Effective étale-descent morphisms in the category of M-ordered sets

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Abstract. A characterization of effective étale-descent morphisms in the category M-**Ord** of M-ordered sets, for a given monoid M, is obtained using the corresponding characterization in **Cat** of small categories.

Keywords. M-ordered sets - Descent data - Effective descent morphisms - Effective étale-descent morphisms - Discrete (op-)fibrations

1 Introduction

In [6] G. Janelidze and M. Sobral gave a complete characterization of the morphisms in the category **Ord** of (pre)ordered sets (denoted by **Preord** in [6]) which are effective for descent with respect to the class of étale morphisms, i.e., discrete fibrations. In [9] M. Sobral characterized the effective descent morphisms in the category **Cat** of small categories with respect to the class of discrete (op-)fibrations. These two works suggested the study of descent theory for the class of étale morphisms in the category M-**Ord** of M-ordered sets for a given monoid M. Using the identification of M-ordered sets as M-normed small categories given by M.M. Clementino, E. Colebunders and W. Tholen in [2], in this paper we present a complete characterization of the effective étale-descent morphisms in M-**Ord**.

2 M-ordered sets as M-normed categories

Given a monoid M, consider the monad

$$\mathbb{M} = (M \times (-), \mu, \eta)$$

on **Set**, with $\mu_X : M \times M \times X \to M \times X$ defined by $(m, n, x) \mapsto (mn, x)$ and $\eta_X : X \to M \times X$ by $x \mapsto (1, x)$, for each set X. The Barr extension [1] $\overline{\mathbb{M}} : \mathbf{Rel} \to \mathbf{Rel}$ is an extension of the monad $\mathbb{M}$, and it is given by

$$(m, x)(\overline{\mathbb{M}}r)(n, y) \Longleftrightarrow m = n \quad \text{and} \quad x(r)y,$$

where $r : X \to Y$ is a relation, $x \in X$, $y \in Y$, and $m, n \in M$. The category $(\overline{\mathbb{M}}, 2)$-**Cat** of $(\overline{\mathbb{M}}, 2)$-categories and functors is the category M-**Ord** of *M-ordered sets* and *equivariant maps*. For a relation $a : M \times X \to X$ one writes $x \xrightarrow{m} y$ instead of $(m, x)(a)y$, that is x is related to y with weight m. As remarked in [2, Section V.1.4], this arrow notation for the structure of an $(\overline{\mathbb{M}}, 2)$-category (X, a) emphasizes that X is actually the object set of a small category, denoted again by X, with hom-sets

$$X(x, y) = \{(x, m, y) \mid m \in M \quad \text{and} \quad x \xrightarrow{m} y\}$$

for $x, y \in X$; moreover this small category comes equipped with a faithful functor

$$\nu_X : X \to M, \quad (x, m, y) \mapsto m,$$

with M considered as a one-object category. Accordingly, the identity morphisms and composition in an M-ordered set X are given by

$$x \xrightarrow{e_M} x \quad \text{and} \quad (x \xrightarrow{m} y \quad \& \quad y \xrightarrow{n} z \Longrightarrow x \xrightarrow{nm} z),$$

while an equivariant map $f : X \to Y$ must satisfy

$$x \xrightarrow{m} y \Longrightarrow f(x) \xrightarrow{m} f(y)$$

for all $x, y \in X$ and $m \in M$. Defining an M-norm to be a functor from the small category X to the category M, we have a full embedding

$$I : (\overline{\mathbb{M}}, 2)\text{-}\mathbf{Cat} \hookrightarrow \mathbf{Cat} \downarrow M, \quad (X, a) \mapsto (X, \nu_X).$$

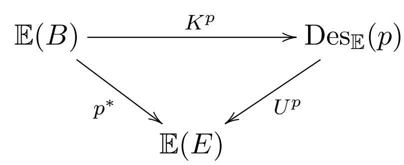
Proposition 2.1 [2, Proposition V.1.4.2] *The functor I is reflective and identifies $(\overline{\mathbb{M}}, 2)$-categories as those small categories over M whose norm is faithful.*

Now let $\mathbb{E}$ be the class of étale morphisms in M-**Ord**. As introduced in [3], a $(\mathbb{T}, 2)$-functor is étale if it is a pullback stable 'discrete fibration' (see [3] for details). For a cartesian monad, as $\mathbb{M}$ is, discrete fibrations are pullback stable, hence étale morphisms are the same as discrete fibrations. Using the arrow notation, an equivariant map $f : X \to Y$ in M-**Ord** is an étale morphism if and only if:

$$\forall x_0 \in X, \quad \forall y_1 \in Y, \quad \forall m \in M : y_1 \xrightarrow{m} f(x_0) \Longrightarrow \exists! x_1 \in f^{-1}(y_1) : x_1 \xrightarrow{m} x_0.$$

The problem concerning the characterization of effective étale-descent morphisms in M-**Ord** can be stated as follows.

Given an equivariant map $p : E \to B$ of M-ordered sets, denote by $\mathbb{E}(B)$ the slice category of étale morphisms over B and by $p^* : \mathbb{E}(B) \to \mathbb{E}(E)$ the pullback functor along p. We have then a commutative (up to isomorphism) diagram



where $\mathrm{Des}_{\mathbb{E}}(p)$ is the category of $\mathbb{E}$-descent data for p, and U^p and K^p are the forgetful and the comparison functor, respectively. By definition, the equivariant map $p : E \to B$ is an *(effective) étale-descent morphism* if the comparison functor K^p is full and faithful (an equivalence of categories). If we replace $\mathbb{E}$ by the class of all morphisms, p is said to be an *effective descent morphism*. For a more detailed presentation of descent theory we refer the Reader to the papers [7] and [8].

3 Characterization of (effective) étale-descent morphisms

Consider the following diagram

$$(\overline{\mathbb{M}}, 2)\text{-}\mathbf{Cat} \xrightarrow{I} \mathbf{Cat} \downarrow M \xrightarrow{U} \mathbf{Cat},$$

where I is the full embedding described in Section 2 and U is the obvious forgetful functor. A functor $F : X \to Y$ in $\mathbf{Cat}$ is called a *discrete fibration* if for every object x in X and every morphism of the form $g : y' \to F(x)$ in Y there exists a unique morphism $f : x' \to x$ in X such that $F(f) = g$. The notion of 'discrete fibration' given in Section 2 for a morphism in M-$\mathbf{Ord}$ coincides with the notion of discrete fibration given above when we consider M-ordered sets as (M-normed) small categories. Moreover, given an equivariant map $p : E \to B$ of M-ordered sets, since I preserves pullbacks, the pullback functor $p^* : \mathbb{E}(B) \to \mathbb{E}(E)$, where $\mathbb{E}$ is the class of étale morphisms in M-$\mathbf{Ord}$, is described by the following diagram

$$\begin{array}{ccc}
E \times_B A & \xrightarrow{\pi_2} & A \\
{\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \alpha} \\
E & \xrightarrow{p} & B \\
{\scriptstyle \nu_E}\downarrow & \swarrow{\scriptstyle \nu_B} & \\
M & &
\end{array}$$

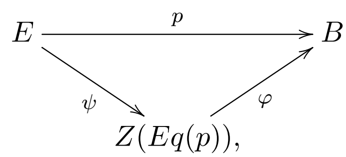
where the square is in $\mathbf{Cat}$. In fact for a discrete fibration $\alpha : A \to B$ of small categories, being in particular a faithful functor, the composition

$$A \xrightarrow{\alpha} B \xrightarrow{\nu_B} M$$

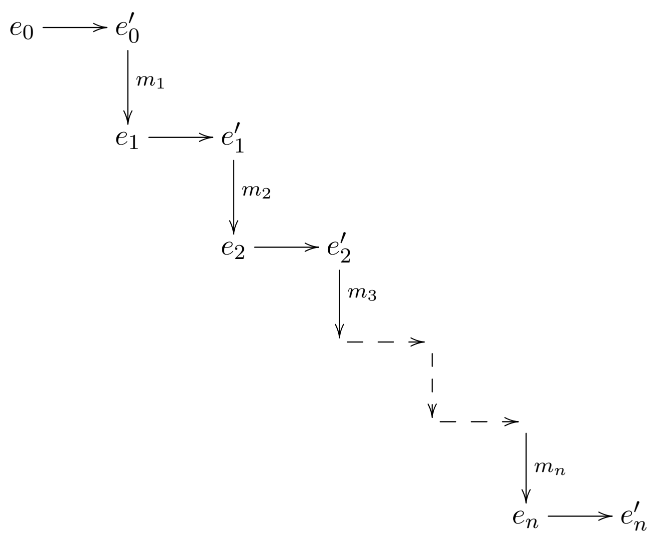
gives an M-valued norm for A making $\alpha : A \to B$ an object in $\mathbb{E}(B)$. Hence the arguments given in [9] leading to the characterization of effective descent morphisms in $\mathbf{Cat}$ with respect to the class of discrete (op-)fibrations can be used to get a characterization of the effective étale-descent morphisms in M-$\mathbf{Ord}$.

Following those arguments, the equivariant map $p : E \to B$ can be then factorized in

Cat in the following way



where $Z(Eq(p))$ is the *category of zigzags* with E as object-set and morphisms given by equivalent classes of zigzags of the form



where $p(e_i) = p(e'_i)$ for $i = 0, \cdots, n$, and $e'_i \xrightarrow{m_{i+1}} e_{i+1}$ in E for $i = 0, \cdots, n-1$. The equivalent classes are given by the smallest equivalence relation $\sim$ for which

$$\begin{array}{ccc} e_0 \longrightarrow e'_0 & & e_0 \\ \big\downarrow m & \sim & \big\downarrow m \\ e'_1 & & e_1 \longrightarrow e'_1. \end{array}$$

For a more detailed presentation of this construction we refer the Reader to the papers [6] and [9]. An n-zigzag as above will be denoted by

$$[e_n, e'_n]m_n \cdots m_2[e_1, e'_1]m_1[e_0, e'_0].$$

The morphism ψ is defined as the identity on objects and $\psi(e_0 \xrightarrow{m} e_1) = [e_0 \xrightarrow{m} e_1]$ on morphisms, while φ on objects acts as p and the image of an equivalent class of an n-zigzag $[e_n, e'_n]m_n \cdots m_2[e_1, e'_1]m_1[e_0, e'_0]$ via φ is

$$p(e_0) \xrightarrow{m_1} p(e'_1) \xrightarrow{m_2} \cdots \xrightarrow{m_n} p(e'_n) = p(e_0) \xrightarrow{m_n \cdots m_2 m_1} p(e'_n).$$

Based on the observations above, the Theorem below immediately follows form the similarly formulated [9, Theorem 2], while the Corollary can be deduced either directly from it, or, easily, from [9, Corollary 3].

Theorem 3.1 *An equivariant map $p : E \to B$ is an effective étale-descent morphism in M-**Ord** if and only if $\varphi : Z(Eq(p)) \to B$ is a full and faithful lax epimorphism in **Cat**.*

Corollary 3.2 *An equivariant map $p : E \to B$ is an effective étale-descent morphism in M-**Ord** if and only if*

(i) For each $p(e) \xrightarrow{k} p(e')$ in B with $k \in M$ there exists a zigzag in $Z(Eq(p))$

$$[e_n, e'_n]m_n \cdots m_2[e_1, e'_1]m_1[e_0, e'_0]$$

with $k = m_n \ldots m_2 m_1$, and such a zigzag is unique up to equivalence.

(ii) every point $b \in B$ is in relation to a point of the image via a right-invertible element of the monoid, i.e., for each $b \in B$ there exist $e \in E$, $n, m \in M$ such that $p(e) \xrightarrow{n} b$ and $b \xrightarrow{m} p(e)$ with $nm = 1$.

Remarks 3.3

1. When $M = 1$, 1-**Ord**=**Ord**, also identified as the full subcategory of **Cat** given by small categories X for which $X \to 1$ is faithful. Hence the characterization of effective étale-descent morphsism in M-**Ord** generalizes the characterization in **Ord** given in [6]. As in **Ord** uniqueness of zigzags encodes the fact that $Z(Eq(p))$ is a (pre)order, here it encodes the property that $Z(Eq(p))$ is an M-ordered set, with norm $Z(Eq(p)) \xrightarrow{\varphi} B \xrightarrow{\nu_B} M$.

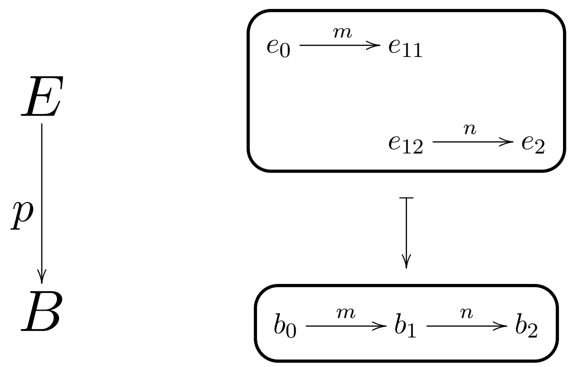
2. The étale-descent morphisms in M-**Ord** are precisely the morphisms for which condition (*ii*) is satisfied.

3. Since φ^{op} is a full and faithful lax epimorphism if and only if the same holds for φ, we conclude, as in the case for **Cat**, that the effective descent morphisms in M-**Ord** with respect to the class of discrete op-fibrations coincide with the effective étale-descent morphisms.

4. Effective descent morphisms in M-**Ord** were characterized in [4, Theorem 1.8] as the equivariant maps $p : E \to B$ such that

$$\forall \quad b_2 \xrightarrow{m} b_1 \xrightarrow{n} b_0 \quad \text{in } B \quad \exists \quad e_2 \xrightarrow{m} e_1 \xrightarrow{n} e_0 \quad \text{in } E : \forall i = 0, 1, 2 \quad p(e_i) = b_i.$$

   Therefore every effective descent morphism is effective for étale-descent. The converse is not true, even for surjective maps, as illustrated by the following modification of [5, Example 8.7]:



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References

[1] M. Barr, Relational Algebras, in: Reports of the Midwest Category Seminar, IV, pp 39-55, *Lecture Notes in Mathematics* **137**, Springer, Berlin. (1970).

[2] M.M. Clementino, E. Colebunders and W. Tholen, Lax algebras as spaces, in: Monoidal Topology, pp 375-466, *Encyclopedia Math. Appl.* **153**, *Cambridge Univ. Press, Cambridge* (2014).

[3] M.M. Clementino, D. Hofmann and G. Janelidze, On exponentiability of étale algebraic homomorphisms, *J. Pure Appl. Algebra* **217** (2013), 1195-1207.

[4] M.M. Clementino, D. Hofmann and A. Montoli, Covering Morphisms in Categories of Relational Algebras, *Appl. Categ. Structures*, **22** (2014), 767-788.

[5] G. Janelidze and M. Sobral, Finite preorders and topological descent I, *J. Pure Appl. Algebra* 175 (2002), 187-205.

[6] G. Janelidze and M. Sobral, Finite preorders and topological descent II: étale descent, *J. Pure Appl. Algebra* **174** (2002), 303-309.

[7] G. Janelidze and W. Tholen, Facets of Descent I, *Appl. Categ. Structures* **2** (1994), 1-37.

[8] G. Janelidze and W. Tholen, Facets of Descent II, *Appl. Categ. Structures* **5** (1997), 229-248.

[9] M. Sobral, Descent for Discrete (Co)fibrations, *Appl. Categ. Structures* **12** (2004), 527-535.