THE STRUCTURE OF LEIBNIZ SUPERALGEBRAS ADMITTING A MULTIPLICATIVE BASIS

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ABSTRACT. In the literature, most of the descriptions of different classes of Leibniz superalgebras $(\mathfrak{L} = \mathfrak{L}_{\bar{0}} \oplus \mathfrak{L}_{\bar{1}}, [\cdot,\cdot])$ have been made by given the multiplication table on the elements of a graded basis $\mathcal{B} = \{v_k\}_{k\in K}$ of $\mathfrak{L}$, in such a way that for any $i, j \in K$ we have $[v_i, v_j] = \lambda_{i,j}[v_j, v_i] \in \mathbb{F}v_k$ for some $k \in K$, where $\mathbb{F}$ denotes the base field and $\lambda_{i,j} \in \mathbb{F}$. In order to give a unifying viewpoint of all these classes of algebras we introduce the category of Leibniz superalgebras admitting a multiplicative basis and study its structure. We show that if a Leibniz superalgebra $\mathfrak{L} = \mathfrak{L}_{\bar{0}} \oplus \mathfrak{L}_{\bar{1}}$ admits a multiplicative basis then it is the direct sum $\mathfrak{L} = \bigoplus_\alpha \mathcal{I}_\alpha$ with any $\mathcal{I}_\alpha = \mathcal{I}_{\alpha,\bar{0}} \oplus \mathcal{I}_{\alpha,\bar{1}}$ a well described ideal of $\mathfrak{L}$ admitting a multiplicative basis inherited from $\mathcal{B}$. Also the $\mathcal{B}$-simplicity of $\mathfrak{L}$ is characterized in terms of J-connections.

Keywords: Leibniz superalgebra, multiplicative basis, infinite dimension, structure theory.

1. INTRODUCTION AND PREVIOUS DEFINITIONS

Leibniz superalgebras appear as an extension of Leibniz algebras (see [4, 5, 10, 13, 14, 15, 16, 17]), in a similar way than Lie superalgebras generalize Lie algebras, motivated in part for its applications in Physics. The present paper is devoted to the study of the structure of Leibniz superalgebras $\mathfrak{L}$ admitting a multiplicative basis over a field $\mathbb{F}$. Since a Leibniz algebra is a particular case of a Leibniz superalgebra (with $\mathfrak{L}_{\bar{1}} = \{0\}$), this work extends the results exhibited in [6]. We would like to remark that the techniques used in this paper also hold in the infinite-dimensional case over arbitrary fields, being adequate enough to provide us a second Wedderburn-type theorem in this general framework (Theorems 2.1 and 3.1). Moreover, although we make use of the ideal $\mathfrak{I}$ which is deeply inherent to Leibniz theory, we believe that our approach can be useful for the knowledge of the structure of wider classes of algebras.

Definition 1.1. A *Leibniz superalgebra* $\mathfrak{L}$ is a $\mathbb{Z}_2$-graded algebra $\mathfrak{L} = \mathfrak{L}_{\bar{0}} \oplus \mathfrak{L}_{\bar{1}}$ over an arbitrary base field $\mathbb{F}$, with its bilinear product denoted by $[\cdot,\cdot]$, whose homogenous elements $x \in \mathfrak{L}_{\bar{i}}, y \in \mathfrak{L}_{\bar{j}}, \bar{i}, \bar{j} \in \mathbb{Z}_2$, satisfy

$$[x, y] \in \mathfrak{L}_{\bar{i}+\bar{j}}$$

$$[x, [y, z]] = [[x, y], z] - (-1)^{\bar{j}\bar{k}}[[x, z], y] \quad \text{(Super Leibniz identity)}$$

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for any homogenous element $z \in \mathfrak{L}_{\overline{k}}, \overline{k} \in \mathbb{Z}_2$.

Remark 1.1. Note that Super Leibniz identity is considered by the *right side* in the sense that the multiplication operators on the right by elements in $\mathfrak{L}_{\overline{0}}$ are derivations on the homogeneous elements, as it is done in the references [4, 5, 10, 13, 17]. However, we could have considered a Super Leibniz identity in which the multiplication operators on the left by elements in $\mathfrak{L}_{\overline{0}}$ would act as derivations on the homogeneous elements, as it is the case in the references [14, 15, 16]. Of course, the development of the present work would have been similar in this case.

Clearly $\mathfrak{L}_{\overline{0}}$ is a Leibniz algebra. Moreover, if the identity $[x, y] = -(-1)^{\overline{ij}}[y, x]$ holds, then Super Leibniz identity becomes Super Jacobi identity and so Leibniz superalgebras generalize also Lie superalgebras, which is of interest in the formalism of mechanics of Nambu [12].

The usual concepts are considered in a graded sense. A *subsuperalgebra* A of $\mathfrak{L}$ is a graded subspace $A = A_{\overline{0}} \oplus A_{\overline{1}}$ satisfying $[A, A] \subset A$. An *ideal* $\mathcal{I}$ of $\mathfrak{L}$ is a graded subspace $\mathcal{I} = \mathcal{I}_{\overline{0}} \oplus \mathcal{I}_{\overline{1}}$ of $\mathfrak{L}$ such that

$$[\mathcal{I}, \mathfrak{L}] + [\mathfrak{L}, \mathcal{I}] \subset \mathcal{I}.$$

The (graded) ideal $\mathfrak{I}$ generated by

$$\{[x, y] + (-1)^{\overline{ij}}[y, x] : x \in \mathfrak{L}_{\overline{i}}, y \in \mathfrak{L}_{\overline{j}}, \overline{i}, \overline{j} \in \mathbb{Z}_2\}$$

plays an important role in the theory since it determines the (possible) non-super Lie character of $\mathfrak{L}$. From definition of ideal $[\mathfrak{I}, \mathfrak{L}] \subset \mathfrak{I}$ and from Super Leibniz identity, it is straightforward to check that this ideal satisfies

$$[\mathfrak{L}, \mathfrak{I}] = 0. \tag{1}$$

Here we note that the usual definition of simple superalgebra lacks of interest in the case of Leibniz superalgebras because would imply the ideal $\mathfrak{I} = \mathfrak{L}$ or $\mathfrak{I} = 0$, being so $\mathfrak{L}$ an abelian (product zero) or a Lie superalgebra respectively (see Equation (1)). Abdykassymova and Dzhumadil'daev introduced in [1, 2] an adequate definition in the case of Leibniz algebras $(L, [\cdot, \cdot])$ by calling simple to the ones such that its only ideals are $\{0\}$, L and the one generated by the set $\{[x, x] : x \in L\}$. Following this vain, we consider the next definition.

Definition 1.2. A Leibniz superalgebra $\mathfrak{L}$ is called *simple* if $[\mathfrak{L}, \mathfrak{L}] \neq 0$ and its only (graded) ideals are $\{0\}, \mathfrak{I}$ and $\mathfrak{L}$.

Observe that we can write

$$\mathfrak{L} = \mathfrak{I} \oplus \neg\mathfrak{I}$$

where $\neg\mathfrak{I} = \neg\mathfrak{I}_{\overline{0}} \oplus \neg\mathfrak{I}_{\overline{1}}$ is a linear complement of $\mathfrak{I} = \mathfrak{I}_{\overline{0}} \oplus \mathfrak{I}_{\overline{1}}$ in $\mathfrak{L}$ (here we adapt this notation in order to standardize the one already used in [7, 8, 9]). Actually $\neg\mathfrak{I}$ is isomorphic as linear space to $\mathfrak{L}/\mathfrak{I}$, the so called corresponding Lie superalgebra of $\mathfrak{L}$. In general, $\neg\mathfrak{I}$ is not an ideal of $\mathfrak{L}$ from $[\mathfrak{I}, \neg\mathfrak{I}] \subset \mathfrak{I}$. Then the multiplication in $\mathfrak{L}$ is represented in the table

	$\mathfrak{I}_{\overline{0}}$	$\neg\mathfrak{I}_{\overline{0}}$	$\mathfrak{I}_{\overline{1}}$	$\neg\mathfrak{I}_{\overline{1}}$
$\mathfrak{I}_{\overline{0}}$	0	$\mathfrak{I}_{\overline{0}}$	0	$\mathfrak{I}_{\overline{1}}$
$\neg\mathfrak{I}_{\overline{0}}$	0	$\mathfrak{I}_{\overline{0}} \oplus \neg\mathfrak{I}_{\overline{0}}$	0	$\mathfrak{I}_{\overline{1}} \oplus \neg\mathfrak{I}_{\overline{1}}$
$\mathfrak{I}_{\overline{1}}$	0	$\mathfrak{I}_{\overline{1}}$	0	$\mathfrak{I}_{\overline{0}}$
$\neg\mathfrak{I}_{\overline{1}}$	0	$\mathfrak{I}_{\overline{1}} \oplus \neg\mathfrak{I}_{\overline{1}}$	0	$\mathfrak{I}_{\overline{0}} \oplus \neg\mathfrak{I}_{\overline{0}}$

Hence, by taking $\mathcal{B}_{\mathfrak{I}_{\bar{i}}}$ and $\mathcal{B}_{\neg\mathfrak{I}_{\bar{i}}}$ bases of $\mathfrak{I}_{\bar{i}}$ and $\neg\mathfrak{I}_{\bar{i}}$, for $\bar{i} \in \mathbb{Z}_2$, respectively, then

$$\mathcal{B} = \underbrace{(\mathcal{B}_{\mathfrak{I}_{\bar{0}}} \dot{\cup} \mathcal{B}_{\mathfrak{I}_{\bar{1}}})}_{\mathcal{B}_{\mathfrak{I}}} \dot{\cup} \underbrace{(\mathcal{B}_{\neg\mathfrak{I}_{\bar{0}}} \dot{\cup} \mathcal{B}_{\neg\mathfrak{I}_{\bar{1}}})}_{\mathcal{B}_{\neg\mathfrak{I}}}$$

is a basis of $\mathfrak{L}$.

Definition 1.3. A basis $\mathcal{B} = \{v_{k,\bar{i}} : k \in K, \ \bar{i} \in \mathbb{Z}_2\}$ of $\mathfrak{L}$ is said to be *multiplicative* if for any $k_1, k_2 \in K, \bar{i}, \bar{j} \in \mathbb{Z}_2$ we have $[v_{k_1,\bar{i}}, v_{k_2,\bar{j}}] \in \mathbb{F}v_{k,\bar{i}+\bar{j}}$ for some $k \in K$.

Example 1.1. Consider the 5-dimensional $\mathbb{Z}_2$-graded vector space $\mathfrak{L} = \mathfrak{L}_{\bar{0}} \oplus \mathfrak{L}_{\bar{1}}$, over a base field $\mathbb{F}$ of characteristic different from 2, with basis $\mathcal{B}_{\mathfrak{I}_{\bar{1}}} = \{e_1, e_2\}, \mathcal{B}_{\neg\mathfrak{I}_{\bar{0}}} = \{u_a, u_b, u_c\}$; where the products on these elements are given by:

$$[u_b, u_a] = -u_c, \quad [u_a, u_b] = u_c, \quad [u_a, u_c] = -2u_a,$$
$$[u_c, u_a] = 2u_a, \quad [u_c, u_b] = -2u_b, \quad [u_b, u_c] = 2u_b,$$
$$[e_1, u_b] = e_2, \quad [e_1, u_c] = -e_1, \quad [e_2, u_a] = e_1, \quad [e_2, u_c] = e_2,$$

and where the omitted products are equal to zero. Then $\mathfrak{L} = \mathfrak{L}_{\bar{0}} \oplus \mathfrak{L}_{\bar{1}}$ becomes a (non-Lie) Leibniz superalgebra admitting $\mathcal{B} = \mathcal{B}_{\mathfrak{I}_{\bar{1}}} \dot{\cup} \mathcal{B}_{\neg\mathfrak{I}_{\bar{0}}}$ as multiplicative basis.

Example 1.2. Let us denote by $\mathbb{N}^*$ the set of non-negative integers. Consider the infinite-dimensional complex $\mathbb{Z}_2$-graded vector space $\mathfrak{L} = \mathfrak{L}_{\bar{0}} \oplus \mathfrak{L}_{\bar{1}}$ with basis $\mathcal{B}_{\mathfrak{I}_{\bar{1}}} = \{e_{(n,k)} : n, k \in \mathbb{N}^* \ \text{ and } \ k \leq n\}, \mathcal{B}_{\neg\mathfrak{I}_{\bar{0}}} = \{e_{(n,-1)}, e_{(n,-2)}, e_{(n,-3)} : n \in \mathbb{N}\}$; with the following table of multiplication:

$$[e_{(n,-1)}, e_{(n,-3)}] = 2e_{(n,-1)}, \ [e_{(n,-3)}, e_{(n,-1)}] = -2e_{(n,-1)},$$
$$[e_{(n,-2)}, e_{(n,-3)}] = -2e_{(n,-2)}, \ [e_{(n,-3)}, e_{(n,-2)}] = 2e_{(n,-2)},$$
$$[e_{(n,-1)}, e_{(n,-2)}] = e_{(n,-3)}, \ [e_{(n,-2)}, e_{(n,-1)}] = -e_{(n,-3)},$$
$$[e_{(n,k)}, e_{(n,-3)}] = (n-2k)e_{(n,k)}, \text{ for } 0 \leq k \leq n;$$
$$[e_{(n,k)}, e_{(n,-2)}] = e_{(n,k+1)}, \text{ for } 0 \leq k \leq n-1;$$
$$[e_{(n,k)}, e_{(n,-1)}] = k(k-n-1)e_{(n,k-1)}, \text{ for } 1 \leq k \leq n;$$

and where the omitted products are equal to zero. Then $\mathfrak{L} = \mathfrak{L}_{\bar{0}} \oplus \mathfrak{L}_{\bar{1}}$ is a (non-Lie) Leibniz superalgebra admitting $\mathcal{B} = \mathcal{B}_{\mathfrak{I}_{\bar{1}}} \dot{\cup} \mathcal{B}_{\neg\mathfrak{I}_{\bar{0}}}$ as multiplicative basis.

Remark 1.2. Observe that if we write

$$\mathcal{B}_{\mathfrak{I}_{\bar{i}}} = \{e_{n,\bar{i}}\}_{n \in I_{\bar{i}}} \text{ and } \mathcal{B}_{\neg\mathfrak{I}_{\bar{i}}} = \{u_{r,\bar{i}}\}_{r \in J_{\bar{i}}}, \text{ for } \bar{i} \in \mathbb{Z}_2.$$

Since $\mathfrak{I}$ is an ideal together with Equation (1) we know that the only possible non-zero products among the elements in $\mathcal{B}$ are:

(1) For $n \in I_{\bar{i}}, r \in J_{\bar{j}}$ and $\bar{i}, \bar{j} \in \mathbb{Z}_2$ we have $[e_{n,\bar{i}}, u_{r,\bar{j}}] \in \mathbb{F}e_{k,\bar{i}+\bar{j}}$ for some $k \in I_{\bar{i}+\bar{j}}$.
(2) For $r \in J_{\bar{i}}$, $s \in J_{\bar{j}}$ and $\bar{i}, \bar{j} \in \mathbb{Z}_2$ we have either $[u_{r,\bar{i}}, u_{s,\bar{j}}] \in \mathbb{F}u_{l,\bar{i}+\bar{j}}$ for some $l \in J_{\bar{i}+\bar{j}}$ or $[u_{r,\bar{i}}, u_{s,\bar{j}}] \in \mathbb{F}e_{n,\bar{i}+\bar{j}}$ for some $n \in I_{\bar{i}+\bar{j}}$.

Lemma 1.1. *Let $(\mathfrak{L}, [\cdot,\cdot])$ be a Leibniz superalgebra over a base field $\mathbb{F}$ of characteristic different to 2. If $\mathcal{B} = \{v_k\}_{k \in K}$ is a graded basis of $\mathfrak{L}$ such that for any $k_1, k_2 \in K$ we have $[v_{k_1}, v_{k_2}] = \lambda_{k_1,k_2}[v_{k_2}, v_{k_1}] \in \mathbb{F}v_k$ for some $k \in K$ and some $\lambda_{k_1,k_2} \in \mathbb{F}$ then $\mathfrak{L}$ admits $\mathcal{B}$ as multiplicative basis.*

Proof. By the definition of $\mathfrak{I}$ we see that it is generated as linear space by $\{v_j : j \in J\}$, for some subset J of K. So we can find a basis $\mathcal{B}_{\mathfrak{I}}$ of $\mathfrak{I}$ formed by elements of $\mathcal{B}$ and a basis $\mathcal{B}_{\neg\mathfrak{I}} := \mathcal{B} \setminus \mathcal{B}_{\mathfrak{I}}$ of $\neg\mathfrak{I}$ which make of $\mathcal{B}$ a multiplicative basis. $\square$

The preceding lemma shows that all commutative (up to a scalar) Leibniz superalgebras admit a multiplicative basis. For instance, this is the case of null-filiforms Leibniz superalgebras, Leibniz superalgebras of maximal nilindex or Leibniz superalgebras with nilindex $n+m+1$ (see [3, 10, 11]).

The paper is organized as follows. In $\mathfrak{S}2$ inspired by the connections of roots developed for split Leibniz algebras and superalgebras in [7, 8], we introduce similar techniques on the index set of the multiplicative basis $\mathcal{B}$. Our purpose is to obtain a powerful tool for the study of this class of superalgebras. By making use of these results we see that any Leibniz superalgebra $\mathfrak{L}$ admitting a multiplicative basis is of the form $\mathfrak{L} = \bigoplus_\alpha \mathcal{I}_\alpha$, where every $\mathcal{I}_\alpha$ is a well described ideal having a multiplicative basis inherited from $\mathcal{B}$. In $\mathfrak{S}3$ the $\mathcal{B}$-simplicity of these ideals is characterized in terms of the J-connection.

2. Decomposition as direct sum of ideals

In what follows $\mathfrak{L} = (\mathfrak{I}_{\bar{0}} \oplus \neg\mathfrak{I}_{\bar{0}}) \oplus (\mathfrak{I}_{\bar{1}} \oplus \neg\mathfrak{I}_{\bar{1}})$ denotes a Leibniz superalgebra over a base field $\mathbb{F}$ admitting a multiplicative basis

$$\mathcal{B} = (\mathcal{B}_{\mathfrak{I}_{\bar{0}}} \dot{\cup} \mathcal{B}_{\mathfrak{I}_{\bar{1}}}) \dot{\cup} (\mathcal{B}_{\neg\mathfrak{I}_{\bar{0}}} \dot{\cup} \mathcal{B}_{\neg\mathfrak{I}_{\bar{1}}}) \tag{2}$$

where $\mathcal{B}_{\mathfrak{I}_{\bar{i}}} = \{e_{n,\bar{i}}\}_{n \in I_{\bar{i}}}$ and $\mathcal{B}_{\neg\mathfrak{I}_{\bar{i}}} = \{u_{r,\bar{i}}\}_{r \in J_{\bar{i}}}$, for $\bar{i} \in \mathbb{Z}_2$, and where, by renaming if necessary, we can suppose $K_{\bar{i}} \cap P_{\bar{j}} = \emptyset$ for any $K, P \in \{I, J\}$, $\bar{i}, \bar{j} \in Z_2$ and $K_{\bar{i}} \neq P_{\bar{j}}$. We begin this section by developing connection techniques among the elements in the index sets $I_{\bar{0}} \dot{\cup} I_{\bar{1}} \dot{\cup} J_{\bar{0}} \dot{\cup} J_{\bar{1}}$ as the main tool in our study. Now, for each $k \in I_{\bar{0}} \dot{\cup} I_{\bar{1}} \dot{\cup} J_{\bar{0}} \dot{\cup} J_{\bar{1}}$, a new assistant variable $\widetilde{k} \notin I_{\bar{0}} \dot{\cup} I_{\bar{1}} \dot{\cup} J_{\bar{0}} \dot{\cup} J_{\bar{1}}$ is introduced and we denote by

$$\widetilde{I}_{\bar{i}} := \{\widetilde{n} : n \in I_{\bar{i}}\} \text{ and } \widetilde{J}_{\bar{i}} := \{\widetilde{r} : r \in J_{\bar{i}}\},$$

for $i \in \mathbb{Z}_2$, the sets consisting of all these new symbols. Also, given any $\widetilde{k} \in \widetilde{K_{\bar{i}}}$, $K \in \{I, J\}$, $i \in \mathbb{Z}_2$, we denote

$$\widetilde{(\widetilde{k})} := k.$$

Finally, we write by $\mathcal{P}(A)$ the power set of a given set A.

Next, we consider an operation which recover, in some sense, certain multiplicative relations among the elements of the basis $\mathcal{B}$:

$$\star : (I_{\bar{0}} \dot{\cup} I_{\bar{1}} \dot{\cup} J_{\bar{0}} \dot{\cup} J_{\bar{1}}) \times (I_{\bar{0}} \dot{\cup} I_{\bar{1}} \dot{\cup} J_{\bar{0}} \dot{\cup} J_{\bar{1}} \dot{\cup} \widetilde{I_{\bar{0}}} \dot{\cup} \widetilde{I_{\bar{1}}} \dot{\cup} \widetilde{J_{\bar{0}}} \dot{\cup} \widetilde{J_{\bar{1}}}) \to \mathcal{P}(I_{\bar{0}} \dot{\cup} I_{\bar{1}} \dot{\cup} J_{\bar{0}} \dot{\cup} J_{\bar{1}}),$$

where for any $\bar{i}, \bar{j} \in \mathbb{Z}_2$ is defined by

- For $n \in I_{\bar{i}}$, $m \in I_{\bar{j}}$,
$$n \star m := \emptyset$$
- For $n \in I_{\bar{i}}$ and $r \in J_{\bar{j}}$,
$$n \star r \begin{cases} \emptyset, & \text{if } [e_{n,\bar{i}}, u_{r,\bar{j}}] = 0 \\ \{m\}, & \text{if } 0 \neq [e_{n,\bar{i}}, u_{r,\bar{j}}] \in \mathbb{F}e_{m,\bar{i}+\bar{j}} \text{ with } m \in I_{\bar{i}+\bar{j}} \end{cases}$$
- For $n \in I_{\bar{i}}$ and $\widetilde{m} \in \widetilde{I}_{\bar{j}}$,
$$n \star \widetilde{m} := \{r \in J_{\bar{i}+\bar{j}} : 0 \neq [e_{m,\bar{j}}, u_{r,\bar{i}+\bar{j}}] \in \mathbb{F}e_{n,\bar{i}}\}$$
- For $n \in I_{\bar{i}}$ and $\widetilde{r} \in \widetilde{J}_{\bar{j}}$,
$$n \star \widetilde{r} := \{s \in J_{\bar{i}+\bar{j}} : 0 \neq [u_{r,\bar{j}}, u_{s,\bar{i}+\bar{j}}] \in \mathbb{F}e_{n,\bar{i}}\} \cup$$
$$\{t \in J_{\bar{i}+\bar{j}} : 0 \neq [u_{t,\bar{i}+\bar{j}}, u_{r,\bar{j}}] \in \mathbb{F}e_{n,\bar{i}}\} \cup \{m \in I_{\bar{i}+\bar{j}} : 0 \neq [e_{m,\bar{i}+\bar{j}}, u_{r,\bar{j}}] \in \mathbb{F}e_{n,\bar{i}}\}.$$

- For $r \in J_{\bar{i}}$, $s \in J_{\bar{j}}$,

$$r \star s := \begin{cases} \emptyset, & \text{if } [u_{r,\bar{i}}, u_{s,\bar{j}}] = 0 \\ \{t\}, & \text{if } 0 \neq [u_{r,\bar{i}}, u_{s,\bar{j}}] \in \mathbb{F}u_{t,\bar{i}+\bar{j}} \\ \{n\}, & \text{if } 0 \neq [u_{r,\bar{i}}, u_{s,\bar{j}}] \in \mathbb{F}e_{n,\bar{i}+\bar{j}} \end{cases}$$

- For $r \in J_{\bar{i}}$ and $\widetilde{n} \in \widetilde{I_{\bar{j}}}$,

$$r \star \widetilde{n} := \emptyset$$

- For $r \in J_{\bar{i}}$ and $\widetilde{s} \in \widetilde{J_{\bar{j}}}$,

$$r \star \widetilde{s} := \{t \in J_{\bar{i}+\bar{j}} : 0 \neq [u_{t,\bar{i}+\bar{j}}, u_{s,\bar{j}}] \in \mathbb{F}u_{r,\bar{i}}\} \cup \{q \in J_{\bar{i}+\bar{j}} : 0 \neq [u_{s,\bar{j}}, u_{q,\bar{i}+\bar{j}}] \in \mathbb{F}u_{r,\bar{i}}\}.$$

The mapping $\star$ is not still adequate to use in an iterative process necessary for our purposes and so we need to introduce the following one:

$$\phi : \mathcal{P}(I_{\bar{0}} \dot\cup I_{\bar{1}} \dot\cup J_{\bar{0}} \dot\cup J_{\bar{1}}) \times (I_{\bar{0}} \dot\cup I_{\bar{1}} \dot\cup J_{\bar{0}} \dot\cup J_{\bar{1}} \dot\cup \widetilde{I_{\bar{0}}} \dot\cup \widetilde{I_{\bar{1}}} \dot\cup \widetilde{J_{\bar{0}}} \dot\cup \widetilde{J_{\bar{1}}})) \to \mathcal{P}(I_{\bar{0}} \dot\cup I_{\bar{1}} \dot\cup J_{\bar{0}} \dot\cup J_{\bar{1}}),$$

as

- $\phi(\emptyset, I_{\bar{0}} \dot\cup I_{\bar{1}} \dot\cup J_{\bar{0}} \dot\cup J_{\bar{1}} \dot\cup \widetilde{I_{\bar{0}}} \dot\cup \widetilde{I_{\bar{1}}} \dot\cup \widetilde{J_{\bar{0}}} \dot\cup \widetilde{J_{\bar{1}}})) := \emptyset$,
- For any $\emptyset \neq K \in \mathcal{P}(I_{\bar{0}} \dot\cup I_{\bar{1}} \dot\cup J_{\bar{0}} \dot\cup J_{\bar{1}})$ and $a \in I_{\bar{0}} \dot\cup I_{\bar{1}} \dot\cup J_{\bar{0}} \dot\cup J_{\bar{1}} \dot\cup \widetilde{I_{\bar{0}}} \dot\cup \widetilde{I_{\bar{1}}} \dot\cup \widetilde{J_{\bar{0}}} \dot\cup \widetilde{J_{\bar{1}}}$,

$$\phi(K, a) := \bigcup_{k \in K} (k \star a) \cup (a \star k).$$

Lemma 2.1. *For any $K \in \mathcal{P}(I_{\bar{0}} \dot\cup I_{\bar{1}} \dot\cup J_{\bar{0}} \dot\cup J_{\bar{1}})$ and $a \in I_{\bar{0}} \dot\cup I_{\bar{1}} \dot\cup J_{\bar{0}} \dot\cup J_{\bar{1}} \dot\cup \widetilde{I_{\bar{0}}} \dot\cup \widetilde{I_{\bar{1}}} \dot\cup \widetilde{J_{\bar{0}}} \dot\cup \widetilde{J_{\bar{1}}})$,*

$$k \in \phi(K, a) \text{ if and only if } \phi(\{k\}, \tilde{a}) \cap K \neq \emptyset. \tag{3}$$

Proof. It is straightforward to observe that for any $k_1, k_2 \in I_{\bar{0}} \dot\cup I_{\bar{1}} \dot\cup J_{\bar{0}} \dot\cup J_{\bar{1}}$ and

$$a \in I_{\bar{0}} \dot\cup I_{\bar{1}} \dot\cup J_{\bar{0}} \dot\cup J_{\bar{1}} \dot\cup \widetilde{I_{\bar{0}}} \dot\cup \widetilde{I_{\bar{1}}} \dot\cup \widetilde{J_{\bar{0}}} \dot\cup \widetilde{J_{\bar{1}}}$$

we have $k_2 \in k_1 \star a \cup a \star k_1$ if and only if $k_1 \in k_2 \star \tilde{a}$. □

Definition 2.1. Let k and k' be elements in the index set $I_{\bar{0}} \dot\cup I_{\bar{1}} \dot\cup J_{\bar{0}} \dot\cup J_{\bar{1}}$. We say k is *connected* to k' if either $k = k'$ or there exists a subset

$$\{k_1, k_2, \dots, k_{n-1}, k_n\} \subset I_{\bar{0}} \dot\cup I_{\bar{1}} \dot\cup J_{\bar{0}} \dot\cup J_{\bar{1}} \dot\cup \widetilde{I_{\bar{0}}} \dot\cup \widetilde{I_{\bar{1}}} \dot\cup \widetilde{J_{\bar{0}}} \dot\cup \widetilde{J_{\bar{1}}}$$

with $n \geq 2$ such that the following conditions hold:

1. $k_1 = k$.
2. $\phi(\{k_1\}, k_2) \neq \emptyset$,
   $\phi(\phi(\{k_1\}, k_2), k_3) \neq \emptyset$,
   $\vdots$
   $\phi(\phi(\cdots(\phi(\{k_1\}, k_2), \cdots), k_{n-2}), k_{n-1}) \neq \emptyset$.
3. $k' \in \phi(\phi(\cdots(\phi(\{k_1\}, k_2), \cdots), k_{n-1}), k_n)$.

The subset $\{k_1, k_2, \dots, k_{n-1}, k_n\}$ is called a *connection* from k to k'.

Proposition 2.1. *The relation $\sim$ in $I_{\bar{0}} \dot\cup I_{\bar{1}} \dot\cup J_{\bar{0}} \dot\cup J_{\bar{1}}$, defined by $k \sim k'$ if and only if k is connected to k', is an equivalence relation.*

Proof. By definition $k \sim k$, that is, the relation $\sim$ is reflexive. Let us see the symmetric character of $\sim$: If $k \sim k'$ with $k \neq k'$ then there exists a connection

$$\{k_1, k_2, \ldots, k_{n-1}, k_n\}$$

from k to k' satisfying Definition 2.1. Let us show that the set

$$\{k', \widetilde{k}_n, \widetilde{k}_{n-1}, \ldots, \widetilde{k}_3, \widetilde{k}_2\}$$

gives rise to a connection from k' to k. Indeed, by taking

$$K := \phi(\cdots(\phi(\{k_1\}, k_2), \cdots), k_{n-1})$$

we can apply the relation given by (3) to the expression

$$k' \in \phi(K, k_n)$$

to get

$$\phi(\{k'\}, \widetilde{k}_n) \cap K \neq \emptyset$$

and so

$$\phi(\{k'\}, \widetilde{k}_n) \neq \emptyset.$$

By taking

$$h \in \phi(\{k'\}, \widetilde{k}_n) \cap K,$$

then

$$h \in K = \phi(\cdots(\phi(\{k_1\}, k_2), \cdots), k_{n-1}),$$

by the relation given by (3) we get

$$\phi(\{h\}, \widetilde{k}_{n-1}) \cap \phi(\cdots(\phi(\{k_1\}, k_2), \cdots), k_{n-2}) \neq \emptyset,$$

but $h \in \phi(\{k'\}, \widetilde{k}_n)$, therefore $\{h\} \subset \phi(\{k'\}, \widetilde{k}_n)$ and consequently

$$\phi(\phi(\{k'\}, \widetilde{k}_n), \widetilde{k}_{n-1}) \cap \phi(\cdots(\phi(\{k_1\}, k_2), \cdots), k_{n-2}) \neq \emptyset.$$

By iterating this process we get

$$\phi(\phi(\cdots(\phi(\{k'\}, \widetilde{k}_n), \cdots), \widetilde{k}_{n-r+1}), \widetilde{k}_{n-r}) \cap$$

$$\phi(\phi(\cdots(\phi(\{k_1\}, k_2), \cdots), k_{n-r-2}), k_{n-r-1}) \neq \emptyset$$

for $0 \leq r \leq n-3$. Observe that this relation in the case $r = n-3$ reads as

$$\phi(\phi(\cdots(\phi(\{k'\}, \widetilde{k}_n), \cdots), \widetilde{k}_4), \widetilde{k}_3) \cap \phi(\{k_1\}, k_2) \neq \emptyset.$$

Since $k_1 = k$, if we write $\widetilde{K} := \phi(\phi(\cdots(\phi(\{\widetilde{k'}\}, \widetilde{k}_n), \cdots), \widetilde{k}_4), \widetilde{k}_3)$, the previous observation allows us to assert $\phi(\{k\}, k_2) \cap \widetilde{K} \neq \emptyset$. Hence the relation (3) applies to get

$$k \in \phi(\phi(\cdots(\phi(\{k'\}, \widetilde{k}_n), \cdots), \widetilde{k}_3), \widetilde{k}_2)$$

and concludes $\sim$ is symmetric.

Finally, let us verify the transitive character of $\sim$. Suppose $k \sim k'$ and $k' \sim k''$. If $k = k'$ or $k' = k''$ it is trivial, so suppose $k \neq k'$ and $k' \neq k''$ and write $\{k_1, \ldots, k_n\}$ for a connection from k to k' and $\{k'_1, \ldots, k'_m\}$ for a connection from k' to k''. Then we clearly see that $\{k_1, \ldots, k_n, k'_2, \ldots, k'_m\}$ is a connection from k' to k''. We have shown the connection relation is an equivalence relation. $\square$

By the above proposition we can consider the next quotient set on the index set $I_{\bar{0}}\dot{\cup}I_{\bar{1}}\dot{\cup}J_{\bar{0}}\dot{\cup}J_{\bar{1}}$,

$$(I_{\bar{0}}\dot{\cup}I_{\bar{1}}\dot{\cup}J_{\bar{0}}\dot{\cup}J_{\bar{1}})/\sim = \{[k] : k \in I_{\bar{0}}\dot{\cup}I_{\bar{1}}\dot{\cup}J_{\bar{0}}\dot{\cup}J_{\bar{1}}\},$$

becoming $[k]$ the set of elements in $I_{\bar{0}}\dot{\cup}I_{\bar{1}}\dot{\cup}J_{\bar{0}}\dot{\cup}J_{\bar{1}}$ which are connected to k.

Our next goal in this section is to associate an ideal $\mathcal{I}_{[k]}$ of $\mathfrak{L}$ to any $[k]$. Fix $k \in I_{\bar{0}}\dot{\cup}I_{\bar{1}}\dot{\cup}J_{\bar{0}}\dot{\cup}J_{\bar{1}}$, we start by defining the linear subspaces $\mathfrak{I}_{[k]} = \mathfrak{I}_{[k],\bar{0}} \oplus \mathfrak{I}_{[k],\bar{1}} \subset \mathfrak{I}$ and $\neg\mathfrak{I}_{[k]} = \neg\mathfrak{I}_{[k],\bar{0}} \oplus \neg\mathfrak{I}_{[k],\bar{1}} \subset \neg\mathfrak{I}$ as follows

$$\mathfrak{I}_{[k],\bar{i}} := \bigoplus_{l\in[k]\cap I_{\bar{i}}} \mathbb{F}e_{l,\bar{i}} \subset \mathfrak{I}_{\bar{i}},$$

$$\neg\mathfrak{I}_{[k],\bar{i}} := \bigoplus_{h\in[k]\cap J_{\bar{i}}} \mathbb{F}u_{h,\bar{i}} \subset \neg\mathfrak{I}_{\bar{i}}$$

for any $\bar{i} \in \mathbb{Z}_2$. Finally, we denote by $\mathcal{I}_{[k]}$ the direct sum of the two subspaces above, that is,

$$\mathcal{I}_{[k]} := (\mathfrak{I}_{[k],\bar{0}} \oplus \mathfrak{I}_{[k],\bar{1}}) \oplus (\neg\mathfrak{I}_{[k],\bar{0}} \oplus \neg\mathfrak{I}_{[k],\bar{1}})$$

Definition 2.2. Let $\mathfrak{L}$ be a Leibniz superalgebra admitting a multiplicative basis $\mathcal{B}$. A subsuperalgebra $A \subset \mathfrak{L}$ admits a multiplicative basis $\mathcal{B}_A$ *inherited from* $\mathcal{B}$ if $\mathcal{B}_A$ is a multiplicative basis of A satisfying $\mathcal{B}_A \subset \mathcal{B}$.

Proposition 2.2. *For any $k \in I_{\bar{0}}\dot{\cup}I_{\bar{1}}\dot{\cup}J_{\bar{0}}\dot{\cup}J_{\bar{1}}$, the linear subspace $\mathcal{I}_{[k]}$ is an ideal of $\mathfrak{L}$ admitting a multiplicative basis inherited from the one of $\mathfrak{L}$.*

Proof. We can write

$$[\mathcal{I}_{[k]}, \mathfrak{L}] = [\mathfrak{I}_{[k]} \oplus \neg\mathfrak{I}_{[k]}, (\bigoplus_{n\in I_{\bar{0}}} \mathbb{F}e_{n,\bar{0}}) \oplus (\bigoplus_{m\in I_{\bar{1}}} \mathbb{F}e_{m,\bar{1}}) \oplus (\bigoplus_{r\in J_{\bar{0}}} \mathbb{F}u_{r,\bar{0}}) \oplus (\bigoplus_{s\in J_{\bar{1}}} \mathbb{F}u_{s,\bar{1}})].$$

In case $[e_{l,\bar{i}}, u_{r,\bar{j}}] \neq 0$ for some $l \in [k]\cap I_{\bar{i}}, r \in J_{\bar{j}}$ and $\bar{i}, \bar{j} \in \mathbb{Z}_2$, we have $0 \neq [e_{l,\bar{i}}, u_{r,\bar{j}}] \in \mathbb{F}e_{p,\overline{i+j}}$ with $p \in I_{\bar{i}+\bar{j}}$ and so $p \in \phi(\{l\}, r) = l \star r$, therefore the connection $\{l, r\}$ gives us $l \sim p$, so $p \in [k]$ and then $0 \neq [e_{l,\bar{i}}, u_{r,\bar{j}}] \in \mathfrak{I}_{[k]}$. Hence we get

$$[\mathfrak{I}_{[k]}, (\bigoplus_{r\in J_{\bar{0}}} \mathbb{F}u_{r,\bar{0}}) \oplus (\bigoplus_{s\in J_{\bar{1}}} \mathbb{F}u_{s,\bar{1}})] \subset \mathfrak{I}_{[k]} \subset \mathcal{I}_{[k]}.$$

In a similar way we have $[\neg\mathfrak{I}_{[k]}, (\bigoplus_{r\in J_{\bar{0}}} \mathbb{F}u_{r,\bar{0}}) \oplus (\bigoplus_{s\in J_{\bar{1}}} \mathbb{F}u_{s,\bar{1}})] \subset \mathcal{I}_{[k]}$ and taking into account Equation (1) we conclude

$$[\mathcal{I}_{[k]}, \mathfrak{L}] \subset \mathcal{I}_{[k]}.$$

On the other hand,

$$[\mathfrak{L}, \mathcal{I}_{[k]}] = [(\bigoplus_{n\in I_{\bar{0}}} \mathbb{F}e_{n,\bar{0}}) \oplus (\bigoplus_{m\in I_{\bar{1}}} \mathbb{F}e_{m,\bar{1}}) \oplus (\bigoplus_{r\in J_{\bar{0}}} \mathbb{F}u_{r,\bar{0}}) \oplus (\bigoplus_{s\in J_{\bar{1}}} \mathbb{F}u_{s,\bar{1}}), \mathfrak{I}_{[k]} \oplus \neg\mathfrak{I}_{[k]}]$$

and in case $0 \neq [e_{n,\bar{i}}, u_{h,\bar{j}}]$ for some $n \in I_{\bar{i}}, h \in [k] \cap J_{\bar{j}}$ and $\bar{i}, \bar{j} \in \mathbb{Z}_2$ we have $[e_{n,\bar{i}}, u_{h,\bar{j}}] \in \mathbb{F}e_{p,\bar{i}+\bar{j}}$ with $p \in I_{\bar{i}+\bar{j}}$. Then $p \in \phi(\{h\}, n) = h \star n$ and we see that the connection $\{h, n\}$ gives us $h \sim p$ and so $[(\bigoplus_{n\in I_{\bar{0}}} \mathbb{F}e_{n,\bar{0}}) \oplus (\bigoplus_{m\in I_{\bar{1}}} \mathbb{F}e_{m,\bar{1}}), \neg\mathfrak{I}_{[k]}] \subset \mathfrak{I}_{[k]} \subset \mathcal{I}_{[k]}$. In a similar way

$$[(\bigoplus_{r\in J_{\bar{0}}} \mathbb{F}u_{r,\bar{0}}) \oplus (\bigoplus_{s\in J_{\bar{1}}} \mathbb{F}u_{s,\bar{1}}), \neg\mathfrak{I}_{[k]}] \subset \mathcal{I}_{[k]}$$

and by Equation (1) then

$$[\mathfrak{L}, \mathcal{I}_{[k]}] \subset \mathcal{I}_{[k]}.$$

Hence $\mathcal{I}_{[k]}$ is an ideal of $\mathfrak{L}$.

Finally, observe that the set

$$\mathcal{B}_{\mathcal{I}_{[k]}} := \{e_{n,\bar{0}} : n \in [k] \cap I_{\bar{0}}\} \dot{\cup} \{e_{m,\bar{1}} : m \in [k] \cap I_{\bar{1}}\} \dot{\cup}$$
$$\{u_{r,\bar{0}} : r \in [k] \cap J_{\bar{0}}\} \dot{\cup} \{u_{s,\bar{1}} : s \in [k] \cap J_{\bar{1}}\}$$

is a multiplicative basis of $\mathcal{I}_{[k]}$ satisfying $\mathcal{B}_{\mathcal{I}_{[k]}} \subset \mathcal{B}$. Hence we see that $\mathcal{I}_{[k]}$ admits a multiplicative basis inherited from the one of $\mathfrak{L}$. □

Corollary 2.1. *If $\mathfrak{L}$ is simple, then there exists a connection between any couple of elements in the index set $I_{\bar{0}} \dot{\cup} I_{\bar{1}} \dot{\cup} J_{\bar{0}} \dot{\cup} J_{\bar{1}}$.*

Proof. The simplicity of $\mathfrak{L}$ implies $[\mathfrak{L}, \mathfrak{L}] \neq 0$ and so $\neg\mathfrak{I} \neq \emptyset$, then at least there exists $r_0 \in J_{\mathrm{i}}, \bar{i} \in \mathbb{Z}_2$, such that $\{u_{r_0,\bar{i}}\} \subset \mathcal{B}_{\neg\mathfrak{I}_{\bar{i}}}$. Applying Proposition 2.2, $\mathcal{I}_{[r_0]}$ is an ideal and by its construction $\mathcal{I}_{[r_0]} \not\subset \mathfrak{I}$, therefore $\mathcal{I}_{[r_0]} = \mathfrak{L}$ being then $[r_0] = I_{\bar{0}} \dot{\cup} I_{\bar{1}} \dot{\cup} J_{\bar{0}} \dot{\cup} J_{\bar{1}}$. That is, any couple of elements in $I_{\bar{0}} \dot{\cup} I_{\bar{1}} \dot{\cup} J_{\bar{0}} \dot{\cup} J_{\bar{1}}$ are connected. □

Theorem 2.1. *A Leibniz superalgebra $\mathfrak{L}$ admitting a multiplicative basis decomposes as the direct sum*

$$\mathfrak{L} = \bigoplus_{[k] \in (I_{\bar{0}} \dot{\cup} I_{\bar{1}} \dot{\cup} J_{\bar{0}} \dot{\cup} J_{\bar{1}})/\sim} \mathcal{I}_{[k]},$$

where any $\mathcal{I}_{[k]} = \mathfrak{I}_{[k]} \oplus \neg\mathfrak{I}_{[k]}$ is one of the ideals, admitting a multiplicative basis inherited from the one of $\mathfrak{L}$, given in Proposition 2.2.

Proof. Since we can write $\mathfrak{L} = \mathfrak{I} \oplus \neg\mathfrak{I}$ and

$$\mathfrak{I} = \bigoplus_{[k] \in (I_{\bar{0}} \dot{\cup} I_{\bar{1}} \dot{\cup} J_{\bar{0}} \dot{\cup} J_{\bar{1}})/\sim} \mathfrak{I}_{[k]}, \quad \neg\mathfrak{I} = \bigoplus_{[k] \in (I_{\bar{0}} \dot{\cup} I_{\bar{1}} \dot{\cup} J_{\bar{0}} \dot{\cup} J_{\bar{1}})/\sim} \neg\mathfrak{I}_{[k]}.$$

From $\mathcal{I}_{[k]} = \mathfrak{I}_{[k]} \oplus \neg\mathfrak{I}_{[k]}$ by definition, we clearly have

$$\mathfrak{L} = \bigoplus_{[k] \in (I_{\bar{0}} \dot{\cup} I_{\bar{1}} \dot{\cup} J_{\bar{0}} \dot{\cup} J_{\bar{1}})/\sim} \mathcal{I}_{[k]}.$$

□

Example 2.1. *Consider the Leibniz superalgebra $\mathfrak{L} = \mathfrak{L}_{\bar{0}} \oplus \mathfrak{L}_{\bar{1}}$ presented in Example 1.1. We have $I_{\bar{1}} = \{1, 2\}$ and $J_{\bar{0}} = \{a, b, c\}$. From the multiplication table of $\mathfrak{L}$ it is not difficult to write the operation $\star$ in a concrete way. For instance, we have*

$$1 \star c = 2 \star a = \{1\} \qquad a \star b = b \star a = \{c\}$$
$$1 \star b = 2 \star c = \{2\} \qquad a \star c = c \star a = \{a\}$$

Then, we can also obtain an explicit expression of the mapping

$$\phi : \mathcal{P}(I_{\bar{0}} \dot{\cup} I_{\bar{0}} \dot{\cup} J_{\bar{0}} \dot{\cup} J_{\bar{1}}) \times (I_{\bar{0}} \dot{\cup} I_{\bar{0}} \dot{\cup} J_{\bar{0}} \dot{\cup} J_{\bar{1}} \dot{\cup} \widetilde{I}_{\bar{0}} \dot{\cup} \widetilde{I}_{\bar{0}} \dot{\cup} \widetilde{J}_{\bar{0}} \dot{\cup} \widetilde{J}_{\bar{1}}) \longrightarrow \mathcal{P}(I_{\bar{0}} \dot{\cup} I_{\bar{0}} \dot{\cup} J_{\bar{0}} \dot{\cup} J_{\bar{1}}).$$

Observe that the connection $\{1, b\}$ gives $1 \sim 2$, with the connection $\{a, b\}$ we have $a \sim c$ and considering $\{b, a\}$ we obtain $b \sim c$. Since $1 \star \tilde{2} = \{b\}$ we get $1 \sim b$ and therefore $(I_{\bar{0}} \dot{\cup} I_{\bar{0}} \dot{\cup} J_{\bar{0}} \dot{\cup} J_{\bar{1}})/\sim = \{[1]\}$ where $[1] = \{1, 2, a, b, c\}$. By Theorem 2.1 we see that $\mathfrak{L} = \mathcal{I}_{[1]}$, where $\mathcal{I}_{[1]}$ is an ideal of $\mathfrak{L}$ with a unique (multiplicative) basis $\{1, 2, a, b, c\}$. In fact, since $\mathfrak{L}$ is a simple (non-Lie) Leibniz superalgebra, by Corollary 2.2 all elements in $I_{\bar{0}} \dot{\cup} I_{\bar{1}} \dot{\cup} J_{\bar{0}} \dot{\cup} J_{\bar{1}}$ are connected and we just have one ideal.

Example 2.2. *Let $\mathfrak{L} = \mathfrak{L}_{\bar{0}} \oplus \mathfrak{L}_{\bar{1}}$ be the Leibniz superalgebra considered in Example 1.2. We have $I = \{(n,k) : n \in \mathbb{N}, 0 \leq k \leq n\}$ and $J = \{(n,-1),(n,-2),(n,-3) : n \in \mathbb{N}\}$. From the multiplication table of $\mathfrak{L}$ it is not difficult to express the operation $\star$ completely. For instance, we have*

$$
\begin{aligned}
&(n,k) \star (n,-3) = \{(n,k)\} \quad k \in I\\
&(n,k) \star (n,-2) = \{(n,k+1)\} \quad k \in \{0,\dots,n-1\}\\
&(n,k) \star (n,-1) = \{(n,k-1)\} \quad k \in \{1,\dots,n\}\\
&(n,-1) \star (n,-2) = (n,-2) \star (n,-1) = \{(n,-3)\}\\
&(n,-1) \star (n,-3) = (n,-3) \star (n,-1) = \{(n,-1)\}\\
&(n,-2) \star (n,-3) = (n,-3) \star (n,-2) = \{(n,-2)\}
\end{aligned}
$$

From here, we can also obtain an explicit expression of the mapping

$$\phi : \mathcal{P}(I_{\bar{0}} \dot\cup I_{\bar{0}} \dot\cup J_{\bar{0}} \dot\cup J_{\bar{1}}) \times (I_{\bar{0}} \dot\cup I_{\bar{0}} \dot\cup J_{\bar{0}} \dot\cup J_{\bar{1}} \dot\cup \widetilde{I_{\bar{0}}} \dot\cup \widetilde{I_{\bar{0}}} \dot\cup \widetilde{J_{\bar{0}}} \dot\cup \widetilde{J_{\bar{1}}}) \longrightarrow \mathcal{P}(I_{\bar{0}} \dot\cup I_{\bar{0}} \dot\cup J_{\bar{0}} \dot\cup J_{\bar{1}}).$$

Observe that the connection $\{(n,-1),\widetilde{(n,-2)}\}$ gives $(n,-1) \sim (n,-3)$, with the connection $\{(n,-2),\widetilde{(n,-2)}\}$ we get $(n,-2) \sim (n,-3)$, the connection $\{(n,k+1),\widetilde{(n,k)}\}$ let us assert $(n,k+1) \sim (n,-2)$ and considering the connection $\{(n,k-1),\widetilde{(n,k)}\}$ we have $(n,k-1) \sim (n,-1)$, for $k \in \{0,\dots,n-1\}$ and $k \in \{1,\dots,n\}$, respectively. Hence,

$$(I_{\bar{0}} \dot\cup I_{\bar{1}} \dot\cup J_{\bar{0}} \dot\cup J_{\bar{1}})/\sim = \{[(n,0)] : n \in \mathbb{N}\}$$

where any

$$[(n,0)] = \{(n,k) : 0 \leq k \leq n\} \cup \{(n,-1),(n,-2),(n,-3)\}$$

and so Theorem 2.1 allows us to assert

$$\mathfrak{L} = \bigoplus_{n \in \mathbb{N}} \mathcal{I}_{[(n,0)]}$$

being any $\mathcal{I}_{[(n,0)]} = \mathcal{I}_{[(n,0)],\bar{0}} \oplus \mathcal{I}_{[(n,0)],\bar{1}}$, with $\mathcal{I}_{[(n,0)],\bar{0}} = \operatorname{span}\{e_{(n,-1)}, e_{(n,-2)}, e_{(n,-3)}\}$ and $\mathcal{I}_{[(n,0)],\bar{1}} = \operatorname{span}\{e_{(n,k)} : 0 \leq k \leq n\}$, an ideal admitting a (multiplicative) basis inherited from the one of $\mathfrak{L}$.

3. THE $\mathcal{B}$-SIMPLE COMPONENTS

In this section our target is to characterize the minimality of the ideals which give rise to the decomposition of $\mathfrak{L}$ in Theorem 2.1, in terms of connectivity properties in the index set $I_{\bar{0}} \dot\cup I_{\bar{1}} \dot\cup J_{\bar{0}} \dot\cup J_{\bar{1}}$. Taking into account Definition 1.2 we introduce the next concept in a natural way.

Definition 3.1. A Leibniz superalgebra $\mathfrak{L}$ admitting a multiplicative basis $\mathcal{B}$ is called $\mathcal{B}$-*simple* if $[\mathfrak{L},\mathfrak{L}] \neq 0$ and its only ideals admitting a multiplicative basis inherited from $\mathcal{B}$ are $\{0\}, \mathfrak{I}$ and $\mathfrak{L}$.

As in the previous section, $\mathfrak{L} = (\mathfrak{I}_{\bar{0}} \oplus \neg\mathfrak{I}_{\bar{0}}) \oplus (\mathfrak{I}_{\bar{1}} \oplus \neg\mathfrak{I}_{\bar{1}})$ denotes a Leibniz superalgebra over an arbitrary base field $\mathbb{F}$ and of arbitrary dimension, admitting a multiplicative basis $\mathcal{B} = (\mathcal{B}_{\mathfrak{I}_{\bar{0}}} \dot\cup \mathcal{B}_{\mathfrak{I}_{\bar{1}}}) \dot\cup (\mathcal{B}_{\neg\mathfrak{I}_{\bar{0}}} \dot\cup \mathcal{B}_{\neg\mathfrak{I}_{\bar{1}}})$ where $\mathcal{B}_{\mathfrak{I}_{\bar{i}}} = \{e_{n,\bar{i}}\}_{n \in I_{\bar{i}}}$ and $\mathcal{B}_{\neg\mathfrak{I}_{\bar{i}}} = \{u_{r,\bar{i}}\}_{r \in J_{\bar{i}}}$, for $\bar{i} \in \mathbb{Z}_2$, and where $K_{\bar{i}} \cap P_{\bar{j}} = \emptyset$ for any $K, P \in \{I, J\}$, $\bar{i}, \bar{j} \in \mathbb{Z}_2$ and $K_{\bar{i}} \neq P_{\bar{j}}$.

We have the opportunity of restricting the connectivity relation to the set $I_{\bar{0}} \dot\cup I_{\bar{1}}$ and to the set $J_{\bar{0}} \dot\cup J_{\bar{1}}$ by just allowing that the connections are formed by elements in $J_{\bar{0}} \dot\cup J_{\bar{1}} \dot\cup \widetilde{J_{\bar{0}}} \dot\cup \widetilde{J_{\bar{1}}}$. Then we say two indexes of $\Upsilon_{\bar{0}} \dot\cup \Upsilon_{\bar{1}}$, where either $\Upsilon \in \{I, J\}$, are J-connected.

Definition 3.2. Let k and k' be two elements in $\Upsilon_{\overline{0}}\dot{\cup}\Upsilon_{\overline{1}}$ with either $\Upsilon = I$ or $\Upsilon = J$. We say k is *J-connected* to k' and we denote by $k \sim_J k'$, if either $k = k'$ or there exists a connection $\{r_1, r_2, \dots, r_n\}$ from k to k' (in the sense of Definition 2.1) such that

$$r_2, \dots, r_n \in J_{\overline{0}}\dot{\cup}J_{\overline{1}}\dot{\cup}\widetilde{J_{\overline{0}}}\dot{\cup}\widetilde{J_{\overline{1}}}.$$

We also say the set $\{r_1, r_2, \dots, r_n\}$ is a *J-connection* from k to k'.

We observe that it is straightforward to verify the arguments in Proposition 2.1 allow us to assert that the relation $\sim_J$ is an equivalence relation in $I_{\overline{0}}\dot{\cup}I_{\overline{1}}$ and in $J_{\overline{0}}\dot{\cup}J_{\overline{1}}$. Therefore

$$(\Upsilon_{\overline{0}}\dot{\cup}\Upsilon_{\overline{1}})/\sim_J = \{[k]_J : k \in \Upsilon_{\overline{0}}\dot{\cup}\Upsilon_{\overline{1}}\}$$

becoming $[k]_J$ the set of elements in $\Upsilon_{\overline{0}}\dot{\cup}\Upsilon_{\overline{1}}$ which are J-connected to k, with either $\Upsilon = I$ or $\Upsilon = J$.

Let us introduce the notion of $\star$-multiplicativity in the framework of Leibniz superalgebras with multiplicative bases, in a similar way to the ones of closed-multiplicativity for split Leibniz algebras, split Leibniz superalgebras and graded Leibniz algebras (see [7, 8, 9] for these notions and examples). From now on, for any $\widetilde{j} \in \widetilde{J_{\overline{i}}}$, $i \in \mathbb{Z}_2$, we denote $u_{\widetilde{j}} = 0$.

Definition 3.3. A Leibniz superalgebra $\mathfrak{L} = \mathfrak{I} \oplus \neg\mathfrak{I}$ admits a *$\star$-multiplicative basis* $\mathcal{B} = \{v_{k,\overline{i}} : k \in K,\ \overline{i} \in \mathbb{Z}_2\}$, which decomposes as in Equation (2), if it is multiplicative and for any $k, r \in I_{\overline{0}}\dot{\cup}I_{\overline{1}}\dot{\cup}J_{\overline{0}}\dot{\cup}J_{\overline{1}}$ and $a \in I_{\overline{0}}\dot{\cup}I_{\overline{1}}\dot{\cup}J_{\overline{0}}\dot{\cup}J_{\overline{1}}\dot{\cup}\widetilde{I_{\overline{0}}}\dot{\cup}\widetilde{I_{\overline{0}}}\dot{\cup}\widetilde{J_{\overline{0}}}\dot{\cup}\widetilde{J_{\overline{1}}}$ such that $k \in r \star a$, then $v_{k,\overline{i}} \in [v_{r,\overline{j}}, \mathfrak{L}_{\overline{i}+\overline{j}}]$.

Proposition 3.1. *Suppose $\mathfrak{L}$ admits a $\star$-multiplicative basis $\mathcal{B}$. If $J_{\overline{0}}\dot{\cup}J_{\overline{1}}$ has all of their elements J-connected, then any nonzero ideal $\mathcal{I} \subset \mathfrak{L}$ with a multiplicative basis inherited from $\mathcal{B}$ such that $\mathcal{I} \not\subset \mathfrak{I}$ satisfies $\mathcal{I} = \mathfrak{L}$.*

Proof. Since $\mathcal{I} \not\subset \mathfrak{I}$ we can take some $r_0 \in J_{\overline{i}_0}$ such that

$$0 \neq u_{r_0,\overline{i}_0} \in \mathcal{I}. \tag{4}$$

for certain $\overline{i}_0 \in \mathbb{Z}_2$. We know that $J_{\overline{0}}\dot{\cup}J_{\overline{1}}$ has all of their elements J-connected. If $J_{\overline{0}}\dot{\cup}J_{\overline{1}} = \{r_0\}$ trivially $\neg\mathfrak{I} \subset \mathcal{I}$. If $|J_{\overline{0}}\dot{\cup}J_{\overline{1}}| > 1$ we take $s \in J_{\overline{j}}$ (with $\overline{j} \in \mathbb{Z}_2$) different from r_0, being then $0 \neq \mathbb{F}u_{s,\overline{j}}$, we can consider a J-connection

$$\{r_0, r_2, \dots, r_n\} \subset J_{\overline{0}}\dot{\cup}J_{\overline{1}}\dot{\cup}\widetilde{J_{\overline{0}}}\dot{\cup}\widetilde{J_{\overline{1}}} \tag{5}$$

from r_0 to s.

We know that

$$\phi(\{r_0\}, r_2) \neq \emptyset$$

and so we can take $a_1 \in \phi(\{r_0\}, r_2) = r_0 \star r_2$. Now, taking into account Equation (4) and the $\star$-multiplicativity of $\mathcal{B}$ we get, if $a_1 \in J_{\overline{i}_0+\overline{j}}$

$$0 \neq u_{a_1,\overline{i}_0+\overline{j}} \in \mathbb{F}[u_{r_0,\overline{i}_0}, u_{l_2,\overline{j}}] \subset \mathcal{I}$$

or, if $a_1 \in I_{\overline{i}_0+\overline{j}}$

$$0 \neq e_{a_1,\overline{i}_0+\overline{j}} \in \mathbb{F}[u_{r_0,\overline{i}_0}, u_{l_2,\overline{j}}] \subset \mathcal{I}$$

for $l_2 = \{r_2, \widetilde{r}_2\} \cap J_{\overline{j}}$ and $\overline{j} \in \mathbb{Z}_2$.

Since $s \in J_{\overline{0}}\dot{\cup}J_{\overline{1}}$, necessarily $\phi(\{r_0\}, r_2) \cap (J_{\overline{0}}\dot{\cup}J_{\overline{1}}) \neq \emptyset$ and we have

$$0 \neq \bigoplus_{r \in \phi(\{r_0\}, r_2) \cap J_{\overline{i}}} \mathbb{F}u_{r,\overline{i}} \subset \mathcal{I}. \tag{6}$$

for any $\bar{i} \in \mathbb{Z}_2$. Since

$$\phi(\phi(\{r_0\}, r_2), r_3) \neq \emptyset$$

we can argue as above, taking into account Equation (6), to get

$$0 \neq \bigoplus_{r \in \phi(\phi(\{r_0\}, r_2), r_3) \cap J_{\bar{i}}} \mathbb{F}u_{r,\bar{i}} \subset \mathcal{I}$$

for $\bar{i} \in \mathbb{Z}_2$. By reiterating this process with the J-connection (5) we obtain

$$0 \neq \bigoplus_{r \in \phi(\phi(\cdots(\phi(r_0,r_2),\cdots),r_{n-1}),r_n) \cap J_{\bar{i}}} \mathbb{F}u_{r,\bar{i}} \subset \mathcal{I}.$$

Since $s \in \phi(\phi(\cdots(\phi(r_0, r_2), \cdots), r_{n-1}), r_n) \cap J_{\bar{j}}$ we conclude $u_{s,\bar{j}} \in \mathcal{I}$ for all $s \in J_{\bar{j}} \setminus \{r_0\}$ and $\bar{j} \in \mathbb{Z}_2$ and so

$$\neg\mathfrak{I} = \bigoplus_{p \in J_{\bar{0}}, q \in J_{\bar{1}}} (\mathbb{F}u_{p,\bar{0}} \oplus \mathbb{F}u_{q,\bar{1}}) \subset \mathcal{I}. \tag{7}$$

Considering $\mathfrak{I} \subset [\mathfrak{I}, \neg\mathfrak{I}] + [\neg\mathfrak{I}, \neg\mathfrak{I}]$ by $\star$-multiplicativity, Equation (7) allows us to assert

$$\mathfrak{I} \subset \mathcal{I}. \tag{8}$$

Finally, since $\mathfrak{L} = \mathfrak{I} \oplus \neg\mathfrak{I}$, Equations (7) and (8) give us $\mathcal{I} = \mathfrak{L}$. □

Proposition 3.2. *Suppose $\mathfrak{L}$ admits a $\star$-multiplicative basis $\mathcal{B}$. If $I_{\bar{0}} \dot{\cup} I_{\bar{1}}$ has all of its elements J-connected, then any nonzero ideal $\mathcal{I} \subset \mathfrak{L}$ with a multiplicative basis inherited from $\mathcal{B}$ such that $\mathcal{I} \subset \mathfrak{I}$ satisfies $\mathcal{I} = \mathfrak{I}$.*

Proof. Taking into account $\mathcal{I} \subset \mathfrak{I}$ we can fix a some $n_0 \in I_{\bar{i}_0}$ satisfying

$$0 \neq e_{n_0,\bar{i}_0} \in \mathcal{I}$$

for certain $\bar{i}_0 \in \mathbb{Z}_2$. Since $I_{\bar{0}} \dot{\cup} I_{\bar{1}}$ has all of its elements J-connected, we can argue from n_0 with the $\star$-multiplicativity of $\mathcal{B}$ as it is done in Proposition 3.1 from r_0 to get $\mathfrak{I} \subset \mathcal{I}$ and then $\mathcal{I} = \mathfrak{I}$. □

Theorem 3.1. *Suppose $\mathfrak{L}$ admits a $\star$-multiplicative basis $\mathcal{B}$. Then $\mathfrak{L}$ is $\mathcal{B}$-simple if and only if $I_{\bar{0}} \dot{\cup} I_{\bar{1}}$ and $J_{\bar{0}} \dot{\cup} J_{\bar{1}}$ have respectively all of their elements J-connected.*

Proof. Suppose $\mathfrak{L}$ is $\mathcal{B}$-simple. We take $n \in I_{\bar{0}} \dot{\cup} I_{\bar{1}}$ and we observe that the linear space $\bigoplus_{m \in I_{\bar{0}} \cap [n]_J, l \in I_{\bar{1}} \cap [n]_J} (\mathbb{F}e_{m,\bar{0}} \oplus \mathbb{F}e_{l,\bar{1}})$ is an ideal of $\mathfrak{L}$ with a multiplicative basis inherited from $\mathcal{B}$. Indeed, we have trivially

$$\Big[\mathfrak{L}, \bigoplus_{m \in I_{\bar{0}} \cap [n]_J, l \in I_{\bar{1}} \cap [n]_J} (\mathbb{F}e_{m,\bar{0}} \oplus \mathbb{F}e_{l,\bar{1}})\Big] + \Big[\bigoplus_{m \in I_{\bar{0}} \cap [n]_J, l \in I_{\bar{1}} \cap [n]_J} (\mathbb{F}e_{m,\bar{0}} \oplus \mathbb{F}e_{l,\bar{1}}), \mathfrak{I}\Big] \subset$$

$$\subset [\mathfrak{L}, \mathfrak{I}] = 0.$$

We only need to prove

$$\Big[\bigoplus_{m \in I_{\bar{0}} \cap [n]_J, l \in I_{\bar{1}} \cap [n]_J} (\mathbb{F}e_{m,\bar{0}} \oplus \mathbb{F}e_{l,\bar{1}}), u_{r,\bar{0}} \oplus u_{s,\bar{1}}\Big] \subset \bigoplus_{m \in I_{\bar{0}} \cap [n]_J, l \in I_{\bar{1}} \cap [n]_J} (\mathbb{F}e_{m,\bar{0}} \oplus \mathbb{F}e_{l,\bar{1}})$$

for any $r \in J_{\bar{0}}$, $s \in J_{\bar{1}}$. In fact, given any $e_{n_0,\bar{i}_0} \in \bigoplus_{m \in I_{\bar{0}} \cap [n]_J, l \in I_{\bar{1}} \cap [n]_J} (\mathbb{F}e_{m,\bar{0}} \oplus \mathbb{F}e_{l,\bar{1}})$ such that $0 \neq [e_{n_0,\bar{i}_0}, u_{t,\bar{j}}] = e_{p,\bar{i}_0+\bar{j}}$, for $u_{t,\bar{j}} \in \{u_{r,\bar{0}}, u_{s,\bar{1}}\}$ and some $p \in I_{\bar{i}_0+\bar{j}}$. We have $p \in n_0 \star t$ and so $\{n_0, t\}$ is a J-connection meaning that $n_0 \sim_J p$. By the symmetry $p \sim_J n_0$ and by transitivity of $p \sim_J n_0 \sim_J n$, and we get

$$e_{p,\bar{i}_0+\bar{j}} \in \bigoplus_{m \in I_{\bar{0}} \cap [n]_J, l \in I_{\bar{1}} \cap [n]_J} (\mathbb{F}e_{m,\bar{0}} \oplus \mathbb{F}e_{l,\bar{1}}).$$

Hence $[e_{n_0,\bar{i}_0}, u_{t,\bar{j}}] \subset \bigoplus_{m \in I_{\bar{0}} \cap [n]_J, l \in I_{\bar{1}} \cap [n]_J} (\mathbb{F}e_{m,\bar{0}} \oplus \mathbb{F}e_{l,\bar{1}})$ as desired. We conclude

$$\bigoplus_{m \in I_{\bar{0}} \cap [n]_J, l \in I_{\bar{1}} \cap [n]_J} (\mathbb{F}e_{m,\bar{0}} \oplus \mathbb{F}e_{l,\bar{1}})$$

is an ideal of $\mathfrak{L}$ endowed with a multiplicative basis inherited from $\mathcal{B}$ (trivial by construction) and so, by $\mathcal{B}$-simplicity, necessarily $\bigoplus_{m \in I_{\bar{0}} \cap [n]_J, l \in I_{\bar{1}} \cap [n]_J} (\mathbb{F}e_{m,\bar{0}} \oplus \mathbb{F}e_{l,\bar{1}}) = \mathfrak{I}$ and consequently any couple of indexes in I are J-connected. Consider now any $r \in J$ and the linear subspace

$$\mathfrak{I} \oplus \bigoplus_{s \in J_{\bar{0}} \cap [r]_J, t \in J_{\bar{1}} \cap [r]_J} (\mathbb{F}u_{s,\bar{0}} \oplus \mathbb{F}u_{t,\bar{1}}).$$

Using a similar argument to the above one we see this linear subspace is actually an ideal of $\mathfrak{L}$ which admits a multiplicative basis inherited from $\mathcal{B}$. From $\mathcal{B}$-simplicity,

$$\mathfrak{I} \oplus \bigoplus_{s \in J_{\bar{0}} \cap [r]_J, t \in J_{\bar{1}} \cap [r]_J} (\mathbb{F}u_{s,\bar{0}} \oplus \mathbb{F}u_{t,\bar{1}}) = \mathfrak{L}$$

which implies in particular

$$\mathfrak{I} \oplus \bigoplus_{s \in J_{\bar{0}} \cap [r]_J, t \in J_{\bar{1}} \cap [r]_J} (\mathbb{F}u_{s,\bar{0}} \oplus \mathbb{F}u_{t,\bar{1}}) = \mathfrak{I} \oplus \bigoplus_{r \in J_{\bar{0}}, q \in J_{\bar{1}}} (\mathbb{F}u_{r,\bar{0}} \oplus \mathbb{F}u_{q,\bar{1}})$$

and so we get any couple of indexes in J are also J-connected.

Conversely, consider $\mathcal{I}$ a nonzero ideal of $\mathfrak{L}$ admitting a multiplicative basis inherited by the one of $\mathfrak{L}$. We have two possibilities for $\mathcal{I}$, either $\mathcal{I} \not\subset \mathfrak{I}$ or $\mathcal{I} \subset \mathfrak{I}$. In the first one, Proposition 3.1 gives us $\mathcal{I} = \mathfrak{L}$, while in the second one Proposition 3.2 shows $\mathcal{I} = \mathfrak{I}$. Therefore in both cases $\mathfrak{L}$ is $\mathcal{B}$-simple. □

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