) over an arbitrary base field \( \mathbb{F} \), with its bilinear product denoted by \([\cdot, \cdot]\), whose homogenous elements \( x \in \mathcal{L}_i, y \in \mathcal{L}_j, i, j \in \mathbb{Z}_2 \), satisfy

\[
[x, y] \in \mathcal{L}_{i+j}
\]

\[
[x, [y, z]] = [[x, y], z] - (-1)^{ij}[[x, z], y]
\]

(Super Leibniz identity)

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for any homogenous element \( z \in \mathcal{L}_\pi, \pi \in \mathbb{Z}_2 \).

**Remark 1.1.** Note that Super Leibniz identity is considered by the right side in the sense that the multiplication operators on the right by elements in \( \mathcal{L}_0 \) are derivations on the homogeneous elements, as it is done in the references [4, 5, 10, 13, 17]. However, we could have considered a Super Leibniz identity in which the multiplication operators on the left by elements in \( \mathcal{L}_0 \) would act as derivations on the homogeneous elements, as it is the case in the references [14, 15, 16]. Of course, the development of the present work would have been similar in this case.

Clearly \( \mathcal{L}_0 \) is a Leibniz algebra. Moreover, if the identity \([x, y] = -(1)^{\pi \bar{\pi}}[y, x]\) holds, then Super Leibniz identity becomes Super Jacobi identity and so Leibniz superalgebras generalize also Lie superalgebras, which is of interest in the formalism of mechanics of Nambu [12].

The usual concepts are considered in a graded sense. A subsuperalgebra \( A \) of \( \mathcal{L} \) is a graded subspace \( A = A_0 \oplus A_1 \) satisfying \([A, A] \subset A\). An ideal \( I \) of \( \mathcal{L} \) is a graded subspace \( I = I_0 \oplus I_1 \) of \( \mathcal{L} \) such that \([I, \mathcal{L}] + [\mathcal{L}, I] \subset I\).

The (graded) ideal \( I \) generated by \[\{[x, y] + (-1)^{\pi \bar{\pi}}[y, x] : x \in \mathcal{L}_\pi, y \in \mathcal{L}_{\bar{\pi}}, \pi, \bar{\pi} \in \mathbb{Z}_2\}\] plays an important role in the theory since it determines the (possible) non-super Lie character of \( \mathcal{L} \). From definition of ideal \([I, \mathcal{L}] \subset I\) and from Super Leibniz identity, it is straightforward to check that this ideal satisfies

\[
[I, \mathcal{L}] = 0.
\]

Here we note that the usual definition of simple superalgebra lacks of interest in the case of Leibniz superalgebras because would imply the ideal \( I = \mathcal{L} \) or \( I = 0 \), being so \( \mathcal{L} \) an abelian (product zero) or a Lie superalgebra respectively (see Equation (1)). Abdikassyymova and Dzhumadil’daev introduced in [1, 2] an adequate definition in the case of Leibniz algebras \((\mathcal{L}, [\cdot, \cdot])\) by calling simple to the ones such that its only ideals are \( \{0\}, L \) and the one generated by the set \([x, x] : x \in L\). Following this vain, we consider the next definition.

**Definition 1.2.** A Leibniz superalgebra \( \mathcal{L} \) is called simple if \([\mathcal{L}, \mathcal{L}] \neq 0 \) and its only (graded) ideals are \( \{0\}, I \) and \( \mathcal{L} \).

Observe that we can write

\[ \mathcal{L} = I \oplus \sim I \]

where \( \sim I = \sim I_0 \oplus \sim I_1 \) is a linear complement of \( I = I_0 \oplus I_1 \) in \( \mathcal{L} \) (here we adapt this notation in order to standardize the one already used in [7, 8, 9]). Actually \( \sim I \) is isomorphic as linear space to \( \mathcal{L}/I \), the so called corresponding Lie superalgebra of \( \mathcal{L} \). In general, \( \sim I \) is not an ideal of \( \mathcal{L} \) from \([I, \sim I] \subset I\). Then the multiplication in \( \mathcal{L} \) is represented in the table

\[
\begin{array}{c|cccc}
  & I_0 & \sim I_0 & I_1 & \sim I_1 \\
\hline
I_0 & 0 & I_0 & 0 & I_0 \\
\sim I_0 & I_0 \oplus \sim I_0 & 0 & \sim I_1 & I_1 \oplus \sim I_1 \\
I_1 & 0 & I_1 & 0 & I_1 \\
\sim I_1 & I_1 \oplus \sim I_1 & 0 & I_0 & I_0 \oplus \sim I_0 \\
\end{array}
\]
Hence, by taking $B_{3\tau}$ and $B_{3\tau}$ bases of $J_\tau$ and $-J_\tau$, for $\tau \in \mathbb{Z}_2$, respectively, then

$$B = (B_{3\tau} \cup B_{3\tau}) \cup (B_{3\tau} \cup B_{3\tau})$$

is a basis of $\mathcal{L}$.

**Definition 1.3.** A basis $B = \{v_k^i : k \in K, i, j \in \mathbb{Z}_2\}$ of $\mathcal{L}$ is said to be multiplicative if for any $k_1, k_2 \in K, i, j \in \mathbb{Z}_2$ we have $[v_k^i, v_k^j] \in Fv_{k^i+j}^\tau$ for some $k \in K$.

**Example 1.1.** Consider the 5-dimensional $\mathbb{Z}_2$-graded vector space $\mathcal{L} = \mathcal{L}_\tau \oplus \mathcal{L}_\tau$, over a base field $F$ of characteristic different from 2, with basis $B_{3\tau} = \{e_1, e_2\}$, $B_{3\tau} = \{u_a, u_b, u_c\}$; where the products on these elements are given by:

$$[u_b, u_a] = -u_c, \quad [u_a, u_b] = u_c, \quad [u_a, u_a] = -2u_a,$$

$$[u_c, u_a] = 2u_a, \quad [u_c, u_b] = -2u_b, \quad [u_b, u_b] = 2u_b,$$

$$[e_1, u_b] = e_2, \quad [e_1, u_c] = -e_1, \quad [e_2, u_b] = e_1, \quad [e_2, u_c] = e_2,$$

and where the omitted products are equal to zero. Then $\mathcal{L} = \mathcal{L}_\tau \oplus \mathcal{L}_\tau$ becomes a (non-Lie) Leibniz superalgebra admitting $B = B_{3\tau} \cup B_{3\tau}$ as multiplicative basis.

**Example 1.2.** Let us denote by $\mathbb{N}^+$ the set of non-negative integers. Consider the infinite-dimensional complex $\mathbb{Z}_2$-graded vector space $\mathcal{L} = \mathcal{L}_\tau \oplus \mathcal{L}_\tau$ with basis $B_{3\tau} = \{e_{(n,k)}: n, k \in \mathbb{N}^+ \text{ and } k \leq n\}$, $B_{3\tau} = \{e_{(n-1,k), e_{(n-2,k)}, e_{(n-3)}} : n \in \mathbb{N}\}$; with the following table of multiplication:

$$[e_{(n-1,k)}, e_{(n-3,k)}] = 2e_{(n-1,k)}, \quad [e_{(n-3,k)}, e_{(n-1,k)}] = -2e_{(n-1,k)},$$

$$[e_{(n-2,k)}, e_{(n-3,k)}] = -2e_{(n-2,k)}, \quad [e_{(n-3,k)}, e_{(n-2,k)}] = 2e_{(n-2,k)},$$

$$[e_{(n,k), e_{(n-3,k)}}] = e_{(n,k+1)}, \quad [e_{(n,k), e_{(n-3,k)}}] = e_{(n,k+1)}, \quad [e_{(n,k), e_{(n-3,k)}}] = -(n-k)e_{(n,k-1)}, \quad \text{for } 1 \leq k \leq n;$$

and where the omitted products are equal to zero. Then $\mathcal{L} = \mathcal{L}_\tau \oplus \mathcal{L}_\tau$ is a (non-Lie) Leibniz superalgebra admitting $B = B_{3\tau} \cup B_{3\tau}$ as multiplicative basis.

**Remark 1.2.** Observe that if we write $B_{3\tau} = \{e_{n}^{\tau} \in \mathcal{L}_{3\tau} \text{ and } B_{3\tau} = \{u_{n}^{\tau} \in \mathcal{L}_{3\tau} \}$ for $\tau \in \mathbb{Z}_2$.

Since $\mathcal{J}$ is an ideal together with Equation (1) we know that the only possible non-zero products among the elements in $B$ are:

1. For $n \in I_\tau, r \in J_\tau$ and $\tau, \tau' \in \mathbb{Z}_2$ we have $[e_{n}^{\tau}, u_{r}^{\tau'}] \in Fv_{n+\tau+r}^{\tau+\tau'}$ for some $k \in I_{\tau+\tau'}$.
2. For $r \in J_\tau, s \in J_\tau$ and $\tau, \tau' \in \mathbb{Z}_2$ we have either $[u_{r}^{\tau}, u_{s}^{\tau'}] \in Fv_{n+\tau+r}^{\tau+\tau'}$ for some $l \in I_{\tau+\tau'}$ or $[u_{r}^{\tau}, u_{s}^{\tau'}] \in Fv_{n+\tau+r}^{\tau+\tau'}$ for some $n \in I_{\tau+\tau'}$.

**Lemma 1.1.** Let $(\mathcal{L}, [\cdot, \cdot])$ be a Leibniz superalgebra over a base field $F$ of characteristic different from 2. If $B = \{v_k : k \in K\}$ is a graded basis of $\mathcal{L}$ such that for any $k_1, k_2 \in K$ we have $[v_k, v_k] = \lambda_k v_k$ for some $k \in K$ and some $\lambda_k, k_2 \in F$ for some $k \in K$ and some $\lambda_k, k_2 \in F$ then $\mathcal{L}$ admits $B$ as multiplicative basis.

**Proof.** By the definition of $\mathcal{J}$ we see that it is generated as linear space by $\{v_j : j \in J\}$, for some subset $J$ of $K$. So we can find a basis $B_{3\tau}$ of $\mathcal{J}$ formed by elements of $B$ and a basis $B_{3\tau} := B \backslash B_{3\tau}$ of $-\mathcal{J}$ which make of $B$ a multiplicative basis. \[\Box\]
The preceding lemma shows that all commutative (up to a scalar) Leibniz superalgebras admit a multiplicative basis. For instance, this is the case of null-filiform Leibniz superalgebras, Leibniz superalgebras of maximal nilindex or Leibniz superalgebras with nilindex \( n + m + 1 \) (see [3, 10, 11]).

The paper is organized as follows. In §2 inspired by the connections of roots developed for split Leibniz algebras and superalgebras in [7, 8], we introduce similar techniques on the index set of the multiplicative basis \( B \). Our purpose is to obtain a powerful tool for the study of this class of superalgebras. By making use of these results we see that any Leibniz superalgebra \( L \) admitting a multiplicative basis is of the form \( L = \bigoplus_{\alpha} L_{\alpha} \), where every \( L_{\alpha} \) is a well described ideal having a multiplicative basis inherited from \( B \). In §3 the \( B \)-simplicity of these ideals is characterized in terms of the \( J \)-connection.

2. DECOMPOSITION AS DIRECT SUM OF IDEALS

In what follows \( L = (\mathcal{F}_{1} \oplus \mathcal{F}_{-1}) \oplus (\mathcal{F}_{-1} \oplus \mathcal{F}_{1}) \) denotes a Leibniz superalgebra over a base field \( F \) admitting a multiplicative basis

\[
(2) \quad B = (\mathcal{B}_{3_{\mathbb{F}}} \cup \mathcal{B}_{3_{\mathbb{F}}}) \cup (\mathcal{B}_{-3_{\mathbb{F}}} \cup \mathcal{B}_{-3_{\mathbb{F}}})
\]

where \( \mathcal{B}_{3_{\mathbb{F}}} = \{ e_{i,j} \}_{i \in I_{3}, j \in J_{3}} \) and \( \mathcal{B}_{-3_{\mathbb{F}}} = \{ u_{i,j} \}_{i \in I_{3}, j \in J_{3}} \), for \( i \in \mathbb{Z}_{2} \), and where, by renaming if necessary, we can suppose \( K_{\gamma} \cap P_{\gamma} = \emptyset \) for any \( K, P \in \{ I, J \}, i, j \in \mathbb{Z}_{2} \) and \( K_{\gamma} \neq P_{\gamma} \).

We begin this section by developing connection techniques among the elements in the index sets \( I_{\gamma} \cup J_{\gamma} \cup \tilde{I}_{\gamma} \cup \tilde{J}_{\gamma} \) as the main tool in our study. Now, for each \( k \in I_{\gamma} \cup J_{\gamma} \cup \tilde{I}_{\gamma} \cup \tilde{J}_{\gamma} \), a new assistant variable \( \tilde{k} \not\in I_{\gamma} \cup J_{\gamma} \cup \tilde{I}_{\gamma} \cup \tilde{J}_{\gamma} \) is introduced and we denote by

\[
\tilde{I}_{i} := \{ \tilde{n} : n \in I_{i} \} \quad \text{and} \quad \tilde{J}_{i} := \{ \tilde{r} : r \in J_{i} \},
\]

for \( i \in \mathbb{Z}_{2} \), the sets consisting of all these new symbols. Also, given any \( \tilde{k} \in \tilde{K}_{\gamma} \), \( K \in \{ I, J \}, i \in \mathbb{Z}_{2} \), we denote

\[
\tilde{\tilde{k}} := k.
\]

Finally, we write by \( \mathcal{P}(A) \) the power set of a given set \( A \).

Next, we consider an operation which recover, in some sense, certain multiplicative relations among the elements of the basis \( B \):

\[
*: (I_{\gamma} \cup J_{\gamma} \cup \tilde{I}_{\gamma} \cup \tilde{J}_{\gamma}) \times (I_{\gamma} \cup J_{\gamma} \cup \tilde{I}_{\gamma} \cup \tilde{J}_{\gamma} \cup \tilde{I}_{\gamma} \cup \tilde{J}_{\gamma} \cup \tilde{I}_{\gamma} \cup \tilde{J}_{\gamma}) \rightarrow \mathcal{P}(I_{\gamma} \cup J_{\gamma} \cup \tilde{I}_{\gamma} \cup \tilde{J}_{\gamma}),
\]

where for any \( i, j \in \mathbb{Z}_{2} \) is defined by

- For \( n \in I_{i}, m \in J_{j} \),
  \[
  n * m := \emptyset
  \]
- For \( n \in I_{i} \) and \( r \in J_{j} \),
  \[
  n * \{ \emptyset, \{ m \} \} := \emptyset, \quad \text{if } [e_{n,\tilde{i}}, u_{r,\tilde{j}}] = 0
  \]
- For \( n \in I_{i} \) and \( \tilde{m} \in \tilde{I}_{j} \),
  \[
  n * \tilde{m} := \{ r \in J_{i+j} : 0 \neq [e_{m,\tilde{i}}, u_{r,\tilde{j}}] \in F e_{n,\tilde{i+j}} \} \subset F e_{n,\tilde{i+j}}
  \]
- For \( n \in I_{i} \) and \( \tilde{r} \in \tilde{J}_{j} \),
  \[
  n * \tilde{r} := \{ s \in J_{i+j} : 0 \neq [u_{r,\tilde{i+j}}, e_{s,\tilde{i+j}}] \in F e_{n,\tilde{i+j}} \} \cup \{ m \in I_{i+j} : 0 \neq [e_{m,\tilde{i+j}}, u_{r,\tilde{j}}] \in F e_{n,\tilde{i+j}} \}.
  \]
For $r \in J_{\mathcal{T}}$, $s \in J_{\mathcal{T}}$,
\[ r \ast s := \begin{cases} 0, & \text{if } [u_{s,\tilde{r}}, u_{s,\tilde{r}}] = 0 \\
\{t\}, & \text{if } 0 \neq [u_{s,\tilde{r}}, u_{s,\tilde{r}}] \in F u_{s,\tilde{r}+\mathcal{T}} \\
\{n\}, & \text{if } 0 \neq [u_{s,\tilde{r}}, u_{s,\tilde{r}}] \in F e_{n,\tilde{r}+\mathcal{T}} \end{cases} \]

For $r \in J_{\mathcal{T}}$ and $\tilde{n} \in \tilde{J}_{\mathcal{T}}$,
\[ r \ast \tilde{n} := \emptyset \]

For $r \in J_{\mathcal{T}}$ and $\tilde{s} \in \tilde{J}_{\mathcal{T}}$,
\[ r \ast \tilde{s} := \{ t \in J_{\mathcal{T}+\mathcal{T}} : 0 \neq [u_{t,\tilde{r}+\mathcal{T}}, u_{s,\tilde{r}}] \in F u_{t,\tilde{r}} \} \cup \{ q \in J_{\mathcal{T}+\mathcal{T}} : 0 \neq [u_{s,\tilde{r}}, u_{q,\tilde{r}+\mathcal{T}}] \in F u_{q,\tilde{r}} \} . \]

The mapping $\ast$ is not still adequate to use in an iterative process necessary for our purposes and so we need to introduce the following one:
\[
\phi : \mathcal{P}(I_{\mathcal{T}} \cup I_{\mathcal{T}} \cup J_{\mathcal{T}}) \times (I_{\mathcal{T}} \cup I_{\mathcal{T}} \cup J_{\mathcal{T}} \cup \tilde{I}_{\mathcal{T}} \cup \tilde{I}_{\mathcal{T}} \cup \tilde{J}_{\mathcal{T}}) \to \mathcal{P}(I_{\mathcal{T}} \cup I_{\mathcal{T}} \cup J_{\mathcal{T}}),
\]
as
\[ \phi(\emptyset, \emptyset) = \emptyset, \]
\[ \phi(a, b) := \bigcup_{k \in K} (a \ast k) \cup (a \ast b). \]

**Lemma 2.1.** For any $K \in \mathcal{P}(I_{\mathcal{T}} \cup I_{\mathcal{T}} \cup J_{\mathcal{T}})$ and $a \in I_{\mathcal{T}} \cup I_{\mathcal{T}} \cup J_{\mathcal{T}} \cup \tilde{I}_{\mathcal{T}} \cup \tilde{I}_{\mathcal{T}} \cup \tilde{J}_{\mathcal{T}}$,
\[
(3) \quad k \in \phi(K, a) \text{ if and only if } \phi(\{k\}, \tilde{a}) \cap K \neq \emptyset.
\]

**Proof.** It is straightforward to observe that for any $k_1, k_2 \in I_{\mathcal{T}} \cup I_{\mathcal{T}} \cup J_{\mathcal{T}}$ and
\[ a \in I_{\mathcal{T}} \cup I_{\mathcal{T}} \cup J_{\mathcal{T}} \cup \tilde{I}_{\mathcal{T}} \cup \tilde{I}_{\mathcal{T}} \cup \tilde{J}_{\mathcal{T}}, \]
we have $k_2 \in k_1 \ast a \cup a \ast k_1$ if and only if $k_1 \in k_2 \ast a$. \hfill \Box

**Definition 2.1.** Let $k$ and $k'$ be elements in the index set $I_{\mathcal{T}} \cup I_{\mathcal{T}} \cup J_{\mathcal{T}}$. We say $k$ is connected to $k'$ if either $k = k'$ or there exists a subset
\[ \{k_1, k_2, \ldots, k_{n-1}, k_n\} \subset I_{\mathcal{T}} \cup I_{\mathcal{T}} \cup J_{\mathcal{T}} \cup \tilde{I}_{\mathcal{T}} \cup \tilde{I}_{\mathcal{T}} \cup \tilde{J}_{\mathcal{T}} \]
with $n \geq 2$ such that the following conditions hold:
1. $k_1 = k$.
2. $\phi(\{k_1\}, k_2) \neq \emptyset, \phi(\{k_1\}, k_3) \neq \emptyset, \ldots$.
3. $k' \in \phi(\{k_1\}, k_2, \ldots, k_{n-1}, n)$. \hfill \Box

The subset $\{k_1, k_2, \ldots, k_{n-1}, k_n\}$ is called a connection from $k$ to $k'$. **Proposition 2.1.** The relation $\sim$ in $I_{\mathcal{T}} \cup I_{\mathcal{T}} \cup J_{\mathcal{T}}$, defined by $k \sim k'$ if and only if $k$ is connected to $k'$, is an equivalence relation.
Proof. By definition $k \sim k$, that is, the relation $\sim$ is reflexive. Let us see the symmetric character of $\sim$: If $k \sim k'$ with $k \neq k'$ then there exists a connection
\[
\{k_1, k_2, \ldots, k_{n-1}, k_n\}
\]
from $k$ to $k'$ satisfying Definition 2.1. Let us show that the set
\[
\{k', k_n, k_{n-1}, \ldots, k_3, k_2\}
\]
gives rise to a connection from $k'$ to $k$. Indeed, by taking
\[
K := \phi(\cdots(\phi(\{k_1\}, k_2), \cdots), k_{n-1})
\]
we can apply the relation given by (3) to the expression
\[
k' \in \phi(K, k_n)
\]
to get
\[
\phi([k'], k_n) \cap K \neq \emptyset
\]
and so
\[
\phi([k'], k_n) \neq \emptyset.
\]
By taking
\[
h \in \phi([k'], k_n) \cap K,
\]
then
\[
h \in K = \phi(\cdots(\phi(\{k_1\}, k_2), \cdots), k_{n-1}),
\]
by the relation given by (3) we get
\[
\phi([h], k_{n-1}) \cap \phi(\cdots(\phi(\{k_1\}, k_2), \cdots), k_{n-2}) \neq \emptyset,
\]
but $h \in \phi([k'], k_n)$, therefore $h \subset \phi(\{k'\}, k_n)$ and consequently
\[
\phi(\phi([k'], k_n), k_{n-1}) \cap \phi(\cdots(\phi(\{k_1\}, k_2), \cdots), k_{n-2}) \neq \emptyset.
\]
By iterating this process we get
\[
\phi(\phi(\cdots(\phi([k'], k_n), \cdots), k_{n-1}), k_{n-2}) \neq \emptyset
\]
for $0 \leq r \leq n - 3$. Observe that this relation in the case $r = n - 3$ reads as
\[
\phi(\phi(\cdots(\phi(\{k\}', k_n), \cdots), k_4), k_3) \cap \phi(\{k_1\}, k_2) \neq \emptyset.
\]
Since $k_1 = k$, if we write $\tilde{K} := \phi(\phi(\cdots(\phi(\{k\}', k_n), \cdots), k_4), k_3)$, the previous observation allows us to assert $\phi(\{k\}, k_2) \cap \tilde{K} \neq \emptyset$. Hence the relation (3) applies to get
\[
k \in \phi(\phi(\cdots(\phi(\{k\}', k_n), \cdots), k_3), k_2)
\]
and concludes $\sim$ is symmetric.

Finally, let us verify the transitive character of $\sim$. Suppose $k \sim k'$ and $k' \sim k''$. If $k = k'$ or $k' = k''$ it is trivial, so suppose $k \neq k'$ and $k' \neq k''$ and write $\{k_1, \ldots, k_n\}$ for a connection from $k$ to $k'$ and $\{k'_1, \ldots, k'_m\}$ for a connection from $k'$ to $k''$. Then we clearly see that $\{k_1, \ldots, k_n, k'_1, \ldots, k'_m\}$ is a connection from $k'$ to $k''$. We have shown the connection relation is an equivalence relation. \qed
By the above proposition we can consider the next quotient set on the index set $I_\mathcal{T} \cup I_\mathcal{T} \cup I_\mathcal{T} \cup I_\mathcal{T}$, 

$$(I_\mathcal{T} \cup I_\mathcal{T} \cup I_\mathcal{T} \cup I_\mathcal{T})/ \sim = \{ [k] : k \in I_\mathcal{T} \cup I_\mathcal{T} \cup I_\mathcal{T} \cup I_\mathcal{T} (\mod k) \},$$

becoming $[k]$ the set of elements in $I_\mathcal{T} \cup I_\mathcal{T} \cup I_\mathcal{T} \cup I_\mathcal{T}$ which are connected to $k$.

Our next goal in this section is to associate an ideal $\mathcal{I}_{\{k\}}$ of $\mathcal{L}$ to any $[k]$. Fix $k \in I_\mathcal{T} \cup I_\mathcal{T} \cup I_\mathcal{T} \cup I_\mathcal{T}$, we start by defining the linear subspaces $\mathcal{J}_{\{k\}} = \mathcal{J}_{\{k\}, \pi} \oplus \mathcal{J}_{\{k\}, \tau} \subset \mathcal{J}$ and $\mathcal{J} = \mathcal{J}_{\{k\}} \oplus \mathcal{J}_{\{k\}, \tau}$ as follows

$$\mathcal{J}_{\{k\}, \pi} := \bigoplus_{i \in [k] \cap I_\tau} \mathcal{F} e_{i, \pi} \subset \mathcal{J},$$

$$\mathcal{J}_{\{k\}, \tau} := \bigoplus_{j \in [k] \cap J_\tau} \mathcal{F} u_{j, \tau} \subset -\mathcal{J}$$

for any $j \in \mathbb{Z}_2$. Finally, we denote by $\mathcal{I}_{\{k\}}$ the direct sum of the two subspaces above, that is,

$$\mathcal{I}_{\{k\}} := (\mathcal{J}_{\{k\}, \pi} \oplus \mathcal{J}_{\{k\}, \tau}) \oplus (-\mathcal{J}_{\{k\}, \pi} \oplus -\mathcal{J}_{\{k\}, \tau})$$

**Definition 2.2.** Let $\mathcal{L}$ be a Leibniz superalgebra admitting a multiplicative basis $\mathcal{B}$. A subsuperalgebra $A \subset \mathcal{L}$ admits a multiplicative basis $\mathcal{B}_A$ inherited from $\mathcal{B}$ if $\mathcal{B}_A$ is a multiplicative basis of $A$ satisfying $\mathcal{B}_A \subset \mathcal{B}$.

**Proposition 2.2.** For any $k \in I_\mathcal{T} \cup I_\mathcal{T} \cup I_\mathcal{T} \cup I_\mathcal{T}$, the linear subspace $\mathcal{I}_{\{k\}}$ is an ideal of $\mathcal{L}$ admitting a multiplicative basis inherited from the one of $\mathcal{L}$.

**Proof.** We can write

$$[\mathcal{I}_{\{k\}}, \mathcal{L}] = [\mathcal{J}_{\{k\}} \oplus -\mathcal{J}_{\{k\}}, (\bigoplus_{n \in I_\pi} \mathcal{F} e_{n, \pi}) \oplus (\bigoplus_{m \in I_\tau} \mathcal{F} e_{m, \tau}) \oplus (\bigoplus_{r \in I_\pi} \mathcal{F} u_{r, \pi}) \oplus (\bigoplus_{s \in I_\tau} \mathcal{F} u_{s, \tau})].$$

In case $[e_{i, \pi}, u_{r, \tau}] \neq 0$ for some $l \in [k] \cap I_\tau$, $r \in J_\tau$ and $i, j \in \mathbb{Z}_2$, we have $0 \neq [e_{i, \pi}, u_{r, \tau}] \in \mathcal{F} e_{p, \pi}$ with $p \in I_\tau + I$ and so $p \in \phi([l], r) = l \ast r$, therefore the connection $\{l, r\}$ gives us $l \sim p$, so $p \in [k]$ and then $0 \neq [e_{i, \pi}, u_{r, \tau}] \in \mathcal{J}_{\{k\}}$. Hence we get

$$[\mathcal{J}_{\{k\}}, (\bigoplus_{r \in I_\pi} \mathcal{F} u_{r, \pi}) \oplus (\bigoplus_{s \in I_\tau} \mathcal{F} u_{s, \tau})] \subset \mathcal{J}_{\{k\}} \subset \mathcal{I}_{\{k\}}.$$

In a similar way we have $[-\mathcal{J}_{\{k\}}, (\bigoplus_{r \in I_\pi} \mathcal{F} u_{r, \pi}) \oplus (\bigoplus_{s \in I_\tau} \mathcal{F} u_{s, \tau})] \subset \mathcal{I}_{\{k\}}$ and taking into account Equation (1) we conclude

$$[\mathcal{I}_{\{k\}}, \mathcal{L}] \subset \mathcal{I}_{\{k\}}.$$

On the other hand,

$$[\mathcal{L}, \mathcal{I}_{\{k\}}] = [(\bigoplus_{n \in I_\pi} \mathcal{F} e_{n, \pi}) \oplus (\bigoplus_{m \in I_\tau} \mathcal{F} e_{m, \tau}) \oplus (\bigoplus_{r \in I_\pi} \mathcal{F} u_{r, \pi}) \oplus (\bigoplus_{s \in I_\tau} \mathcal{F} u_{s, \tau}), \mathcal{J}_{\{k\}} \oplus -\mathcal{J}_{\{k\}}]$$

and in case $0 \neq [e_{n, \pi}, u_{h, \tau}]$ for some $n \in I_\tau, h \in [k] \cap J_\tau$ and $i, j \in \mathbb{Z}_2$ we have $[e_{n, \pi}, u_{h, \tau}] \in \mathcal{F} e_{i, \pi}$ with $p \in I_\pi + I$. Then $p \in \phi([h], n) = h \ast n$ and we see that the connection $\{h, n\}$ gives us $h \sim p$ and so $[\bigoplus_{n \in I_\pi} \mathcal{F} e_{n, \pi}) \oplus (\bigoplus_{m \in I_\tau} \mathcal{F} e_{m, \tau})], -\mathcal{J}_{\{k\}] \subset \mathcal{J}_{\{k\}} \subset \mathcal{I}_{\{k\}}$. In a similar way

$$[(\bigoplus_{r \in I_\pi} \mathcal{F} u_{r, \pi}) \oplus (\bigoplus_{s \in I_\tau} \mathcal{F} u_{s, \tau}), -\mathcal{J}_{\{k\}] \subset \mathcal{I}_{\{k\}}$$
and by Equation (1) then

$$[\mathcal{L}, \mathcal{I}_{[k]}] \subset \mathcal{I}_{[k]}.$$ 

Hence $\mathcal{I}_{[k]}$ is an ideal of $\mathcal{L}$.

Finally, observe that the set

$$B_{\mathcal{I}_{[k]}} := \{ e_{n,\sigma} : n \in [k] \cap I_\sigma \} \cup \{ e_{m,\tau} : m \in [k] \cap I_\tau \} \cup \{ u_{r,\sigma} \cup u_{s,\tau} : s \in [k] \cap J_\tau \}$$

is a multiplicative basis of $\mathcal{I}_{[k]}$ satisfying $B_{\mathcal{I}_{[k]}} \subset B$. Hence we see that $\mathcal{I}_{[k]}$ admits a multiplicative basis inherited from the one of $\mathcal{L}$.

\begin{flushright}
\Box
\end{flushright}

**Corollary 2.1.** If $\mathcal{L}$ is simple, then there exists a connection between any couple of elements in the index set $I_\sigma \cup I_\tau \cup J_\sigma \cup J_\tau$.

**Proof.** The simplicity of $\mathcal{L}$ implies $[\mathcal{L}, \mathcal{L}] \neq 0$ and so $\mathcal{I} \neq \emptyset$, then at least there exists $r_0 \in J_\tau \cap \mathcal{I}$, such that $\{ u_{r_0,\sigma} \} \subset B_{\mathcal{I}_{[k]}}$. Applying Proposition 2.2, $\mathcal{I}_{[r_0]}$ is an ideal and by its construction $\mathcal{I}_{[r_0]} \subseteq \mathcal{I}$, therefore $\mathcal{I}_{[r_0]} = \mathcal{L}$ being then $[r_0] = I_\sigma \cup I_\tau \cup J_\sigma \cup J_\tau$. That is, any couple of elements in $I_\sigma \cup I_\tau \cup J_\sigma \cup J_\tau$ are connected.

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**Theorem 2.1.** A Leibniz superalgebra $\mathcal{L}$ admitting a multiplicative basis decomposes as the direct sum

$$\mathcal{L} = \bigoplus_{[k] \in \{I_\sigma \cup I_\tau \cup J_\sigma \cup J_\tau\}/\sim} \mathcal{I}_{[k]},$$

where any $\mathcal{I}_{[k]} = \mathcal{I}_{[k]} \oplus \mathcal{J}_{[k]}$ is one of the ideals, admitting a multiplicative basis inherited from the one of $\mathcal{L}$, given in Proposition 2.2.

**Proof.** Since we can write $\mathcal{L} = \mathcal{I} \oplus \mathcal{J}$ and

$$\mathcal{J} = \bigoplus_{[k] \in \{I_\sigma \cup I_\tau \cup J_\sigma \cup J_\tau\}/\sim} \mathcal{J}_{[k]}, \quad \mathcal{I} = \bigoplus_{[k] \in \{I_\sigma \cup I_\tau \cup J_\sigma \cup J_\tau\}/\sim} -\mathcal{J}_{[k]},$$

From $\mathcal{I}_{[k]} = \mathcal{I}_{[k]} \oplus \mathcal{J}_{[k]}$ by definition, we clearly have

$$\mathcal{L} = \bigoplus_{[k] \in \{I_\sigma \cup I_\tau \cup J_\sigma \cup J_\tau\}/\sim} \mathcal{I}_{[k]}.$$ 

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**Example 2.1.** Consider the Leibniz superalgebra $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2$ presented in Example 1.1. We have $I_2 = \{1, 2\}$ and $J_2 = \{a, b, c\}$. From the multiplication table of $\mathcal{L}$ it is not difficult to write the operation $\ast$ in a concrete way. For instance, we have

$$\begin{align*}
1 \ast c &= 2 \ast a = \{1\} & a \ast b &= b \ast a = \{c\} \\
1 \ast b &= 2 \ast c = \{2\} & a \ast c &= c \ast a = \{a\}
\end{align*}$$

Then, we can also obtain an explicit expression of the mapping

$$\phi : \mathcal{P}(I_2 \cup J_2 \cup J_2 \cup J_2) \times (I_2 \cup J_2 \cup J_2 \cup J_2) \rightarrow \mathcal{P}(I_2 \cup J_2 \cup J_2 \cup J_2).$$

Observe that the connection $\{1, b\}$ gives $1 \sim 2$, with the connection $\{a, b\}$ we have $a \sim c$ and considering $\{b, a\}$ we obtain $b \sim c$. Since $1 \sim 2 = \{b\}$ we get $1 \sim b$ and therefore $I_2 \cup J_2 \cup J_2 \cup J_2 \sim \{[1]\}$ where $[1] = \{1, 2, a, b, c\}$. By Theorem 2.1 we see that $\mathcal{L} = \mathcal{L}_{[1]}$, where $\mathcal{L}_{[1]}$ is an ideal of $\mathcal{L}$ with a unique (multiplicative) basis $\{1, 2, a, b, c\}$. In fact, since $\mathcal{L}$ is a simple (non-Lie) Leibniz superalgebra, by Corollary 2.2 all elements in $I_2 \cup J_2 \cup J_2 \cup J_2$ are connected and we just have one ideal.
Example 2.2. Let $\mathfrak{L} = \mathfrak{L}_T \oplus \mathfrak{L}_T$ be the Leibniz superalgebra considered in Example 1.2. We have $I = \{(n,k) : n \in \mathbb{N}, 0 \leq k \leq n\}$ and $J = \{(n,-1), (n,-2), (n,-3) : n \in \mathbb{N}\}$. From the multiplication table of $\mathfrak{L}$ it is not difficult to express the operation $\ast$ completely. For instance, we have
\[
\begin{align*}
(n,k) \ast (n,-3) &= \{(n,k)\} \quad k \in I \\
(n,k) \ast (n,-2) &= \{(n,k+1)\} \quad k \in \{0, \ldots, n-1\} \\
(n,k) \ast (n,-1) &= \{(n,k-1)\} \quad k \in \{1, \ldots, n\} \\
(n,-1) \ast (n,-2) &= \{(n,-2) \ast (n,-1)\} = \{(n,-3)\} \\
(n,-1) \ast (n,-3) &= \{(n,-3) \ast (n,-1)\} = \{(n,-1)\} \\
(n,-2) \ast (n,-3) &= \{(n,-3) \ast (n,-2)\} = \{(n,-2)\}
\end{align*}
\]
From here, we can also obtain an explicit expression of the mapping
\[
\phi : \mathcal{P}(I_T \cup J_T \cup J_T | J_T) \times (I_T \cup J_T \cup J_T \cup J_T \cup J_T \cup J_T) \rightarrow \mathcal{P}(I_T \cup J_T \cup J_T \cup J_T).
\]
Observe that the connection $\{(n,-1), (n,-2)\}$ gives $n, -1 \sim (n,-3)$, with the connection $\{(n,-2), (n,-3)\}$ we get $n, -2 \sim (n,-3)$, the connection $\{(n,k+1), (n,k)\}$ let us assert $n, k+1 \sim (n,-2)$ and considering the connection $\{(n,k-1), (n,k)\}$ we have $n, k-1 \sim (n,-1)$ for $k \in \{0, \ldots, n-1\}$ and $k \in \{1, \ldots, n\}$, respectively. Hence,
\[
(I_T^0 \cup J_T^0 \cup J_T^0) / \sim = \{(n,0) : n \in \mathbb{N}\}
\]
where any
\[
\begin{align*}
(n,0) &= \{(n,k) : 0 \leq k \leq n\} \cup \{(n,-1), (n,-2), (n,-3)\}
\end{align*}
\]
and so Theorem 2.1 allows us to assert
\[
\mathfrak{L} = \bigoplus_{n \in \mathbb{N}} I_{(n,0)}
\]
being any $I_{(n,0)} = I_{(n,0),\pi} \oplus I_{(n,0),\tau}$ with $I_{(n,0),\pi} = \text{span}\{e_{(n,-1)}, e_{(n,-2)}, e_{(n,-3)}\}$ and $I_{(n,0),\tau} = \text{span}\{e_{(n,k)} : 0 \leq k \leq n\}$, an ideal admitting a (multiplicative) basis inherited from the one of $\mathfrak{L}$.

3. The $B$-simple components

In this section our target is to characterize the minimality of the ideals which give rise to the decomposition of $\mathfrak{L}$ in Theorem 2.1, in terms of connectivity properties in the index set $I_T \cup J_T \cup J_T \cup J_T$. Taking into account Definition 1.2 we introduce the next concept in a natural way.

Definition 3.1. A Leibniz superalgebra $\mathfrak{L}$ admitting a multiplicative basis $B$ is called $B$-simple if $[\mathfrak{L}, \mathfrak{L}] \neq 0$ and its only ideals admitting a multiplicative basis inherited from $B$ are $\{0\}$, $\mathfrak{J}$ and $\mathfrak{L}$.

As in the previous section, $\mathfrak{L} = (\mathfrak{J}_T \oplus \mathfrak{J}_T) \oplus (\mathfrak{J}_T \oplus \mathfrak{J}_T)$ denotes a Leibniz superalgebra over an arbitrary base field $\mathbb{F}$ and of arbitrary dimension, admitting a multiplicative basis $\mathcal{B} = (B_{T_\pi} \cup B_{T_\pi}) \cup (B_{T_\pi} \cup B_{T_\pi})$ where $B_{T_\pi} = \{e_{n,\pi}\}_{n \in \mathbb{F}_T}$ and $B_{T_\pi} = \{e_{n,\pi}\}_{n \in \mathbb{F}_T}$, for $\pi \in \mathbb{Z}_2$, and where $K_T \cap P_T = 0$ for any $K, P \in \{I, J\}$, $\pi \in \mathbb{Z}_2$ and $K_T \neq I_T$.

We have the opportunity of restricting the connectivity relation to the set $J_T^0 \cup J_T^0$ and to the set $J_T^0 \cup J_T^0$ by just allowing that the connections are formed by elements in $J_T^0 \cup J_T^0 \cup J_T^0 \cup J_T^0$. Then we say two indexes of $\mathfrak{L}$, where either $\mathfrak{Y} \in \{I, J\}$, are $J$-connected.
Definition 3.2. Let $k$ and $k'$ be two elements in $\Upsilon_1 \cup \Upsilon_2$ with either $\Upsilon = I$ or $\Upsilon = J$ We say $k$ is $J$-connected to $k'$ and we denote by $k \sim_J k'$, if either $k = k'$ or there exists a connection $\{r_1, r_2, \ldots, r_n\}$ from $k$ to $k'$ (in the sense of Definition 2.1) such that

\[ r_2, \ldots, r_n \in \tilde{J}_{\Upsilon_1} \cup J_{\Upsilon_1} \cup \tilde{J}_{\Upsilon_2} \cup J_{\Upsilon_2}. \]

We also say the set $\{r_1, r_2, \ldots, r_n\}$ is a $J$-connection from $k$ to $k'$.

We observe that it is straightforward to verify the arguments in Proposition 2.1 allow us to assert that the relation $\sim_J$ is an equivalence relation in $I_{\Upsilon_1} \cup J_{\Upsilon_1}$ and in $J_{\Upsilon_1} \cup J_{\Upsilon_2}$. Therefore

\[ (\Upsilon_{\Upsilon_1} \cup \Upsilon_2)/ \sim_J = \{[k], j : k \in \Upsilon_{\Upsilon_1} \cup \Upsilon_2\} \]

becoming $[k]_J$ the set of elements in $\Upsilon_{\Upsilon_1} \cup \Upsilon_2$ which are $J$-connected to $k$, with either $\Upsilon = I$ or $\Upsilon = J$.

Let us introduce the notion of $*$-multiplicativity in the framework of Leibniz superalgebras with multiplicative bases, in a similar way to the ones of closed-multiplicativity for split Leibniz algebras, split Leibniz superalgebras and graded Leibniz algebras (see [7, 8, 9] for these notions and examples). From now on, for any $j \in J_{\Upsilon_1}$, $i \in \mathbb{Z}_2$, we denote $u_j^i = 0$.

Definition 3.3. A Leibniz superalgebra $\mathcal{L} = \mathcal{J} \oplus \mathcal{J}$ admits a $*$-multiplicative basis $\mathcal{B} = \{v_{k, \tilde{i}} : k \in K, \tilde{i} \in \mathbb{Z}_2\}$, which decomposes as in Equation (2), if it is multiplicative and for any $k, r \in I_{\Upsilon_1} \cup J_{\Upsilon_1} \cup J_{\Upsilon_2} \cup \tilde{J}_{\Upsilon_1}$ and $a \in I_{\Upsilon_1} \cup J_{\Upsilon_1} \cup J_{\Upsilon_2} \cup \tilde{J}_{\Upsilon_1}$, $J_{\Upsilon_2} \cup \tilde{J}_{\Upsilon_1}$ such that $k \in r \ast a$, then $r_{k, \tilde{i}} \in \{v_{k, \tilde{i}}, \mathcal{J}_{\Upsilon_1} \}$.

Proposition 3.1. Suppose $\mathcal{L}$ admits a $*$-multiplicative basis $\mathcal{B}$. If $\tilde{J}_{\Upsilon_1} \cup J_{\Upsilon_1}$ has all of their elements $J$-connected, then any nonzero ideal $\mathcal{I} \subset \mathcal{L}$ with a multiplicative basis inherited from $\mathcal{B}$ such that $\mathcal{I} \not\subseteq \mathcal{J}$ satisfies $\mathcal{I} = \mathcal{L}$.

Proof. Since $\mathcal{I} \not\subset \mathcal{J}$ we can take some $r_0 \in J_{\Upsilon_0}$ such that

\[ 0 \neq u_{r_0, \tilde{r}_0} \in \mathcal{I}. \]

for certain $\tilde{r}_0 \in \mathbb{Z}_2$. We know that $\tilde{J}_{\Upsilon_1} \cup J_{\Upsilon_1}$ has all of their elements $J$-connected. If $\tilde{J}_{\Upsilon_1} \cup J_{\Upsilon_1} = \{r_i\}$ trivially $\mathcal{J} \subset \mathcal{I}$. If $|\tilde{J}_{\Upsilon_1} \cup J_{\Upsilon_1}| > 1$ we take $s \in J_{\Upsilon_1}$ (with $\tilde{r} \in \mathbb{Z}_2$) different from $r_0$, being then $0 \neq F u_{s, \tilde{r}}$, we can consider a $J$-connection

\[ \{r_0, r_2, \ldots, r_n\} \subset \tilde{J}_{\Upsilon_1} \cup J_{\Upsilon_1} \]

from $r_0$ to $s$.

We know that

\[ \phi(\{r_0, r_2\}) \neq \emptyset \]

and so we can take $a_1 \in \phi(\{r_0, r_2\}) = r_0 \ast r_2$. Now, taking into account Equation (4) and the $*$-multiplicativity of $\mathcal{B}$ we get, if $a_1 \in J_{\Upsilon_0} \cup \tilde{J}_{\Upsilon_0} + \tilde{J}_{\Upsilon_1}$

\[ 0 \neq e_{a_1, \tilde{r}_0 + \tilde{J}_{\Upsilon_1}} \in F[u_{r_0, \tilde{r}_0}, u_{r_2, \tilde{J}_{\Upsilon_1}}] \subset \mathcal{I} \]

or, if $a_1 \in J_{\Upsilon_0} \cup \tilde{J}_{\Upsilon_1}$

\[ 0 \neq e_{a_1, \tilde{r}_0 + \tilde{J}_{\Upsilon_1}} \in F[u_{r_0, \tilde{r}_0}, u_{r_2, \tilde{J}_{\Upsilon_1}}] \subset \mathcal{I} \]

for $\tilde{r}_2 \in \{\tilde{r}_2, \tilde{J}_{\Upsilon_1}\}$ and $\tilde{J} \in \mathbb{Z}_2$.

Since $s \in \tilde{J}_{\Upsilon_1} \cup J_{\Upsilon_1}$, necessarily $\phi(\{r_0, r_2\}) \cap (\tilde{J}_{\Upsilon_1} \cup J_{\Upsilon_1}) \neq \emptyset$ and we have

\[ 0 \neq \bigoplus_{r \in \phi(\{r_0, r_2\}) \cap J_{\Upsilon_1}} F u_{r, \tilde{r}} \subset \mathcal{I}. \]
for any $\bar{i} \in \mathbb{Z}_2$. Since
\[
\phi(\phi(\{r_0\}, r_2), r_3) \neq \emptyset
\]
we can argue as above, taking into account Equation (6), to get
\[
0 \neq \bigoplus_{r \in \phi(\{r_0\}, r_2), r_3} F_{\nu, r} \subset \mathcal{I}
\]
for $\bar{i} \in \mathbb{Z}_2$. By reiterating this process with the $J$-connection (5) we obtain
\[
0 \neq \bigoplus_{r \in \phi(\cdots(\phi(r_0, r_2), \cdots), r_{n-1}), r_n) \cap \mathcal{J}} F_{\nu, r} \subset \mathcal{I}.
\]
Since $s \in \phi(\cdots(\phi(r_0, r_2), \cdots), r_{n-1}), r_n) \cap \mathcal{J}$ we conclude $u_{s, \mathcal{J}} \in \mathcal{I}$ for all $s \in \mathcal{J} \setminus \{r_0\}$ and $\mathcal{J} \in \mathbb{Z}_2$ and so
\[
\neg \mathcal{J} = \bigoplus_{p \in \mathcal{J}, q \in \mathcal{J}} (F_{\nu, p} \oplus F_{\nu, q}) \subset \mathcal{I}.
\]
Considering $\mathcal{J} \subset [\mathcal{J}, \neg \mathcal{J}] + [-\mathcal{J}, \neg \mathcal{J}]$ by $*$-multiplicativity, Equation (7) allows us to assert
\[
(8) \quad \mathcal{J} \subset \mathcal{I}.
\]
Finally, since $\mathcal{J} = \mathcal{J} \oplus \neg \mathcal{J}$, Equations (7) and (8) give us $\mathcal{I} = \mathcal{J}$. $\square$

**Proposition 3.2.** Suppose $\mathcal{L}$ admits a $*$-multiplicative basis $\mathcal{B}$. If $I_{\mathcal{P}} \mathcal{J} I_{\mathcal{P}}$ has all of its elements $J$-connected, then any nonzero ideal $\mathcal{I} \subset \mathcal{L}$ with a multiplicative basis inherited from $\mathcal{B}$ such that $\mathcal{I} \subset \mathcal{J}$ satisfies $\mathcal{I} = \mathcal{J}$.

**Proof.** Taking into account $\mathcal{I} \subset \mathcal{J}$ we can fix a some $n_0 \in I_{\mathcal{I}}$ satisfying
\[
0 \neq e_{n_0, \mathcal{I}} \in \mathcal{I}
\]
for certain $\mathcal{I}_0 \in \mathbb{Z}_2$. Since $I_{\mathcal{P}} \mathcal{J} I_{\mathcal{P}}$ has all of its elements $J$-connected, we can argue from $n_0$ with the $*$-multiplicativity of $\mathcal{B}$ as it is done in Proposition 3.1 from $r_0$ to get $\mathcal{I} \subset \mathcal{J}$ and then $\mathcal{I} = \mathcal{J}$. $\square$

**Theorem 3.1.** Suppose $\mathcal{L}$ admits a $*$-multiplicative basis $\mathcal{B}$. Then $\mathcal{L}$ is $\mathcal{B}$-simple if and only if $I_{\mathcal{P}} \mathcal{J} I_{\mathcal{P}}$ and $J_{\mathcal{P}} \mathcal{J} J_{\mathcal{P}}$ have respectively all of their elements $J$-connected.

**Proof.** Suppose $\mathcal{L}$ is $\mathcal{B}$-simple. We take $n \in I_{\mathcal{P}} \mathcal{J} I_{\mathcal{P}}$ and we observe that the linear space
\[
\bigoplus_{m \in I_{\mathcal{P}} \cap \mathcal{J} \cap \mathcal{J}, \ell \in I_{\mathcal{P}} \cap \mathcal{J} \cap \mathcal{J}} (F_{e_{n, \mathcal{P}} \oplus F_{e_{\ell, \mathcal{P}}}})
\]
is an ideal of $\mathcal{L}$ with a multiplicative basis inherited from $\mathcal{B}$. Indeed, we have trivially
\[
[e_{n, \mathcal{P}} \oplus F_{e_{\ell, \mathcal{P}}}, u_{r, \mathcal{P}} \oplus u_{s, \mathcal{P}}] \subset \bigoplus_{m \in I_{\mathcal{P}} \cap \mathcal{J} \cap \mathcal{J}, \ell \in I_{\mathcal{P}} \cap \mathcal{J} \cap \mathcal{J}} (F_{e_{n, \mathcal{P}} \oplus F_{e_{\ell, \mathcal{P}}}}), \mathcal{J} \subset [\mathcal{L}, \mathcal{J}] = 0.
\]
We only need to prove
\[
\bigoplus_{m \in I_{\mathcal{P}} \cap \mathcal{J} \cap \mathcal{J}, \ell \in I_{\mathcal{P}} \cap \mathcal{J} \cap \mathcal{J}} (F_{e_{n, \mathcal{P}} \oplus F_{e_{\ell, \mathcal{P}}}, u_{r, \mathcal{P}} \oplus u_{s, \mathcal{P}}}) \subset \bigoplus_{m \in I_{\mathcal{P}} \cap \mathcal{J} \cap \mathcal{J}, \ell \in I_{\mathcal{P}} \cap \mathcal{J} \cap \mathcal{J}} (F_{e_{n, \mathcal{P}} \oplus F_{e_{\ell, \mathcal{P}}}})
\]
for any $n \in I_{\mathcal{P}} \cap \mathcal{J} \cap \mathcal{J}$ and $\mathcal{J} \cap \mathcal{J}$ such that $0 \neq e_{n, \mathcal{P}} \oplus u_{r, \mathcal{P}} \oplus u_{s, \mathcal{P}}$ for $u_{r, \mathcal{P}} \oplus u_{s, \mathcal{P}}$ and some $p \in I_{\mathcal{P}} \cap \mathcal{J} \cap \mathcal{J}$. We
have $p \in n_0 \star t$ and so $\{n_0, t\}$ is a $J$-connection meaning that $n_0 \sim_J p$. By the symmetry $p \sim_J n_0$ and by transitivity of $p \sim_J n_0 \sim_J n$, and we get

$$e_{p, n_0 + t} \in \bigoplus_{m \in I_{r^n}[n], r \in I_{r^n}[n]} (F_{m, \bar{p}} \oplus F_{e_1, \bar{T}}).$$

Hence $[e_{n_0, t_0}, u_{i, T}] \subset \bigoplus_{m \in I_{r^n}[n], r \in I_{r^n}[n]} (F_{m, \bar{p}} \oplus F_{e_1, \bar{T}})$ as desired. We conclude

$$B \oplus \bigoplus_{s \in I_{r^n}[r], t \in I_{r^n}[r]} (F_{u_{s, \bar{p}}} \oplus F_{u_{i, \bar{T}}}),$$

is an ideal of $L$ endowed with a multiplicative basis inherited from $B$ (trivial by construction) and so, by $B$-simplicity, necessarily $m \in I_{r^n}[n], r \in I_{r^n}[n]$ and consequently any couple of indexes in $I$ are $J$-connected. Consider now any $r \in J$ and the linear subspace

$$J \oplus \bigoplus_{s \in I_{r^n}[r], t \in I_{r^n}[r]} (F_{u_{s, \bar{p}}} \oplus F_{u_{i, \bar{T}}}) = L$$

which implies in particular

$$J \oplus \bigoplus_{s \in I_{r^n}[r], t \in I_{r^n}[r]} (F_{u_{s, \bar{p}}} \oplus F_{u_{i, \bar{T}}}) = J \oplus \bigoplus_{r \in I_{r^n}[r], q \in I_{r^n}[r]} (F_{u_{s, \bar{p}}} \oplus F_{u_{q, \bar{T}}})$$

and so we get any couple of indexes in $J$ are also $J$-connected.

Conversely, consider $I$ a nonzero ideal of $L$ admitting a multiplicative basis inherited by the one of $L$. We have two possibilities for $I$, either $I \varsubsetneq J$ or $I \subseteq J$. In the first one, Proposition 3.1 gives us $I = L$, while in the second one Proposition 3.2 shows $I = J$. Therefore in both cases $L$ is $B$-simple.

\[ \square \]

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