LAX ORTHOGONAL FACTORISATION SYSTEMS

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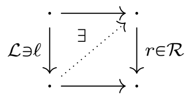
Abstract. This paper introduces *lax orthogonal algebraic weak factorisation systems* on 2-categories and describes a method of constructing them. This method rests in the notion of *simple 2-monad*, that is a generalisation of the simple reflections studied by Cassidy, Hébert and Kelly. Each simple 2-monad on a finitely complete 2-category gives rise to a lax orthogonal algebraic weak factorisation system, and an example of a simple 2-monad is given by completion under a class of colimits. The notions of KZ *lifting operation*, *lax natural lifting operation* and *lax orthogonality* between morphisms are studied.

1. Introduction

This paper contains four main contributions: the introduction of *lax orthogonal algebraic weak factorisation systems* (AWFSs); the introduction of the concept of KZ *diagonal fillers* and the study of their relationship to lax orthogonal AWFSs; the introduction of *simple 2-monads*, and the proof that each such induces an AWFSs; the proof that 2-monads given by completion under a class of colimits are simple and the study of the induced factorisations.

Weak factorisation systems form the basic ingredient of *Quillen model structures* [?], and, as the name indicates, are a weakening of the ubiquitous orthogonal factorisation systems. A weak factorisation system (WFS) on a category consists of two classes of morphisms $\mathcal{L}$ and $\mathcal{R}$ satisfying two properties: every morphism can be written as a composition of a morphism in $\mathcal{L}$ followed by one in $\mathcal{R}$, and for any commutative square, with vertical morphisms in $\mathcal{L}$ and $\mathcal{R}$ as depicted in (1.1), there exists a diagonal filler. One says that r has the right lifting property with respect to ℓ and that ℓ has the left lifting property with respect to r.



(1.1)

When r is the unique map to the terminal object, one usually says that C is injective with respect to ℓ.

In order to unify the study of injectivity with respect to different classes of continuous maps between T_0 topological spaces, Escardó [?] employed lax idempotent 2-monads, also known as KZ 2-monads, on poset-enriched categories – these are the same as 2-categories whose hom-categories are posets. For example, if T is such a lax idempotent 2-monad, the T-algebras can be described as the objects A that are

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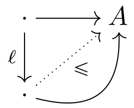
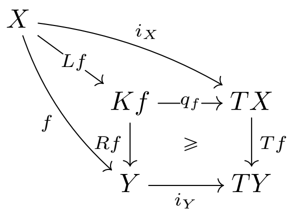
injective to all the morphisms ℓ such that $T\ell$ is a coretract left adjoint – also known as a T-embedding. A central point is that not only each morphism $\mathrm{dom}(\ell) \to A$ has an extension along ℓ, but moreover it has a *least* extension: one that is smallest amongst all extensions.



The assignment that sends a morphism to its least extension can be described in terms of the 2-monad T, so one no longer has the *property* of the existence of at least one extension, but the *algebraic structure* that constructs the extension. If one wants to describe WFSs in this context, instead of just injectivity, one is led to consider algebraic weak factorisation systems (AWFSs), to which we shall return in this introduction.

Injective continuous maps. One of the basic examples that fit in the framework of [?] is that of the filter monad on the category of T_0 spaces, that assigns to each space its space of filters of open sets. It was shown in [?] that the algebras for this monad are the topological spaces that arise as continuous lattices with the Scott topology. These spaces were known to be precisely those injective with respect to subspace embeddings [?]. In [?] this and other related results are generalised, characterising those continuous maps of T_0 spaces that have the right lifting property with respect to different classes of embeddings, and exhibiting for each a WFS in the category of T_0 spaces. A morphism $f\colon X \to Y$ is factorised through the subspace $Kf \subseteq TX \times Y$ of those (φ, y) such that $Tf(\varphi) \leqslant \{U \in \mathcal{O}(Y) : y \in U\}$. The space TX can be the topological space of filters of open sets of X or a variant of it, and $i_X\colon X \to TX$ the inclusion of X as the set of principal filters, $i_X(x) = \{U \in \mathcal{O}(X) : x \in U\}$. The space Kf fits in a diagram as displayed. The maps q_f and Rf send $(\varphi, y) \in Kf$ to $\varphi \in TX$ and $y \in Y$ respectively. The inequality symbol inside the square denotes the fact that $Tf \cdot q_f \leqslant i_Y \cdot Rf$.



Central to the arguments in [?] is the fact that the monad $f \mapsto Rf$ is lax idempotent or KZ. This property is intimately linked with the fact that Lf is always an embedding of the appropriate variant – eg, when TX is the space of all filters of open sets, then Lf is a topological embedding.

The construction of the factorisation of maps just described resembles the classical case of simple reflections [?]. One of the aims of the present paper is to show that both constructions are particular instances of a more general one.

Algebraic weak factorisation systems. Algebraic weak factorisation systems (AWFSs) were introduced in [?] with the name *natural weak factorisation systems*, with a distributive axiom later added in [?]. The theory of AWFSs has been developed in [?] and [?], especially with respect to their relationship to double categories and to cofibrant generation. The present paper takes the theory in a new direction, that of AWFSs on 2-categories whose lifting operations, or diagonal fillers, have a universal property with respect to 2-cells.

Many of the factorisation systems that occur in practice provide a construction for the factorisation of an arbitrary morphism. Such a structure on a category $\mathcal{C}$ is called a *functorial factorisation* and can be described in several equivalent ways: as a functor $\mathcal{C}^{\mathbf{2}} \to \mathcal{C}^{\mathbf{3}}$ that is compatible with domain and codomain; as a codomain-preserving – ie with identity codomain component – pointed endofunctor $\Lambda\colon 1 \Rightarrow R$ of $\mathcal{C}^{\mathbf{2}}$; as a domain-preserving copointed endofunctor $\Phi\colon L \Rightarrow 1$ of $\mathcal{C}^{\mathbf{2}}$. Then, a morphism f factors as $f = Rf \cdot Lf$. Any such functorial factorisation has an underlying WFS $(\mathcal{L}, \mathcal{R})$ where $\mathcal{L}$ consists of those morphisms that admit an (L, Φ)-coalgebra structure and $\mathcal{R}$ of those that admit an (R, Λ)-algebra structure. One usually wants, however, to guarantee that $Lf \in \mathcal{L}$ and $Rf \in \mathcal{R}$, for which one requires extra data in the form of a comultiplication that makes (L, Φ) into a comonad L and a multiplication that makes (R, Λ) into a monad R. The pair (L, R) together with an extra distributivity condition is called an AWFS.

The underlying WFS of an AWFS (L, R) is an orthogonal factorisation system precisely when L and R are idempotent [?]; for this, it is enough if either is idempotent [?].

All the above constructions can be performed on 2-categories instead of categories. Two morphisms $\ell\colon A \to B$ and $r\colon C \to D$ in a 2-category $\mathscr{K}$ are *lax orthogonal* when the comparison morphism

$$\mathscr{K}(B, C) \longrightarrow \mathscr{K}(A, C) \times_{\mathscr{K}(A,D)} \mathscr{K}(B, D)$$

has a left adjoint coretract. – In the the usual definitions of weak orthogonality and orthogonality this morphism must be an epimorphism and, respectively, an isomorphism. – This left adjoint provides diagonal fillers that moreover satisfy a universal property with respect to 2-cells. A choice of diagonal fillers like these that is in addition natural with respect to ℓ and r we call a KZ *lifting operation*.

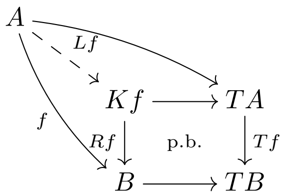
When the 2-category $\mathscr{K}$ is locally a preorder, the lax orthogonality of ℓ and r reduces to the statement, encountered before in this introduction, that for each commutative square (1.1) there exists a least diagonal filler.

The notion of AWFS on a 2-category we choose is the straightforward generalisation of the usual notion of AWFS on a category. If $\mathscr{K}$ is a 2-category, an AWFS on $\mathscr{K}$ consists of a 2-comonad L and a 2-monad R on $\mathscr{K}^{\mathbf{2}}$ that form an AWFS on the underlying category of $\mathscr{K}^{\mathbf{2}}$, and that satisfy $\operatorname{cod} L = \operatorname{dom} R$ as 2-functors; the definition can be found in Section 4.c.

An interesting question is what is the property on an AWFS that corresponds to the existence of KZ lifting operations. The answer is that both the 2-comonad and the 2-monad of the AWFS must be *lax idempotent* – proved in Theorem 9.10. Equivalently, either the 2-comonad *or* the 2-monad must be lax idempotent – proved in Section 5. This last statement mirrors the case of AWFSs whose underlying WFS is orthogonal, for which, as mentioned earlier, it is enough that either the comonad *or* the monad be idempotent.

A basic example of a lax idempotent AWFS is the one that factors a functor $f\colon A \to B$ as a left adjoint coretract $A \to f \downarrow B$ followed by the split opfibration $f \downarrow B \to B$. We refer to this AWFS as the coreflection–opfibration AWFS.

Simple reflections. The paper [?] studies the relationship between orthogonal factorisation systems, abbreviated OFSs, and reflections. Every OFS $(\mathcal{E}, \mathcal{M})$ on a category $\mathcal{C}$ induces a reflection on $\mathcal{C}$ as long as $\mathcal{C}$ has a terminal object 1; the reflective subcategory is $\mathcal{M}/1$, the full subcategory of those objects X such that $X \to 1$ belongs to $\mathcal{M}$. Under certain hypotheses, a reflection, or an idempotent monad T on $\mathcal{C}$, induces an OFS. One of the possible hypotheses is that T be *simple*, which means that for any morphism f the dashed morphism into the pullback depicted below is inverted by T. The factorisation of f is then given by $f = Rf \cdot Lf$, and the left class of morphisms consists of those which are inverted by T.



There is an alternative way of describing simple reflections which seems to be absent from the literature. Suppose that T is an idempotent monad on $\mathcal{C}$ and denote by T-Iso the category of morphisms in $\mathcal{C}$ that are inverted by T. This category fits in a pullback square

$$\begin{array}{ccc} \mathsf{T}\text{-Iso} & \longrightarrow & \mathrm{Iso} \\ {\scriptstyle U}\downarrow & & \downarrow \\ \mathcal{C}^{\mathbf{2}} & \xrightarrow{T^{\mathbf{2}}} & \mathcal{C}^{\mathbf{2}} \end{array}$$

where both vertical functors are full inclusions.

Proposition 1.2. *The reflection* T *is simple if and only if* $U\colon \mathsf{T}\text{-Iso} \hookrightarrow \mathcal{C}^{\mathbf{2}}$ *is a coreflective subcategory.*

One way of expressing the construction of the OFS from T is the following. On any category $\mathcal{A}$ we have the OFS (Iso, Mor), with left class the isomorphisms and right class all morphisms. Isomorphisms are the coalgebras for the idempotent comonad L' on $\mathcal{A}^{\mathbf{2}}$ given by $L'(f) = 1_{\mathrm{dom}(f)}$. If $F \dashv U\colon \mathcal{A} \hookrightarrow \mathcal{C}$ is the adjunction induced by the reflection T, the copointed endofunctor (L, Φ) defined by pullback along the unit of the adjunction satisfies the property that the rectangle on the right hand side below is a pullback. In other words, (L, Φ)-coalgebras are those morphisms that are inverted by F; equivalently $(L, \Phi)\text{-Coalg} \cong \mathsf{T}\text{-Iso}$.

$$\begin{array}{ccc} L & \longrightarrow & U^{\mathbf{2}}L'F^{\mathbf{2}} \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\Phi'} \\ 1 & \longrightarrow & U^{\mathbf{2}}F^{\mathbf{2}} \end{array} \qquad \begin{array}{ccc} (L,\Phi)\text{-Coalg} & \longrightarrow & (L',\Phi')\text{-Coalg} \\ \downarrow & \text{p.b.} & \downarrow \\ \mathcal{C}^{\mathbf{2}} & \xrightarrow{F^{\mathbf{2}}} & \mathcal{A}^{\mathbf{2}} \end{array}$$

Any morphism that is inverted by T is orthogonal to Tf and therefore to its pullback Rf; in particular, Lf satisfies this if the reflection is simple. Therefore, we obtain an OFS when T is simple, with left class those morphisms that are inverted by T.

Simple 2-adjunctions and AWFSs. The above analysis can be adapted to the case where categories are substituted by 2-categories and OFSs by lax orthogonal AWFSs. Reflections are substituted by lax idempotent 2-monads, idempotent (co)monads by lax idempotent 2-(co)monads, the simple reflections by appropriately defined simple 2-adjunctions or simple 2-monads. The reflective subcategory Iso of the arrow category is substituted by the lax idempotent 2-comonad whose algebras are coretract left adjoints, while Mor is substituted by the free split opfibration 2-monad. A version of the main theorem of Section 11, appropriately modified for this introduction, states:

Theorem. *If the 2-adjunction* $F \dashv U\colon \mathscr{K} \to \mathscr{A}$ *is simple, and* $\mathscr{K}, \mathscr{A}$ *are 2-categories with enough finite limits, then there is a lax orthogonal AWFS* (L, R) *on* $\mathscr{K}$ *whose* L*-coalgebras are morphisms* f *of* $\mathscr{K}$ *with a coretract adjunction* $Ff \dashv r$ *in* $\mathscr{A}$*.*

The notion of simple 2-adjunction is central to the theorem, and occupies most of Sections 10 and 11.

A *simple 2-monad* is one whose associated free algebra 2-adjunction is simple. When all the 2-categories involved are in fact categories, lax idempotent 2-monads reduce to reflections and our concept of simple 2-monad to the one of simple reflection. Therefore, we know that there are lax idempotent 2-monads that are not simple, as [?] gives examples of reflections that are not simple.

Examples and further work. The main example treated in the present article arises from categories with colimits. Given a class of colimits, there exists a 2-monad on **Cat** whose algebras are categories with chosen colimits of that class. We show that these 2-monads are simple, giving rise to lax orthogonal AWFSs on **Cat**. Even though the left morphisms of this factorisation system are described in general in the theorem above, the right class of morphisms is more difficult to pin down. We carefully investigate the right class of morphisms and show that they do not coincide with the obvious candidates: the opfibrations whose fibers have chosen colimits of the given class and whose push-forward functors between fibres preserve them.

There are a number of examples of lax orthogonal AWFSs on locally preordered 2-categories, including that on the (2-)category of T_0 topological spaces mentioned earlier in this introduction, that we have had to leave out of this article for reasons of space. These will appear in a companion paper that will concentrate in the case of locally ordered 2-categories, which is still rich enough to encompass a large number of examples and relates to a rich literature on the subject of injectivity in order-enriched categories.

Description of sections. Sections 2 and 3 can be regarded as a fairly self-contained recount of the basic definitions and properties of AWFSs.

We put together at the beginning of Section 4 some facts about lax idempotent 2-(co)monads, one of our main tools, before introducing lax orthogonal AWFS, our main subject of study.

Section 5 proves that in order for an AWFS to be lax orthogonal it suffices that either the 2-monad or the 2-comonad be lax idempotent.

Sections 6 and 7 recount the notions of lifting operations and diagonal fillers, with their relationship to AWFSs. Our approach uses modules or profunctors and appears to be novel. In a 2-category one can consider the usual lifting operations, but also lax natural ones. We define lax natural and KZ diagonal fillers in Section 8 and prove that lax orthogonal AWFSs give rise to KZ diagonal fillers. Lax orthogonal functorial factorisations are briefly considered.

In Section 9 we characterise lax orthogonal AWFSs as those AWFS (L, R) for which R-algebras are algebraically KZ injective to all L-coalgebras, or equivalently, for which natural KZ diagonal fillers exist for squares from L-coalgebras to R-algebras.

Section 10 introduces the concept of simple adjunction of 2-functors. One of our main results, the construction of a lax idempotent AWFS from a simple adjunction, can be found in Section 11.

Section 12 studies the case when the simple 2-adjunction is the free algebra adjunction induced by a 2-monad, that we call a *simple* 2-monad, as it generalises the notion of simple reflection [?]. Conditions that guarantee that a lax idempotent 2-monad is simple are provided.

Section 13 studies the example of – enriched – categories and completion under colimits. We show that for a class of colimits Φ, the 2-monad whose algebras are categories with *chosen* colimits of that class is simple, whence inducing a lax orthogonal AWFS (L, R). We prove in Section 13.d that R-algebras are always split opfibrations with fibrewise chosen Φ-colimits and that the converse does not always hold. The article concludes with a short section that comments on further work and examples.

2. Background on algebraic weak factorisation systems

In the last few years there has been much interest in algebraic weak factorisation systems (AWFSs) mainly due to their connection to Quillen's model categories and the small object argument, but also due to the homotopical approach to type theory (homotopy type theory). The basic theory of AWFSs appeared in [**?**] with the name of *natural weak factorisation system*, and was later expanded in [**?**], especially with respect to the construction of cofibrantly generated AWFS. Further study appeared recently in [**?**]. From Section 4 onwards, the present paper expands the theory in another direction, that of AWFS in 2-categories whose lifting operations, or diagonal fillers, satisfy a universal property with respect to 2-cells. Before all that we need to collect present basics of the theory of AWFS, mostly following [**?**, **?**].

2.a. **The definition of AWFS.** We denote by $\mathbf{2}$ the category with two objects 0 and 1 and only one non-identity arrow $0 \to 1$, and by $\mathbf{3}$ the category with three objects and three non-identity morphisms $0 \to 1 \to 2$. Given a category $\mathcal{C}$ consider the functors $d_0, d_1, d_2 \colon \mathcal{C}^{\mathbf{3}} \to \mathcal{C}^{\mathbf{2}}$ that send a pair of composable morphisms $(f \colon A \to B, g \colon B \to C)$ in $\mathcal{C}$ to: $d_0(f,g) = f$, $d_1(f,g) = g \cdot f$, $d_2(f,g) = g$.

When displaying diagrams, we shall denote an object $f \in \mathcal{C}^{\mathbf{2}}$ by a vertical arrow and a morphism $(h,k) \colon f \to g$ in $\mathcal{C}^{\mathbf{2}}$ by a commutative square, as shown.

$$
\begin{array}{c}
\cdot \\
{\scriptstyle f}\big\downarrow \\
\cdot
\end{array}
\qquad\qquad
\begin{array}{ccc}
\cdot & \xrightarrow{h} & \cdot \\
{\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\
\cdot & \xrightarrow[k]{} & \cdot
\end{array}
$$

Definition 2.1. A *functorial factorisation* in $\mathcal{C}$ is a section of the composition functor $d_1 \colon \mathcal{C}^{\mathbf{3}} \to \mathcal{C}^{\mathbf{2}}$. This means that for each morphism $(h,k) \colon f \to g$ in $\mathcal{C}^{\mathbf{2}}$ we have a factorisation, functorial in (h,k), as depicted.

$$
\begin{array}{ccc}
A & \xrightarrow{h} & C \\
{\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\
B & \xrightarrow[k]{} & D
\end{array}
\quad\longmapsto\quad
\begin{array}{ccc}
A & \xrightarrow{h} & C \\
{\scriptstyle Lf}\big\downarrow & & \big\downarrow{\scriptstyle Lg} \\
Kf & \xrightarrow{K(h,k)} & Kg \\
{\scriptstyle Rf}\big\downarrow & & \big\downarrow{\scriptstyle Rg} \\
B & \xrightarrow{k} & D
\end{array}
\tag{2.2}
$$

A functorial factorisation as above induces a pointed endofunctor $\Lambda \colon 1 \Rightarrow R$ and a copointed endofunctor $\Phi \colon L \Rightarrow 1$ on $\mathcal{C}^{\mathbf{2}}$. The endofunctor L is given by $f \mapsto Lf$, and the component of the copoint Φ at the object f is depicted on the left hand side of (2.a). Similarly, $f \mapsto Rf$, and the component of the point Λ at the object f is depicted on the right hand side of (2.a). We note that the domain component of Φ and the codomain component of Λ are identities, which implies $\mathrm{dom}L = \mathrm{dom}$ and $\mathrm{cod}R = \mathrm{cod}$, as functors $\mathcal{C}^{\mathbf{2}} \to \mathcal{C}$. We say that (L, Φ) is *domain preserving* and that (R, Λ) is *codomain preserving.*

$$
\begin{array}{ccc}
A & = & A \\
{\scriptstyle Lf}\big\downarrow & & \big\downarrow{\scriptstyle f} \\
Kf & \xrightarrow[Rf]{} & B
\end{array}
\qquad\qquad
\begin{array}{ccc}
A & \xrightarrow{Lf} & Kf \\
{\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle Rf} \\
B & = & B
\end{array}
$$

Conversely, either a domain preserving copointed endofunctor (L, Φ) or a codomain preserving pointed endofunctor (R, Λ) on $\mathcal{C}^{\mathbf{2}}$ define a functorial factorisation, in the first case by setting $Rf = \mathrm{cod}(\Phi_f)$, and in the second case by setting $Lf = \mathrm{dom}(\Lambda_f)$.

Definition 2.3. An *algebraic weak factorisation system* [**?**, **?**] is a functorial factorisation where the copointed endofunctor $\Phi\colon L \Rightarrow 1$ is equipped with a comultiplication $\Sigma\colon L \Rightarrow L^2$, making it into a comonad L, and the pointed endofunctor $\Lambda\colon 1 \Rightarrow R$ is equipped with a multiplication $\Pi\colon R^2 \Rightarrow R$, making it into a monad R, plus a distributivity condition. The components of this comultiplication and multiplication will be denoted as follows.

$$\Sigma_f = \begin{tikzcd} A \arrow[r, equal] \arrow[d, "Lf"'] & A \arrow[d, "L^2 f"] \\ Kf \arrow[r, "\sigma_f"] & KLf \end{tikzcd} \qquad \Pi_f = \begin{tikzcd} KRf \arrow[r, "\pi_f"] \arrow[d, "R^2 f"'] & Kf \arrow[d, "Rf"] \\ B \arrow[r, equal] & B \end{tikzcd}$$

Furthermore, the monad and comonad must be related by the distributivity condition introduced in [**?**] that asserts that the natural transformation $\Delta\colon LR \Rightarrow RL$ with components

$$\Delta_f = \begin{tikzcd} \cdot \arrow[r, "\sigma_f"] \arrow[d, "LRf"'] \arrow[dr, "1"] & \cdot \arrow[d, "RLf"] \\ \cdot \arrow[r, "\pi_f"'] & \cdot \end{tikzcd} \tag{2.4}$$

is a distributive law, ie that the diagrams shown below commute. In fact, the two triangles automatically commute as a consequence of the comonad and monad axioms for L and R.

$$\begin{tikzcd} LR \arrow[rr, "\Delta"] \arrow[dr, "\Phi R"'] & & RL \arrow[dl, "R\Phi"] \\ & R & \end{tikzcd} \qquad \begin{tikzcd} & L \arrow[dl, "L\Lambda"'] \arrow[dr, "\Lambda L"] & \\ LR \arrow[rr, "\Delta"] & & RL \end{tikzcd}$$

$$\begin{tikzcd} LR \arrow[rr, "\Delta"] \arrow[d, "\Sigma R"'] & & RL \arrow[d, "R\Sigma"] \\ L^2R \arrow[r, "L\Delta"] & LRL \arrow[r, "\Delta L"] & RL^2 \end{tikzcd} \qquad \begin{tikzcd} LR^2 \arrow[r, "\Delta R"] \arrow[d, "L\Pi"'] & RLR \arrow[r, "R\Delta"] & R^2L \arrow[d, "\Pi L"] \\ LR \arrow[rr, "\Delta"] & & RL \end{tikzcd}$$

One of the ideas behind this definition is that the L-coalgebras have the left lifting property with respect to the R-algebras, as explained below. An L-coalgebra structure on a morphism $f\colon A \to B$, respectively, an R-algebra structure on f, is given by morphisms in $\mathcal{C}^{\mathbf{2}}$ of the form

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "f"'] & A \arrow[d, "Lf"] \\ B \arrow[r, "s"'] & Kf \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Kf \arrow[r, "p"] \arrow[d, "Rf"'] & A \arrow[d, "f"] \\ B \arrow[r, equal] & B \end{tikzcd}$$

The domain and codomain components depicted by equality symbols are identity morphisms as a consequence of the counit axiom of the comonad L, respectively unit axiom of the monad R. These axioms also imply $Rf \cdot s = 1_B$ and $p \cdot Lf = 1_A$.

Continuing, given a morphism (h, k) in $\mathcal{C}^{\mathbf{2}}$ as in (2.2), we get a diagonal filler as depicted.

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "Lf"] \arrow[dd, "f"'] & & C \arrow[r, equal] \arrow[d, "Lg"'] & C \arrow[dd, "g"] \\ & Kf \arrow[r, "K(h,k)"] \arrow[d, "Rf"] & Kg \arrow[ur, "p"] \arrow[dr, "Rg"'] & \\ B \arrow[r, equal] \arrow[ur, "s"] & B \arrow[rr, "k"] & & B \end{tikzcd} \tag{2.5}$$

Remark 2.6. Every AWFS (L, R) has an underlying WFS $(\mathcal{L}, \mathcal{R})$, where $\mathcal{L}$ consists of those morphisms of $\mathcal{C}$ that admit a coalgebra structure for the copointed endofunctor (L, Φ) and $\mathcal{R}$ consists of those morphisms that admit an algebra structure for the pointed endofunctor (R, Λ). To verify this, one can observe that both $\mathcal{L}$ and $\mathcal{R}$ are closed under retracts, and that each morphism f factors as $f = Rf \cdot Lf$, where Lf admits the (L, Φ)-coalgebra structure $\Sigma_f : Lf \to L^2 f$ and Rf admits the (R, Λ)-algebra structure $\Pi_f : R^2 f \to Rf$.

2.b. **Orthogonal factorisations as AWFSs.** We continue with some more background, in this case, the characterisation of orthogonal factorisation systems in terms of the associated AWFS. Clearly, any orthogonal factorisation system $(\mathcal{E}, \mathcal{M})$ in a category $\mathcal{C}$ induces an AWFS. This is a consequence of the uniqueness of the factorisations. One can easily characterise the AWFS obtained in this way.

Proposition 2.7 ([**?**, Thm 3.2])**.** *The following are equivalent for an AWFS* (L, R)*:*

- *The comonad* L *and the monad* R *are idempotent.*
- *The underlying WFS is an OFS.*

Furthermore, if R is idempotent, then so is L, a proof of which can be found in [**?**].

2.c. **Right morphisms form a fibration.** This section collects some of the material of [**?**, §3.4] that will be crucial later on.

A functor $P : \mathcal{A} \to \mathcal{C}^{\mathbf{2}}$ is a *discrete pullback-fibration* if it is just like a discrete fibration except that only pullback squares have cartesian liftings. More explicitly, for each $a \in \mathcal{A}$ and each pullback square $(h, k) : f \to P(a)$ – f is the pullback of the morphism $P(a)$ along k – there exists a unique morphism $\alpha : \bar{a} \to a$ in $\mathcal{A}$ such that $P\alpha = (h, k)$.

Lemma 2.8. *Suppose the category* $\mathcal{C}$ *has pullbacks. For any codomain preserving monad* R *on* $\mathcal{C}^{\mathbf{2}}$*, the codomain functor exhibits* R-Alg *as a discrete pullback-fibration over* $\mathcal{C}$*.*

To give an idea of the proof, suppose that $g : C \to D$ has an R-algebra structure $p_g : Kg \to C$, and that

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B & \xrightarrow{k} & D \end{array}$$

is a pullback square. Then, the R-algebra structure on f is given by the morphism $p_f : Kf \to A$ induced by the universal property of pullbacks and the equality displayed below. This is the unique algebra structure that makes (h, k) a morphism of algebras.

$$\begin{array}{ccccc} Kf & \xrightarrow{p_f} & A & \xrightarrow{h} & C \\ {\scriptstyle Rf}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ B & = & B & \xrightarrow{k} & D \end{array} \quad = \quad \begin{array}{ccccc} Kf & \xrightarrow{K(h,k)} & Kg & \xrightarrow{p_g} & C \\ {\scriptstyle Rf}\downarrow & & {\scriptstyle Rg}\downarrow & & \downarrow{\scriptstyle g} \\ B & \xrightarrow{k} & D & = & D \end{array}$$

2.d. **Miscellaneous remarks.** Before moving to the next section and the subject of double categories, we collect three observations that will be of use later on. We use the adjunctions $\mathrm{cod} \dashv \mathrm{id} \dashv \mathrm{dom} : \mathcal{C}^{\mathbf{2}} \to \mathcal{C}$, the first of which has identity counit and the second has identity unit.

Remark 2.9. Suppose given a functorial factorisation, with associated copointed endofunctor (L, Φ) and pointed endofunctor (R, Λ). The identity natural transformation $1_{\mathcal{C}} = \mathrm{dom} \cdot L \cdot \mathrm{id}$ corresponds under $\mathrm{id} \dashv \mathrm{dom}$ to a natural transformation

$(1, L)$ with f-component equal to the morphism depicted on the right hand side below.

$$\mathcal{C}^2 \underset{L}{\overset{\mathrm{id}\cdot\mathrm{dom}}{\Downarrow (1,L)}} \mathcal{C}^2 \qquad \begin{array}{ccc} \mathrm{dom}\, f & = & \mathrm{dom}\, f \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle Lf} \\ \mathrm{dom}\, f & \xrightarrow[Lf]{} & Kf \end{array}$$

Remark 2.10. Given a functorial factorisation in $\mathcal{C}$ with associated copointed endofunctor (L, Φ) and pointed endofunctor (R, Λ), denote by $V\colon (R,\Lambda)\text{-Alg} \to \mathcal{C}^{\mathbf{2}}$ the corresponding forgetful functor from the category of algebras for the pointed endofunctor (R, Λ). Define a natural transformation as the composition of two transformations, as displayed.

$$\begin{array}{ccc} (R,\Lambda)\text{-Alg} & \xrightarrow{V} & \mathcal{C}^{\mathbf{2}} \\ {\scriptstyle V}\downarrow & {\scriptstyle (1,p)}\Nearrow & \downarrow{\scriptstyle \mathrm{id}\cdot\mathrm{dom}} \\ \mathcal{C}^{\mathbf{2}} & \xrightarrow[L]{} & \mathcal{C}^{\mathbf{2}} \end{array} \qquad (1,p)\colon L\cdot V \Longrightarrow \mathrm{id}\cdot\,\mathrm{dom}\cdot R\cdot V \Longrightarrow \mathrm{id}\cdot\,\mathrm{dom}\cdot V$$

The first arrow is the mate of the identity natural transformation $\mathrm{cod}\cdot L = \mathrm{dom}\cdot R$ under the adjunction $\mathrm{cod} \dashv \mathrm{id}$. The second arrow is the application of the (R, Λ)-algebra structure of $R \cdot V \Rightarrow V$. Explicitly, the component of $(1, p)$ on an (R, Λ)-algebra (f, p_f) is

$$\begin{array}{ccc} \mathrm{dom}\, f & = & \mathrm{dom}\, f \\ {\scriptstyle Lf}\downarrow & & \downarrow{\scriptstyle 1} \\ Kf & \xrightarrow[p_f]{} & \mathrm{dom}\, f \end{array}$$

Remark 2.11. The pasting along L of the transformation $(1, L)$ of Remark 2.9 with the transformation $(1, p)$ of Remark 2.10 is the identity. This is a consequence of the unit axiom for (R, Λ)-algebras: if (f, p_f) is an algebra, then $p_f \cdot Lf = 1$.

3. Double categories of algebras and coalgebras

This section collects remarks on double categories and AWFSs, due to R Garner. The definition of AWFSs used in [**?**] differs of the original one [**?**] in the requirement of an extra distributivity condition: the transformation $\Delta\colon LR \Rightarrow RL$ displayed in (2.4) should be a mixed distributive law. This condition is what makes possible the definition of a composition of R-algebras and of L-coalgebras, as we proceed to explain.

The standard category object in $\mathbf{Cat}^{\mathrm{op}}$ displayed on the left below induces a category object in $\mathbf{Cat}$, that is, a double category, displayed in the centre, that we may call the double category of squares and denote by $\mathbb{S}\mathrm{q}(\mathcal{C})$. Objects of $\mathbb{S}\mathrm{q}(\mathcal{C})$ are those of $\mathcal{C}$, vertical and horizontal morphisms are morphisms of $\mathcal{C}$, while 2-cells in $\mathbb{S}\mathrm{q}(\mathcal{C})$ are commutative squares in $\mathcal{C}$.

$$\mathbf{3} \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow}\; \mathbf{2} \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow}\; \mathbf{1} \qquad \mathcal{C}^{\mathbf{3}} \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \mathcal{C}^{\mathbf{2}} \;\substack{\xrightarrow{\mathrm{cod}}\\ \xleftarrow{\mathrm{id}}\\ \xrightarrow{\mathrm{dom}}}\; \mathcal{C} \qquad \begin{array}{ccc} \mathsf{R}\text{-Alg} & \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow} & \mathcal{C} \\ {\scriptstyle V}\downarrow & & \| \\ \mathcal{C}^{\mathbf{2}} & \substack{\xrightarrow{\mathrm{cod}}\\ \xleftarrow{\mathrm{id}}\\ \xrightarrow{\mathrm{dom}}} & \mathcal{C} \end{array} \tag{3.1}$$

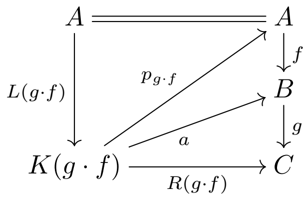
The central result of this section is the following.

Proposition 3.2. *If* R *is a codomain-preserving monad on* $\mathcal{C}$*, there is a bijection between* AWFS*s with monad* R *and extensions of the diagram on the right hand side of* (3.1) *to a double functor, by which we mean extensions of the reflexive graph*

$\mathsf{R}\text{-Alg} \rightrightarrows \mathcal{C}$ *to a category object that makes* (3.1) *into a functor internal to* **Cat** *– a double functor into* $\mathbb{S}\mathrm{q}(\mathcal{C})$.

Below we give an indication of the proof of this proposition; a more detailed account can be found in [**?**, §3].

3.a. **From AWFSs to double categories.** If (L, R) is an AWFS on $\mathcal{C}$, R-algebras can be composed, in the sense that if $f\colon A \to B$ and $g\colon B \to C$ are R-algebras, then an R-algebra structure for $g \cdot f$ can be constructed from the AWFS. Explicitly, if $(p_f, 1_B)\colon Rf \to f$ and $(p_g, 1_C)\colon Rg \to g$ are the R-algebra structures of f and g, with respective domain components $p_f\colon Kf \to A$ and $p_g\colon Kg \to B$, then the R-algebra structure $(p_{g\cdot f}, 1_C)\colon R(g \cdot f) \to g \cdot f$ is constructed in the following manner. The reader will recall from Section 2, especially from diagram (2.5), that any morphism $f \to g$ from an L-coalgebra to an R-algebra has a canonical diagonal filler. Consider the canonical diagonal filler a of the square $(f, R(g \cdot f))\colon L(g \cdot f) \to g$, and then the canonical diagonal filler $p_{g\cdot f}$ of the square $(1_A, a)\colon L(g \cdot f) \to f$, as depicted in the diagram. It can be shown without much problem that this is an R-algebra structure on $g \cdot f$.

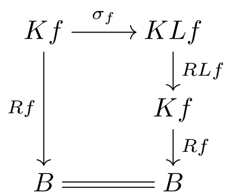


We write $(g, p_g) \bullet (f, p_f)$ for the R-algebra $(g \cdot f, p_{g\cdot f})$ described. This operation on pairs of composable R-algebras can be shown to be associative and has identities $(1_A, R1_A)$, so there is a double category $\mathsf{R}\text{-}\mathbb{A}\mathrm{lg}$. The forgetful functor from R-algebras forms part of a double functor, depicted on the right hand of (3.1).

3.b. **From double categories to AWFSs.** Suppose that R is a codomain-preserving monad on $\mathcal{C}^{\mathbf{2}}$, with associated codomain preserving copointed endofunctor (L, Φ) on $\mathcal{C}^{\mathbf{2}}$. Each double category structure on $\mathsf{R}\text{-Alg} \rightrightarrows \mathcal{C}$ that is compatible with the composition of morphisms in $\mathcal{C}$ induces a comultiplication $\Sigma\colon L \Rightarrow L^2$ that makes $\mathsf{L} = (L, \Phi, \Sigma)$ into a comonad and (L, R) an AWFS. In the following we explain how to construct the comultiplication from the double category structure.

If (f, p_f) and (g, p_g) are R-algebras with $\mathrm{cod}(f) = \mathrm{dom}(g)$, the double category structure provides for a vertical composition $(g, p_g) \bullet (f, p_f) = (g \cdot f, p_g \bullet p_f)$ with underlying morphism $g \cdot f$. The identities for the vertical composition are the R-algebras $(1, R1)$. Morphisms of R-algebras can be vertically composed too: given such morphisms $(h, k)\colon f \to g$ and $(k, \ell)\colon f' \to g'$, then (h, ℓ) is a morphism $f' \bullet f \to g' \bullet g$.

The comultiplication $\Sigma_f = (1, \sigma_f)\colon Lf \to L^2 f$ can be constructed from the double category structure in the following manner. Consider the morphism of R-algebras $(\sigma_f, 1)\colon Rf \to Rf \bullet RLf$ that corresponds under free R-algebra adjunction to the morphism $(L^2 f, 1)\colon f \to Rf \cdot RLf$ in $\mathcal{C}^{\mathbf{2}}$.



$$\sigma_f \cdot Lf = L^2 f$$

The detail of the proof that the components σ_f yield a comultiplication for L can be found in [**?**, Prop. 4].

4. Lax orthogonal AWFSs

This section introduces the fundamental definition of this work, lax orthogonal AWFSs, and describes the most basic 2-categorical example. Before all that, we shall recall some facts about lax idempotent 2-monads.

4.a. **2-monads.** We shall assume throughout the paper that the reader is familiar with the basic notions of 2-category, 2-functor, 2-natural transformation and modification. Familiarity with 2-(co)monads shall also be assumed, but we can take this opportunity to remind the reader of the definitions; a complete account can be found in [**?**]. A 2-monad $\mathsf{T} = (T, i, m)$ on a 2-category $\mathscr{K}$ is a 2-functor $T\colon \mathscr{K} \to \mathscr{K}$ with 2-natural transformations $i\colon 1_{\mathscr{K}} \Rightarrow T$, called the unit, and $m\colon T^2 \Rightarrow T$, called the multiplication, that satisfy the usual axioms of a monad; in other words, the underlying functor of T with the underlying natural transformations of i and m form an ordinary monad on the underlying category of $\mathscr{K}$. The definition of 2-comonad is dual.

An algebra for the 2-monad T is, by definition, an algebra for its underlying monad. This amounts to an object A with a morphism $a\colon TA \to A$ that satisfies the usual algebra axioms – the 2-cells play no role here. We shall usually be concerned with the so-called *strict morphisms* of T-algebras, which are the morphisms of algebras for the underlying monad of T; ie a strict morphism $(A, a) \to (B, b)$ is a morphism $f\colon A \to B$ in $\mathscr{K}$ such that $b \cdot Tf = f \cdot a$. However, the 2-dimensional aspect of $\mathscr{K}$ enable us to speak of *lax morphisms*, which are morphisms $f\colon A \to B$ equipped with a 2-cell $\bar{f}\colon b \cdot Tf \Rightarrow f \cdot a$ that must satisfy certain coherence axioms. For example, (lax) monoidal functors are examples of lax morphisms for a certain 2-monad. There is a dual notion of *oplax morphism*, which is a morphism $f\colon A \to B$ with a 2-cell $\bar{f}\colon f \cdot a \Rightarrow b \cdot Tf$ that must satisfy coherence axioms. A lax morphism $(f, \bar{f})$ whose 2-cell $\bar{f}$ is invertible is said to be a *pseudomorphism*.

The four types of morphisms described in the previous paragraph are the morphisms of four 2-categories, all with the T-algebras as objects: $\mathsf{T}\text{-Alg}_s$ has the strict morphisms as morphisms; $\mathsf{T}\text{-Alg}_\ell$ has the lax morphisms as morphisms; $\mathsf{T}\text{-Alg}_c$ has the oplax morphisms as morphisms; and $\mathsf{T}\text{-Alg}$ has the pseudomorphisms as morphisms.

A useful fact about adjunctions and (op)lax morphisms is the so-called *doctrinal adjunction* theorem, of which we state the version that we will use later.

Proposition 4.1. *Let T be a 2-monad on $\mathscr{K}$. An oplax morphism $(f, \bar{f})\colon (A, a) \to (B, b)$ between T-algebras has a left adjoint in the 2-category $\mathsf{T}\text{-Alg}_c$ if and only if f has a left adjoint in $\mathscr{K}$ and $\bar{f}$ is invertible.*

4.b. **Lax idempotent 2-monads.** An essential part of our definition of lax orthogonal AWFS is the concept of a lax idempotent 2-monad, or KZ 2-monad, that we recount in this section. We begin by introducing some space-saving terminology. Suppose given an adjunction $f \dashv g$ in a 2-category, with unit $\eta\colon 1 \Rightarrow g \cdot f$ and counit $\varepsilon\colon f \cdot g \Rightarrow 1$. We say that $f \dashv g$ is a *retract (coretract)* adjunction when the counit (unit) is an identity 2-cell.

Definition 4.2. A 2-monad $\mathsf{T} = (T, i, m)$ on a 2-category $\mathscr{K}$ is *lax idempotent*, or *Kock-Zöberlein*, or simply KZ, if any of the following equivalent conditions hold.

(i) $Ti \dashv m$ with identity unit (coretract adjunction).
(ii) $m \dashv iT$ with identity counit (retract adjunction).

(iii) Each T-algebra structure $a\colon TA \to A$ on an object A is part of an adjunction $a \dashv i_A$ with identity counit (retract adjunction).
(iv) There is a modification $\delta\colon Ti \Rightarrow iT$ satisfying $\delta \cdot i = 1$ and $m \cdot \delta = 1$.
(v) The forgetful 2-functor $U_\ell\colon \mathsf{T}\text{-Alg}_\ell \to \mathscr{K}$ is fully faithful.
(vi) For any pair of T-algebras A, B, every morphism $f\colon UA \to UB$ in $\mathscr{K}$ admits a unique structure of a lax morphism of T-algebras.
(vii) For any morphism $f\colon X \to A$ into a T-algebra (A, a), the identity 2-cell exhibits $a \cdot Tf$ as a left extension of f along i_X.

The conditions (i), (ii) and (iv) appeared in [**?**] and [**?**], albeit in a slightly different context; [**?**] shows the equivalence of these three conditions. The proof of the equivalence of the whole list, in the case of a 2-monad, can be found in [**?**].

Being lax idempotent can be regarded as a property of the 2-monad, since, for example, there can exist at most one counit for the adjunction in (i).

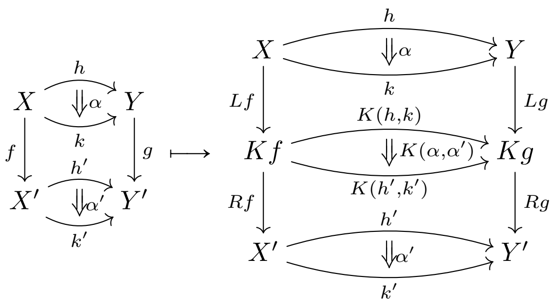
It may be useful to say a few words about how to obtain a left extension from the modification δ. If $f\colon X \to A$ and $g\colon TX \to A$ are morphisms into a T-algebra (A, a), and $\alpha\colon f \Rightarrow g \cdot i_X$ a 2-cell, then the corresponding 2-cell $a \cdot Tf \Rightarrow g$ is constructed as $(a \cdot Tg \cdot \delta_X) \cdot (a \cdot T\alpha)$.

Definition 4.3. A 2-comonad $\mathsf{G} = (G, e, d)$ on $\mathscr{K}$ is *lax idempotent*, or KZ, if the 2-monad $(G^{\mathrm{op}}, e^{\mathrm{op}}, d^{\mathrm{op}})$ on $\mathscr{K}^{\mathrm{op}}$ is lax idempotent. This means that we have conditions dual to the ones spelled out above for 2-monads; eg adjunctions $eG \dashv d \dashv Ge$, a modification $\delta\colon Ge \Rightarrow eG$, etc. We state one of the conditions in full: given a morphism $f\colon A \to X$ from a G-coalgebra (A, s), the identity 2-cell exhibits $Gf \cdot s\colon A \to GX$ as a left lifting of f through e_X.

4.c. **Definition and basic properties of lax orthogonal AWFSs.** A lax orthogonal AWFS will be, first of all, an AWFS on a 2-category. We shall start, thus, with the definition of 2-functorial factorisations and AWFSs on 2-categories.

Definition 4.4. A *2-functorial factorisation* on a 2-category $\mathscr{K}$ is a 2-functor that is a section of the 2-functor $\mathscr{K}^{\mathbf{3}} \to \mathscr{K}^{\mathbf{2}}$ that sends a pair of composable morphisms to its composition.

A 2-functorial factorisation on $\mathscr{K}$ induces a functorial factorisation $f \mapsto Rf \cdot Lf$ on the underlying ordinary category of $\mathscr{K}$, and in addition it factorises 2-cells, as depicted.



There is a bijection between the family of 2-functorial factorisations and the family of copointed endo-2-functors $\Phi\colon L \Rightarrow 1$ of $\mathscr{K}^{\mathbf{2}}$ with $\operatorname{dom}\Phi_f = 1_{\operatorname{dom}(f)}$; and also the family of pointed endo-2-functors $\Lambda\colon 1 \Rightarrow R$ with $\operatorname{cod}\Lambda_f = 1_{\operatorname{cod}(f)}$, for all $f \in \mathscr{K}^{\mathbf{2}}$.

Definition 4.5. An AWFS on a 2-category $\mathscr{K}$ consists of a pair (L, R) formed by a 2-comonad and a 2-monad on $\mathscr{K}^{\mathbf{2}}$ satisfying the same properties as AWFSs on categories. More explicitly,

- the domain of the counit $\Phi\colon L \Rightarrow 1$ is an identity morphism;

- the codomain of the unit $\Lambda\colon 1 \Rightarrow R$ is an identity morphism;
- Both (L, Φ) and (R, Λ) must give rise to the same 2-functorial factorisation on $\mathscr{K}$;
- the 2-natural transformation Δ of (2.4) must be a distributive law between the underlying comonad of L and the underlying monad of R on the ordinary underlying category of $\mathscr{K}$.

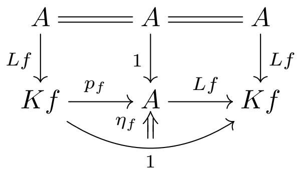
Definition 4.6. An AWFS (L, R) in a 2-category $\mathscr{K}$ is *lax orthogonal* if the 2-comonad L and the 2-monad R are lax idempotent.

We will later see in Section 5 that it is enough to require that either L *or* R be lax idempotent.

Remark 4.7. It was observed in Remark 2.11 that the transformation $(1, p)\colon L \cdot V \to \mathrm{id} \cdot \mathrm{dom} \cdot V$ of Remark 2.10 has as right inverse $(1, L) \cdot V$, where $(1, L)$ is the transformation of Remark 2.9. We claim that, when the 2-monad R is lax idempotent, we also have a retract adjunction in the 2-category $\mathbf{2\text{-}Cat}(\mathsf{R}\text{-}\mathrm{Alg}_s, \mathscr{K}^{\mathbf{2}})$ of 2-functors, 2-natural transformations and modifications

$$(1, p) \dashv (1, L) \cdot V \colon \mathrm{id} \cdot \mathrm{dom} \cdot V \Longrightarrow L \cdot V.$$

The counit of this adjunction is the identity modification, and the unit has components



for $f \in \mathsf{R}\text{-}\mathrm{Alg}_s$, where η_f is the domain component of the unit of the adjunction $(p_f, 1) \dashv (Lf, 1)$ provided by the fact that R is lax idempotent – numeral (iii) of Definition 4.2. The fact that this defines a modification with components $(1, \eta_f)$ follows, and clearly satisfies the triangle identities.

4.d. **A basic example.** There is a lax orthogonal AWFS that will play the role analogous to the role that the OFS $(\mathrm{Iso}, \mathrm{Mor})$ plays in the context of simple reflections – as explained in Section 1. The next few pages give a complete description of this basic example of a lax orthogonal AWFS.

Every functor $f\colon A \to B$ factors as $Lf\colon A \to Kf = (f \downarrow B)$ followed by $Rf\colon Kf \to B$, where $Lf(a) = (a, 1\colon f(a) \to f(a), f(a))$, and $Rf(a, \beta\colon f(a) \to b, b) = b$. The associated pointed endofunctor R on $\mathbf{Cat}^{\mathbf{2}}$ given by $f \mapsto Rf$ underlies the free split opfibration monad R. Precisely the same factorisation can be constructed in any 2-category $\mathscr{K}$ with the necessary comma objects. At this point one could deduce that there is an AWFS (L, R) by observing that split opfibrations compose and the results cited in Section 3, and furthermore, one could use the results of Section 5 to prove that the AWFS is lax orthogonal. Instead, we shall give an explicit description of the comonad and its coalgebras, as they will become important in later sections.

Given a 2-category $\mathscr{K}$ we can perform two constructions to obtain new 2-categories. The first is the 2-category $\mathbf{Lari}(\mathscr{K})$, whose objects are morphisms f in $\mathscr{K}$ equipped with a right adjoint coretract r_f, ie a left adjoint right inverse or LARI, in the terminology used in [**?**]; we may write an object of this 2-category as (f, r), omitting the unit and counit of the adjunction, since the unit is an identity 2-cell and the counit is, therefore, the unique 2-cell that satisfies the adjunction triangle axioms for $f \dashv r$. A morphism $(f, r) \to (f', r')$ in $\mathbf{Lari}(\mathscr{K})$ is a morphism $(h, k)\colon f \to f'$ in $\mathscr{K}^{\mathbf{2}}$ such that $r' \cdot k = h \cdot r$. It is not difficult to show that (h, k) is automatically compatible with the counits of the adjunctions: if the counits are ε and ε', then $\varepsilon' \cdot k = k \cdot \varepsilon$. The 2-cells between morphisms in $\mathbf{Lari}(\mathscr{K})$ are just the 2-cells in between them in $\mathscr{K}^{\mathbf{2}}$. There is a forgetful 2-functor $\mathbf{Lari}(\mathscr{K}) \to \mathscr{K}^{\mathbf{2}}$, and LARIs can be composed via the usual composition of adjunctions, so if (f, r) and (f', r') are LARIs with f and f' composable morphisms, then $(f' \cdot f, r \cdot r')$ is canonically a LARI. In this way, $\mathbf{Lari}(\mathscr{K})$ has a double category structure, and furthermore, the composition is obviously compatible with 2-cells, so the double category structure extends to an internal category in the category of 2-categories.

The second construction is a 2-category $\mathbf{OpFib}(\mathscr{K})$ of split opfibrations in $\mathscr{K}$, by which we mean morphisms f in $\mathscr{K}$ such that each functor $\mathscr{K}(-, f)$ is a split opfibration in $[\mathscr{K}^{\mathrm{op}}, \mathbf{Cat}]$. There is a forgetful 2-functor $\mathbf{OpFib}(\mathscr{K}) \to \mathscr{K}^{\mathbf{2}}$ and the composition of two split opfibrations is canonically a split opfibration, so $\mathbf{OpFib}(\mathscr{K})$ is an internal category in the category of 2-categories.

If the 2-category $\mathscr{K}$ has enough comma objects, then there is an AWFS (L, R) on $\mathscr{K}$ that satisfies $\mathbf{OpFib}(\mathscr{K}) \cong \mathsf{R}\text{-Alg}_s$, and, as we will see in Proposition 4.11, $\mathbf{Lari}(\mathscr{K}) \cong \mathsf{L}\text{-Coalg}_s$. Let us first say a few words about R. The free split opfibration on f is given by a comma object as depicted on the left hand side of (4.8). The unit of R has components $\Lambda_f = (Lf, 1)$, where $Lf \colon A \to Kf$ is the unique morphism such that $Rf \cdot Lf = f$, $q_f \cdot Lf = 1$ and $\nu_f \cdot Lf = 1$. The multiplication $\Pi_f = (\pi_f, 1)$ is given by the unique morphism $\pi_f \colon KRf \to Kf$ satisfying the three equalities depicted on the right hand side.

$$\begin{array}{ccc} Kf & \xrightarrow{q_f} & A \\ {\scriptstyle Rf}\downarrow & \not\Leftarrow \nu_f & \downarrow{\scriptstyle f} \\ B & = & B \end{array} \qquad q_f \cdot \pi_f = q_f \cdot q_{Rf} \qquad Rf \cdot \pi_f = R^2 f \qquad \nu_f \cdot \pi_f = \nu_{Rf}(\nu_f \cdot q_{Rf}) \tag{4.8}$$

We remark that Lf comes equipped with an adjunction $Lf \dashv q_f$ with identity unit, where $q_f \colon Kf \to A$ is the projection. The counit $\omega_{Lf} \colon Lf \cdot q_f \Longrightarrow 1$ is the 2-cell induced by the universal property of comma objects and the conditions

$$q_f \cdot \omega_{Lf} = 1 \colon q_f \Longrightarrow q_f \quad \text{and} \quad Rf \cdot \omega_{Lf} = \nu_f \colon Rf \cdot Lf \cdot q_f = f \cdot q_f \Longrightarrow Rf.$$

The copointed endo-2-functor L underlies a 2-comonad with comultiplication $\Sigma \colon L \Rightarrow L^2$, defined by the following equality and the universal property of comma objects.

$$\begin{array}{ccccc} Kf & \xrightarrow{\sigma_f} & KLf & \xrightarrow{q_{Lf}} & A \\ & & {\scriptstyle RLf}\downarrow & \not\Leftarrow \nu_{Lf} & \downarrow{\scriptstyle Lf} \\ & & Kf & = & Kf \end{array} \;=\; \begin{array}{ccc} Kf & \xrightarrow{q_f} & A \\ \| & \not\Leftarrow \omega_{Lf} & \downarrow{\scriptstyle Lf} \\ Kf & = & Kf \end{array} \tag{4.9}$$

The 2-monad R is well-known to be lax idempotent. To see that the comonad L is lax idempotent, one can exhibit an adjunction $\Phi_{Lf} \dashv \Sigma_f$ with identity counit. The existence of an adjunction $RLf \dashv \sigma_f$, with identity counit follows from Remark 4.10 below. The fact that this adjunction yields an adjunction $\Phi_{Lf} \dashv \Sigma_f$ in $\mathscr{K}^{\mathbf{2}}$ can be readily checked. We leave the verification of the distributivity law between the 2-comonad and 2-monad to the reader.

Remark 4.10. Given a comma object as exhibited on the left below, each adjunction $\ell \dashv r$ induces a retract adjunction $p \dashv s$, where s is defined by the equality on the right hand side.

$$\begin{array}{ccc} \ell \downarrow t & \xrightarrow{q} & A \\ {\scriptstyle p}\downarrow & \not\Leftarrow \nu & \downarrow{\scriptstyle \ell} \\ X & \xrightarrow[t]{} & B \end{array} \qquad\qquad \begin{array}{ccccc} X & \xrightarrow{s} & \ell \downarrow t & \xrightarrow{q} & A \\ & & {\scriptstyle p}\downarrow & \not\Leftarrow \nu & \downarrow{\scriptstyle \ell} \\ & & X & \xrightarrow[t]{} & B \end{array} \;=\; \begin{array}{ccccc} X & \xrightarrow{t} & B & \xrightarrow{r} & A \\ & & & {\scriptstyle 1}\searrow \;\not\Leftarrow \varepsilon & \downarrow{\scriptstyle \ell} \\ & & & & B \end{array}$$

The unit $\eta\colon 1 \Rightarrow s \cdot p$ is the unique 2-cell satisfying $p \cdot \eta = 1$ and

$$\ell \downarrow t \underset{s\cdot p}{\overset{1}{\underset{\Downarrow\eta}{\rightrightarrows}}} \ell \downarrow t \xrightarrow{q} A \quad = \quad \begin{array}{ccccc} \ell \downarrow t & \xrightarrow{q} & A & = & A \\ {\scriptstyle p}\downarrow & \not{\Downarrow}\,\nu & \downarrow{\scriptstyle \ell} & \Downarrow & \nearrow{\scriptstyle r} \\ X & \xrightarrow[t]{} & B & & \end{array}$$

We make a final observation that will be of use later on. Suppose that the unit of $\ell \dashv r$ is an identity and $h\colon Z \to \ell \downarrow t$ is any morphism such that $\nu \cdot h$ is an identity 2-cell. Then $\eta \cdot h$ is an identity 2-cell.

Proposition 4.11. *Let* (L, R) *be the* AWFS *described above in this section.*

1. *There is an isomorphism over* $\mathscr{K}^{\mathbf{2}}$ *between* (L,Φ)-Coalg *(the 2-category of coalgebras for the copointed endo-2-functor* (L,Φ)*) and the 2-category with*
   - *Objects* (f, v, ξ) *where* $f\colon A \rightleftarrows B : v$ *and* $\xi\colon f \cdot v \Rightarrow 1$ *that satisfy* $v \cdot f = 1_A$ *and* $\xi \cdot f = 1$ *– strong deformation retracts of* B.
   - *Morphisms* $(f, v, \xi) \to (f', v', \xi')$*, morphisms* $(h, k)\colon f \to f'$ *in* $\mathscr{K}^{\mathbf{2}}$ *such that* $h \cdot v = v' \cdot k$ *and* $\xi' \cdot k = k \cdot \xi$.
   - *2-cells* $(h, k) \Rightarrow (\bar{h}, \bar{k})\colon (f, v, \xi) \to (f', v', \xi')$*, 2-cells* $(\alpha, \beta)\colon (h, k) \Rightarrow (\bar{h}, \bar{k})$ *in* $\mathscr{K}^{\mathbf{2}}$ *such that* $\alpha \cdot v = v' \cdot \beta$.
2. *There is an isomorphism over* $\mathscr{K}^{\mathbf{2}}$ *between* L-Coalg_s *and the 2-category* $\mathbf{Lari}(\mathscr{K})$.
3. *Cofree* L*-coalgebras correspond to the coretract adjunctions* $Lf \dashv q_f$.
4. *The double category structure on* L-Coalg_s *induced by this* AWFS *is that of* $\mathbf{Lari}(\mathscr{K})$*, ie given by composition of coretract adjunctions.*

Proof. The reader would recall from (4.8) the definition of Kf as a comma object. There is a bijection between morphisms $s\colon B \to Kf$ such that $Rf \cdot s = 1_B$ and morphisms $v\colon B \to A$ equipped with a 2-cell $\xi\colon f \cdot v \Rightarrow 1_B$; the bijection is given by composing with the comma object ν_f, ie $v = q_f \cdot s$ and $\xi = \nu_f \cdot s$. Under this bijection, the condition $s \cdot f = Lf$, which means that $(1, s)$ is a morphism $f \to Lf$, translates into $\xi \cdot f = 1$. This completes the description of (L,Φ)-coalgebras.

Next we translate the condition $\sigma_f \cdot s = K(1, s) \cdot s$, that is the coassociativity axiom for that makes an (L,Φ)-coalgebra into an L-coalgebra. Denote the counit of $Lf \dashv q_f$ by ω_f, and recall that σ_f is defined by (4.9). The morphism $\sigma_f \cdot s$ corresponds under the universal property of the comma object ν_{Lf} to the 2-cell

$$\begin{array}{ccccccc} B & \xrightarrow{s} & Kf & \xrightarrow{\sigma_f} & KLf & \xrightarrow{q_{Lf}} & A \\ & & & & {\scriptstyle RLf}\downarrow & \not{\Downarrow}\,\nu_{Lf} & \downarrow{\scriptstyle Lf} \\ & & & & Kf & = & Kf \end{array} \quad = \quad \begin{array}{ccccc} B & \xrightarrow{s} & Kf & \xrightarrow{q_f} & A \\ & & \| & \not{\Downarrow}\,\omega_{Lf} & \downarrow{\scriptstyle Lf} \\ & & Kf & = & Kf \end{array} \tag{4.12}$$

while $K(1, s) \cdot s$ corresponds to the 2-cell displayed below.

$$\begin{array}{ccccccc} B & \xrightarrow{s} & Kf & \xrightarrow{K(1,s)} & KLf & \xrightarrow{q_{Lf}} & A \\ & & & & {\scriptstyle RLf}\downarrow & \not{\Downarrow}\,\nu_{Lf} & \downarrow{\scriptstyle Lf} \\ & & & & Kf & = & Kf \end{array} \quad = \quad \begin{array}{ccccccc} B & \xrightarrow{s} & Kf & \xrightarrow{q_f} & A & & \\ & & {\scriptstyle Rf}\downarrow & \not{\Downarrow}\,\nu_f & \downarrow{\scriptstyle f} & & \\ & & B & = & B & \xrightarrow[s]{} & Kf \end{array}$$

$$= \quad \begin{array}{ccccc} B & \xrightarrow{v} & A & & \\ & \diagdown\!\diagdown \;\not{\Downarrow}\,\xi & \downarrow{\scriptstyle f} & & \\ & & B & \xrightarrow[s]{} & Kf \end{array} \tag{4.13}$$

Therefore, s is a coalgebra precisely when (4.12) equals (4.13). These are both 2-cells between morphisms with codomain Kf, and as such they are equal if and only if their respective compositions with the projections Rf and q_f coincide. Their composition with Rf yield respectively

$$Rf \cdot \omega_{Lf} \cdot s = \nu_f \cdot s = \xi \qquad \text{and} \qquad Rf \cdot s \cdot \xi = \xi$$

while their composition with q_f yield respectively

$$q_f \cdot \omega_f \cdot s = 1 \qquad \text{and} \qquad q_f \cdot s \cdot \xi = v \cdot \xi.$$

It follows that s is coassociative if and only if $v \cdot \xi = 1$, completing the description of L-coalgebras as coretract adjunctions $f \dashv v$.

We now describe the morphisms of (L, Φ)-coalgebras from $(1, s)\colon f \to Lf$ to $(1, s')\colon f' \to Lf'$. Such a morphism is a morphism $(h, k)\colon f \to f'$ in $\mathscr{K}^{\mathbf{2}}$ satisfying $s' \cdot k = K(h, k) \cdot s$. Composing with the comma object $\nu_{f'}$, this equality translates into $v' \cdot k = h \cdot v$ and $\xi' \cdot k = k \cdot \xi$. A morphism of L-coalgebras is just a morphism between the underlying (L, Φ)-coalgebras.

A 2-cell between morphisms (h, k), $(\bar{h}, \bar{k})\colon (f, s) \to (f', s')$ of (L, Φ)-algebras is a pair of 2-cells $\alpha\colon h \Rightarrow \bar{h}$ and $\beta\colon k \Rightarrow \bar{k}$ satisfying $K(\alpha, \beta) \cdot s = s' \cdot \beta$. This is an equality of 2-cells between 1-cells with codomain Kf', so it holds if and only if it does after composing with the projections Rf' and $q_{f'}$. The composition of this equality with Rf' yields $\beta = \beta$ – no information here – while its composition with $q_{f'}$ yields $\alpha \cdot v = \beta \cdot v'$. This completes the description of (L, Φ)-Coalg. When (f, s) and (f', s') are L-algebras, with associated coretract adjunctions (f, v, ξ) and (f', v', ξ'), this latter equality is void too, since its mate automatically holds. Explicitly,

$$\big(h \cdot v \xrightarrow{\alpha \cdot v} \bar{h} \cdot v\big) = \big(h \cdot v = v' \cdot k \xrightarrow{v' \cdot \beta} v' \cdot \bar{k} = \bar{h} \cdot v\big)$$

holds if and only if it does after precomposing with f and composing with the unit $1 = v \cdot f$ of $f \dashv v$:

$$\alpha = \alpha \cdot v \cdot f = \big(h = h \cdot v \cdot f = v' \cdot k \cdot f \xrightarrow{v' \cdot \beta \cdot f} v' \cdot \bar{k} \cdot f = \bar{h} \cdot v \cdot f = \bar{h}\big).$$

But this latter equality automatically holds, by $\beta \cdot f = f' \cdot \alpha$. This shows that 2-cells in L-Coalg$_s$ are simply 2-cells in $\mathscr{K}^{\mathbf{2}}$.

Finally, we prove the fourth statement of the proposition. The 2-category of L-coalgebras is equipped with an obvious composition: that of coretract adjunctions. Any such composition corresponds to a unique multiplication $\bar{\Pi}\colon R^2 \to R$ that makes $(R, \Lambda, \bar{\Pi})$ a 2-monad and satisfies the distributivity condition – see Section 3 for the details. We have to show that $\bar{\Pi}$ equals the multiplication Π of the free split opfibration 2-monad.

By the comments at the end of Section 3.b, or rather the dual version of those comments, $\bar{\Pi}_f = (\bar{\pi}_f, 1)$ is defined by the property that $(1, \bar{\pi}_f)$ is the unique morphism of L-coalgebras from $L(Rf) \bullet Lf$ to Lf that composed with the counit $\Phi_f = (1, Rf)\colon Lf \to f$ yields the morphism $(1, R^2 f)\colon LRf \cdot Lf \to f$ in $\mathscr{K}^{\mathbf{2}}$.

$$\begin{array}{ccc} A & = & A \\ {\scriptstyle Lf}\downarrow & & \\ Kf & & \downarrow{\scriptstyle Lf} \\ {\scriptstyle LRf}\downarrow & & \\ KLf & \xrightarrow{\bar{\pi}_f} & Kf \end{array} \tag{4.14}$$

By the previous parts of the present proposition, to say that $(1, \bar{\pi}_f)$ is a morphism of L-coalgebras is equivalent to saying that 1_A and $\bar{\pi}_f$ form a commutative square with the right adjoints of Lf and of $LRf \cdot Lf$. It is worth keep in mind that composition of L-coalgebras that induces $\bar{\pi}_f$ is the usual composition of coretract adjunctions, so the right adjoint of Lf is q_f and the right adjoint of $LRf \cdot Lf$

is $q_f \cdot q_{Rf}$. It follows that, to say that $(1, \bar{\pi}_f)$ is a morphism of L-coalgebras is equivalent to requiring the following equality.

$$q_f \cdot \bar{\pi}_f = q_f \cdot q_{Rf}$$

So far we have unravelled the definition of $\bar{\pi}_f$. In order to deduce that $\bar{\pi}_f$ equals the multiplication π_f of the free split opfibration 2-monad, it suffices to verify that $(1, \pi_f)$ is too a morphism of L-coalgebras of the form (4.14) and $\Phi_f \cdot (1, \pi_f) = (1, R^2 f)$. The latter equation always holds, as it is $Rf \cdot \pi_f = R^2 f$, as remarked in the equation (4.8). The fact that $(1, \pi_f)$ is a morphism of L-coalgebras is, by the same argument applied to $\bar{\pi}_f$, the condition $q_f \cdot \pi_f = q_f \cdot q_{Rf}$, which holds again by (4.8). By uniqueness of $\bar{\pi}_f$, we obtain $\pi_f = \bar{\pi}_f$ and thus the vertical composition of LARIs coincides with that of L-coalgebras. □

Remark 4.15. In general, for a copointed endofunctor (G, ε) on a category $\mathcal{C}$, and a retraction $r\colon Y \twoheadrightarrow X$ with section s in $\mathcal{C}$, each (G, ε)-coalgebra structure $\delta\colon Y \to GY$ on Y induces another on X. This induced coalgebra structure is $(Gr)\cdot\delta\cdot s\colon X \to GX$. Later, in the proof of Proposition 8.19, we shall need the description of this construction in the case of the copointed endo-2-functor (L, Φ) of Proposition 4.11. Let (f, v, ξ) be a coalgebra and $(r_0, r_1)\colon f \to \bar{f}$ a retraction on $\mathscr{K}^{\mathbf{2}}$ with section (s_0, s_1). The induced coalgebra structure $(\bar{f}, \bar{v}, \bar{\xi})$ is given by $\bar{v} = r_0 \cdot v \cdot s_1$ and

$$\bar{f} \cdot \bar{v} = \bar{f} \cdot r_0 \cdot v \cdot s_1 = r_1 \cdot f \cdot v \cdot s_1 \xRightarrow{r_1 \cdot \xi \cdot s_1} r_1 \cdot s_1 = 1.$$

5. The 2-comonad is lax idempotent if the 2-monad is so

In this section we show that, in order for an AWFS on a 2-category to be lax orthogonal, it suffices that *either* its 2-monad *or* its 2-comonad be lax idempotent. This result can be seen as a two-dimensional generalisation of the fact that an AWFS on a category is orthogonal if either its monad or its comonad is idempotent – a fact that is explained in [?]. However, the proof, as it is to be expected, is more involved. Incidentally, our proof uses the double category structure on $\mathsf{R}\text{-Alg}_s$ mentioned in Section 3.

Theorem 5.1. *The 2-comonad of an AWFS on a 2-category is lax idempotent provided the 2-monad is lax idempotent.*

Proof. Denote the AWFS on the 2-category $\mathscr{K}$ by (L, R), where L is a 2-comonad with counit $\Phi\colon L \Rightarrow 1$ and comultiplication $\Sigma\colon L \Rightarrow L^2$, and R is a 2-monad with unit $\Lambda\colon 1 \Rightarrow R$ and multiplication $\Pi\colon R^2 \Rightarrow R$. We will verify one of the equivalent conditions that make L a lax idempotent 2-comonad – the corresponding conditions for a 2-monad are mentioned in Section 4.b – namely, that there exist coretract adjunctions $\Sigma_f \dashv L\Phi_f$ whose counits form a modification in $f\colon A \to B$. In Section 3.b we mentioned that $(\sigma_f, 1_B)$ is a morphism of R-algebras $Rf \to Rf \bullet RLf$, where the codomain is the vertical composition of the R-algebras RLf and Rf. Consider the morphism

$$RLf \xrightarrow{R\Phi_f} Rf \xrightarrow{(\sigma_f, 1_B)} Rf \bullet RLf$$

which is, by Section 4.b, a left extension along Λ_{Lf} of its composition with the unit Λ_{Lf}

$$(\sigma_f, 1_B) \cdot R\Phi_f \cdot \Lambda_{Lf} = (\sigma_f, 1_B) \cdot \Lambda_f \cdot \Phi_f = (L^2 f, Rf)\colon Lf \longrightarrow Rf \cdot RLf.$$

The morphism $(1_{KLf}, Rf)\colon RLf \to Rf \cdot RLf$ in $\mathscr{K}^{\mathbf{2}}$ satisfies $(1_{KLf}, Rf) \cdot \Lambda_{Lf} = (L^2 f, Rf)$ too, therefore the universal property of left extensions gives a unique 2-cell $(\sigma_f, 1_B) \cdot R\Phi_f \Rightarrow (1_{KLf}, Rf)$ in $\mathscr{K}^{\mathbf{2}}$ whose composition with Λ_{Lf} is the identity

2-cell. This forces the 2-cell to be of the form

$$(\varepsilon_f, 1_{1_B})\colon (\sigma_f, 1_B)\cdot R\Phi_f \Longrightarrow (1_{KLf}, Rf) \tag{5.2}$$

for a 2-cell in $\mathscr{K}$

$$\varepsilon_f\colon \sigma_f \cdot K(1_A, Rf) \Longrightarrow 1_{KLf}\colon KLf \to KLf$$

since the codomain component of Λ_f is an identity. This definition makes $(\varepsilon_f, 1_{1_B})$, and hence ε_f, a modification in f, a fact that can be verified by using the universal property of left extensions.

We now proceed to prove that ε_f is the counit of a coretract adjunction $\sigma_f \dashv K(1_A, Rf)$ in $\mathscr{K}$, for which we must show three conditions:

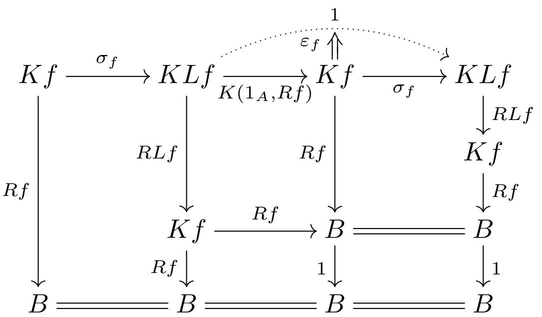
$$\varepsilon_f \cdot \sigma_f = 1 \qquad K(1_A, Rf)\cdot \varepsilon_f = 1 \qquad \varepsilon_f \cdot L^2 f = 1. \tag{5.3}$$

The first two conditions are the triangle identities of the adjunction, while the last one means that ε_f is a 2-cell in $\mathscr{K}^{\mathbf{2}}$.

Consider the morphism of R-algebras

$$Rf \xrightarrow{(\sigma_f, 1_B)} Rf \bullet RLf \xrightarrow{(Rf, 1_B)\bullet R\Phi_f} 1_B \bullet Rf \xrightarrow{1_B \bullet (\sigma_f, 1_B)} 1_B \bullet Rf \bullet RLf$$

that can be depicted in the way of the following diagram – of solid arrows – where, as always, objects of $\mathscr{K}^{\mathbf{2}}$ are represented by vertical arrows and morphisms of $\mathscr{K}^{\mathbf{2}}$ by commutative squares.



(5.4)

This morphism is equal to $(\sigma_f, 1_B)$, since $K(1_A, Rf)\cdot \sigma_f = 1$. Now consider the dotted identity arrow and the 2-cell ε_f in (5.4), observing that it defines a 2-cell

$$(\varepsilon_f, 1)\colon (\sigma_f, 1)\cdot R\Phi_f \Longrightarrow 1$$

in $\mathscr{K}^{\mathbf{2}}$, and, upon precomposing with σ_f, a 2-cell

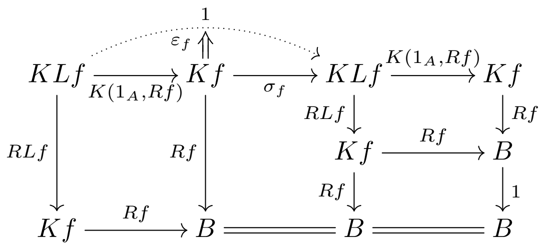
$$(\varepsilon_f \cdot \sigma_f, 1_{1_B})\colon (\sigma_f, 1_B) \Longrightarrow (\sigma_f, 1_B) \tag{5.5}$$

with equal domain and codomain. This 2-cell (5.5) precomposed with $\Lambda_f\colon f \to Rf$ equals the identity, for $\sigma_f \cdot Lf = L^2 f$ and $\varepsilon_f \cdot L^2 f = 1$ by definition of ε_f. Since $(\sigma_f, 1)$ is a left extension of $(\sigma_f, 1)\cdot \Lambda_f$ along Λ_f, we must have $(\varepsilon_f \cdot \sigma_f, 1_{1_B}) = 1$, the first of the equalities (5.3).

In order to prove the second equality of (5.3), consider the morphism of R-algebras

$$RLf \xrightarrow{R\Phi_f} RL \xrightarrow{(\sigma_f, 1_B)} Rf \bullet RLf \xrightarrow{(Rf, 1_B)\bullet R\Phi_f} 1_B \bullet Rf$$

that can be depicted as in the following diagram – of solid arrows.



This morphism equals $(K(1_A, Rf), 1)$, since $K(1_A, Rf) \cdot \sigma_f = 1$ by the counit axiom of the comonad L. If we now consider the dotted identity arrow, the 2-cell ε_f induces an endo-2-cell

$$(K(1_A, Rf) \cdot \varepsilon_f, 1_{Rf}) \colon (K(1_A, Rf), Rf) \Longrightarrow (K(1_A, Rf), Rf), \tag{5.6}$$

which, by definition of ε_f (5.2), equals the identity when precomposed with Λ_{Lf}. The morphism $(K(1_A, Rf), Rf)$ is a morphism of R-algebras, and hence a left extension along Λ_{Lf}, from where we deduce that (5.6) must be the identity 2-cell. That is, $K(1_A, Rf) \cdot \varepsilon_f = 1$, the second equality of (5.3).

All that remains to verify is $\varepsilon_f \cdot L^2 f = 1$, but this is part of the definition of ε_f, completing the proof. □

6. Lifting operations

We turn to the second part of the article where we put the emphasis on *lifting operations* and their relationship to AWFSs. In this section and the next we leave the case of 2-categories and return to the framework of ordinary categories. After setting out our own approach to lifting operations, we recall a number of notions known for ordinary AWFS. This is a necessary step previous to extending these notions to lax idempotent AWFSs from Section 8 onwards.

6.a. Background on modules.

As a preamble to the next section, let us briefly remind the reader about the language of modules or profunctors, which will be heavily used henceforth.

Definition 6.1. A *module* or *profunctor* ϕ from a category $\mathcal{A}$ to a category $\mathcal{B}$, denoted by $\phi \colon \mathcal{A} \to\!\!\!\!\!\!\mapsto \mathcal{B}$, is a functor $\mathcal{B}^{\mathrm{op}} \times \mathcal{A} \to \mathbf{Set}$, and a module morphism is a natural transformation. Given another module $\psi \colon \mathcal{B} \to\!\!\!\!\!\!\mapsto \mathcal{C}$, the composition $\psi \cdot \phi$ is defined by the coend formula $(\psi \cdot \phi)(c, a) = \int^B \psi(c, b) \times \phi(b, a)$; the identity $1_{\mathcal{A}}$ for this composition is given by $1_{\mathcal{A}}(a, a') = \mathcal{A}(a, a')$. In this way we obtain a bicategory **Mod**.

Each functor $F \colon \mathcal{A} \to \mathcal{B}$ induces two modules F_* and F^* given by $F_*(b, a) = \mathcal{B}(b, Fa)$ and $F^*(a, b) = \mathcal{B}(Fa, b)$. Furthermore, there is an adjunction $F_* \dashv F^*$ with unit and counit given by components

$$\mathcal{A}(a, a') \xrightarrow{F} \mathcal{B}(Fa, Fa') \cong F^* \cdot F_*(a, a')$$

$$F_* \cdot F^*(b, b') = \int^A \mathcal{B}(b, Fa) \times \mathcal{B}(Fa, b') \xrightarrow{\mathrm{comp}} \mathcal{B}(b, b').$$

The coend form of the Yoneda lemma implies that $(\psi \cdot F_*)(a, c) \cong \psi(c, Fa)$ and $F^* \cdot \chi(a, d) \cong \chi(Fa, d)$, whenever these compositions of modules are defined.

Similarly, if $\alpha \colon F \Rightarrow G$ is a natural transformation between functors $\mathcal{A} \to \mathcal{B}$, then there are morphisms of modules $\alpha_* \colon F_* \to G_*$ and $\alpha^* \colon G^* \to F^*$, with components

$$\alpha_*(b, a) = \mathcal{B}(b, \alpha_a) \qquad \alpha^*(a, b) = \mathcal{B}(\alpha_a, b).$$

6.b. **Lifting operations.** Fix a category $\mathcal{C}$. Recall that there are adjunctions

$$\mathrm{cod} \dashv \mathrm{id} \dashv \mathrm{dom} \colon \mathcal{C}^{\mathbf{2}} \longrightarrow \mathcal{C} \tag{6.2}$$

the first of which has identity counit and the second of which has identity unit.

Define a module (profunctor) $\mathsf{Diag} \colon \mathcal{C}^{\mathbf{2}} \longrightarrow\!\!\!\!\!\!\shortmid\;\; \mathcal{C}^{\mathbf{2}}$ in the following way. Given two morphisms f, g in $\mathcal{C}$, $\mathsf{Diag}(f, g)$ is the set of commutative diagrams of the form

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ {\scriptstyle f}\downarrow & \overset{d}{\nearrow} & \downarrow{\scriptstyle g} \\ B & \xrightarrow[k]{} & D \end{array} \tag{6.3}$$

The action of $\mathcal{C}^{\mathbf{2}}$ on either side is simply by pasting the appropriate commutative square.

Lemma 6.4. *There are isomorphisms of modules between* Diag *and the following four modules* $\mathcal{C}^{\mathbf{2}} \longrightarrow\!\!\!\!\!\!\shortmid\;\; \mathcal{C}^{\mathbf{2}}$*.*

$$\mathrm{cod}^* \cdot \mathrm{dom}_* \qquad \mathrm{id}_* \cdot \mathrm{id}^* \qquad (\mathrm{id} \cdot \mathrm{dom})_* \qquad (\mathrm{id} \cdot \mathrm{cod})^*$$

Proof. The bijection $\mathsf{Diag}(f, g) \cong (\mathrm{cod}^* \cdot \mathrm{dom}_*)(f, g) \cong \mathcal{C}(\mathrm{cod}(f), \mathrm{dom}(g))$ is the obvious one, that sends a commutative square with a diagonal filler d as in (6.3) to the morphism d. Due to the adjunctions (6.2), $\mathrm{cod}^* \cong \mathrm{id}_*$ and $\mathrm{id}^* \cong \mathrm{dom}_*$, and we obtain isomorphisms of $\mathrm{cod}^* \cdot \mathrm{dom}_*$ with $(\mathrm{id} \cdot \mathrm{dom})_*$, and with $\mathrm{id}_* \cdot \mathrm{id}^*$, and with $\mathrm{cod}^* \cdot \mathrm{id}^* \cong (\mathrm{id} \cdot \mathrm{cod})^*$. ☐

The second isomorphism is the one induced by the fact that $\mathrm{id}^* \cong \mathrm{dom}_*$, as seen in (6.2). The isomorphism $\mathsf{Diag}(f, g) \cong (\mathrm{id} \cdot \mathrm{dom})_*(f, g) = \mathcal{C}^{\mathbf{2}}(f, 1_{\mathrm{dom}\, g})$ is given by

$$\begin{array}{ccc} \cdot & \xrightarrow{h} & \cdot \\ {\scriptstyle f}\downarrow & \overset{d}{\nearrow} & \downarrow{\scriptstyle g} \\ \cdot & \xrightarrow[k]{} & \cdot \end{array} \quad \longmapsto \quad \begin{array}{ccc} \cdot & \xrightarrow{h} & \cdot \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle 1_{\mathrm{dom}\, g}} \\ \cdot & \xrightarrow[d]{} & \cdot \end{array}$$

The counit of $\mathrm{id}_* \dashv \mathrm{id}^*$ is a module morphism

$$\mathsf{Diag} \longrightarrow 1_{\mathcal{C}^{\mathbf{2}}} \tag{6.5}$$

whose component at (f, g) sends the element (6.3) to the outer commutative square. It corresponds, under $\mathsf{Diag} \cong (\mathrm{id} \cdot \mathrm{dom})_*$, to the module morphism induced by the natural transformation with f-component

$$\mathrm{id} \cdot \mathrm{dom} \Longrightarrow 1_{\mathcal{C}^{\mathbf{2}}} \qquad \begin{array}{ccc} \cdot & = & \cdot \\ {\scriptstyle 1_{\mathrm{dom}\, f}}\downarrow & & \downarrow{\scriptstyle f} \\ \cdot & \xrightarrow[f]{} & \cdot \end{array}$$

Definition 6.6. Let $(\mathcal{A}, U)$, $(\mathcal{B}, V)$ be two objects of $\mathbf{Cat}/\mathcal{C}^{\mathbf{2}}$, and define a module

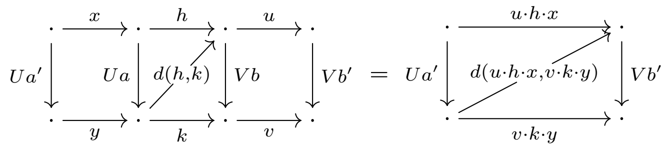
$$\mathsf{Diag}(U, V) \colon \mathcal{B} \xrightarrow{V_*} \mathcal{C}^{\mathbf{2}} \xrightarrow{\mathsf{Diag}} \mathcal{C}^{\mathbf{2}} \xrightarrow{U^*} \mathcal{A}. \tag{6.7}$$

The module morphism $\mathsf{Diag} \to 1$ induces another $\mathsf{Diag}(U, V) \to U^* \cdot V_*$. A *lifting operation* for U, V is a section for this module morphism, and amounts to a choice, for each square in $\mathcal{C}$ of the form

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ {\scriptstyle Ua}\downarrow & & \downarrow{\scriptstyle Vb} \\ B & \xrightarrow[k]{} & D \end{array} \qquad a \in \mathcal{A},\ b \in \mathcal{B} \tag{6.8}$$

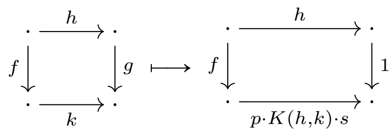
of a diagonal filler $d(h, k)$, in such a way that it is natural with respect to composition on either side. To expand on this point, suppose given morphisms $\alpha \colon a' \to a$

in $\mathcal{A}$ sent by U to the square $U\alpha = (x, y)\colon Ua' \to Ua$, and $\beta\colon b \to b'$ in $\mathcal{B}$ sent by V to $V\beta = (u, v)\colon Vb \to Vb'$. The naturality of $d(h, k)$ means that the equality below holds.

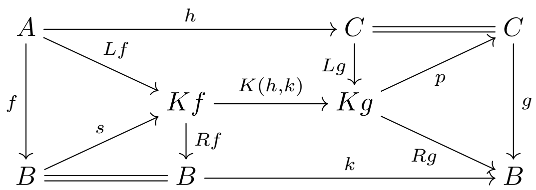


Example 6.9. A functorial factorisation system, with associated copointed endofunctor (L, Φ) and pointed endofunctor (R, Λ), gives rise to a lifting operation for the forgetful functors $U\colon (L, \Phi)\text{-Coalg} \to \mathcal{C}^{\mathbf{2}}$ and $V\colon (R, \Lambda)\text{-Alg} \to \mathcal{C}^{\mathbf{2}}$. The section to the module morphism $\mathsf{Diag}(U, V) \to U^* \cdot V_*$ has component at an object (f, g) of $(L, \Phi)\text{-Coalg} \times (R, \Lambda)\text{-Alg}$ described in the following terms. If $(1, s)\colon f \to Lf$ is the coalgebra structure of f and $(p, 1)\colon Rg \to g$ the algebra structure of g, the component is given by

$$\mathcal{C}^{\mathbf{2}}(f, g) \xrightarrow{L} \mathcal{C}^{\mathbf{2}}(Lf, Lg) \xrightarrow{\mathcal{C}^{\mathbf{2}}(s,1)} \mathcal{C}^{\mathbf{2}}(f, Lg) \xrightarrow{\mathcal{C}^{\mathbf{2}}(1,(1,p))} \mathcal{C}^{\mathbf{2}}(f, 1_{\mathrm{dom}(g)}) \tag{6.10}$$

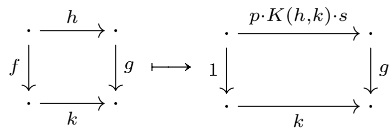


The reader would have noticed that the diagonal filler $p \cdot K(h, k) \cdot s$ so obtained is the same one mentioned in Section 2.a and that we reproduce below.



There is an equivalent description of (6.10) that, instead of using $\mathcal{C}^{\mathbf{2}}(f, 1_{\mathrm{dom}(g)}) \cong \mathsf{Diag}(U, V)(f, g)$, uses $\mathcal{C}^{\mathbf{2}}(1_{\mathrm{cod}(f)}, g) \cong \mathsf{Diag}(U, V)(f, g)$. From this point of view, the section takes the form

$$\mathcal{C}^{\mathbf{2}}(f, g) \xrightarrow{R} \mathcal{C}^{\mathbf{2}}(Rf, Rg) \xrightarrow{\mathcal{C}^{\mathbf{2}}(1,p)} \mathcal{C}^{\mathbf{2}}(Rf, g) \xrightarrow{\mathcal{C}^{\mathbf{2}}((s,1),1)} \mathcal{C}^{\mathbf{2}}(1_{\mathrm{cod}(f)}, g)$$



Example 6.11. A functorial factorisation corresponds to an orthogonal factorisation system when $\mathsf{Diag}(U, V) \to U^* \cdot V_*$ is invertible.

Remark 6.12. Let us now assume that in Definition 6.6 U has a right adjoint G. Then, the module (6.7) is isomorphic to $(G \cdot \mathrm{id} \cdot \mathrm{dom} \cdot V)_*$, $U^* \cdot V_*$ is isomorphic to $(G \cdot V)_*$, and the module morphism $U^* \cdot \mathsf{Diag} \cdot V_* \to U^* \cdot V_*$ corresponds to the natural transformation

$$G \cdot \mathrm{id} \cdot \mathrm{dom} \cdot V \Longrightarrow G \cdot V \tag{6.13}$$

induced by the counit of the adjunction $\mathrm{id} \dashv \mathrm{dom}$, ie the transformation with component at $b \in \mathcal{B}$

$$G(1, Vb)\colon G1_{\mathrm{dom}(Vb)} \longrightarrow GVb.$$

Now suppose that the functor U is the forgetful functor $U\colon \mathsf{L}\text{-Coalg} \to \mathcal{C}^{\mathbf{2}}$, for a comonad L, and still denote by G its right adjoint. Denote by $F_{\mathsf{L}}\colon \mathcal{C}^{\mathbf{2}} \to \mathrm{Kl}(\mathsf{L})$ the Kleisli construction of L. The natural transformation (6.13), belonging to the full image of G, can be described as a morphism in $[\mathcal{B}, \mathrm{Kl}(\mathsf{L})]$

$$F_{\mathsf{L}} \cdot \mathrm{id} \cdot \mathrm{dom} \cdot V \Longrightarrow F_{\mathsf{L}} \cdot V. \tag{6.14}$$

Proposition 6.15. *Given a comonad L on $\mathcal{C}^{\mathbf{2}}$, lifting operations for the functors $U\colon \mathsf{L}\text{-Coalg} \to \mathcal{C}^{\mathbf{2}}$ and $V\colon \mathcal{B} \to \mathcal{C}^{\mathbf{2}}$ are in bijective correspondence with sections of the natural transformation* (6.14)*.*

Proof. The proof is an application of Remark 6.12 to the case when the right adjoint is the universal functor into the Kleisli category of the comonad. □

Example 6.16. As we saw in Example 6.9, each AWFS (L, R) on $\mathcal{C}$ induces a lifting operation which corresponds to a section of (6.14), by Proposition 6.15. We can describe in explicit terms this section as follows.

If $V\colon \mathsf{R}\text{-Alg} \to \mathcal{C}^{\mathbf{2}}$ is the forgetful functor, consider the transformation $(1,p)\colon L \cdot V \Rightarrow \mathrm{id} \cdot \mathrm{dom} \cdot V$ as in Remark 2.10, and denote by $\theta\colon F_{\mathsf{L}}V \longrightarrow F_{\mathsf{L}}\mathrm{id} \cdot \mathrm{dom} \cdot V$ the associated morphism in $[\mathsf{R}\text{-Alg}, \mathrm{Kl}(\mathsf{L})]$. It is easy to check that θ is the required section: $F_{\mathsf{L}}(1,g) \cdot \theta_g$ is, as a morphism in $\mathcal{C}^{\mathbf{2}}$,

$$(1,g) \cdot (1,p) = (1, Rg) = \Phi_g.$$

7. The universal category with lifting operations

R. Garner defined in [?] for each functor $U\colon \mathcal{A} \to \mathcal{C}^{\mathbf{2}}$ a category $\mathcal{A}^{\pitchfork}$ and a functor $U^{\pitchfork}\colon \mathcal{A}^{\pitchfork} \to \mathcal{C}^{\mathbf{2}}$ as follows. The objects of $\mathcal{A}^{\pitchfork}$ are pairs (g, ϕ^g), where $g \in \mathcal{C}^{\mathbf{2}}$ and ϕ^g is an assignment of a diagonal filler for each square

$$\begin{array}{ccc} \cdot & \xrightarrow{h} & \cdot \\ {\scriptstyle Ua}\downarrow & \phi^g(a,h,k) & \downarrow{\scriptstyle g} \\ \cdot & \xrightarrow[k]{} & \cdot \end{array} \tag{7.1}$$

which are compatible with morphisms $U\alpha\colon Ua' \to Ua$, in the sense that

$$\phi^g(a,h,k) \cdot \mathrm{cod}(U\alpha) = \phi^g(a', h \cdot \mathrm{dom}(U\alpha), k \cdot \mathrm{cod}(U\alpha)).$$

A morphism $(g, \phi^g) \to (e, \phi^e)$ is a morphism $(u,v)\colon g \to e$ in $\mathcal{C}^{\mathbf{2}}$ such that $u \cdot \phi^g(a,h,k) = \phi^e(a, u \cdot h, v \cdot k)$, for all (h,k).

The functor $U^{\pitchfork}$ just forgets the lifting operations, or in other words, $U^{\pitchfork}(g, \phi^g) = g$. There is a canonical lifting operation from U to $U^{\pitchfork}$, namely the lifting operation that given a commutative square $(h,k)\colon Ua \to U^{\pitchfork}(g,\phi^g) = g$ picks out the diagonal $\phi^g(a,h,k)$, as in the diagram (7.1). Furthermore, $U^{\pitchfork}$ equipped with this lifting operation is universal among functors into $\mathcal{C}^{\mathbf{2}}$ that are equipped with a lifting operation against U.

The category $\mathcal{A}^{\pitchfork}$ and the functor $U^{\pitchfork}$ can be constructed as a certain limit in **Cat**, of the form

$$\begin{array}{ccccccc} & \overset{U^{\pitchfork}}{\nearrow} & \mathcal{C}^{\mathbf{2}} & \xrightarrow{Y} & \mathcal{P}(\mathcal{C}^{\mathbf{2}}) & \overset{\mathcal{P}(U^*)}{\searrow} & \\ \mathcal{A}^{\pitchfork} & & & \Downarrow & & & \mathcal{P}(\mathcal{A}) \\ & \underset{U^{\pitchfork}}{\searrow} & \mathcal{C}^{\mathbf{2}} & \xrightarrow[\widehat{\mathsf{Diag}}_{\mathcal{C}}]{} & \mathcal{P}(\mathcal{C}^{\mathbf{2}}) & \underset{\mathcal{P}(U^*)}{\nearrow} & \end{array}$$

where $\mathcal{P}(\mathcal{X})$ denotes the presheaf category on $\mathcal{X}$, and $\widehat{\mathsf{Diag}}_{\mathcal{C}}$ is the functor associated to Diag. Equally well, $U^{\pitchfork}$ is a certain enhanced limit, in the sense of [?].

We continue with some further observations from [?]. The universal property of $U^{\pitchfork}$ implies that lifting operations for the pair of functors $U\colon \mathcal{A} \to \mathcal{C}^{\mathbf{2}} \leftarrow \mathcal{B} : V$ are in bijection with functors $\mathcal{B} \to \mathcal{A}^{\pitchfork}$ over $\mathcal{C}^{\mathbf{2}}$. In particular, each AWFS (L, R) in $\mathcal{C}$

gives rise to a canonical functor $\mathsf{R}\text{-Alg} \to \mathsf{L}\text{-Coalg}^{\pitchfork}$. Furthermore, this functor is fully faithful, as we proceed to show. Let $(p,1)\colon Rg \to g$ and $(p',1)\colon Rg' \to g'$ be two R-algebra structures, and $(u,v)\colon g \to g'$ a morphism in $\mathsf{L}\text{-Coalg}^{\pitchfork}$. We know that the chosen diagonal filler of the square $(1,Rg)\colon Lg \to g$ is p, and similarly for g' and p', so we have $u \cdot p = p' \cdot K(u,v)$. Hence, $(p',1) \cdot R(u,v) = (u,v) \cdot (p,1)$, so (u,v) is a morphism of R-algebras.

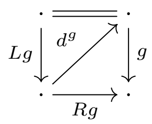
Lemma 7.2. *Given a functor $U\colon \mathcal{A} \to \mathcal{C}^{\mathbf{2}}$, an adjunction $U \dashv G$, and $g \in \mathcal{C}^{\mathbf{2}}$, there is a bijection between the structure of an object $(g,\phi^g) \in \mathcal{A}^{\pitchfork}$ and sections s of $G(1,g)\colon G(1_{\operatorname{dom} g}) \to Gg$ in $\mathcal{A}$. If $(f,\phi^f) \in \mathcal{A}^{\pitchfork}$ is another object, with associated section t, morphisms $(g,\phi^g) \to (f,\phi^f)$ in $\mathcal{A}^{\pitchfork}$ are in bijection with morphisms $(h,k)\colon g \to f$ in $\mathcal{C}^{\mathbf{2}}$ such that $G(h,h) \cdot s = t \cdot G(h,k)$.*

Proof. See discussion before Proposition 6.15. □

Lemma 7.3. *Assume the conditions of Lemma 7.2. Then, for any full subcategory $\mathcal{F} \subset \mathcal{A}$ containing the full image of G, the functor $\mathcal{A}^{\pitchfork} \to \mathcal{F}^{\pitchfork}$ induced by the inclusion is an isomorphism.*

Proof. Denote by $J\colon \mathcal{F} \hookrightarrow \mathcal{A}$ the fully faithful inclusion functor, and by H the right adjoint to UJ, observing that $JH = G$. An object of $\mathcal{F}^{\pitchfork}$ is a lifting operation for the functors UJ and $g\colon \mathbf{1} \to \mathcal{C}^{\mathbf{2}}$, ie a section to the module morphism $(UJ)^* \cdot \mathsf{Diag} \cdot g_* \to (UJ)^* \cdot g_*$. The same data can be equally given by a section to the morphism $H(1,g)\colon H(1_{\operatorname{dom}(g)}) \to Hg$ in $\mathcal{F}$; or a section to the image of this morphism under the fully faithful J. But $JH = G$, so we simply have a section of $G(1,g)$, which is precisely an object of $\mathcal{A}^{\pitchfork}$ by Lemma 7.2. This shows that $\mathcal{A}^{\pitchfork} \to \mathcal{F}^{\pitchfork}$ is bijective on objects. The proof that it is fully faithful is along the same lines, and is left to the reader. □

Corollary 7.4. *If L is a domain preserving comonad on $\mathcal{C}^{\mathbf{2}}$, the category $\mathsf{L}\text{-Coalg}^{\pitchfork}$ can be described as the category with objects pairs (g,d^g) satisfying the commutativity of*



(7.5)

and morphisms $(g,d^g) \to (f,d^f)$ morphisms $(h,k)\colon g \to f$ in $\mathcal{C}^{\mathbf{2}}$ such that $h \cdot d^g = d^f \cdot K(h,k)$. If $\mathcal{F} \subset \mathsf{L}\text{-Coalg}$ is a full subcategory containing the cofree L-coalgebras, the induced functor $\mathsf{L}\text{-Coalg}^{\pitchfork} \longrightarrow \mathcal{F}^{\pitchfork}$ over $\mathcal{C}^{\mathbf{2}}$ is an isomorphism.

Proof. An object of $\mathsf{L}\text{-Coalg}^{\pitchfork}$ can be described, by Lemma 7.2, as a morphism g of $\mathcal{C}$ equipped with a section $s = (s_0, s_1)$ to $L(1,g)\colon L1_{\operatorname{dom} g} \to Lg$ that is a morphism of L-coalgebras. In fact, $s_0 = 1$ since $L(1,g)$ has identity domain component. The morphism of coalgebras $(1,s)\colon Lg \to L1_{\operatorname{dom} g}$ corresponds to a unique morphism $(1,d^g)\colon Lg \to 1_{\operatorname{dom} g}$ in $\mathcal{C}^{\mathbf{2}}$, where $d^g = R1_{\operatorname{dom} g} \cdot s_1$, by the cofree coalgebra adjunction. Similarly, the equality of coalgebra morphisms $1_{Lg} = L(1,g) \cdot s$ can be translated into

$$\Phi_g = \Phi_g \cdot L(1,g) \cdot s = (1,g) \cdot \Phi_{1_{\operatorname{dom} g}} \cdot s;$$

the domain component of each morphism in this string of equalities is an identity morphism, while the codomain component yields $Rg = g \cdot R1_{\operatorname{dom} g} \cdot s_1 = g \cdot d^g$. So far we have proven that $1 = L(1,g) \cdot s$ is equivalent to the commutativity of the bottom triangle in (7.5). Again by the free coalgebra adjunction, the top triangle in (7.5), namely $d^g \cdot Lg = 1$, is equivalent to $s_1 \cdot Lg = L(1_{\operatorname{dom} g})$, which says that $(1, s_1)$ is a morphism $Lg \to L1_{\operatorname{dom} g}$. This completes the description of the objects of $\mathsf{L}\text{-Coalg}^{\pitchfork}$.

We now prove the part of the statement relating to morphisms. Suppose that $(h,k)\colon (g,\phi^g) \to (f,\phi^f)$ is a morphism in $\mathsf{L}\text{-Coalg}^{\pitchfork}$ and $s\colon Lg \to L1_{\mathrm{dom}_g}$ and $t\colon Lf \to L1_{\mathrm{dom}\, f}$ the sections of $L(1,g)$ and $L(1,f)$ provided by Lemma 7.2. By the same lemma, the condition of (h,k) being a morphism in $\mathsf{L}\text{-Coalg}^{\pitchfork}$ is equivalent to the equality $L(h,h)\cdot s = t\cdot L(h,k)$, which is equivalent to

$$(h,h)\cdot \Phi_g \cdot s = \Phi_f \cdot L(h,h)\cdot s = \Phi_f \cdot t \cdot L(h,k).$$

The domain component of this equality is trivial, and so it is equivalent to the equality of its codomain component, which is

$$h\cdot d^g = h\cdot Rg\cdot s = Rf\cdot t\cdot K(h,k) = d^f \cdot K(h,k).$$

The last sentence of the statement follows from Lemma 7.3, completing the proof. □

8. KZ lifting operations

Section 6.b described the algebraic structure that provides a lifting operation, and the category $\mathcal{A}^{\pitchfork}$, in terms of modules. This section introduces variations of these notions that are suitable to lax orthogonal factorisations.

8.a. **Lifting operations in 2-categories.** Before introducing the main definitions of this section, let us remind the reader about some facts around **Cat**-modules. A **Cat**-module ϕ from a 2-category $\mathscr{A}$ to another $\mathscr{B}$, denoted by $\phi\colon \mathscr{A} \to\!\!\!\!\!\!+\ \mathscr{B}$, is a 2-functor $\mathscr{B}^{\mathrm{op}}\times\mathscr{A} \to \mathbf{Cat}$. A difference with the case of modules between ordinary categories is that for **Cat**-modules there is a 2-category $\mathbf{Cat\text{-}Mod}(\mathscr{A},\mathscr{B})$ of **Cat**-modules $\mathscr{A} \to\!\!\!\!\!\!+\ \mathscr{B}$: the morphisms are 2-natural transformations and the 2-cells are the modifications.

Given a **Cat**-module $\phi\colon \mathscr{B} \to\!\!\!\!\!\!+\ \mathscr{C}$, and 2-functors $F\colon \mathscr{A}\to\mathscr{B}$ and $G\colon \mathscr{D}\to\mathscr{C}$, we write $\phi\cdot F_*\colon \mathscr{A} \to\!\!\!\!\!\!+\ \mathscr{C}$ and $G^*\cdot\phi\colon \mathscr{B} \to\!\!\!\!\!\!+\ \mathscr{D}$ for the modules defined by the formulas

$$(\phi\cdot F_*)(c,a) = \phi(c,Fa) \quad\text{and}\quad (G^*\cdot\phi)(d,b) = \phi(Gd,b).$$

In a completely analogous way to the case of **Set**-modules or profunctors addressed in Section 6.a, we have the following facts:

- Each 2-functor $F\colon \mathscr{A}\to\mathscr{B}$ induces a pair of **Cat**-modules $F_*\colon \mathscr{A} \to\!\!\!\!\!\!+\ \mathscr{B}$ and $F^*\colon \mathscr{B} \to\!\!\!\!\!\!+\ \mathscr{A}$, by the formulas $F_*(b,a) = \mathscr{B}(b,Fa)$ and $F^*(a,b) = \mathscr{B}(Fa,b)$.
- Each 2-natural transformation $\alpha\colon F\Rightarrow G$ induces a morphism of **Cat**-modules $\alpha_*\colon F_*\to G_*$ by $\alpha_*(b,a) = \mathscr{B}(b,\alpha_a)$.
- If F, $G\colon \mathscr{A}\to\mathscr{B}$ are 2-functors, each morphism of **Cat**-modules $F_*\to G_*$ is of the form α_* for a unique 2-natural transformation $\alpha\colon F\Rightarrow G$.

The forgetful 2-functor from $\mathbf{Cat\text{-}Mod}(\mathscr{A},\mathscr{B})$ to $\mathbf{Cat}^{|\mathscr{B}|\times|\mathscr{A}|}$ that sends a module ϕ to the family of categories $\{\phi(a,b)|a\in|\mathscr{A}|, b\in|\mathscr{B}|\}$ is 2-monadic, by the usual argument: its left adjoint is given by left Kan extension along the inclusion of objects $|\mathscr{B}|\times|\mathscr{A}| \hookrightarrow \mathscr{B}^{\mathrm{op}}\times\mathscr{A}$, and it is conservative. We denote the associated 2-monad by T.

We now substitute the category $\mathcal{C}$ in Section 6.b by a 2-category $\mathscr{K}$, and make the modules into **Cat**-enriched modules. So $\mathsf{Diag}(f,g)$ is now the category with objects commutative squares with a diagonal filler as depicted in (6.3), with morphisms, from an object with diagonal d to another with diagonal d', given by 2-cells $d\Rightarrow d'$ in $\mathscr{K}$; in other words, $\mathsf{Diag}(f,g)$ is isomorphic to the category $\mathscr{K}(\operatorname{cod} f, \operatorname{dom} g)$. Given 2-functors $U\colon \mathscr{A}\to\mathscr{K}^{\mathbf{2}}$ and $V\colon \mathscr{B}\to\mathscr{K}^{\mathbf{2}}$, we define $\mathsf{Diag}(U,V)$ in the same way as we did in Section 6.b in the case of ordinary categories, ie $\mathsf{Diag}(U,V)(a,b) = \mathscr{K}^{\mathbf{2}}(Ua,Vb)$, with the difference that now the modules are **Cat**-enriched.

Definition 8.1. (1) A *lifting operation* for a pair of 2-functors U, V into $\mathscr{K}^{\mathbf{2}}$ is a **Cat**-module that is a section of $\mathsf{Diag}(U,V) \to U^* \cdot V_*$; in other words, it is a section in $\mathbf{Cat}\text{-}\mathbf{Mod}(\mathscr{A},\mathscr{B})$ – the reader may want to compare with Definition 6.6.

(2) A *lax natural lifting operation* for U, V is a section of $\mathsf{Diag}(U,V) \to U^* \cdot V_*$ in $\mathsf{T}\text{-Alg}_c$, the 2-category of T-algebras and oplax morphisms, also known as colax morphisms, for the 2-monad T on $\mathbf{Cat}^{|\mathscr{B}|\times|\mathscr{A}|}$ whose algebras are **Cat**-modules – an explicit description can be found below.

An object of the 2-category $\mathsf{T}\text{-Alg}_c$ is a **Cat**-module $\varphi\colon \mathscr{A} \to\!\!\!\!\!\to \mathscr{B}$, while a morphism $t\colon \varphi \to \psi$ is a morphism of the underlying matrices that is oplax with respect to the action of $\mathscr{A}$ and $\mathscr{B}$. This means that, given a morphism $f\colon a \to a'$ in $\mathscr{A}$, and $g\colon b \to b'$ in $\mathscr{B}$, there is extra data

$$\begin{array}{ccc} \varphi(b,a') & \xrightarrow{t(b,a')} & \psi(b,a') \\ {\scriptstyle \varphi(g,f)}\big\downarrow & \bar{t}_{f,g} \Nearrow & \big\downarrow{\scriptstyle \psi(g,f)} \\ \varphi(b',a) & \xrightarrow[t(b',a)]{} & \psi(b'a) \end{array}$$

satisfying coherence axioms.

Each component $U^* \cdot V_*(a,b) \to \mathsf{Diag}(U,V)(a,b)$ of the section of Definition 8.1 gives a diagonal filler for each square $Ua \to Vb$ in $\mathscr{K}^{\mathbf{2}}$. The oplax morphism structure on the section can be described as follows. Suppose the morphisms $\alpha\colon a' \to a$ in $\mathscr{A}$ and $\beta\colon b \to b'$ in $\mathscr{B}$ are mapped by U and V to commutative squares in $\mathscr{K}$

$$\begin{array}{ccc} A' & \xrightarrow{x} & A \\ {\scriptstyle Ua'}\big\downarrow & & \big\downarrow{\scriptstyle Ua} \\ B' & \xrightarrow{y} & B \end{array} \quad \text{and} \quad \begin{array}{ccc} C & \xrightarrow{u} & C' \\ {\scriptstyle Vb}\big\downarrow & & \big\downarrow{\scriptstyle Vb'} \\ D & \xrightarrow{v} & D' \end{array}$$

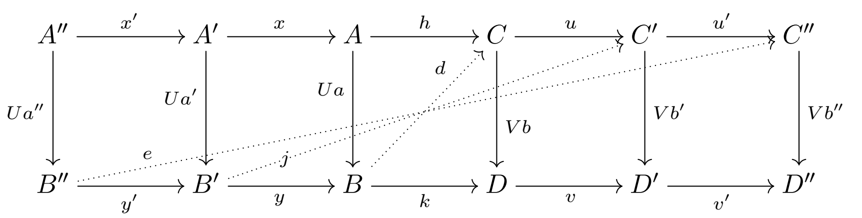
Consider the diagonal fillers given by the respective components of the section:

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ {\scriptstyle Ua}\big\downarrow & \overset{d}{\nearrow} & \big\downarrow{\scriptstyle Vb} \\ B & \xrightarrow[k]{} & C \end{array} \quad \text{and} \quad \begin{array}{ccc} A' & \xrightarrow{u\cdot h\cdot x} & C' \\ {\scriptstyle Ua'}\big\downarrow & \overset{j}{\nearrow} & \big\downarrow{\scriptstyle Vb'} \\ B' & \xrightarrow[v\cdot k\cdot y]{} & D' \end{array}$$

Then, the oplax morphism structure on $U^* \cdot V_* \to \mathsf{Diag}(U,V)$ provides a 2-cell $\omega = \omega(\alpha,\beta)\colon j \Rightarrow u\cdot d\cdot y$, satisfying $(Vb')\cdot\omega = 1$, $\omega\cdot(Ua') = 1$, and coherence conditions that we proceed to describe. Suppose given an object d of $\mathsf{Diag}(U,V)(a,b)$ as above, and morphisms in $\mathscr{A}$ and $\mathscr{B}$

$$a'' \xrightarrow{\alpha'} a' \xrightarrow{\alpha} a \quad \text{and} \quad b \xrightarrow{\beta} b' \xrightarrow{\beta'} b''$$

we have the following diagram, where the dashed arrows are chosen diagonal fillers.



The condition corresponding to the associativity axiom of the oplax morphism $U^* \cdot V_* \to \mathsf{Diag}(U,V)$ says that

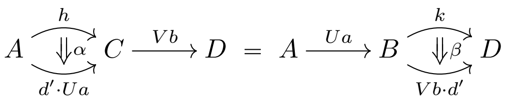
$$\big(e \xrightarrow{\omega(\alpha\cdot\alpha',\beta'\cdot\beta)} u'\cdot u\cdot d\cdot y\cdot y'\big) = \big(e \xrightarrow{\omega(\alpha',\beta')} u'\cdot j\cdot y' \xrightarrow{u'\cdot\omega(\alpha,\beta)\cdot y'} u'\cdot u\cdot d\cdot y\cdot y'\big)$$

The axiom corresponding to the unit axiom of the oplax morphism $U^* \cdot V_* \to \mathsf{Diag}(U,V)$ says that $\omega(1,1) = 1$.

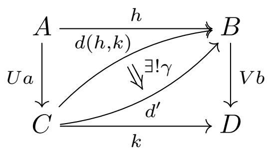
8.b. **KZ lifting operations.** Having introduced in the previous section lifting operations and lax natural lifting operations, we now introduce a version of lifting operation that corresponds to lax orthogonal AWFSs, namely, KZ lifting operations.

Definition 8.2. A KZ *lifting operation* in $\mathscr{K}$ for the 2-functors U, V is a left adjoint section to the morphism $\mathsf{Diag}(U,V) \to U^* \cdot V_*$ in the 2-category **Cat-Mod**$(\mathscr{A},\mathscr{B})$.

In more explicit terms, a KZ lifting operation is given by, for each square (6.8) in $\mathscr{K}$, a diagonal filler $d(h,k)$, with the following universal property. For any $d' \colon B \to C$ and any pair of 2-cells α, β satisfying

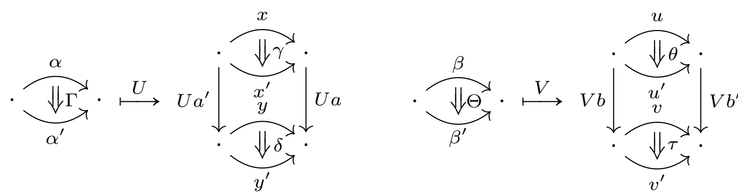


there exists a unique 2-cell $\gamma \colon d(h,k) \Rightarrow d'$ such that $\gamma \cdot Ua = \alpha$ and $Vb \cdot \gamma = \beta$.

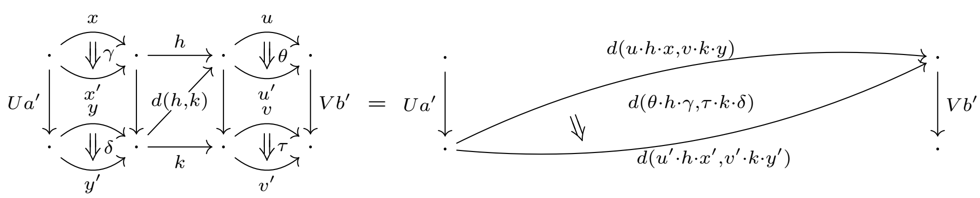


(8.3)

This universal property makes, by the usual argument, $d(h,k)$ functorial in (h,k): for any pair of 2-cells $\mu \colon h \Rightarrow h'$ and $\kappa \colon k \Rightarrow k'$ such that $Vb \cdot \mu = \kappa \cdot Ua$, there is a 2-cell $d(\mu,\kappa) \colon d(h,k) \Rightarrow d(h',k')$. Furthermore, the diagonal fillers $d(h,k)$ must be 2-natural in a and b, in the following sense. Suppose that Γ is a 2-cell in $\mathscr{A}$ and Θ a 2-cell in $\mathscr{B}$, sent by U and V to 2-cells in $\mathscr{K}^{\mathbf{2}}$ as depicted.



The 2-naturality of the lifting operation means that there is an equality of 2-cells $\theta \cdot d(h,k) \cdot \delta = d(\theta \cdot h \cdot \gamma, \tau \cdot k \cdot \delta)$.



Definition 8.4. A *lax natural* KZ *lifting operation* in $\mathscr{K}$ for the 2-functors U, V is a left adjoint section to the morphism $\mathsf{Diag}(U,V) \to U^* \cdot V_*$ in the 2-category $\mathbf{Cat}^{|\mathscr{B}|\times|\mathscr{A}|}$.

This means that a lax natural KZ lifting operation is given by a left adjoint section for each component

$$\mathsf{Diag}(U,V)(a,b) \longrightarrow \mathscr{K}^{\mathbf{2}}(Ua,Vb) \qquad a \in \mathscr{A},\ b \in \mathscr{B}.$$

More explicitly, it is given by a choice, for each square $(h,k) \colon Ua \to Vb$, of a diagonal filler $d(h,k) \colon \operatorname{cod}(Ua) \to \operatorname{dom}(Vb)$ with the property that 2-cells $d(h,k) \Rightarrow d' \colon \operatorname{cod}(Ua) \to \operatorname{dom}(Vb)$ are in bijection with 2-cells $(h,k) \Rightarrow (d' \cdot Ua, Vb \cdot d')$. In other words, the same universal property of KZ lifting operation, except that the chosen diagonals $d(h,k)$ need not be natural in a, b.

Remark 8.5. A lax natural KZ lifting operation equates to providing, for each $a \in \mathscr{A}$ and $b \in \mathscr{B}$, with $Ua\colon A \to B$ and $Vb\colon C \to D$, a left adjoint section of the usual comparison functor

$$\mathscr{K}(B,C) \longrightarrow \mathscr{K}(A,C) \times_{\mathscr{K}(A,D)} \mathscr{K}(B,D).$$

However, the presentation using modules effortlessly yields more, as discussed below.

Proposition 8.6. *(1) Each KZ lifting operation is also a lax natural KZ lifting operation.*
(2) Each lax natural KZ lifting operation is also a lax natural lifting operation.

Proof. The first part of the statement follows from the existence of a forgetful 2-functor

$$\mathbf{Cat}\text{-}\mathbf{Mod}(\mathscr{A},\mathscr{B}) \longrightarrow \mathbf{Cat}^{|\mathscr{A}|\times|\mathscr{B}|}. \tag{8.7}$$

Then, given 2-functors $U\colon \mathscr{A} \to \mathscr{K}^{\mathbf{2}}$ and $V\colon \mathscr{B} \to \mathscr{K}^{\mathbf{2}}$, a left adjoint section to the canonical morphism $\mathsf{Diag}(U,V) \to U^* \cdot V_*$ in $\mathbf{Cat}\text{-}\mathbf{Mod}(\mathscr{A},\mathscr{B})$ is also a left adjoint section in $\mathbf{Cat}^{|\mathscr{A}|\times|\mathscr{B}|}$.

For the second part of the statement, one needs to use the fact that the 2-functor (8.7) is monadic; this means that there is a 2-monad T on $\mathbf{Cat}^{|\mathscr{A}|\times|\mathscr{B}|}$ and that $\mathbf{Cat}\text{-}\mathbf{Mod}(\mathscr{A},\mathscr{B})$ is the 2-category $\mathsf{T}\text{-}\mathrm{Alg}_s$ of T-algebras and strict morphisms. A lax natural KZ lifting operation is a left adjoint section of (8.7) in the 2-category $\mathbf{Cat}^{|\mathscr{A}|\times|\mathscr{B}|}$. Since (8.7) is a strict morphism of T-algebras, the *doctrinal adjunction* Proposition 4.1 ensures that (8.7) has a left adjoint in $\mathsf{T}\text{-}\mathrm{Alg}_c$, which is the definition of lax natural lifting operation – Definition 8.1. ☐

Lemma 8.8. *KZ lifting operations are unique up to canonical isomorphism. More precisely, if the diagonal filler $d(h,k)$ and $d'(h,k)$ define two KZ lifting operations between 2-functors $U\colon \mathscr{A} \to \mathscr{K}^{\mathbf{2}} \leftarrow \mathscr{B}\colon V$, then there exists a unique 2-cell $\gamma(h,k)\colon d(h,k) \Rightarrow d'(h,k)$ such that $Vb\cdot\gamma(h,k) = 1$ and $\gamma(h,k)\cdot Ua = 1$, as depicted in* (8.3)*. Furthermore, γ is invertible.*

Proof. This is a direct consequence of the universal property of KZ lifting operations. ☐

Proposition 8.9. *KZ lifting operations for the 2-functors $U\colon \mathscr{A} \to \mathscr{K}^{\mathbf{2}}$ and $V\colon \mathscr{B} \to \mathscr{K}^{\mathbf{2}}$ are, if U has a right adjoint G, in bijective correspondence with left adjoint sections of the morphism $G \cdot \mathrm{id} \cdot \mathrm{dom} \cdot V \to G \cdot V$ induced by the counit of $\mathrm{id} \dashv \mathrm{dom}$ – with components $G(1, Vb)$ – in the 2-category $[\mathscr{B},\mathscr{A}]$ of 2-functors $\mathscr{B} \to \mathscr{A}$.*

Proof. By the comments about **Cat**-modules at the beginning of Section 8.a and same argument deployed in Remark 6.12, the **Cat**-module transformation

$$U^* \cdot \mathsf{Diag} \cdot V_* \longrightarrow U^* \cdot V_* \tag{8.10}$$

corresponds to the 2-natural transformation of the statement

$$G \cdot \mathrm{id} \cdot \mathrm{dom} \cdot V \Longrightarrow G \cdot V. \tag{8.11}$$

Since the 2-functor from $[\mathscr{B},\mathscr{A}]$ to $\mathbf{Cat}\text{-}\mathbf{Mod}(\mathscr{B},\mathscr{A})$ that sends F to F_* is full and faithful – an isomorphism on hom-categories – (8.10) has a left adjoint coretract if and only if (8.11) does. ☐

Proposition 8.12. *Given a lax orthogonal* AWFS (L,R) *on* $\mathscr{K}$*, the 2-natural transformation* $F^{\mathsf{L}}\cdot \mathrm{id}\cdot \mathrm{dom}\cdot V\Rightarrow F^{\mathsf{L}}\cdot V$ *induced by the counit of* $\mathrm{id}\dashv\mathrm{dom}$ *has a left adjoint section in* $[\mathsf{R}\text{-}\mathrm{Alg}_s,\mathsf{L}\text{-}\mathrm{Coalg}_s]$*, where* $F^{\mathsf{L}}\colon\mathscr{K}^{\mathbf{2}}\to\mathsf{L}\text{-}\mathrm{Coalg}_s$ *is the cofree coalgebra 2-functor and* V *the forgetful 2-functor from* $\mathsf{L}\text{-}\mathrm{Coalg}_s$*.*

Proof. Given an R-algebra structure $(p_g,1)\colon Rg\to g$ we need to exhibit a coretract adjunction in $\mathsf{L}\text{-}\mathrm{Coalg}_s$ with right adjoint $L(1,g)\colon L1_{\mathrm{dom}(g)}\to Lg$. We know from Remark 4.7 that there is a coretract adjunction $(1,p_g)\dashv(1,Lg)$, whose unit we denote by η_g; the same remark points out that these adjunctions are 2-natural in (g,p_g). Together with the adjunction $\Sigma_g\dashv L\Phi_g$ that exhibits L as lax idempotent, we obtain

$$L(1,p_g)\cdot\Sigma_g\dashv L\Phi_g\cdot L(1,Lg)=L(1,g).$$

The unit of this composition of adjunctions is

$$1=L\Phi_g\cdot\Sigma_g\xrightarrow{L\Phi_g\cdot L(\eta_g)\cdot\Sigma_g}L\Phi_g\cdot L(1,Lg)\cdot L(1,p)\cdot\Sigma_g=1,$$

which is the identity since $\Phi_g\cdot\eta_g=1$ – again by Remark 4.7. □

Theorem 8.13. *Each lax orthogonal* AWFS (L,R) *on the 2-category* $\mathscr{K}$ *induces*

(1) A KZ *lifting operation for* $\mathsf{L}\text{-}\mathrm{Coalg}_s\to\mathscr{K}^{\mathbf{2}}$ *and* $\mathsf{R}\text{-}\mathrm{Alg}_s\to\mathscr{K}^{\mathbf{2}}$*.*
(2) A lax natural KZ *lifting operation for* $U_\ell\colon\mathsf{L}\text{-}\mathrm{Coalg}_\ell\to\mathscr{K}^{\mathbf{2}}$ *and* $V_\ell\colon\mathsf{R}\text{-}\mathrm{Alg}_\ell\to\mathscr{K}^{\mathbf{2}}$*.*

Moreover, the diagonal fillers are those given by the AWFS *in the usual way –* (2.5)*.*

Proof. The first part is a direct consequence of Propositions 8.9 and 8.12. The second part means that there must exist a left adjoint coretract to each functor

$$\mathsf{Diag}(U_\ell,V_\ell)((f,s),(g,p))=\mathscr{K}(\mathrm{cod}(f),\mathrm{dom}(g))\longrightarrow\mathscr{K}^{\mathbf{2}}(f,g)\qquad(8.14)$$

where (f,s) is an L-coalgebra and (g,p) an R-algebra. We know that such a left adjoint coretract does exist, by the first part of the statement, and the proof is complete. □

Remark 8.15. It may be useful to exhibit the counit of the coretract adjunction in the proof of Theorem 8.13, in which (8.14) is the right adjoint, even though it is not necessary to prove that result. Let d be a diagonal filler for a square

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B & \xrightarrow{k} & D \end{array}$$

from an L-coalgebra (f,s) to an R-algebra (g,p). The diagram on the left below shows the equality $K(h,k)=K(1_C,Rg)\cdot K(1_C,Lg)\cdot K(h,d)$, while the diagram on the right shows that $p=R1_C\cdot K(1_C,p)\cdot\sigma_g$.

$$\begin{array}{ccccccc} \cdot & \xrightarrow{h} & \cdot & = & \cdot & = & \cdot \\ {\scriptstyle Lf}\downarrow & & \downarrow{\scriptstyle L1_C} & & \downarrow{\scriptstyle L^2g} & & \downarrow{\scriptstyle Lg} \\ \cdot & \xrightarrow{K(h,d)} & \cdot & \xrightarrow{K(1_C,Lg)} & \cdot & \xrightarrow{K(1_C,Rg)} & \cdot \\ {\scriptstyle Rf}\downarrow & & \downarrow{\scriptstyle R1_C} & & \downarrow{\scriptstyle RLg} & & \downarrow{\scriptstyle Rg} \\ \cdot & \xrightarrow{d} & \cdot & \xrightarrow{Lg} & \cdot & \xrightarrow{Rg} & \cdot \end{array}\qquad\begin{array}{ccccc} \cdot & = & \cdot & = & \cdot \\ {\scriptstyle Lg}\downarrow & & \downarrow{\scriptstyle L^2g} & & \downarrow{\scriptstyle L1_C} \\ \cdot & \xrightarrow{\sigma_g} & \cdot & \xrightarrow{K(1_C,p)} & \cdot \\ & \diagdown\!\!\diagdown & \downarrow{\scriptstyle RLg} & & \downarrow{\scriptstyle R1_C} \\ & & \cdot & \xrightarrow{p} & \cdot \end{array}$$

Using these equalities, one can show that the counit $p \cdot K(h,k) \cdot s \Rightarrow d$ given by the KZ lifting operation can be described by

$$p \cdot K(h,k) \cdot s = R1_C \cdot K(1_C, p) \cdot \sigma_g \cdot K(1_C, Rg) \cdot K(1_C, Lg) \cdot K(h,d) \cdot s \Longrightarrow \\ \Longrightarrow R1_C \cdot K(1_C, p) \cdot K(1_C, Lg) \cdot K(h,d) \cdot s = R1_C \cdot K(h,d) \cdot s = d \cdot Rf \cdot s = d \tag{8.16}$$

where the unlabelled arrow is the one induced by the counit $\sigma_g \cdot K(1, Rg) \Rightarrow 1_{Kg}$ that endows the comonad L with its lax idempotent structure.

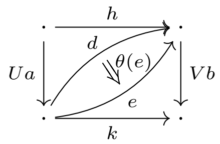
Theorem 8.13 (2) can be rephrased by saying that the usual lifting operation for (L, R) is, when both L and R are lax idempotent, lax natural with respect to all morphisms in $\mathscr{K}$.

8.c. **Lax orthogonal functorial factorisations.** We have seen in the previous sections that the lifting operation of a lax orthogonal AWFS has the extra structure of a KZ lifting operation. One could ask what extra structure is inherited from a lax orthogonal AWFS to its underlying WFS. Since we work with algebraic factorisations, we have at our disposal not only mere WFSs but functorial factorisations, and it is for these that we answer the question.

Let $\mathscr{A}$, $\mathscr{B}$ be 2-categories and $\mathscr{X}$ be $\mathbf{Cat}\text{-}\mathbf{Mod}(\mathscr{B}, \mathscr{A})$. Denote by M the 2-monad (M, Λ^M, Π^M) on $\mathscr{X}^{\mathbf{2}}$ whose algebras are morphisms in $\mathscr{X}$ equipped with a left adjoint coretract. A dual of M has been described in Section 4.d; more precisely, if L is the 2-comonad of Proposition 4.11, whose algebras are morphisms equipped with a right adjoint retract defined on the 2-category $(\mathscr{X}^{\mathrm{op}})^{\mathbf{2}} \cong (\mathscr{X}^{\mathbf{2}})^{\mathrm{op}}$, then M is L^{op}. An algebra for the pointed endo-2-functor (M, Λ^M) is a morphism $\alpha\colon \phi \to \psi$ equipped with a coretract $\sigma\colon \psi \to \phi$ and a 2-cell $m\colon \sigma \cdot \alpha \Rightarrow 1$ such that $\sigma \cdot m = 1$. This is a dual form of Proposition 4.11 (1).

Definition 8.17. Consider 2-functors U and V from $\mathscr{A}$ and $\mathscr{B}$ into $\mathscr{K}^{\mathbf{2}}$. A *lax orthogonality structure* on U, V is an (M, Λ^M)-coalgebra structure on the morphism of $\mathbf{Cat}$-modules $U^* \cdot \mathsf{Diag} \cdot V_* \to U^* \cdot V_*$. Consider a functorial factorisation on $\mathscr{K}$ with associated copointed endo-2-functor (L, Φ) and associated pointed endo-2-functor (R, Λ). A lax orthogonality structure on the functorial factorisation is one on U, V, for U the forgetful 2-functor from (L, Φ)-coalgebras and V the forgetful 2-functor from (R, Λ)-algebras.

Explicitly, a lax orthogonality structure as in the definition is a choice of 2-natural diagonal fillers $d(a,b)(h,k)\colon \operatorname{cod}(Ua) \to \operatorname{dom}(Vb)$ that is functorial on squares $(h,k)\colon Ua \to Vb$, and 2-natural in $a \in \mathscr{A}$, $b \in \mathscr{B}$. Furthermore, for any diagonal filler e of (h,k) we are given a 2-cell $\theta(a,b)(e)\colon d(a,b)(h,k) \Rightarrow e$ that is 2-natural in e and a modification on a, b.

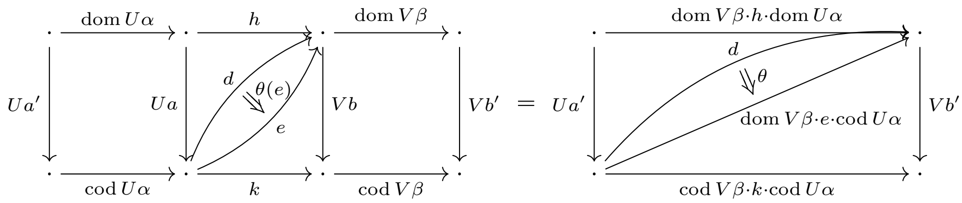


The 2-cells $\theta(a,b)(e)$ must satisfy $(Vb) \cdot \theta(a,b)(e) = 1_k$ and $\theta(a,b)(e) \cdot (Ua) = 1_h$. Naturality in e means that for each 2-cell $\epsilon\colon e \Rightarrow \bar{e}$ the equality

$$\big(\theta(a,b)(\bar{e})\big)\big(d(a,b)(\epsilon \cdot Ua, Vb \cdot \epsilon)\big) = \epsilon\theta(a,b)(e)$$

holds. The modification property for θ means that, if $\alpha\colon a' \to a$ and $\beta\colon b \to b'$ are morphisms in $\mathscr{A}$ and $\mathscr{B}$, then

$$\operatorname{dom}(V\beta) \cdot \theta(a,b)(e) \cdot \operatorname{cod}(U\alpha) = \theta(a',b')(\operatorname{dom}(V\beta) \cdot e \cdot \operatorname{cod}(U\alpha)).$$



Observe that there is no reason why θ should satisfy the extra property that the endo-2-cell $\theta(a,b)(d(a,b)(h,k))$ of $d(a,b)(h,k)$ be an identity 2-cell.

Remark 8.18. In the particular instance when $\mathscr{A} = \mathscr{B} = \mathbf{1}$, the 2-functors U and V pick out morphisms $f\colon A \to B$ and $g\colon C \to D$ in $\mathscr{K}$, and a lax orthogonality structure for f, g can be described simply as a functor D that is a section of the canonical comparison functor H into the pullback, together with a natural transformation $\theta\colon DH \Rightarrow 1$ that satisfies $H\theta = 1$. This structure can be described as a choice of a diagonal filler $D(h,k)$ for each square (h,k) and a 2-cell $\theta(e)\colon D(h,k) \Rightarrow e$ for any other diagonal filler e, that satisfies $g \cdot \theta(e) = 1$ and $\theta(e) \cdot f = 1$.

$$\mathscr{K}(B,C) \underset{D}{\overset{H}{\rightleftarrows}} \mathscr{K}(A,C) \times_{\mathscr{K}(A,D)} \mathscr{K}(B,D)$$

Proposition 8.19. *The underlying 2-functorial factorisation of a lax orthogonal* AWFS *carries a canonical lax orthogonal structure, whose diagonal fillers are those induced by the 2-functorial factorisation in the usual way – as in Example 6.9.*

Proof. For an AWFS (L,R), consider the forgetful functors U and V from, respectively, the 2-categories of (L,Φ)-coalgebras and (R,Λ)-algebras. Denote by $\bar{\Sigma} = (1,s)\colon U \Rightarrow LU$ the L-coalgebra structure of U, and $\bar{\Pi} = (p,1)\colon RV \Rightarrow V$ the R-algebra structure of V. In this proof we use the notation introduced in the second paragraph of this section: $\mathscr{X}$ is the 2-category of **Cat**-modules from L-Coalg$_s$ to R-Alg$_s$, and $\mathsf{M} = (M,\Lambda^M,\Pi^M)$ the 2-monad on $\mathscr{X}^{\mathbf{2}}$ whose algebras are right adjoint retracts.

We can form two objects of $\mathscr{X}^{\mathbf{2}}$ depicted as the vertical arrows in the square below, induced by the **Cat**-module morphism $\mathsf{Diag} \to 1$ introduced in (6.5). The morphisms of **Cat**-modules $\bar{\Sigma}^*\colon (LU)^* \to U^*$ and $\bar{\Lambda}_*\colon (RV)_* \to V_*$ induce a morphism in $\mathscr{X}^{\mathbf{2}}$, depicted as the commutative diagram in $\mathscr{X}$ below.

$$\begin{array}{ccc} U^*L^*\mathsf{Diag}R_*V_* & \xrightarrow{\bar{\Sigma}^*\mathsf{Diag}\bar{\Pi}_*} & U^*\mathsf{Diag}V_* \\ \downarrow & & \downarrow \\ U^*L^*R_*V_* & \xrightarrow{\bar{\Sigma}^*\bar{\Pi}_*} & U^*V_* \end{array} \tag{8.20}$$

This morphism is a retraction in $\mathscr{X}^{\mathbf{2}}$, since $\bar{\Sigma}^*$ and $\bar{\Pi}_*$ are retractions with respective sections Φ^* and Λ_*. Theorem 8.13 implies that the object of $\mathscr{X}^{\mathbf{2}}$ depicted by the leftmost vertical arrow in the diagram carries a structure of an M-algebra. Hence the object on the right hand side, as a retract of an M-algebra, carries an (M,Λ^M)-algebra structure that makes the retraction (8.20) a morphism of (M,Λ^M)-algebras. It remains to show that the section of $U^*\cdot\mathsf{Diag}\cdot V_* \to U^*\cdot V_*$ so obtained is equal to that induced by the functorial factorisation, as described in Example 6.9, for which we appeal to Remark 4.15. The induced section is

$$U^*V_* \xrightarrow{U^*\Phi^*\Lambda_*V_*} U^*L^*R_*V_* \to U^*L^*\mathsf{Diag}_{\mathscr{K}}R_*V_* \xrightarrow{(1,s)^*\mathsf{Diag}(p,1)_*} U^*\mathsf{Diag}V_*$$

where the middle morphism is the KZ lifting operation for LU, RV. One can verify that the diagonal filler of a square $(h,k)\colon f \to g$, where (f,s) is an (L,Φ)-coalgebra and (g,p) and (R,Λ)-algebra, is $p\cdot d\cdot s$ where d is the diagonal filler of

$(Lg \cdot h, k \cdot Rf)\colon Lf \to Rg$. But $d = K(h,k)$, so $p \cdot d \cdot s$ is precisely the diagonal filler induced by the functorial factorisation. □

9. Algebraic KZ Injectivity

In previous sections we have visited the construction of the universal category $\mathcal{A}^{\pitchfork}$ with lifting operations against a functor $\mathcal{A} \to \mathcal{C}^{\mathbf{2}}$, and the fact that, for any AWFS (L,R) on $\mathcal{C}$, each R-algebra comes equipped with a lifting operation against L-coalgebras; in other words, the existence of a functor $\mathsf{R}\text{-Alg} \to \mathsf{L}\text{-Coalg}^{\pitchfork}$. In this section we concentrate in the analogous constructions adapted to the case of lax orthogonal AWFSs, where KZ lifting operations will play an important role.

The reader would recall from Section 7, and originally from [?], the definition of the free category with a lifting operation $U^{\pitchfork}\colon \mathcal{A}^{\pitchfork} \to \mathcal{C}^{\mathbf{2}}$ for $U\colon \mathcal{A} \to \mathcal{C}^{\mathbf{2}}$. If $U\colon \mathscr{A} \to \mathscr{K}^{\mathbf{2}}$ is a 2-functor instead, $\mathscr{A}^{\pitchfork}$ has objects (g,ϕ) where ϕ is a section of the morphism $U^* \cdot \mathsf{Diag} \cdot g_* \to U^* \cdot g_*$ in the 2-category $\mathbf{Cat\text{-}Mod}(\mathbf{1},\mathscr{A})$, which is isomorphic to $[\mathscr{A}^{\mathrm{op}},\mathbf{Cat}]$. Morphisms $(g,\phi) \to (g',\phi')$ are those morphisms $(u,v)\colon g \to g'$ in $\mathscr{K}^{\mathbf{2}}$ that are compatible with the sections, while 2-cells $(u,v) \Rightarrow (\bar u,\bar v)$ are pairs of 2-cells $\alpha\colon u \to \bar u$ and $\beta\colon v \to \bar v$ in $\mathscr{K}$ such that the equality below holds – we omit the dots that denote composition to save space.

$$U^*g_* \xrightarrow{\phi} U^*\mathsf{Diag}g_* \underset{U^*\mathsf{Diag}(\bar u,\bar v)_*}{\overset{U^*\mathsf{Diag}(u,v)_*}{\Downarrow}} U^*\mathsf{Diag}g'_* \;=\; U^*g_* \underset{U^*(\bar u,\bar v)_*}{\overset{U^*(u,v)_*}{\Downarrow}} U^*g'_* \xrightarrow{\phi'} U^*\mathsf{Diag}g'_*$$

In more elementary terms, $\alpha \cdot \phi(a,h,k) = \phi'(a, \alpha \cdot h, \beta \cdot k)$, for each $a \in \mathscr{A}$ and each square $(h,k)\colon Ua \to g$. The 2-functor $U^{\pitchfork}\colon \mathscr{A}^{\pitchfork} \to \mathscr{K}^{\mathbf{2}}$ is the obvious one, analogous to the case of ordinary categories.

Next we introduce a different construction, the universal 2-category with a KZ lifting operation.

Definition 9.1. Given a 2-functor $U\colon \mathscr{A} \to \mathscr{K}^{\mathbf{2}}$ define another $U^{\pitchfork_{\mathrm{KZ}}}\colon \mathscr{A}^{\pitchfork_{\mathrm{KZ}}} \to \mathscr{K}^{\mathbf{2}}$ in the following manner.

- Its objects are morphisms $g \in \mathscr{K}^{\mathbf{2}}$ that are *algebraically* KZ *injective to* U, by which we mean that they are equipped with a KZ lifting operation for the 2-functors U, $g\colon \mathbf{1} \to \mathscr{K}^{\mathbf{2}}$; ie a left adjoint coretract to the morphism of **Cat**-modules $U^* \cdot \mathsf{Diag} \cdot g_* \to U^* \cdot g_*$. Hence, an object of $\mathscr{A}^{\pitchfork_{\mathrm{KZ}}}$ is an object of $\mathscr{A}^{\pitchfork}$ equipped with the extra structure of a coretract adjunction.
- A morphism $g \to g'$ in $\mathscr{A}^{\pitchfork_{\mathrm{KZ}}}$ is a morphism (h,k) in $\mathscr{K}^{\mathbf{2}}$ such that in the diagram below not only the square formed with the right adjoints commutes – this always holds – but moreover the diagram represents a morphism of adjunctions; ie the square formed by the vertical arrows and the horizontal left adjoints commutes, and the vertical morphisms are compatible with the counits.

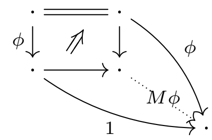
$$\begin{array}{ccc} U^* \cdot \mathsf{Diag} \cdot g_* & \underset{\longrightarrow}{\overset{\longleftarrow}{\bot}} & U^* \cdot g_* \\ {\scriptstyle U^*\cdot\mathsf{Diag}\cdot(h,k)_*}\downarrow & & \downarrow{\scriptstyle U^*\cdot(h,k)_*} \\ U^* \cdot \mathsf{Diag} \cdot g'_* & \underset{\longrightarrow}{\overset{\longleftarrow}{\bot}} & U^* \cdot g'_* \end{array} \tag{9.2}$$

- The 2-cells in $\mathscr{A}^{\pitchfork_{\mathrm{KZ}}}$ are those of $\mathscr{K}^{\mathbf{2}}$. Observe that any such 2-cell is automatically compatible with the left adjoints in (9.2) – by Proposition 4.11 (2). There are obvious forgetful 2-functors $\mathscr{A}^{\pitchfork_{\mathrm{KZ}}} \to \mathscr{A}^{\pitchfork}$ and $\mathscr{A}^{\pitchfork_{\mathrm{KZ}}} \to \mathscr{K}^{\mathbf{2}}$, the first of which is locally fully faithful.

Dually, given a 2-functor $V\colon \mathscr{B} \to \mathscr{K}^{\mathbf{2}}$ define ${}^{\pitchfork_{\mathrm{KZ}}}V\colon {}^{\pitchfork_{\mathrm{KZ}}}\mathscr{B} \to \mathscr{K}^{\mathbf{2}}$ by ${}^{\pitchfork_{\mathrm{KZ}}}\mathscr{B} = (\mathscr{B}^{\mathrm{op}})^{\pitchfork_{\mathrm{KZ}}}$ and ${}^{\pitchfork_{\mathrm{KZ}}}V = (V^{\mathrm{op}})^{\pitchfork_{\mathrm{KZ}}}$. Here we use the obvious isomorphism $(\mathscr{K}^{\mathbf{2}})^{\mathrm{op}} \cong$

$(\mathscr{K}^{\mathrm{op}})^{\mathbf{2}}$. More explicitly, objects of ${}^{\pitchfork_{\mathrm{KZ}}}\mathscr{B}$ are $f \in \mathscr{K}^{\mathbf{2}}$ equipped with a KZ lifting operation for the 2-functors $f\colon \mathbf{1} \to \mathscr{K}^{\mathbf{2}}$, V.

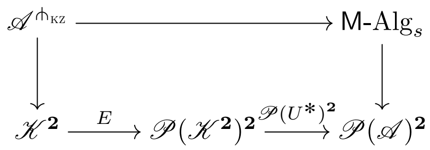
Remark 9.3. There is a concise way of describing $\mathscr{A}^{\pitchfork_{\mathrm{KZ}}}$. Let M be the 2-monad on the 2-category $\mathscr{P}(\mathscr{A})^{\mathbf{2}}$ whose algebras are right adjoint retract morphisms in $\mathscr{P}(\mathscr{A}) = [\mathscr{A}^{\mathrm{op}}, \mathbf{CAT}]$. This 2-monad can be described by performing the construction of the 2-monad of Section 4.d starting from the 2-category $\mathscr{P}(\mathscr{A})^{\mathrm{op}}$. More explicitly, if ϕ is a morphism in $\mathscr{P}(\mathscr{A})$, then $M\phi$ is the morphism with domain the co-comma object depicted and whose composition with this co-comma object is an identity 2-cell.



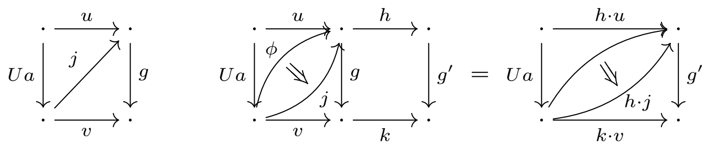
The **Cat**-module morphism $\mathsf{Diag} \longrightarrow 1_{\mathscr{K}^{\mathbf{2}}}$ can be equivalently described as a 2-functor

$$E\colon \mathscr{K}^{\mathbf{2}} \longrightarrow \mathscr{P}(\mathscr{K}^{\mathbf{2}})^{\mathbf{2}}$$

that sends $g \in \mathscr{K}^{\mathbf{2}}$ to $\mathsf{Diag}(-,g) \to \mathscr{K}^{\mathbf{2}}(-,g)$. Then $\mathscr{A}^{\pitchfork_{\mathrm{KZ}}}$ is the pullback of the 2-category of M-algebras along $\mathscr{P}(U^*)^{\mathbf{2}}E$.



Remark 9.4. One can express the compatibility of the morphism (h,k) in $\mathscr{A}^{\pitchfork_{\mathrm{KZ}}}$ with the counits required in Definition 9.1 in terms of diagonal fillers. Given a diagonal filler j as on the left hand side below, the counit provides for a 2-cell $\varepsilon_j\colon \phi(a,u,v) \Rightarrow j$. The compatibility means that $h \cdot \varepsilon_j = \varepsilon_{h\cdot j}$.



These constructions are functorial, in the sense that if $F\colon (\mathscr{A}, U) \to (\mathscr{B}, V)$ is a 2-functor over $\mathscr{K}^{\mathbf{2}}$, there is another 2-functor $F^{\pitchfork_{\mathrm{KZ}}}\colon (\mathscr{B}^{\pitchfork_{\mathrm{KZ}}}, V^{\pitchfork_{\mathrm{KZ}}}) \to (\mathscr{A}^{\pitchfork_{\mathrm{KZ}}}, U^{\pitchfork_{\mathrm{KZ}}})$, which sends $g \in \mathscr{K}^{\mathbf{2}}$ equipped with a KZ lifting operation for V, g to the induced choice for $U = VF, g$. A 2-functor ${}^{\pitchfork_{\mathrm{KZ}}}F$ can be similarly defined.

Remark 9.5. Given $V\colon \mathscr{B} \to \mathscr{K}^{\mathbf{2}}$, there is an isomorphism of categories between 2-functors $\mathscr{B} \to \mathscr{A}^{\pitchfork_{\mathrm{KZ}}}$ over $\mathscr{K}^{\mathbf{2}}$ and KZ lifting operations for the pair of 2-functors U, V. Similarly, there is an isomorphism of categories between 2-functors $\mathscr{A} \to {}^{\pitchfork_{\mathrm{KZ}}}\mathscr{B}$ over $\mathscr{K}^{\mathbf{2}}$ and KZ lifting operations for the pair of 2-functors U, V. We hence have a natural isomorphism of *sets*

$$\mathbf{2\text{-}Cat}/\mathscr{K}^{\mathbf{2}}((\mathscr{A}, U), (\mathscr{B}^{\pitchfork_{\mathrm{KZ}}}, V^{\pitchfork_{\mathrm{KZ}}})) \cong \mathbf{2\text{-}Cat}/\mathscr{K}^{\mathbf{2}}((\mathscr{B}, V), ({}^{\pitchfork_{\mathrm{KZ}}}\mathscr{A}, {}^{\pitchfork_{\mathrm{KZ}}}U))$$

and an adjunction between $(-)^{\pitchfork_{\mathrm{KZ}}}$ and ${}^{\pitchfork_{\mathrm{KZ}}}(-)$.

The unit and counit of this adjunction – or rather, both units – are 2-functors $N_U\colon \mathscr{A} \to {}^{\pitchfork_{\mathrm{KZ}}}(\mathscr{A}^{\pitchfork_{\mathrm{KZ}}})$ and $M_U\colon \mathscr{A} \to ({}^{\pitchfork_{\mathrm{KZ}}}\mathscr{A})^{\pitchfork_{\mathrm{KZ}}}$ commuting with the functors into $\mathscr{K}^{\mathbf{2}}$. The first one corresponds to the tautological KZ lifting operation for the pair of 2-functors $U, U^{\pitchfork_{\mathrm{KZ}}}$, and the second one to the tautological KZ lifting operation for ${}^{\pitchfork_{\mathrm{KZ}}}U, U$.

Example 9.6. In the case when U is the 2-functor $f\colon \mathbf{1} \to \mathscr{K}^{\mathbf{2}}$ that picks out a morphism f, the objects of the 2-category $f^{\pitchfork_{\mathrm{KZ}}}$ are morphisms *algebraically* KZ *injective with respect to* f. This is a slight abuse of language, as a morphism can be algebraically KZ injective to f in more than one way – but two such are, of course, isomorphic.

Lemma 9.7. *Given a 2-functor* $U\colon \mathscr{A} \to \mathscr{K}^{\mathbf{2}}$, *a 2-adjunction* $U \dashv G$ *and* $g \in \mathscr{K}^{\mathbf{2}}$, *there is an isomorphism of 2-categories over* $\mathscr{K}^{\mathbf{2}}$ *between* $\mathscr{A}^{\pitchfork_{\mathrm{KZ}}}$ *and the 2-category described by:*

- *Objects are coretract adjunctions* $\ell_g \dashv G(1,g)\colon G(1_{\mathrm{dom}(g)}) \to Gg$ *in* $\mathscr{A}$.
- *Morphisms from* $\ell_g \dashv G(1,g)$ *to* $\ell_{\bar{g}} \dashv G(1,\bar{g})$ *are morphisms* $(h,k)\colon g \to \bar{g}$ *in* $\mathscr{K}^{\mathbf{2}}$ *such that* $G(h,k)$ *defines a morphism of adjunctions:* $G(h,k)\cdot \ell_g = \ell_{\bar{g}} \cdot G(h,k)$ *and* $G(h,k)$ *commutes with the counits.*
- *2-cells* $(h,k) \Rightarrow (\bar{h},\bar{k})$ *are 2-cells in* $\mathscr{K}^{\mathbf{2}}$, *with no additional conditions.*

Proof. By Proposition 8.9 there is a bijection between objects of $\mathscr{A}^{\pitchfork_{\mathrm{KZ}}}$ and coretract adjunctions as in the statement. The description of the morphisms and 2-cells is a direct translation from the ones of $\mathscr{A}^{\pitchfork_{\mathrm{KZ}}}$ – Definition 9.1. □

Lemma 9.8. *Assume the conditions of Lemma 9.7. Then, for any full sub-2-category* $\mathscr{F} \subset \mathscr{A}$ *containing the full image of* G, *the functor* $\mathscr{A}^{\pitchfork_{\mathrm{KZ}}} \to \mathscr{F}^{\pitchfork_{\mathrm{KZ}}}$ *induced by the inclusion is an isomorphism.*

Proof. If we denote by $J\colon \mathscr{F} \hookrightarrow \mathscr{A}$ the inclusion and $H = JG\colon \mathscr{A} \to \mathscr{F}$ the right adjoint of UJ, Lemma 9.7 allows us to describe $\mathscr{F}^{\pitchfork_{\mathrm{KZ}}}$ as the 2-category with objects coretract adjunctions $\ell_g \dashv H(1,g)\colon H(1_{\mathrm{dom}(g)}) \to Hg$ in $\mathscr{F}$. But to give this retract adjunction in $\mathscr{F}$ is equivalent to giving a retract adjunction $\ell_g \dashv G(1,g)$ in $\mathscr{A}$. The rest of the proof is similarly easy. □

Corollary 9.9. *If* (L,R) *is a lax orthogonal* AWFS *on* $\mathscr{K}$, *there exists a 2-functor*

$$\mathsf{R}\text{-}\mathrm{Alg}_s \longrightarrow \mathsf{L}\text{-}\mathrm{Coalg}_s^{\pitchfork_{\mathrm{KZ}}} \text{ over } \mathsf{L}\text{-}\mathrm{Coalg}_s.$$

Proof. Proposition 8.12 together with Lemma 9.7 imply that, in order to define the 2-functor on objects, we may send an R-algebra $(p,1)\colon Rg \to g$ to a corectract adjunction $\ell \dashv L(1,g)$ in L-Coalg_s. The adjunction is $L(1,p)\cdot \Sigma_g \dashv L(1,g)$, which is the composition of the adjunctions $\Sigma_g \dashv L\Phi_g$ and $L(1,p) \dashv L(1,Lg)$. On morphisms and 2-cells, the 2-functor is defined by the identity. □

Theorem 9.10. *The following are equivalent for an* AWFS (L,R) *on a 2-category.*

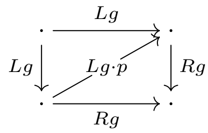
(1) (L,R) *is a lax orthogonal* AWFS.
(2) There is a KZ *lifting operation for the forgetful 2-functors from* L*-coalgebras and from* R*-algebras.*
(3) There is a 2-functor $\mathsf{R}\text{-}\mathrm{Alg}_s \to \mathsf{L}\text{-}\mathrm{Coalg}_s^{\pitchfork_{\mathrm{KZ}}}$ *making* (9.10) *commutative. Furthermore, this 2-functor is essentially unique.*
(4) There is a 2-functor $\mathsf{R}\text{-}\mathrm{Alg}_s \to \mathscr{F}^{\pitchfork_{\mathrm{KZ}}}$ *making the outer diagram in* (9.10) *commutative, for any full sub-2-category* $\mathscr{F} \subset \mathsf{L}\text{-}\mathrm{Coalg}_s$ *containing the cofree* L*-coalgebras. Furthermore, this 2-functor is essentially unique.*

$$\begin{array}{ccccc}
\mathsf{R}\text{-}\mathrm{Alg}_s & \dashrightarrow & \mathsf{L}\text{-}\mathrm{Coalg}_s^{\pitchfork_{\mathrm{KZ}}} & \longrightarrow & \mathscr{F}^{\pitchfork_{\mathrm{KZ}}} \\
& \searrow & \downarrow & & \downarrow \\
& & \mathsf{L}\text{-}\mathrm{Coalg}_s^{\pitchfork} & \longrightarrow & \mathscr{F}^{\pitchfork}
\end{array}$$

Proof. There is a bijection between structures in (2) and those in (3), by definition of $\mathscr{A}^{\pitchfork_{\mathrm{KZ}}}$, in which case both are essentially unique since KZ lifting operations are unique up to isomorphism – Lemma 8.8. The equivalence of (3) and (4) follows from Lemma 9.8, while that of (1) and (3) was already explained above.

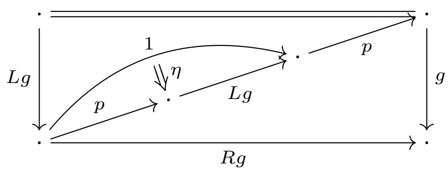
We now proceed to prove (3)⇒(1). As it has been our convention, we will denote by $\mathscr{K}$ the base 2-category, and by U and V the forgetful 2-functors from the 2-categories of L-coalgebras and R-algebras, respectively.

Let (g, p) be an R-algebra. Its image in $\mathsf{L}\text{-Coalg}_s^{\pitchfork}$ can be given as in Corollary 7.4, again by (g, p). By hypothesis, (g, p) carries a structure of an object of $\mathsf{L}\text{-Coalg}_s^{\pitchfork_{\mathrm{KZ}}}$. By definition $p \cdot Lg = 1$ and $g \cdot p = Rg$. Consider the diagonal



and note that Rg is an object of $\mathsf{L}\text{-Coalg}_s^{\pitchfork_{\mathrm{KZ}}}$, and that the chosen diagonal filler of the outer square is the identity morphism. It follows the existence of a unique 2-cell $\eta\colon 1 \Rightarrow Lg \cdot p$ such that $\eta \cdot Lg = 1$ and $Rg \cdot \eta = 1$. The first of these two equalities is one of the triangle identities required to obtain a retract adjunction $p \dashv Lg$. The second of these equalities tells us that, if we can prove the other triangle identity, we obtain not only an adjunction in $\mathscr{K}$ but also a retract adjunction $(p, 1) \dashv \Lambda_g$ in $\mathscr{K}^{\mathbf{2}}$.

We now show that $p \cdot \eta = 1$. Consider the pasting below.



The chosen diagonal filler of the outer diagram is p, and $p \cdot \eta$ is an endo-2-cell of p. In addition, $g \cdot p \cdot \eta = Rg \cdot \eta = 1$ and $p \cdot \eta \cdot Lg = 1$. By the universal property of KZ lifting operations spelled out immediately after Definition 8.2, it must be $p \cdot \eta = 1$. This finishes the proof that R-algebra structures are left adjoint retracts to the components of the unit of R, ie that R is lax idempotent.

One can show that L is lax idempotent either by appealing to Theorem 5.1 or by a duality argument. By taking opposite 2-categories, and taking into account the isomorphism $(\mathscr{K}^{\mathrm{op}})^{\mathbf{2}} \cong (\mathscr{K}^{\mathbf{2}})^{\mathrm{op}}$, the 2-functor $\mathsf{L}\text{-Coalg}_s \to {}^{\pitchfork_{\mathrm{KZ}}}\mathsf{R}\text{-Alg}_s$, which exists by Remark 9.5, transforms into a 2-functor $\mathsf{L}^{\mathrm{op}}\text{-Alg}_s \to \mathsf{R}^{\mathrm{op}}\text{-Coalg}_s^{\pitchfork_{\mathrm{KZ}}}$ that commutes with the 2-functors into $\mathsf{R}^{\mathrm{op}}\text{-Coalg}_s^{\pitchfork}$. By the proof above we know that L^{op} is a lax idempotent 2-monad on $(\mathscr{K}^{\mathbf{2}})^{\mathrm{op}}$, which is to say that L is a lax idempotent 2-comonad. □

Theorem 9.10 has a dual statement of the following form: an AWFS (L, R) is lax orthogonal if and only if there exists an – essentially unique – 2-functor $\mathsf{L}\text{-Coalg}_s \to {}^{\pitchfork_{\mathrm{KZ}}}\mathsf{R}\text{-Alg}_s$ commuting with the respective forgetful functors into ${}^{\pitchfork}\mathsf{R}\text{-Alg}_s$.

Remark 9.11. For a lax orthogonal AWFS (L, R), objects of $\mathsf{L}\text{-Coalg}_s^{\pitchfork_{\mathrm{KZ}}}$ are in bijection with normal pseudo-R-algebras. Indeed, the proof of Theorem 9.10 shows that they are in bijection with retract adjunctions $(p, 1) \dashv \Lambda_g$ in $\mathscr{K}^{\mathbf{2}}$, which are precisely normal pseudo-R-algebras

10. Simple 2-adjunctions and lax idempotent 2-monads

This section introduces the notion of simple 2-adjunction, which can be thought as a lax version of that of simple reflection studied in [**?**].

In the same way that one can define a strict monoidal category as a category with a bifunctor $(- \otimes -)$ that is associative and has a unit object, we may define a *strict monoidal 2-category* as a 2-category $\mathscr{A}$ with a 2-functor $\otimes\colon \mathscr{A} \times \mathscr{A} \to \mathscr{A}$

that is associative and has a unit object I. A monoid in $\mathscr{A}$ is a monoid in its underlying strict monoidal category, ie an object T with a multiplication and unit morphisms that satisfy the usual monoid axioms. The main example for us will be $\mathscr{A} = \mathrm{End}(\mathscr{B})$, the endo-2-morphisms of a 2-category $\mathscr{B}$, where a monoid is a 2-monad.

Definition 10.1. A *lax idempotent monoid* in a strict monoidal 2-category $\mathscr{A}$ is a monoid $j\colon I \to T \leftarrow T\otimes T\colon m$ that satisfies conditions analogous to those of Definition 4.2 numerals (i), (ii) and (iv). These are, in turn,

- $T\otimes j \dashv m$ with identity unit;
- $m \dashv j\otimes T$ with identity counit;
- there is a 2-cell $\delta\colon T\otimes j \Rightarrow j\otimes T\colon T \to T\otimes T$ that satisfies $\delta\cdot j = 1$ and $m\cdot\delta = 1$.

We can now make our first statement of the section. The reader would have noticed that the monoidal 2-categories need not be strict in order for the results to hold, but we keep the strictness hypothesis for simplicity.

Lemma 10.2. *Let $\mathscr{A}$ be a monoidal 2-category and $\mathscr{C} \subseteq \mathscr{A}$ a coreflective 2-category, closed under the monoidal structure, and (T, i, m) a monoid in $\mathscr{A}$. If $\alpha\colon S \to T$ is the coreflection of T into $\mathscr{C}$, then S carries a structure of a monoid (S, j, n) making α a monoid morphism. Assume further that $\alpha\otimes S\colon S\otimes S \to T\otimes S$ is the coreflection of $T\otimes S$. Then S is lax idempotent if there exists a coretract adjunction*

$$\big(T \xrightarrow{T\otimes j} T\otimes S\big) \dashv \big(T\otimes S \xrightarrow{T\otimes\alpha} T\otimes T \xrightarrow{m} T\big). \tag{10.3}$$

Proof. The unit $j\colon I \to S$ and multiplication $n\colon S\otimes S \to S$ are defined by $\alpha\cdot j = i$ and $\alpha\cdot n = m\cdot(\alpha\otimes\alpha)$. We shall define a 2-cell $\delta\colon S\otimes j \Rightarrow j\otimes S\colon S \to S\otimes S$. From the fact that $\alpha\otimes S$ is a coreflection, it follows that to give δ is equally well to give a 2-cell $\delta'\colon (T\otimes j)\cdot\alpha \Rightarrow i\otimes S$, and by the adjunction (10.3), to give a 2-cell $\delta''\colon \alpha \Rightarrow m\cdot(T\otimes\alpha)\cdot(i\otimes S)$, which we choose to be the identity.

The axiom $\delta\cdot j = 1$ of a lax idempotent monoid follows from the triangle identity $\varepsilon\cdot(T\otimes j) = 1$, where ε is the counit of (10.3): we show that $\delta'\cdot j = 1$ below.

$$\delta'\cdot j = ((T\otimes j)\cdot m\cdot(T\otimes\alpha)\cdot\delta'\cdot j)(\varepsilon\cdot(j\otimes S)\cdot j) = \varepsilon\cdot(T\otimes j)\cdot i = 1.$$

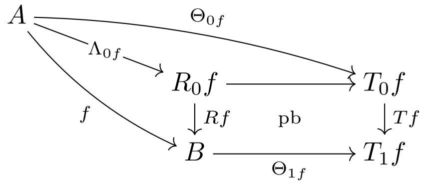
It only rests to verify the axiom $n\cdot\delta = 1$. By the coreflection α, we have to show $1 = \alpha\cdot n\cdot\delta = m\cdot(\alpha\otimes\alpha)\cdot\delta = m\cdot(T\otimes\alpha)\cdot\delta' = \delta''$, which holds by our choice of δ''. □

Before continuing, it is convenient to introduce some notation. Each endo-2-functor S of $\mathscr{K}^{\mathbf{2}}$ corresponds under the isomorphism $\mathrm{End}(\mathscr{K}^{\mathbf{2}}) = [\mathscr{K}^{\mathbf{2}}, \mathscr{K}^{\mathbf{2}}] \cong [\mathscr{K}^{\mathbf{2}}, \mathscr{K}]^{\mathbf{2}}$ to a pair of 2-functors S_0, $S_1\colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}$ with a 2-natural transformation $S_0 \Rightarrow S_1$. We denote the component of this natural transformation at f by $Sf\colon S_0 f \to S_1 f$. A morphism $S \to T$ in $\mathrm{End}(\mathscr{K}^{\mathbf{2}})$ corresponds to a pair of 2-natural transformations $S_0 \Rightarrow T_0$ and $S_1 \Rightarrow T_1$, compatible with $S_0 \Rightarrow S_1$ and $T_0 \Rightarrow T_1$.

A version for categories and functors, as opposite to 2-categories and 2-functors, of the following lemma is contained in [**?**, Prop 4.7].

Lemma 10.4. *If $\mathscr{K}$ has pushouts, then the category of codomain-preserving pointed endo-2-functors $1\backslash\mathrm{End}_{\mathrm{cod}}(\mathscr{K}^{\mathbf{2}})$ is a coreflective sub-2-category of the 2-category of pointed endofunctors $1\backslash\mathrm{End}(\mathscr{K}^{\mathbf{2}})$. The coreflection of a 2-monad has a canonical structure of a codomain-preserving 2-monad that makes the coreflection counit a monad morphism.*

Given a pointed endo-2-functor (T, Θ), its codomain-preserving coreflection (R, Λ) is given by the following pullback square, while the point $\Lambda\colon 1 \Rightarrow R$ is induced by the universal property. The natural transformation $R \Rightarrow T$ with components given by the pullback square is the counit of the coreflection.



Remark 10.5. For future reference, we state that the coreflection $R \Rightarrow T$ of a monad T on $\mathcal{C}^{\mathbf{2}}$ into a codomain-preserving monad R is a monad morphism.

Definition 10.6. Suppose given the following data.

- A 2-adjunction $F \dashv U\colon \mathscr{A} \to \mathscr{K}$, whose counit we denote by $e\colon FU \Rightarrow 1$.
- A 2-monad P on $\mathscr{A}^{\mathbf{2}}$ with multiplication $m\colon P^2 \Rightarrow P$.
- The codomain-preserving coreflection of the 2-monad $U^{\mathbf{2}}\mathsf{P}F^{\mathbf{2}}$, that we denote by $\alpha\colon \mathsf{S} \to U^{\mathbf{2}}\mathsf{P}F^{\mathbf{2}}$, and whose unit we denote $j\colon 1 \Rightarrow S$.

The 2-adjunction is said to be *simple* with respect to P if there is a coretract adjunction in the 2-category $[\mathscr{K}^{\mathbf{2}}, \mathscr{A}^{\mathbf{2}}]$, with components at $f \in \mathscr{K}^{\mathbf{2}}$

$$\big(PFf \xrightarrow{PF^{\mathbf{2}}j_f} PFSf\big) \dashv \big(PFSf \xrightarrow{PF^{\mathbf{2}}\alpha_f} PFUPFf \xrightarrow{PePF^{\mathbf{2}}f} PPFf \xrightarrow{mFf} PFf\big).$$

Lemma 10.7. *Given a simple 2-adjunction as in Definition 10.6, the codomain-preserving reflection* S *is a lax idempotent 2-monad.*

Proof. Let us denote by T the 2-monad $U^{\mathbf{2}}PF^{\mathbf{2}}$. By the construction of the coreflection S as a pullback, it is clear that $\alpha T\colon SS \to TS$ is the coreflection of TS. Lemma 10.2 tells us that S will be lax idempotent if we have a coretract adjunction in $[\mathscr{K}^{\mathbf{2}}, \mathscr{K}^{\mathbf{2}}]$

$$\big(T \xrightarrow{Tj} TS\big) \dashv \big(TS \xrightarrow{T\alpha} TT \longrightarrow T\big).$$

Such an adjunction is obtained from the one of Definition 10.6 by applying $U^{\mathbf{2}}$. $\square$

Let us now make an observation that shall be needed later on.

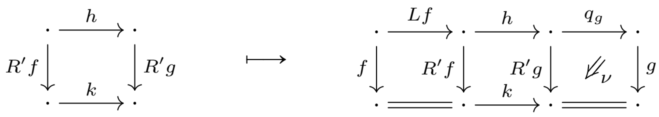
Remark 10.8. Given $F \dashv U$, P and the codomain-preserving coreflection $\alpha\colon S \to U^{\mathbf{2}}PF^{\mathbf{2}}$ as in Lemma 10.7, we claim that the composition of the multiplication with the counit α

$$SS \longrightarrow S \xrightarrow{\alpha} U^{\mathbf{2}}PF^{\mathbf{2}}$$

factors through $U^{\mathbf{2}}mF^{\mathbf{2}}$, where m is the multiplication of P. Indeed, since α is a 2-monad (strict) morphism, we know that

$$(SS \to S \xrightarrow{\alpha} U^{\mathbf{2}}PF^{\mathbf{2}}) = (SS \xrightarrow{\alpha\alpha} U^{\mathbf{2}}PF^{\mathbf{2}}U^{\mathbf{2}}PF^{\mathbf{2}} \to U^{\mathbf{2}}PPF^{\mathbf{2}} \to U^{\mathbf{2}}PF^{\mathbf{2}}).$$

Below we describe Definition 10.6 in a particular case of interest, but before let us recall a few facts about the Kleisli construction for the free split opfibration 2-monad R' on $\mathscr{A}^{\mathbf{2}}$, for a 2-category $\mathscr{A}$ with lax limits of morphisms. This Kleisli construction can be described as the inclusion 2-functor of $\mathscr{A}^{\mathbf{2}}$ into the 2-category $\mathrm{Lax}[\mathbf{2}, \mathscr{A}]$ of 2-functors from $\mathbf{2}$ to $\mathscr{A}$ and lax transformations between them. Morphisms between free R'-algebras are in bijection with morphisms in $\mathrm{Lax}[\mathbf{2}, \mathscr{A}]$, and the bijection is given as displayed below, a fact we shall soon employ.



(10.9)

Proposition 10.10. *Let $F \dashv U \colon \mathscr{A} \to \mathscr{K}$ be a 2-adjunction, where $\mathscr{A}$ has comma objects and $\mathscr{K}$ has pullbacks, and R' be the free split opfibration 2-monad on $\mathscr{A}^{\mathbf{2}}$. Denote by R the codomain-preserving coreflection of $U^{\mathbf{2}}\mathsf{R}'F^{\mathbf{2}}$. The 2-adjunction is simple with respect to the coreflection–opfibration* AWFS *precisely when there are coretract adjunctions $FLf \dashv q_{Ff}\cdot e_{K'Ff}\cdot F\tau_f$ 2-natural in f, where these morphisms are those defined in* (10.11).

Proof. In this proof we shall denote the unit and counit of $F \dashv U$ by i and e, respectively, and the comparison adjoint of the Kleisli construction of a 2-monad P on $\mathscr{A}^{\mathbf{2}}$ by $C \colon \mathrm{Kl}(\mathsf{P}) \to \mathsf{P}\text{-}\mathrm{Alg}_s$. In a moment we will use the well-known fact that C is full and faithful and its replete image is the full subcategory of free algebras. The definition of simple adjunction consists of a coretract adjunction in $[\mathscr{K}^{\mathbf{2}}, \mathscr{A}^{\mathbf{2}}]$ between 2-natural transformations whose components are strict morphisms of P-algebras between free P-algebras in $\mathscr{A}^{\mathbf{2}}$. Therefore, the said coretract adjunction is the image of a coretract adjunction in $[\mathscr{K}^{\mathbf{2}}, \mathrm{Kl}(\mathsf{P})]$ under the 2-functor

$$[1, C] \colon [\mathscr{K}^{\mathbf{2}}, \mathrm{Kl}(\mathsf{P})] \longrightarrow [\mathscr{K}^{\mathbf{2}}, \mathsf{P}\text{-}\mathrm{Alg}_s].$$

When P is the free split opfibration 2-monad R', its Kleisli construction is isomorphic to the inclusion of $\mathscr{A}^{\mathbf{2}}$ into $\mathrm{Lax}[\mathbf{2}, \mathscr{A}]$, by the comments before the present proposition. We can use the correspondence between morphisms of free R'-algebras and morphisms in $\mathrm{Lax}[\mathbf{2}, \mathscr{A}]$ described in (10.9) to deduce the form of the lifting to $[\mathscr{K}^{\mathbf{2}}, \mathrm{Lax}[\mathbf{2}, \mathscr{A}]]$ of the coretract adjunction in $[\mathscr{K}^{\mathbf{2}}, \mathscr{A}^{\mathbf{2}}]$ that exhibits $F \dashv U$ as a simple 2-adjunction. The lifting has component at $f \in \mathscr{K}^{\mathbf{2}}$ displayed below, where ν is the comma object that defines R'.

$$\begin{array}{ccc} FA & \xrightarrow{FLf} & FKf \\ {\scriptstyle Ff}\big\downarrow & & \big\downarrow{\scriptstyle FRf} \\ FB & = & FB \end{array} \quad \dashv \quad \begin{array}{ccccccc} FKf & \xrightarrow{F\tau_f} & FUK'Ff & \xrightarrow{e} & K'Ff & \xrightarrow{q_{Ff}} & FA \\ \big\downarrow{\scriptstyle FRf} & & \big\downarrow{\scriptstyle FUR'Ff} & & \big\downarrow{\scriptstyle R'Ff} & \Swarrow_{\nu} & \big\downarrow{\scriptstyle Ff} \\ FB & \xrightarrow{Fi_B} & FUFB & \xrightarrow{e} & FB & = & FB \end{array} \tag{10.11}$$

This adjunction consists of a coretract adjunction as in the statement of this proposition, plus the requirement that its counit, say α_f, is a 2-cell in $\mathrm{Lax}[\mathbf{2}, \mathscr{A}]$; ie

$$FRf \cdot \alpha_f = \nu \cdot e_{K'Ff} \cdot F\tau_f. \tag{10.12}$$

Thus, the direct part of the statement is trivial. To prove the converse, we will show that if FLf has a right adjoint retract in $\mathscr{K}$ as in the statement, then (10.12) automatically holds. As a consequence of the adjunction, the 2-cell on the left hand side of (10.12) is the unique 2-cell β, with the appropriate domain and codomain, such that $\beta \cdot FLf = 1$. We must verify that the 2-cell on the right hand side of (10.12) satisfies the same property. By definition of Lf,

$$e_{K'Ff} \cdot F\tau_f \cdot FLf = e_{K'Ff} \cdot FUL'(Ff) \cdot Fi_A = L'(Ff) \cdot e_{FA} \cdot Fi_A = L'(Ff),$$

from where it is clear that $\nu \cdot e_{K'Ff} \cdot F\tau_f \cdot FLf = \nu \cdot L'(Ff) = 1$, concluding the proof. □

Lemma 10.7 yields:

Corollary 10.13. *If $F \dashv U \colon \mathscr{A} \to \mathscr{K}$ is a 2-adjunction simple with respect to the free split opfibration 2-monad R' on $\mathscr{A}^{\mathbf{2}}$, then the codomain-preserving coreflection R of $U^{\mathbf{2}}\mathsf{R}'\mathsf{F}^{\mathbf{2}}$ is lax idempotent.*

Remark 10.14. There is a 2-monad morphism with components $(\tau_f, i_{\mathrm{cod}(f)}) \colon Rf \to UR'Ff$, by Remark 10.5. Taking the mate along $F \dashv U$, we obtain an opmorphism of 2-monads $(\hat{\tau}_f, 1_{F\,\mathrm{cod}(f)}) \colon FRf \to R'Ff$.

Recall from Lemma 2.8 that the codomain functor is a fibration from (the underlying category of) $\mathsf{R}\text{-Alg}_s$ to $\mathcal{C}$. In particular the category of split opfibrations in a 2-category $\mathscr{K}$ with lax limits of morphisms is a fibration over $\mathscr{K}$.

Theorem 10.15. *Assume given a 2-adjunction $F \dashv U\colon \mathscr{A} \to \mathscr{K}$ between 2-categories equipped with chosen lax limits of morphisms and pullbacks, strictly preserved by U. If the 2-monad R is as in Proposition 10.10, then there is a canonical 2-functor into the category of split opfibrations in $\mathscr{K}$ that commutes with the forgetful functors into $\mathscr{K}^{\mathbf{2}}$.*

$$\mathsf{R}\text{-Alg}_s \longrightarrow \mathbf{OpFib}_s(\mathscr{K})$$

Proof. Denote by $\mathsf{R}'_{\mathscr{A}}$ and $\mathsf{R}'_{\mathscr{K}}$ the free split opfibration 2-monad on $\mathscr{A}^{\mathbf{2}}$ and $\mathscr{K}^{\mathbf{2}}$ respectively. Clearly $U^{\mathbf{2}}\mathsf{R}'_{\mathscr{A}} = \mathsf{R}'_{\mathscr{K}}U^{\mathbf{2}}$, and there is a monad morphism $\mathsf{R}'_{\mathscr{K}} \to U^{\mathbf{2}}\mathsf{R}'_{\mathscr{A}}F^{\mathbf{2}}$. Since R is by definition the codomain-preserving coreflection of $U^{\mathbf{2}}\mathsf{R}'_{\mathscr{A}}F^{\mathbf{2}}$, there exists a 2-monad morphism $\mathsf{R} \to \mathsf{R}'_{\mathscr{K}}$, which induces the 2-functor of the statement. □

Remark 10.16. The 2-monad R on $\mathscr{K}^{\mathbf{2}}$ of Proposition 10.10 has a slightly more elementary description that will be useful later in our work. The associated 2-functorial factorisation $f = Rf \cdot Lf\colon A \to B$ can be described as follows. The morphism Rf is given by the comma object displayed below, and Lf is the unique morphism such that $\mu_f \cdot Lf = 1$.

$$\begin{array}{ccc} A & \xrightarrow{i_A} & UFA \\ & \searrow^{Lf} & \\ & Kf \xrightarrow{q_f} UFA & \\ \searrow^{f} & Rf\downarrow \quad \not\Downarrow \mu_f \quad \downarrow UFf & \\ & B \xrightarrow[i_B]{} UFB & \end{array} \tag{10.17}$$

This is so because μ_f is related to the comma object ν that appears in (10.11) via the equality below.

$$\begin{array}{ccc} Kf & \xrightarrow{q_f} & UFA \\ {\scriptstyle Rf}\downarrow & \not\Downarrow \mu_f & \downarrow{\scriptstyle UFf} \\ B & \xrightarrow[i_B]{} & UFB \end{array} \;=\; \begin{array}{ccccc} Kf & \xrightarrow{\tau_f} & UK'Ff & \xrightarrow{Uq_{Ff}} & UFA \\ {\scriptstyle Rf}\downarrow & \text{pb} & \downarrow{\scriptstyle UR'Ff} & \not\Downarrow U\nu_{Ff} & \downarrow{\scriptstyle UFf} \\ B & \xrightarrow[i_B]{} & UFB & = & UFB \end{array} \tag{10.18}$$

The multiplication is given by a morphism $\pi_f\colon R^2 f \to Rf$ that satisfies the equality below.

$$\begin{array}{ccccc} KRf & \xrightarrow{\pi_f} & Kf & \xrightarrow{q_f} & UFA \\ {\scriptstyle R^2 f}\downarrow & & \downarrow{\scriptstyle Rf} & \not\Downarrow \mu_f & \downarrow{\scriptstyle UFf} \\ B & = & B & \xrightarrow[i_B]{} & UFB \end{array} \;=\; \begin{array}{ccccccc} KRf & \xrightarrow{q_{Rf}} & UFKf & \xrightarrow{UFq_f} & UFUFA & \xrightarrow{Ue_{FA}} & UFA \\ {\scriptstyle R^2 f}\downarrow & \not\Downarrow \mu_{Rf} & \downarrow{\scriptstyle UFRf} & \not\Downarrow UF\mu_f & \downarrow{\scriptstyle UFUFf} & & \downarrow{\scriptstyle UFf} \\ B & \xrightarrow[i_B]{} & UFB & \xrightarrow[UFi_B]{} & UFUFB & \xrightarrow[Ue_{FB}]{} & UFB \end{array} \tag{10.19}$$

11. AWFSs through simple adjunctions

If $(\mathsf{L}', \mathsf{R}')$ is an AWFS on $\mathcal{A}$, and $F \dashv U\colon \mathcal{A} \to \mathcal{C}$ an adjunction, we obtain a *transferred right algebraic weak factorisation system* $((L, \Phi), \mathsf{R})$ in $\mathcal{C}$. The monad R

is the codomain-preserving coreflection of the monad $U^{\mathbf{2}}\mathsf{R}'F^{\mathbf{2}}$ on $\mathcal{C}^{\mathbf{2}}$ – Lemma 10.4. This means that Rf is given by the pullback in $\mathcal{C}$ depicted on the left hand side.

$$\begin{array}{ccc} Kf & \longrightarrow & UK'Ff \\ {\scriptstyle Rf}\downarrow & & \downarrow{\scriptstyle UR'Ff} \\ B & \xrightarrow{i_B} & UFB \end{array} \qquad \begin{array}{ccccc} A & \xrightarrow{i_A} & UFA & & \\ & \searrow^{Lf} & & \searrow^{UL'Ff} & \\ & & Kf & \xrightarrow{\tau_f} & UK'Ff \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle Rf} & & \downarrow{\scriptstyle UR'Ff} \\ & & B & \xrightarrow{i_B} & UFB \end{array}$$

The associated 2-functorial factorisation is given by $f = Rf \cdot Lf\colon A \to B$ where Lf is as in the diagram on the right hand side above. The corresponding domain-preserving copointed endofunctor (L, Φ) on $\mathcal{C}^{\mathbf{2}}$ sends f to Lf and $\Phi_f = (1, Rf)\colon Lf \to f$, and can be constructed as the pullback on the left below. As a consequence, the diagram on the right is a pullback, as can be easily shown.

$$\begin{array}{ccc} L & \longrightarrow & U^{\mathbf{2}}L'F^{\mathbf{2}} \\ {\scriptstyle \Phi}\downarrow & & \downarrow{\scriptstyle U^{\mathbf{2}}\Phi' F^{\mathbf{2}}} \\ 1 & \xrightarrow{i^{\mathbf{2}}} & U^{\mathbf{2}}F^{\mathbf{2}} \end{array} \qquad \begin{array}{ccc} (L,\Phi)\text{-Coalg} & \longrightarrow & (L',\Phi')\text{-Coalg} \\ \downarrow & & \downarrow \\ \mathcal{C}^{\mathbf{2}} & \xrightarrow{F^{\mathbf{2}}} & \mathcal{A}^{\mathbf{2}} \end{array} \tag{11.1}$$

The above considerations hold not only in the case of categories and functors but also in the case of 2-categories, 2-adjunctions, etc, which we assume for the rest of the section. We also assume that the comonad L' on the 2-category $\mathscr{A}^{\mathbf{2}}$ has as coalgebras the left adjoint coretracts in the 2-category $\mathcal{A}$, which we assume to have comma objects; the 2-category L'-Coalg$_s$ can be written in the notation used in [?] as $\mathbf{Lari}(\mathcal{A})$, where LARI stands for left adjoint right inverse. The resulting factorisation $f = Rf \cdot Lf$ can be described by the following diagram, where the 2-cell μ is a comma object.

$$\begin{array}{ccccc} A & & & & \\ & \searrow^{Lf} & & & \\ & & Kf & \xrightarrow{q_f} & UF \\ & & {\scriptstyle Rf}\downarrow & \Downarrow \mu_f & \downarrow{\scriptstyle UFf} \\ & & B & \xrightarrow[i_B]{} & UFB \end{array} \quad = \quad \begin{array}{ccc} A & \xrightarrow{i_A} & UFA \\ {\scriptstyle f}\searrow & = & \downarrow{\scriptstyle UFf} \\ B & \xrightarrow[i_B]{} & UF \end{array}$$

Definition 11.2. If $F\colon \mathscr{K} \to \mathscr{A}$ is a 2-functor, define a 2-category

$$\begin{array}{ccc} F\text{-}\mathbf{Emb} & \longrightarrow & \mathbf{Lari}(\mathscr{A}) \\ {\scriptstyle G}\downarrow & \text{pb} & \downarrow \\ \mathscr{K}^{\mathbf{2}} & \xrightarrow[F^{\mathbf{2}}]{} & \mathscr{A}^{\mathbf{2}} \end{array}$$

whose objects may be called *F-embeddings* – the terminology is widely used in the context of categories enriched in posets, as topological embeddings are F-embeddings for a certain choice of F. Explicitly, an object of F-**Emb** is a morphism f in $\mathscr{K}$ equipped with a right adjoint retract r_f for Ff in $\mathscr{A}$, with counit $\alpha_f\colon Ff \cdot r_f \Rightarrow 1$. A morphism $(f, r_f, \alpha_f) \to (g, r_g, \alpha_g)$ is a morphism $(h, k)\colon f \to g$ in $\mathscr{K}^{\mathbf{2}}$ such that (Fh, Fk) is a morphism in $\mathbf{Lari}(\mathscr{A})$; this means that $Fh \cdot r_f = r_g \cdot Fk$. (It is not hard to show that the compatibility with the counits, expressed in $Fk \cdot \alpha_f = \alpha_g \cdot Fk$, is automatically satisfied.)

Remark 11.3. The composition of LARIs described in Section 4.d induces a composition on F-**Emb** and the projections $\mathscr{K}^{\mathbf{2}} \leftarrow F$-**Emb** $\to \mathbf{Lari}(\mathscr{K})$ preserve it. Explicitly, if (f, r) and (f', r') are objects of F-**Emb** with f and f' composable morphisms of $\mathscr{K}$, then $(f' \cdot f, r \cdot r')$ has a canonical structure of an F-embedding arising from $F(f' \cdot f) \cong Ff' \cdot Ff \dashv r \cdot r'$.

The pullback square that defines F-**Emb** can be factorised as the pasting of two pullback squares, one of which we have already met in (11.1). In particular, each $f \in F$-**Emb** has an (L, Φ)-coalgebra structure.

$$\begin{array}{ccc} F\text{-}\mathbf{Emb} & \longrightarrow & \mathbf{Lari}(\mathscr{A}) \\ \downarrow & & \downarrow \\ (L,\Phi)\text{-Coalg} & \longrightarrow & (L',\Phi')\text{-Coalg} \\ \downarrow & & \downarrow \\ \mathscr{K}^{\mathbf{2}} & \xrightarrow{F^{\mathbf{2}}} & \mathscr{A}^{\mathbf{2}} \end{array} \tag{11.4}$$

Theorem 11.5. *Suppose given a 2-adjunction $F \dashv U\colon \mathscr{A} \to \mathscr{K}$ where $\mathcal{A}$ has comma objects and $\mathscr{K}$ has pullbacks. Then the following are equivalent.*

(1) The 2-adjunction is simple.
*(2) The copointed endo-2-functor (L, Φ) can be extended to a comonad L and F-**Emb** is isomorphic to L-Coalg$_s$ over (L, Φ)-Coalg.*
*(3) The forgetful 2-functor $G\colon F$-**Emb** $\to \mathscr{K}^{\mathbf{2}}$ has a right adjoint and the induced comonad on $\mathscr{K}^{\mathbf{2}}$ has underlying copointed endo-2-functor (L, Φ).*

Proof. (1)⇒(3) The hypothesis tells us that there are adjunctions $FLg \dashv r_{Lg}$ in $\mathscr{A}$, where $r_{Lg} = e_{FA} \cdot Fq_g\colon FKf \to FA$, with counits $\varepsilon_{Lg}\colon FLf \cdot r_{Lg} \Rightarrow 1$ that are modifications in $g \in \mathscr{K}^{\mathbf{2}}$. This defines a 2-functor $J\colon \mathscr{K}^{\mathbf{2}} \to F$-**Emb** that sends g to $(Lg, r_{Lg}, \varepsilon_{Lg})$. We shall show that J is a right adjoint to G.

Suppose that (f, r_f, α_f) is an object of F-**Emb**, and construct a morphism s_f as the unique morphism satisfying the displayed equality.

$$\begin{array}{ccccc} B & \xrightarrow{s_f} & Kf & \xrightarrow{q_f} & UFA \\ & & {\scriptstyle Rf}\downarrow & \Downarrow \mu_f & \downarrow{\scriptstyle UFf} \\ & & B & \xrightarrow[i_B]{} & UFB \end{array} \quad = \quad \begin{array}{ccccc} B & \xrightarrow{i_B} & UFB & \xrightarrow{Ur_f} & UFA \\ & & & \Downarrow U\alpha_f & \downarrow{\scriptstyle UFf} \\ & & & \searrow^{1} & UFB \end{array} \tag{11.6}$$

To be more precise, $q_f \cdot s_f = Ur_f \cdot i_Y$, $Rf \cdot s_f = 1$ and $\mu_f \cdot s_f = U\alpha_f \cdot i_Y$. We claim that

$$\begin{array}{ccc} A & = & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Lf} \\ B & \xrightarrow{s_f} & Kf \end{array} \qquad \text{is a morphism in } F\text{-}\mathbf{Emb}.$$

We have to show the compatiblity of s_f with the right adjoints, which means that $r_{Lf} \cdot Fs_f = r_f$ should hold, as it indeed does, as witnessed by

$$r_{Lf} \cdot Fs_f = e_{FA} \cdot Fq_f \cdot Fs_f = e_{FA} \cdot F(q_f \cdot s_f) = e_{FA} \cdot F(Ur_f \cdot i_B) = r_f.$$

The morphisms $\Psi_{(f,r_f,\alpha_f)} = (1, s_f)\colon f \to Lf$ in F-**Emb** form a 2-natural transformation $\Psi\colon 1_{F\text{-}\mathbf{Emb}} \Rightarrow JU$. On the other hand, we already have a 2-natural transformation $\Phi\colon UJ = L \Rightarrow 1_{\mathscr{K}^{\mathbf{2}}}$ with components $\Phi_g = (1, Rg)$. We now proceed to show that these are the unit and the counit of a 2-adjunction $G \dashv J$. We start by considering

$$G \overset{G\Psi}{\Longrightarrow} GJG = LG \overset{\Phi G}{\Longrightarrow} G$$

and evaluating on $(f, r_f, \alpha_f) \in F$-**Emb** to obtain $(1, Rf) \cdot (1, s_f) = (1, Rf, \cdot s_f) = (1, 1)$, so one of the triangle identities holds. The other triangle identity involves the composition

$$J \overset{\Psi J}{\Longrightarrow} JGJ \overset{J\Psi}{\Longrightarrow} J$$

which evaluated on $g\colon X \to Y$ in $\mathscr{K}^{\mathbf{2}}$ gives

$$\begin{array}{ccccc} \cdot & = & \cdot & = & \cdot \\ {\scriptstyle Lg}\downarrow & & {\scriptstyle L_g^2}\downarrow & & \downarrow{\scriptstyle Lg} \\ \cdot & \xrightarrow{s_{Lg}} & \cdot & \xrightarrow{K(1,Rg)} & \cdot \end{array}$$

where s_{Lg} is defined according to (11.6). We have to show that $K(1, Rg) \cdot s_{Lg} = 1$; since the codomain is the comma object Kg, this equality is equivalent to the conjunction of the three conditions

$$Rg \cdot K(1, Rg) \cdot s_{Lg} = Rg \qquad q_g \cdot K(1, Rg) \cdot s_{Lg} = q_g \qquad \mu_g \cdot K(1, Rg) \cdot s_{Lg} = \mu_g$$

where $\mu_g \colon UFg \cdot q_g \Rightarrow i_Y \cdot Rg$ is the universal 2-cell of the comma object. The first of these equalities always holds, since $Rg \cdot K(1, Rg) \cdot s_{Lg} = Rg \cdot RLg \cdot s_{Lg} = Rg$. The second equality also holds, by definition of r_{Lg}:

$$q_g \cdot K(1, Rg) \cdot s_{Lg} = q_{Lg} \cdot s_{Lg} = Ur_{Lg} \cdot i_{Kg} = Ue_{FX} \cdot UFq_g \cdot i_{Kg} = q_g.$$

The third and last equality can be rewritten by using what we know of the definition of s_f, as

$$\mu_g = \mu_g \cdot K(1, Rg) \cdot s_{Lg} = UFRg \cdot \mu_{Lg} \cdot s_{Lg} = UFRg \cdot U\alpha_{Lg} \cdot i_Y \tag{11.7}$$

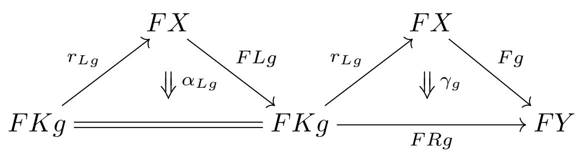
and in order to prove (11.7) we may consider the 2-cell γ_g that is the transpose of μ_g under the 2-adjunction $F \dashv U$.

$$\begin{array}{ccc} Kg & \xrightarrow{q_g} & UFX \\ {\scriptstyle Rg}\downarrow & \swarrow\!\!\!\!\not\;\, \mu_g & \downarrow{\scriptstyle UFg} \\ Y & \xrightarrow[i_Y]{} & UFY \end{array} \qquad \longleftrightarrow \qquad \begin{array}{ccc} FUFX & \xrightarrow{e_{FX}} & FX \\ {\scriptstyle Fq_g}\uparrow & \Downarrow \gamma_g & \downarrow{\scriptstyle Fg} \\ FKg & \xrightarrow[FRg]{} & FY \end{array}$$

We will soon use the fact that the equality $\mu_g \cdot Lg = 1$ translates to $\gamma_g \cdot FLg = 1$. Transposing (11.7) under $F \dashv U$ yields the equivalent equation

$$\gamma_g = FRg \cdot \alpha_{Lg}$$

that we prove by considering the pasting diagram depicted below



and calculating

$$FRg \cdot \alpha_{Lg} = \big(\gamma_g \cdot FLg \cdot r_{Lg}\big)\big(FRg \cdot \alpha_{Lg}\big) = \big(Fg \cdot r_{Lg} \cdot \alpha_{Lg}\big)\gamma_g = \gamma_g$$

where we have used $\gamma_g \cdot FLg = 1$ and $r_{Lg} \cdot \alpha_{Lg} = 1$. This completes this part of the proof.

(3)⇒(2) is clear.

(2)⇒(1) Suppose that $\mathsf{L} = (L, \Phi, \Sigma)$ is a 2-comonad with category of coalgebras and strict maps isomorphic to F-**Emb** over (L, Φ)-Coalg. The comultiplication has components $\Sigma_f = (1, \sigma_f)\colon Lf \to L^2 f$. Since Σ_f is a (L, Φ)-coalgebra structure on Lf, and the commutativity of the top square in (11.4), the morphism $\sigma_f \colon Kf \to KLf$ and the right adjoint r_{Lf} of FLf are related by

$$r_{Lf} = e_{FA} \cdot Fq_{Lf} \cdot F\sigma_f \colon FKLf \longrightarrow FA.$$

We have to show that $r_{Lf} = e_{FA} \cdot Fq_f$, for which it will suffice to show that

$$q_f = q_{Lf} \cdot \sigma_f.$$

We make use of the counit axiom $1 = L\Phi_f \cdot \Sigma_f$, whose codomain component is $1_{Kf} = K(1, Rf) \cdot \sigma_f$. Composing the latter with q_f we obtain the required equality: $q_f = q_f \cdot K(1, Rf) \cdot \sigma_f = q_{Lf} \cdot \sigma_f$. □

Theorem 11.8. *If the 2-adjunction $F \dashv U \colon \mathscr{A} \to \mathscr{K}$ is simple, where $\mathscr{A}$ has comma objects and $\mathscr{K}$ has pullbacks, then there exists an* AWFS (L, R) *on $\mathscr{K}$ that extends the 2-functorial factorisation $f = Rf \cdot Lf$. Furthermore,*

- *this* AWFS *is lax idempotent; and,*
- *the 2-category $\mathsf{L}\text{-Coalg}_s$ is isomorphic over $\mathscr{K}^{\mathbf{2}}$ to the 2-category F-**Emb** of Definition 11.2.*
- *the 2-monad R is the codomain-preserving reflection of $U^{\mathbf{2}}\mathsf{R}'\mathsf{F}^{\mathbf{2}}$, where R' is the free split opfibration 2-monad on $\mathscr{A}^{\mathbf{2}}$.*

Proof. The category F-**Emb** has an obvious composition, mentioned in Remark 11.3, so $\mathsf{L}\text{-Coalg}_s$ does too. This already generates a 2-monad $\mathsf{R} = (R, \Lambda, \Phi)$ with $\Lambda_f = (Lf, 1)$. We shall show that R is the codomain preserving coreflection of the monad $U^{\mathbf{2}}\mathsf{R}'\mathsf{F}^{\mathbf{2}}$ considered in Section 10, where R' is the free split opfibration 2-monad on $\mathscr{A}$. Both this coreflection, that we denote by $\bar{\mathsf{R}}$, and R have the same underlying 2-functor and unit; this is because the description of the respective 2-functorial factorisations in Remark 10.16 and just before Definition 11.2 coincide. It remains to prove that they share the same multiplication. Denote by $\bar{\Pi}$ the multiplication of $\bar{\mathsf{R}}$ and by Π that of R. We know from Section 3.b, or, more precisely, by a dual case to that explained in that section, that Π can be described in terms of the composition of L-coalgebras as the unique morphism of L-coalgebras

$$\begin{array}{ccc} A & = & A \\ {\scriptstyle Lf}\downarrow & & \\ Kf & & \downarrow{\scriptstyle Lf} \\ {\scriptstyle LRf}\downarrow & & \\ KRf & \xrightarrow{\pi_f} & Kf \end{array}$$

that composed with the counit $(1, Rf) \colon Lf \to f$ equals the morphism $(1, R^2 f)$ in $\mathscr{K}^{\mathbf{2}}$; the L-coalgebra structure of $L(Rf) \cdot Lf$ is that given by the composition of coalgebras, ie the composition of F-**Emb**. To show $\Pi = \bar{\Pi}$ we only need to show that $(1, \bar{\pi}_f)$ satisfies the same properties. We certainly know that $Rf \cdot \bar{\pi}_f = R^2 f$, so it remains to show that $(1, \bar{\pi}_f)$ is a strict morphism of L-coalgebras, or, what is the same, that it is a morphism in F-**Emb**. We know that there are coretract adjunctions $FLf \dashv r_{Lf}$ and $FL(Rf) \dashv r_{LRf}$ in $\mathscr{A}$. To say that $(1, \bar{\pi}_f)$ is a morphism in F-**Emb** is to say that

$$r_{Lf} \cdot F\bar{\pi}_f = r_{Lf} \cdot r_{LRf},$$

or, substituting the right adjoints r by their expressions given by the simplicity of $F \dashv U$,

$$\begin{aligned} e_{FA} \cdot Fq_f \cdot F\bar{\pi}_f &= e_{FA} \cdot Fq_f \cdot e_{Kf} \cdot Fq_{Lf} = \\ &= e_{FA} \cdot e_{FUFA} \cdot FUFq_f \cdot Fq_{Lf} = e_{FA} \cdot FUe_{FA} \cdot FUFq_f \cdot Fq_{Lf}. \end{aligned} \tag{11.9}$$

Taking the transpose of each side under $F \dashv U$, the equality is equivalent to

$$q_f \cdot \bar{\pi}_f = Ue_{FA} \cdot UFq_f \cdot q_{Lf}$$

which is precisely the equality satisfied by $\bar{\pi}_f$ as mentioned in Remark 10.16. This completes the proof. □

We now look at the fibrant replacement 2-monad associated to the AWFS constructed.

Corollary 11.10. *Suppose that in Theorem 11.5 the 2-category $\mathscr{K}$ has a terminal object* 1*, and that* $i_1\colon 1 \to UF1$ *is a right adjoint of* $UF1 \to 1$*. Then the restriction of* R *to* $\mathscr{K}/1 \cong \mathscr{K}$ *– the fibrant replacement 2-monad of* (L, R) *– is isomorphic to* UF*.*

Proof. Let us denote by $f\colon A \to 1$ the unique morphism into the terminal object, and by R_1 the restriction of R to $\mathscr{K}/1$. We shall show that in the comma object

$$\begin{array}{ccc} Kf & \xrightarrow{q_A} & UFA \\ {\scriptstyle Rf}\downarrow & \not\Leftarrow & \downarrow{\scriptstyle UFf} \\ 1 & \xrightarrow[i_1]{} & UF1 \end{array}$$

the projection q_A is an isomorphism. For any morphism $x\colon X \to UFA$, there exists a unique 2-cell $UFf \cdot x \Rightarrow i_1 \cdot !$, as these are in bijection, by mateship along $i_1 \dashv f$, with endo-2-cells of $X \to 1$, of which there is only one. Hence $\mathscr{K}(X, q_A)$ is an isomorphism, for each X, and thus q_A is an isomorphism. Since $q_A \cdot Lf = i_A$, and the compatibility of q with the multiplication of R and UF exhibited by (10.19), namely

$$q_A \cdot \pi_f = Ue_{FA} \cdot UFq_f \cdot q_{Kf},$$

we have that q is a 2-monad isomorphism $q\colon R_1 \to UF$. □

We conclude the section with the following lemma, which will be of use in later sections. Corollary 11.10 says that for any morphism $b\colon 1 \to B$ from the terminal object of $\mathscr{K}$ the fibre A_b of any R-algebra $g\colon A \to B$ – ie the pullback of g along b – has a structure of a T-algebra, for the monad T induced by $F \dashv U$.

$$\begin{array}{ccc} A_b & \xrightarrow{z_b} & A \\ {\scriptstyle !}\downarrow & & \downarrow{\scriptstyle g} \\ 1 & \xrightarrow[b]{} & B \end{array}$$

Furthermore, (z_b, b) is a morphism of R-algebras.

Lemma 11.11. *Assume the conditions of Corollary 11.10, and denote by* T *the monad generated by* $F \dashv U$*. Given* $g\colon A \to B$ *and* $b\colon 1 \to B$*, the morphism*

$$(Kg)_b \xrightarrow{z_b} Kg \xrightarrow{q_g} TA$$

is a morphism of T*-algebras.*

Proof. Denote by $a\colon T(Kg)_b \to (Kg)_b$ the T-algebra structure given by Corollary 11.10. We are to show that the following rectangle commutes.

$$\begin{array}{ccccc} T(Kg)_b & \xrightarrow{Tz_b} & TKg & \xrightarrow{Tq_b} & T^2A \\ {\scriptstyle a}\downarrow & & & & \downarrow{\scriptstyle m_A} \\ (Kg)_b & \xrightarrow{z_b} & Kg & \xrightarrow{q_b} & TA \end{array} \tag{11.12}$$

In order to do so, consider the string of equalities displayed below, the first of which holds since (z_b, b) is a morphism of R-algebras; the second holds by definition of π_g, depicted in (10.19); the next equality reflects the definition of $K(z_b, b)$ and the fact that the restriction of R to $\mathscr{K}/1$ is T – Corollary 11.10.

$$\begin{array}{ccccccc} T(Kg)_b & \xrightarrow{a} & (Kg)_b & \xrightarrow{z_b} & Kg & \xrightarrow{q_g} & TA \\ \downarrow & & \downarrow & & {\scriptstyle Rg}\downarrow & \not\Leftarrow & \downarrow{\scriptstyle Tg} \\ 1 & = & 1 & \xrightarrow[b]{} & B & \xrightarrow[i_B]{} & TB \end{array} \quad = \quad \begin{array}{ccccccc} T(Kg)_b & \xrightarrow{K(z_b,b)} & KRg & \xrightarrow{\pi_g} & Kg & \xrightarrow{q_g} & TA \\ \downarrow & & {\scriptstyle R^2g}\downarrow & & {\scriptstyle Rg}\downarrow & \not\Leftarrow & \downarrow{\scriptstyle Tg} \\ 1 & \xrightarrow[b]{} & B & = & B & \xrightarrow[i_B]{} & TB \end{array} \quad =$$

$$
= \begin{array}{ccccccccc}
T(Kg)_b & \xrightarrow{K(z_b,b)} & KRg & \longrightarrow & TKg & \xrightarrow{Tq_g} & T^2A & \xrightarrow{m_A} & TA \\
\downarrow & & \downarrow{\scriptstyle R^2g} & \Swarrow & \downarrow{\scriptstyle TRg} & \Swarrow & \downarrow{\scriptstyle T^2g} & & \downarrow{\scriptstyle Tg} \\
1 & \xrightarrow[b]{} & B & \xrightarrow[i_B]{} & TB & \xrightarrow[Ti_B]{} & T^2B & \xrightarrow[m_B]{} & TB
\end{array} =
$$

$$
= \begin{array}{ccccccccc}
T(Kg)_b & \xrightarrow{1} & T(Kg)_b & \xrightarrow{Tz_b} & TKg & \xrightarrow{Tq_g} & T^2A & \xrightarrow{m_A} & TA \\
\downarrow & \Swarrow_! & \downarrow & & \downarrow{\scriptstyle TRg} & \Swarrow & \downarrow{\scriptstyle T^2g} & & \downarrow{\scriptstyle Tg} \\
1 & \xrightarrow[i_1]{} & T1 & \xrightarrow[Tb]{} & TB & \xrightarrow[Ti_B]{} & T^2B & \xrightarrow[m_B]{} & TB
\end{array}
$$

It is now clear that (11.12) commutes, completing the proof. □

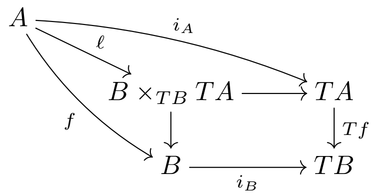
12. Simple 2-monads

A 2-monad T on a 2-category $\mathscr{K}$ with lax limits of morphisms is said to be *simple* if the usual Eilenberg-Moore adjunction $F \dashv U \colon \mathsf{T}\text{-Alg}_s \to \mathscr{K}$ is simple with respect to the coreflection–opfibration AWFS on $\mathsf{T}\text{-Alg}_s$ – in the sense of Definition 10.6. To make this definition more explicit, consider the factorisation of a morphism $f \colon A \to B$ in $\mathscr{K}$ depicted in (10.17), and recall from Proposition 10.10 that the simplicity of $F \dashv U$ amounts to the existence of a certain coretract adjunction in $\mathsf{T}\text{-Alg}_s$; namely

$$TLf \dashv m_{K'Ff} \cdot Tq_f \tag{12.1}$$

where m is the multiplication of T and the rest of the notation is as in the diagram (10.18). This adjunction must be an adjunction in $\mathsf{T}\text{-Alg}_s$ – a condition that is redundant when, for example, T is lax idempotent, as it will often be in our examples.

Remark 12.2. At this point it is useful to consider the meaning of simple 2-monads and the previous proposition when the 2-category is locally discrete, ie just a category $\mathcal{C}$. In this case comma objects are just pullbacks, and the coreflection–opfibration factorisation becomes the orthogonal factorisation (Iso, Mor) that factors a morphism f as the identity followed by f. To say that a monad T on $\mathcal{C}$ is simple is to say that the image of the comparison morphism ℓ, which goes from the naturality square of i to the pullback in (12.2), is sent to an isomorphism by the free T-algebra functor. Equivalently, one can say that $T\ell$ is an isomorphism. Observe that, when T is a reflection, this gives the definition of *simple reflection* in the sense of [**?**].



We consider in this section some properties that guarantee that a 2-monad is simple, thus inducing a transferred AWFS. We make the blanket assumption that the 2-category $\mathscr{K}$ has pullbacks and cotensor products with **2**, and therefore comma objects.

Notation 12.3. If $T \colon \mathscr{A} \to \mathscr{B}$ is a 2-functor and $\lim D$ the limit of a 2-functor D into $\mathscr{A}$, there is a "comparison" morphism $T(\lim D) \to \lim TD$. We are interested in the limits that are comma objects. Given a cospan $f \colon A \to C \leftarrow B : g$, if α and β are the universal 2-cells of the comma objects $f \downarrow g$ and $Tf \downarrow Tg$, then the comparison morphism

$$k \colon T(f \downarrow g) \longrightarrow Tf \downarrow Tg, \tag{12.4}$$

is defined by the equality

$$\begin{array}{ccccc} T(f\downarrow g) & \xrightarrow{k} & Tf\downarrow Tg & \longrightarrow & TA \\ & & \downarrow & \Swarrow_{\beta} & \downarrow{\scriptstyle Tf} \\ & & TB & \xrightarrow[Tg]{} & TC \end{array} \quad = \quad \begin{array}{ccc} T(f\downarrow g) & \longrightarrow & TA \\ \downarrow & \Swarrow_{T(\alpha)} & \downarrow{\scriptstyle Tf} \\ TB & \xrightarrow[Tg]{} & TC \end{array}$$

Proposition 12.5 (Simplicity criterion)**.** *A 2-monad $\mathsf{T} = (T, i, m)$ is simple if it is lax idempotent and satisfies the following property: given morphisms f, u, v as displayed below, composition with the comparison morphism k defined in Notation 12.3 induces a bijection between 2-cells ξ as on the left and 2-cells ζ as on the right. In other words, for each ζ there exists a unique ξ such that $k \cdot \xi = \gamma$.*

$$\begin{array}{ccc} A & \xrightarrow{\quad u \quad} & \\ {\scriptstyle v}\downarrow & \Downarrow\xi & \searrow \\ Tf\downarrow i_B & \xrightarrow[i]{} & T(Tf\downarrow i_B) \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\quad k\cdot u \quad} & \\ {\scriptstyle v}\downarrow & \Downarrow\zeta & \searrow \\ Tf\downarrow i_B & \xrightarrow[k\cdot i]{} & TTf\downarrow Ti_B \end{array}$$

Proof. As we have done before, we will denote by R' the free split opfibration 2-monad on $\mathsf{T}\text{-}\mathrm{Alg}_s^{\mathbf{2}}$. Let R be the 2-monad on $\mathscr{K}^{\mathbf{2}}$ induced from R' by the simple adjunction $F \dashv U$, according to Theorem 11.8. By the same theorem, R is the codomain-preserving coreflection of the 2-monad $U^{\mathbf{2}}\mathsf{R}'F^{\mathbf{2}}$. The right part Rf of the factorisation of a morphism f in $\mathscr{K}$ induced by R is given by a comma object

$$\begin{array}{ccc} Kf & \xrightarrow{q_f} & TA \\ {\scriptstyle Rf}\downarrow & \Swarrow_{\mu_f} & \downarrow{\scriptstyle Tf} \\ B & \xrightarrow[i_B]{} & TB \end{array} \tag{12.6}$$

and the left part $Lf\colon A \to Kf$ is the unique morphism such that $\mu_f \cdot Lf = 1$. As explained at the beginning of the present section, we must exhibit a coretract adjunction (12.1) in $\mathscr{K}$; this adjunction is automatically an adjunction in $\mathsf{T}\text{-}\mathrm{Alg}_s$, since T is lax idempotent.

In order to define a counit $\alpha\colon TLf \cdot m_{K'Ff} \cdot Tq_f \Longrightarrow 1$ we can give its transpose under the free T-algebra 2-adjunction, which is a 2-cell $\bar\alpha\colon TLf \cdot q_f \Longrightarrow i_{Kf}$ in $\mathscr{K}$.

The morphism k of (12.4) is the unique such that satisfies the equality

$$\begin{array}{ccc} TKf & \xrightarrow{Tq_f} & T^2A \\ {\scriptstyle TRf}\downarrow & \Swarrow_{T\mu_f} & \downarrow{\scriptstyle T^2f} \\ TB & \xrightarrow[Ti_B]{} & T^2B \end{array} \quad = \quad \begin{array}{ccccc} TKf & \xrightarrow{k} & T^2f\downarrow Ti_B & \xrightarrow{d_0} & T^2A \\ & & {\scriptstyle d_1}\downarrow & \Swarrow_{\theta} & \downarrow{\scriptstyle T^2f} \\ & & TB & \xrightarrow[Ti_B]{} & T^2B \end{array}$$

To give $\bar\alpha$ is to equally give a pair of 2-cells, corresponding to $d_0 \cdot k \cdot \bar\alpha$ and $d_1 \cdot k \cdot \bar\alpha$:

$$\bar\alpha_1\colon Tq_f \cdot TLf \cdot q_f = Ti_A \cdot q_f \Longrightarrow i_{TA} \cdot q_f = Tq_f \cdot i_{Kf} \tag{12.7}$$

$$\bar\alpha_2\colon TRf \cdot TLf \cdot q_f = Tf \cdot q_f \Longrightarrow i_B \cdot Rf = TRf \cdot i_{Kf} \tag{12.8}$$

compatible with θ, in the sense that the following two compositions of 2-cells must be equal.

$$T^2f \cdot Ti_A \cdot q_f \xrightarrow{T^2f\cdot\bar\alpha_1} T^2f \cdot Tq_f \cdot i_{Kf} = T^2f \cdot d_0 \cdot k \cdot i_{Kf} \xrightarrow{\theta\cdot k\cdot i_{Kf}} Ti_B \cdot d_1 \cdot k \cdot i_{Kf}$$

$$T^2f\cdot d_0\cdot k\cdot TLf\cdot q_f \xrightarrow{\theta\cdot k\cdot TLf\cdot q_f} Ti_B\cdot d_1\cdot k\cdot TLf\cdot q_f = Ti_B\cdot Tf\cdot q_f \xrightarrow{Ti_B\cdot\bar\alpha_2} Ti_B\cdot i_B\cdot Rf$$

Set $\bar\alpha_1 = \delta_A \cdot q_f$ and $\bar\alpha_2 = \mu_f$, where $\delta\colon Ti \Rightarrow iT$ is the modification given by the lax idempotent structure of T. We must verify the pair of 2-cells displayed above are equal. Using that $\theta \cdot k = T\mu_f$, the verification takes the following form, where the first equality is the modification property for δ and the 2-naturality of i, the second is the interchange law in a 2-category, the third holds since $\delta \cdot i = 1$, and the last holds since $\mu_f \cdot Lf = 1$.

$$\big(T\mu_f \cdot i_{Kf}\big)\big(T^2 f \cdot \delta_A \cdot q_f\big) = \big(i_{TB} \cdot \mu_f\big)\big(\delta_B \cdot Tf \cdot q_f\big) = \big(\delta_B \cdot i_B \cdot Rf\big)\big(Ti_B \cdot \mu_f\big) = \\ = Ti_B \cdot \mu_f = (Ti_B \cdot \mu_f)(T\mu_f \cdot TLf \cdot q_f) \quad (12.9)$$

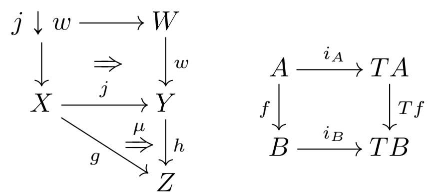
It remains to verify the triangle identities of an adjunction. One of them is $m_A \cdot Tq_f \cdot \alpha = 1$, equivalent to $m_A \cdot Tq_f \cdot \bar{\alpha} = 1$, and by definition of $\bar{\alpha}$, equivalent to $m_A \cdot \bar{\alpha}_1 = 1$. This latter equality clearly holds, since $m_A \cdot \delta = 1$,

Up to now we have only used the hypothesis in the case when v is an identity morphism. In order to prove the other triangle identity $\alpha \cdot (TLf) = 1$ we shall need the hypothesis in its general form, more precisely, for $v = Lf$. The triangle equality is equivalent to $\bar{\alpha} \cdot Lf = 1$, which holds since $\delta_A \cdot q_f \cdot Lf = \delta_A \cdot i_A = 1$ and $\mu_f \cdot Lf = 1$, finishing the proof. □

The proposition will be usually used in the following, less powerful form.

Corollary 12.10. *A 2-monad $\mathsf{T} = (T, i, m)$ is simple if it is lax idempotent and composing with* (12.4) *induces a bijection between 2-cells $u \Rightarrow i_{f\downarrow g} \cdot v$ and $k \cdot u \Rightarrow k \cdot i_{f\downarrow g} \cdot v$, where $f\colon A \to B \leftarrow C : g$ are arbitrary morphisms.*

Let $\mu\colon h \cdot j \Rightarrow g$ be a left extension in a 2-category with comma objects. Recall that μ is a *pointwise left extension* if, whenever pasted with a comma object as depicted on the left hand side below, the resulting 2-cell is a left extension. Recall that if a 2-monad $\mathsf{T} = (T, i, m)$ is lax idempotent then the identity 2-cell below exhibits Tf as a left extension – not necessarily a pointwise extension – of $i_B \cdot f$ along i_A – Section 4.b.

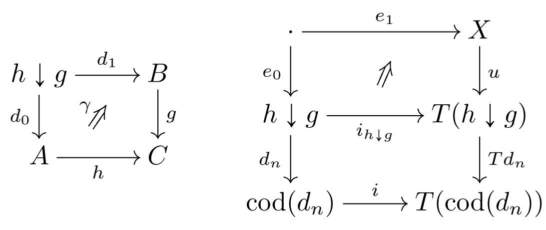


(12.11)

Theorem 12.12. *A lax idempotent 2-monad T is simple if it satisfies:*

- *the identity 2-cell on the right hand side of* (12.11) *exhibits Tf as a pointwise left extension of $i_B \cdot f$ along i_A, for all f;*
- *and the components of the unit $i\colon 1 \to T$ are fully faithful.*

Proof. We will verify the hypothesis of Corollary 12.10. Given a comma object $h \downarrow g$ depicted on the left below, denote by $k\colon T(h \downarrow g) \to Th \downarrow Tg$ the comparison morphism. Given a morphism $u\colon X \to T(h \downarrow g)$, we consider the diagram on the right hand side, where the unlabelled 2-cell is a comma object. This pasting exhibits $(Td_n) \cdot u$ as a left extension, since Td_n is a pointwise left extension.



Given a morphism $v\colon X \to h \downarrow g$, we will show that 2-cells $\alpha\colon k \cdot u \Rightarrow k \cdot i_{h\downarrow g} \cdot v$ are in bijection with 2-cells $u \Rightarrow i_{h\downarrow g} \cdot v$.

We begin by observing that 2-cells α are in bijection with pairs of 2-cells

$$\alpha_0\colon (Td_0) \cdot u \Rightarrow (Td_0) \cdot i_{h\downarrow g} \cdot v \qquad \text{and} \qquad \alpha_1 : (Td_1) \cdot u \Rightarrow (Td_1) \cdot i_{h\downarrow g} \cdot v$$

compatible with $T\gamma$ in the sense that $(Tg \cdot \alpha_1)(T\gamma \cdot u) = (T\gamma \cdot i_{h\downarrow g} \cdot v)(Th \cdot \alpha_0)$ holds.

By the universal property of extensions, α_n is in bijection with 2-cells $i \cdot d_n \cdot e_0 \Rightarrow (Td_n) \cdot i_{h\downarrow g} \cdot v \cdot e_1 = i_{\mathrm{cod}(d_n)} \cdot d_n \cdot v \cdot e_1$, and since i has fully faithful components, with 2-cells $\beta_n \colon d_n \cdot e_0 \Rightarrow d_n \cdot v \cdot e_1$. The compatibility between α_0, α_1, and $T\gamma$ translates into $(g \cdot \beta_1)(\gamma \cdot e_0) = (\gamma \cdot v \cdot e_1)(h \cdot \beta_0)$. By the universal property of γ, the pair β_0, β_1 is in bijection with 2-cells $e_0 \Rightarrow v \cdot e_1$, and thus with 2-cells $i_{h\downarrow g} \cdot e_0 \Rightarrow i_{h\downarrow g} \cdot v \cdot e_1$. Finally, by the description of u as a left extension, these 2-cells are in bijection with 2-cells $u \Rightarrow i_{h\downarrow g} \cdot v$, as required. □

The theorem can be used to prove that, for a class of **Set**-colimits, the 2-monad on **Cat** whose algebras are categories with chosen colimits of that class is simple. Section 13 proves this fact in another way, that applies to enriched categories.

13. Example: completion of V-categories under a class of colimits

This section is divided in four parts. The first compiles the basic facts about completions under a class of colimits that will be needed to prove. In the second part, it is shown that the 2-monad whose algebras are V-categories with chosen colimits of a class is simple, therefore inducing a lax orthogonal AWFS (L, R) on V-**Cat**. The third part exhibits the example when the colimits involved are initial objects. The last part shows that the algebras for R are, at least when $V = \mathbf{Set}$, split opfibrations whose fibres are equipped with chosen colimits and whose push forward functors strictly preserve them. Intuition should dictate that this type of split opfibration should coincide with the R-algebras; however, in general they do not, as we show at the end of the section.

Notation 13.1. In this section we will tend to think of categories enriched in V as objects of a 2-category, in this case V-**Cat**. Instead of denoting V-categories by calligraphic letters, we opt to maintain the notation we have used for 2-categories, where objects are denoted by capital roman letters and morphisms by lowercase letters. As a result, V-categories will be denoted by A, B, etc, and V-functors by $f \colon A \to B$, etc.

13.a. **Completion under colimits.** Throughout the section V will be a base of enrichment, in our case, a complete and cocomplete symmetric monoidal closed category. As argued in [**?**], the usual notion of colimit is not well adapted to the context of enriched categories and must be extended to that of *weighted colimit*. A weight is just a V-functor $\phi \colon J^{\mathrm{op}} \to V$ with J a small V-category. Then a ϕ-weighted colimit of a functor $G \colon J \to C$ is expressed by a V-natural isomorphism

$$C(\mathrm{colim}(\phi, G), c) \cong [J^{\mathrm{op}}, V](\phi, C(G-, C)).$$

The free completion of a V-category C under small colimits can be constructed as the V-category $\mathscr{P}C$ with objects small presheaves – ie V-functors $C^{\mathrm{op}} \to V$ that are a left Kan extension of its own restriction to a small subcategory of C^{op} – and enriched homs given by $\mathscr{P}C(\phi, \psi) = \int_c [\phi c, \psi c]$. This extends to a pseudomonad on V-**Cat**, whose unit has components the Yoneda embedding $y_C \colon C \to \mathscr{P}C$, and whose multiplication we denote by $m^{\mathscr{P}}$. A number of properties of $\mathscr{P}C$, in particular its completeness, are studied in [**?**].

A class of colimits is a set of weights $\Phi = \{\phi_i \colon J_i^{\mathrm{op}} \to \mathbf{Cat}\}_{i \in I}$. The free completion of C under colimits of the class Φ, or Φ-colimits, can be constructed as the smallest full sub-V-category of $\mathscr{P}C$ that is closed under Φ-colimits and contains the representable presheaves. We follow the notation of [**?**] and denote this V-category by ΦC. One obtains a pseudomonad, also denoted by Φ, with unit the corestricted

Yoneda embedding y_C and multiplication m^Φ, together with a pseudomonad morphism $\Phi \to \mathscr{P}$ that has fully faithful components.

Given a class of colimits Φ, [?] shows the existence of a 2-monad on V-**Cat**, that we denote by T_Φ, whose algebras are the V-categories with chosen Φ-colimits, and whose strict morphisms are V-functors that strictly preserve these. Furthermore, the same article shows that this is a lax idempotent 2-monad. Earlier, less general, versions of this monad appeared, for example, in [?].

It is convenient to recall some aspects of the construction of T_Φ from [?]. Part of this construction is an equivalence $t_A\colon T_\Phi A \to \Phi A$ for each V-category A, which form a pseudonatural transformation $\mathsf{T} \to \Phi$, and moreover, a pseudomonad morphism.

Lemma 13.2. *Denote by Φ a small class of V-enriched colimits and the associated pseudomonad on V-**Cat**. Let $f\colon A \to B$ be a fully faithful V-functor into a Φ-cocomplete V-category, and denote by $\tilde{f}\colon \Phi(A) \to B$ a left Kan extension of f along the corestricted Yoneda embedding $y_A\colon A \to \Phi A$. Then the morphisms*

$$\Phi(A)(\phi, y_A(a)) \longrightarrow B(\tilde{f}(\phi), f(a)) \tag{13.3}$$

induced by $\tilde{f}$ are isomorphisms for all $\phi \in \Phi(A)$ and $a \in A$.

Proof. The morphism (13.3) can be written as the composition of $\Phi(A)(\phi, f)$ from $\Phi(A)(\phi, y_A(a))$ to $\Phi(A)(\phi, B(f-, f(a)))$ and the isomorphism between the latter and $B(\mathrm{colim}(\phi, f), f(a))$. The result is an isomorphism since f is full and faithful and $\tilde{f}(\phi) \cong \mathrm{colim}(\phi, f)$. □

An explicit description of $\mathsf{T}_\Phi A$ is only possible in particular instances. In theory, one can give an inductive description of the objects, but in practice this is not very useful. Instead, we will use ΦA and its relationship to $\mathsf{T}_\Phi A$.

13.b. **Simplicity of completion under a class colimits.** Before proving that the 2-monads T_Φ are simple, we need the following easy lemma.

Lemma 13.4. *Suppose given commutative diagrams of V-functors whose horizontal arrows u, v and w are full and faithful.*

$$\begin{array}{ccc} A & \xrightarrow{u} & A' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ C & \xrightarrow{v} & C' \end{array} \qquad \begin{array}{ccc} B & \xrightarrow{w} & B' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ C & \xrightarrow{v} & C' \end{array}$$

Then, the V-functor $h\colon (f \downarrow g) \longrightarrow (f' \downarrow g')$, defined on objects by $(a, b, \alpha) \mapsto (u(a), w(b), v(\alpha))$, is full and faithful.

Proof. Of the routes one may take to prove this result, we choose a fairly direct one. The diagram displayed on the left exhibits $f \downarrow g$ as the comma category $(v \cdot f) \downarrow (v \cdot g)$; this is a direct consequence of the construction of comma categories and the fact that v is full and faithful.

$$\begin{array}{ccccc} f\downarrow g & \longrightarrow & A & & \\ \downarrow & \swArrow & \downarrow{\scriptstyle f} & & \\ B & \xrightarrow[g]{} & C & \xrightarrow[v]{} & C' \end{array} \qquad \begin{array}{ccccc} (f'\cdot u)\downarrow(g'\cdot w) & \rightarrowtail & \bullet & \longrightarrow & A \\ \downarrow & \text{pb} & \downarrow & \text{pb} & \downarrow{\scriptstyle u} \\ \bullet & \rightarrowtail & f'\downarrow g' & \longrightarrow & A' \\ \downarrow & \text{pb} & \downarrow & \swArrow & \downarrow{\scriptstyle f'} \\ B & \underset{w}{\rightarrowtail} & B' & \xrightarrow[g']{} & C' \end{array}$$

Using the commutativity of the diagrams in the statement, we see that $f \downarrow g$ can be constructed as $(f' \cdot u) \downarrow (g' \cdot w)$, and the latter comma category can be constructed from $f' \downarrow g'$ by taking pullbacks, as shown in the diagram on the right. Since full and faithful V-functors are stable under pullback, all the arrows denoted $\rightarrowtail$ are fully faithful functors. The composition of the isomorphism $f \downarrow g \cong (f' \cdot u) \downarrow (g' \cdot w)$ with the diagonal $(f' \cdot u) \downarrow (g' \cdot w) \longrightarrow f' \downarrow g'$ is precisely the V-functor $f \downarrow g \longrightarrow f' \downarrow g'$ of the statement, which is, therefore, full and faithful. □

Let Φ be a small class of colimits, and T_Φ the 2-monad on V-**Cat** whose algebras are small V-categories with chosen colimits of the class Φ.

Theorem 13.5. *The 2-monads T_Φ are simple – in the sense of Section 12 – therefore inducing a lax orthogonal* AWFS $(\mathsf{L}_\Phi, \mathsf{R}_\Phi)$ *on V-**Cat**.*

Proof. Let the 2-monad T in Corollary 12.10 be T_Φ, and assume that we are given V-functors f and g as in the statement of Corollary 12.10. The V-category $Tf \downarrow Tg$ is Φ-cocomplete, as the forgetful 2-functor from T-algebras creates comma objects. The comparison morphism of the corollary is the left Kan extension of the V-functor $h \colon f \downarrow g \to Tf \downarrow Tg$ induced by i_A and i_B. Since h is full and faithful, then k is full and faithful on homs of the form $T(f \downarrow g)(u, i_{f\downarrow g}(v))$ by Lemma 13.2, so we have indeed the bijection of 2-cells required in Corollary 12.10. □

13.c. **Completion under initial objects.** Suppose that the base of enrichment is the category **Set** of sets, so we work with locally small categories, and that the class of colimits has only one member, $\Phi = \{\varnothing \to \mathbf{Set}\}$. Then, Φ-colimits are initial objects, and the 2-monad T_Φ can be described as having endo-2-functor T_Φ that sends a category X to the category constructed by adding to X an object 0 and adding one arrow $0 \to x$ for all $x \in X$. Then the morphism $Rf \colon Kf \to B$, the right part of the factorisation of f, can be described as the split opfibration with fibre over $b \in B$ equal to $T_\Phi(f/b)$ and with push-forward functor $T_\Phi(f/b) \to T_\Phi(f/b')$ induced by a morphism $\beta \colon b \to b'$ equal to $T_\Phi(\beta_*)$ where $\beta_* \colon f/b \to f/b'$ is the push-forward functor of the free split opfibration f/B.

$$\begin{array}{ccc} Kf & \longrightarrow & T_\Phi A \\ {\scriptstyle Rf}\big\downarrow & \not\Swarrow & \big\downarrow{\scriptstyle T_\Phi f} \\ B & \underset{i_B}{\longrightarrow} & T_\Phi B \end{array}$$

An R-algebra is a split opfibration $A \to B$ whose fibres A_b, for $b \in B$, are categories equipped with a chosen initial object, and whose push-forward functors $\beta_* \colon A_b \to A_{b'}$, for $\beta \colon b \to b'$ in B, strictly preserve the initial objects.

13.d. **Split opfibrations with fibrewise chosen Φ-colimits.** Given a small class of **Set**-colimits Φ, denote by $\mathbf{OpFib}_s$-$\mathbf{\Phi}$-$\mathbf{Colim}_s$ the 2-category with objects split opfibrations in **Cat** whose fibres are small categories with chosen colimits of the class Φ and whose push-forward functors strictly preserve these. Morphisms from $p \colon E \to B$ to $p' \colon E' \to B'$ are strict morphisms $(h, k) \colon p \to p'$ of split fibrations – indicated by the first s in the notation – such that the restriction of h to fibres strictly preserves the chosen Φ-colimits – indicated by the second s in the notation. The 2-cells are those of $\mathbf{Cat}^\mathbf{2}$.

Theorem 13.6. *Let Φ be a class of* **Set***-colimits and (L, R) be the lax orthogonal* AWFS *induced by the completion under Φ-colimits. There is a 2-functor over $\mathbf{Cat}^\mathbf{2}$*

$$\mathsf{R}\text{-}\mathrm{Alg}_s \longrightarrow \mathbf{OpFib}_s\text{-}\Phi\text{-}\mathbf{Colim}_s.$$

Proof. We have shown in Theorem 13.5 that the 2-monad T_Φ, whose algebras are categories with chosen Φ-colimits, is simple; this means that the free algebra adjunction $F_\Phi \dashv U_\Phi \colon \mathsf{T}_\Phi\text{-}\mathrm{Alg} \to \mathbf{Cat}^\mathbf{2}$ is simple. The 2-monad R is, by construction, the codomain-preserving coreflection of $U_\Phi R' F_\Phi$, where R' is the split opfibration 2-monad on $\mathsf{T}_\Phi\text{-}\mathrm{Alg}_s$. We are, thus, in a position of applying Theorem 10.15 to deduce the existence of a 2-functor from $\mathsf{R}\text{-}\mathrm{Alg}_s$ to the 2-category $\mathbf{OpFib}_s$ of split opfibrations; this means that each R-algebra is a split opfibration and each morphism of R-algebras is a morphism of split opfibrations. It remains to prove that:

(a) the fibres of any R-algebra are equipped with chosen Φ-colimits;
(b) the push-forward functors between fibres strictly preserve them;
(c) any morphism of R-algebras strictly preserves them.

The fibre of $g\colon A \to B$ over $b \in B$ is the pullback of g along $b\colon \mathbf{1} \to B$; thus, this fibre is an R-algebra $A_b \to \mathbf{1}$. We know from Corollary 11.10 that the restriction of R to $\mathbf{Cat}/\mathbf{1} \cong \mathbf{Cat}$ is isomorphic to T_Φ, so the fibres of R-algebras are T_Φ-algebras, and the restriction of any morphism of R-algebras to fibres is a morphism of T_Φ-algebras. This verifies the literals (a) and (c).

It remains to prove (b), ie that for any morphism $\beta\colon b \to b'$ in B the push-forward functor $A_b \to A_{b'}$ preserves the chosen colimits. The strategy we follow to prove this claim is an usual one: it suffices to prove it for free R-algebras and use that any R-algebra is a canonical coequaliser of free ones.

There is a split coequaliser in $\mathbf{Cat}/B$

$$KRg \underset{L(Rg)}{\overset{\pi_g}{\underset{K(p,1)}{\rightrightarrows}}\!\!\!\!\leftarrow} Kg \underset{Lg}{\overset{p}{\rightleftarrows}} A$$

– where p denotes the R-algebra structure of g – which then lifts to a (non-split) coequaliser in the 2-category of split opfibrations. Taking the fibre over $b \in B$ of this split coequaliser, we obtain a coequaliser in $\mathsf{T}\text{-}\mathrm{Alg}_s$ that splits in $\mathbf{Cat}$. In particular, for any functor d into A_b

$$\operatorname{colim}(\phi, d) = p_b(\operatorname{colim}(\phi, (Lg)_b \cdot d)) \tag{13.7}$$

because p strictly preserves the chosen colimits. Taking fibres over the domain b and the codomain b' of the morphism β in B, we have a commutative square in $\mathbf{Cat}$ where all the categories have chosen Φ-colimits and the horizontal functors strictly preserve them.

$$\begin{array}{ccc} (Kg)_b & \xrightarrow{p_b} & A_b \\ {\scriptstyle\beta_*}\downarrow & & \downarrow{\scriptstyle\beta_*} \\ (Kg)_{b'} & \xrightarrow{p_{b'}} & A_b \end{array} \tag{13.8}$$

Therefore, if the push-forward functors of Rg preserve the chosen Φ-colimits, then so do the ones of g, as shown by the following string of equalities. By commutativity of the square, $\beta_* \cdot p_b$ strictly preserves Φ-colimits, and

$$\beta_*(\operatorname{colim}(\phi, d)) = \beta_* p_b(\operatorname{colim}(\phi, (Lg)_b \cdot d)) = \\ = \operatorname{colim}(\phi, \beta_* \cdot p_b \cdot (Lg)_b \cdot d) = \operatorname{colim}(\phi, \beta_* \cdot d). \tag{13.9}$$

The first equality holds by (13.7), the second equality holds because the diagram (13.8) shows that $\beta_* \cdot p_b$ strictly preserves Φ-colimits, and the last equality is a consequence of $p \cdot Lg = 1$.

We now prove that the push-forward functors of a free R-algebra Rg strictly preserve chosen Φ-colimits. By the description of Kg as a comma object (12.6), its objects are triples (x, b, ξ) where $x \in TA$, $b \in B$ and $\xi\colon (Tg)(x) \to i_B(b)$ is a morphism in TB. If we denote by $z_b\colon (Kg)_b \to Kg$ the inclusion of the fibre over $b \in B$ and $q_g\colon Kg \to TA$ the projection of the comma object, we showed in Lemma 11.11 that $q_g \cdot z_b\colon (Kg)_b \to TA$ strictly preserves Φ-colimits. It is clear that the triangle on the left hand side commutes, since $\beta_*(x, b, \xi) = (x, b', i_B(\beta) \cdot \xi)$.

$$\begin{array}{ccc} (Kg)_b & \xrightarrow{q_g \cdot z_b} & TA \\ {\scriptstyle \beta_*}\downarrow & \nearrow_{q_g \cdot z_{g'}} & \\ (Kg)_{b'} & & \end{array} \qquad \begin{array}{ccc} Tg/i_B(b) & \xrightarrow{\mathrm{pr}_b} & TA \\ {\scriptstyle \beta_*}\downarrow & \nearrow_{\mathrm{pr}_{b'}} & \\ Tg/i_B(b') & & \end{array}$$

But $(Kg)_b$ is the slice category $Tg/i_B(b)$, and $q_g \cdot z_b$ is the projection into TA, so we have a commutative triangle as in the right hand side. We can now apply the Lemma 13.10, that follows the present proof, to deduce that $\mathrm{pr}_b = \beta_* \cdot \mathrm{pr}_{b'}$ preserves and creates chosen Φ-colimits, so β_* strictly preserves chosen Φ-colimits.

This concludes the proof, since the 2-cells are automatically taken care of because the two 2-categories of the statement are locally full and faithful over $\mathbf{Cat}^\mathbf{2}$. □

Lemma 13.10. *(1) Let C be a category with Φ-colimits, $H\colon C \to E$ a Φ-cocontinuous functor, e an object of E and $Q\colon H/e \to C$ the projection. For any functor $D\colon J \to H/e$ and any colimiting cylinder $\eta\colon \phi \Rightarrow C(QD-, c)$ with $\phi \in \Phi$, there exists a unique $\epsilon\colon Hc \to e$ in E and a unique colimiting cylinder $\nu\colon \phi \Rightarrow H/e(D-, (c, \epsilon))$ such that $Q(\nu) = \eta$.*

(2) Moreover, if C is equipped with chosen Φ-colimits, then there exists a unique choice of Φ-colimits on H/e that is strictly preserved by Q.

(3) Suppose $S\colon A \to H/e$ is a functor, where A has chosen Φ-colimits. Then S strictly preserves Φ-colimits if and only if $QS\colon A \to C$ does so.

Proof. Since H preserves colimits, the top horizontal natural transformation in the diagram is a colimiting cylinder. The functor $D\colon J \to H/e$ can be given by a natural transformation δ from the constant functor on the terminal set to $D(HQD-, e)$. By the universal property of the colimit Hc, there exists a unique morphism $\epsilon\colon Hc \to e$ that makes the diagram commute.

$$\begin{array}{ccccc} \phi & \overset{\eta}{\Longrightarrow} & C(QD-, c) & \overset{H}{\Longrightarrow} & E(HQD-, Hc) \\ & \searrow^{!} & & & \vdots\, {\scriptstyle E(1,\epsilon)} \\ & & \Delta 1 & \overset{\delta}{\Longrightarrow} & E(HQD-, e) \end{array}$$

The functor $H/e(D-, (c, \epsilon))$ is the equaliser of the natural transformations

$$C(QD-, c) \overset{H}{\Rightarrow} E(HQD-, Hc) \overset{E(1,\epsilon)}{\Longrightarrow} E(HQD-, e) \tag{13.11}$$

$$C(QD-, c) \Longrightarrow \Delta 1 \overset{\delta}{\Rightarrow} E(HQD-, e) \tag{13.12}$$

from where it follows that η factors uniquely through a certain natural transformation $\nu\colon \phi \Rightarrow H/e(D-, (c, \epsilon))$. This transformation can easily be shown to have the universal property of a colimiting cylinder, a verification that we leave to the reader. In particular, (c, ϵ) is a colimit of D weighted by ϕ.

To prove the second part of the statement, if η exhibits c as $\mathrm{colim}(\phi, DQ)$, then we can choose the colimit $\mathrm{colim}(\phi, D)$ as (c, ϵ), and this is the unique possible choice that makes Q preserve this colimit in a strict way, by the argument of the previous paragraph. The last part of the statement easily follows from the second part. □

In many instances, the 2-functor of Theorem 13.6 is an isomorphism. For example, it is not hard to verify this when Φ is the class for initial objects $\{\varnothing \to \mathbf{Set}\}$; see Section 13.c.

Proposition 13.13. *The 2-functor of Theorem 13.6 is not always an isomorphism.*

Proof. To save space, let us write $\mathscr{F}$ instead of $\mathbf{OpFib}_s$-Φ-$\mathbf{Colim}_s$. One can without much problem show that the forgetful 2-functor $\mathscr{F} \to \mathbf{Cat}^\mathbf{2}$ is monadic, but instead we will consider the free object of $\mathscr{F}$ over a functor $g\colon A \to \mathbf{1} + \mathbf{1}$; the codomain is the discrete category with two objects that we denote by $*$ and $\bullet$. Another way of describing g is as a pair of categories A_* over $\{*\}$ and $A_\bullet$ over $\{\bullet\}$. It is not hard to see that the free object of $\mathscr{F}$ on g is $\tilde{g}\colon \tilde{A} \to \mathbf{1}+\mathbf{1}$, with $\tilde{A}_* = T_\Phi(A_*)$ and $\tilde{A}_\bullet = T_\Phi(A_\bullet)$; this is due to the fact that a split opfibration over a discrete category amounts to just a functor. The details of this point are left to the reader. The universal property of $\tilde{g}$ implies the existence of a unique morphism $(h,1)\colon \tilde{g} \to Rg$ such that

$$
\begin{array}{ccc}
A & \xrightarrow{Lg} & Kg \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle Rg} \\
\mathbf{1}+\mathbf{1} & = & \mathbf{1}+\mathbf{1}
\end{array}
\quad = \quad
\begin{array}{ccccc}
A_* + A_\bullet & \xrightarrow{i_{A_*}+i_{A_\bullet}} & T_\Phi(A_*) + T_\Phi(A_\bullet) & \xrightarrow{h} & Kg \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle Rg} \\
\mathbf{1}+\mathbf{1} & = & \mathbf{1}+\mathbf{1} & = & \mathbf{1}+\mathbf{1}
\end{array}
$$

If the 2-functor of Theorem 13.6 were an isomorphism, the morphism h would be an isomorphism, as both $\tilde{g}$ and Rg would be the free object on g. To complete the proof we must give an example where h is not an isomorphism.

Consider the class of colimits with one sole element $\Phi = \{\Delta\varnothing\colon \mathbf{1} \to \mathbf{Set}\}$ consisting of the functor that picks out the empty set. The colimit of a functor $v\colon 1 \to \mathbf{Set}$ that picks out a set v, weighted by $\Delta\varnothing$ – known as the *tensor product* of v by $\varnothing$ – is $\mathrm{colim}(\Delta\varnothing, v) = \varnothing$. The completion of a small category A under these colimits consists of the full subcategory $\Phi A \subseteq [A^{\mathrm{op}}, \mathbf{Set}]$ defined by the representables together with the initial object. A choice of Φ-colimits on a category A amounts to an assignment of an initial object $0(a) \in A$ for each object $a \in A$.

We can explicitly describe the 2-monad T associated to Φ. If A is a category, let $T_\Phi(A)$ have objects of the form $(a,n) \in \mathrm{ob}\, A \times \mathbb{N}$, and have morphisms defined by the following two clauses: there is a full and faithful functor $i_A\colon A \to T_\Phi(A)$ given on objects by $a \mapsto (a,0)$; and, each object (a,n) for $n > 0$ is an initial object. We equip $T_\Phi(A)$ with the chosen Φ-colimits given by $0(a,n) = (a, n+1)$.

The fibre $(Kg)_*$ of Rg over $*$ is $T_\Phi(A) \downarrow i_{\mathbf{1}+\mathbf{1}}(*)$. In particular some of its objects are of the form $((a,n),\xi)$ where $a \in \mathrm{ob}\, A$, $n > 0$ and $\xi\colon T_\Phi(g)(a,n) = (g(a),n) \to i_{\mathbf{1}\mathbf{1}}(*)$ is a morphism in $T_\Phi(\mathbf{1}+\mathbf{1})$. The domain of ξ is an initial object, so ξ carries no information at all. On the other hand, the restriction of the morphism h to fibres $h_*\colon T_\Phi(A_*) \to (Kg)_*$ does not reach objects of form $((a,n),\xi) \in (Kg)_*$ unless $a \in A_*$. Therefore, h_*, and thus h, is not surjective on objects, completing the proof. $\square$

14. Further work and examples

There are a number of examples and theoretical questions that have been left out of the present article and will benefit from a fuller explanation in forthcoming companion articles. Examples pertaining to the world of topological spaces will be treated in *Lax orthogonal factorisations in topology.* As a way of illustration, we mention the lax orthogonal AWFS arising from the filter monad on the category of T_0 topological spaces; this AWFS has an underlying WFS that was mentioned in our Introduction and studied in [**?**], where information about the filter monad can be found. This factorisation $f = Rf \cdot Lf$ has the property that Lf is a topological embedding and Rf is "fibrewise a continuous lattice" in the appropriate sense.

The main theoretical aspect of lax orthogonal AWFSs left out from the present article is the cofibrant generation thereof. This will have a full treatment in the forthcoming paper *Cofibrantly KZ-generated algebraic weak factorisation systems.*

References

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