A ductility model for steel connections

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Abstract

A model for the evaluation of the ductility of steel connections loaded in bending is presented in this paper. In the context of the component method, whereby a joint is modelled as an assembly of springs (components) and rigid links, using an elastic post-buckling analogy to the bi-linear elastic–plastic behaviour of each component, a general analytical model is proposed that yields the maximum rotation of the connection. Despite the complexity of the various connection types, the proposed model is able to provide analytical solutions for the moment–rotation response of a steel connection by using appropriate transformation criteria for assemblies of components in series and in parallel. The model is applied to typical beam-to-column connections, showing good agreement with numerical results. © 2000 Elsevier Science Ltd. All rights reserved.

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1. Introduction

The evaluation of the ductility of steel and composite connections requires an incremental non-linear analysis. In the context of the component method [1], whereby a joint is modelled as an assembly of springs (components) and rigid links, and concentrating on beam-to-column and beam-to-beam joints, several components contribute to the overall response of the connection, namely: (i) column web in shear, (ii) column web in compression, (iii) column web in tension, (iv) column flange in

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Nomenclature

\( \mathbf{h} \) Position of the centre of rotation
\( h_i \) Position of component \( i \) with respect to the centre of rotation
\( i \) Number of bolt rows; component
\( k_e \) Initial elastic stiffness
\( k_{ei} \) Component \( i \) initial elastic stiffness
\( k_{e,eq} \) Equivalent elastic stiffness for assemblies of components (series or parallel)
\( k_p \) Post-limit stiffness
\( k_{pi} \) Component \( i \) post-limit stiffness
\( k_{p,eq} \) Equivalent post-limit for assemblies of components (series or parallel)
\( q_{1,\theta} \) Total rotation of the joint
\( q_2 \) Rotation of rigid links (compression zone)
\( q_3 \) Rotation of rigid links (tension zone)
\( q_4 \) Axial displacement of the connection
\( z \) Lever arm
\( z_i \) Distance between the compression member and bolt row \( i \) in tension
\( K \) Stiffness (general)
\( L \) Length of rigid links
\( L_i \) Length of rigid links for component \( i \)
\( F \) Force
\( F_c \) Axial force (compression zone)
\( F_i \) Axial force (tension zone)
\( F^C \) Strength (limit load)
\( F_{i}^C \) Component \( i \) strength
\( M_{j,Rd} \) Flexural resistance
\( P_{j} \) Critical point
\( P^B \) Twice the limit load
\( P_{eq}^B \) Twice the limit load for equivalent component
\( P_{i}^B \) Twice the limit load for component \( i \)
\( Q_1 \) Total displacement (level of applied force)
\( Q_{1,i} \) Displacement of elastic spring \( i \)
\( Q_{2,i} \) Rotation of rigid links of length \( L_i \)
\( S_{j,ini} \) Initial stiffness of the connection
\( \epsilon \) Relative error
\( \phi_i \) Joint rotation when the first component reaches its elastic limit
\( \phi_f \) Joint rotation at failure
\( \Delta \) Total (axial) displacement
\( \Delta_i \) Total displacement of component \( i \)
\( \Delta_{e,i} \) Elastic displacement of component \( i \)
\( \Delta_{p,i} \) Plastic displacement of component \( i \)
\[
\Delta^c \quad \text{Total displacement for the compression zone}
\]
\[
\Delta^f \quad \text{Collapse displacement}
\]
\[
\Delta^t \quad \text{Total displacement for the tensile zone}
\]
\[
\Delta^y \quad \text{Yield displacement}
\]
\[
\Delta^f_i \quad \text{Collapse displacement for component } i
\]
\[
\Delta^y_i \quad \text{Yield displacement for component } i
\]
\[
\Phi_i \quad \text{Component } i \text{ ductility index}
\]
\[
\Phi_{\text{joint}} \quad \text{Joint ductility index}
\]

bending, (v) end-plate in bending, (vi) flange cleat in bending, (vii) beam flange in compression, (viii) beam web in tension or compression, (ix) plate in tension or compression, (x) bolts in tension, (xi) bolts in shear, (xii) bolts in bearing and (xiii) welds. Each are characterised by a bi-linear force-displacement curve [2], typically represented in Fig. 1, where \(k_e, k_p, F_C, \Delta^y\) and \(\Delta^f\) denote, respectively, the initial elastic stiffness, the post-limit stiffness, the strength, the yield displacement and the collapse displacement of the component, \(P_B\) being defined as twice the limit load \(F_C\).

Steel joints may present a variety of geometries, with different numbers of bolt rows and connecting parts. Because of this variety of configurations, joint models may range from a simple three-component model as in a welded beam-to-column connection, shown in Fig. 2, to a complex extended end-plate multi bolt-row beam-to-column connection, illustrated in Fig. 3. Despite the differences in these joint models, they share some basic features, namely the subdivision into a tension zone, a compression zone and a shear zone.

In a recent paper [2], a numerical procedure for the assessment of ductility of steel connections was proposed, which involved the identification of the “yield” sequence of the various components, the definition of a component ductility index, \(\Phi_i\) and the evaluation of the corresponding joint ductility index, \(\Phi_{\text{joint}}\), respectively defined by

![Fig. 1. Typical force-displacement diagram for generic component.](image_url)
where $\phi_f$ denotes the joint rotation at failure and $\phi_1$ the joint rotation when the first component reaches its elastic limit.

It is the purpose of the present paper to propose a general ductility model for the evaluation of joint behaviour subject to bending.

2. Ductility model

2.1. Introduction

The analysis of connections is currently based on mechanical models of extensional springs and rigid links, as shown in Figs. 2 and 3. Because of the large number
of components that such configurations may present, obtaining analytical solutions requires simplification of the mechanical model. Since all connection models are composed of a tension zone, a compression zone and a shear zone, it is possible to define for any configuration a simple substitute model, consisting of equivalent springs which retain all the original relevant characteristics. This simplified model, illustrated in Fig. 4(b), exhibits the same behaviour as the original one — Fig. 4(a) — and consists of a tension zone and a compression/shear zone. Referring to Fig. 4(b), the lever arm \( z \) is defined as the distance between the tension zone and the compression zone, \( h \) is defined as the distance between the tension zone and the centre of rotation.

In order to obtain analytical solutions for the complex system from Fig. 4(b), an incremental non-linear procedure is required. Using the approach presented in [3]

\[
(z' = z_i \cos \phi) \quad \quad \quad (h' = h \cos \phi; \quad z' = z \cos \phi)
\]

Fig. 4. General substitute model for steel connections. (a) Original model; (b) Equivalent model; (c) Equivalent elastic model.
of the equivalent elastic model of Fig. 4(b), analytical solutions are obtained, reproducing the non-linear moment–rotation response of the connection. The basic building block consists of the two degree-of-freedom system of Fig. 5(a), which consists of one elastic spring with stiffness $k_e$ and a second elastic spring with stiffness $k_p$ and resistance $F^C = P^B/2$, applied as a pre-compression, the degrees-of-freedom being defined as follows

$Q_1$ — total displacement;  
$Q_2$ — rotation of rigid links.

Clearly, this model exhibits distinct behaviour in tension and compression, yielding the following equilibrium solutions for compressive and tensile loading, respectively:

$$
\begin{align*}
F &= k_e Q_1 \\
F &= \frac{k_e}{k_e + k_p} \left( k_p Q_1 + \frac{P^B}{2} \right)
\end{align*}
$$

Fig. 5. Equivalent elastic system for elasto-plastic spring. (a) Spring in compression; (b) Spring in tension.
Effectively reproducing the original elastic–plastic behaviour of each component. For a component loaded in tension, the equivalent system of Fig. 5(b) yields identical response in tension (bi-linear) and compression (linear).

2.2. Equivalent elastic system for elastic–plastic springs in series

Examination of Fig. 4(a) shows that the tensile or compressive zones of a steel joint often comprise a sequence of components assembled in series. Dealing with the simpler substitute model of Fig. 4(c) requires replacing the real assembly of components in series with a single equivalent spring, as shown in Fig. 6(a). It is thus necessary to define an equivalent elastic system to be able to apply the procedure developed in [4] of replacing each bi-linear spring with an assembly of two elastic springs, as illustrated in Fig. 6(b).

The $2n+1$ degree-of-freedom system of Fig. 6(a) comprises the following degrees-of-freedom:

\[ Q_1 \] — total displacement;  
\[ Q_{1i} \] — displacement of elastic spring $i$;  
\[ Q_{2i} \] — rotation of rigid links of length $L_i$,  

yielding $2^n$ equilibrium paths. Numbering all components according to increasing limit force $F_i^C$, the following first $n+1$ equilibrium solutions are obtained, next reproduced with reference to [4]

![Diagram](a)  

![Diagram](b)

Fig. 6. Equivalent transformation for assembly of components in series. (a) Single equivalent elastic-plastic spring; (b) Equivalent elastic system transformation.
$S_1$: Fundamental (linear elastic) solution:

\[
\begin{align*}
Q_2 &= 0 \\
Q_{2_2} &= 0 \\
\ldots &\Rightarrow F = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{k_{ei}} Q_1 \\
Q_{2_{n-1}} &= 0 \\
Q_{2_n} &= 0
\end{align*}
\] (3)

$S_j$: Equilibrium solution $j$ ($2 \leq j \leq n+1$):

\[
F^{(j)} = \frac{1}{\sum_{i=1}^{n} \frac{1}{k_{ei}} + \sum_{i=1}^{j-1} \frac{1}{k_{pi}}} \left( Q_1 + \sum_{i=1}^{j-1} \frac{P_i^B}{2k_{pi}} \right)
\] (4)

corresponding to the force-displacement diagram of Fig. 7.

Again, following [4], it is thus possible to establish the simpler equivalent elastic system of Fig. 6(b), where $k_{e,eq}$ denotes the equivalent initial elastic stiffness of the
assembly of components in series, \( k_{p,eq} \), the corresponding post-limit stiffness and \( P_{B,eq} \), twice the limit force for the equivalent component, using an appropriate transformation criterium, next described:

2.2.1. Transformation criterium 1: equivalent coincident post-buckling stiffness

In this case, both systems yield identical results for the initial stiffness and the \( j \)th component, the equivalent properties being given by

\[
\begin{align*}
k_{e,eq} &= \frac{1}{\sum_{i=1}^{n} k_{ei}} \\
k_{p,eq} &= \frac{1}{\sum_{i=1}^{n} k_{pi}} \\
P_{B,eq} &= k_{p,eq} \sum_{i=1}^{j-1} \frac{P_{B}}{k_{pi}} 
\end{align*}
\]

(5)

2.2.2. Transformation criterium 2: equivalent secant post-buckling stiffness

In this case, the equivalent model should exhibit the same elastic stiffness, already derived in Eq. (5), while the equivalent post-limit stiffness should be defined as the straight line between the lowest and the \( j \)th critical loads, giving

\[
\begin{align*}
k_{e,eq} &= \frac{1}{\sum_{i=1}^{n} k_{ei}} \\
k_{p,eq} &= \frac{k_{e,eq}}{k_{e,eq} - m_{(eq-Crit.2)}} \\
P_{B,eq} &= P_{B1} 
\end{align*}
\]

(6)

where

\[
m_{(eq-Crit.2)} = \frac{P_{Bj} - P_{B1}}{(P_{Bj} - P_{B1}) \sum_{i=1}^{n} k_{ei} - \sum_{i=1}^{j-1} \frac{P_{B} + P_{Bj} - P_{B1}}{k_{pi}} \sum_{i=1}^{j-1} \frac{1}{k_{pi}}} 
\]

(7)

corresponding \( P_{Bj} \) to twice the \( j \)th critical load.

2.2.3. Transformation criterium 3: equivalent lower-bound minimum post-buckling stiffness

Again in this case, the equivalent model should exhibit the same elastic stiffness and the same equivalent pre-compression as the previous case, the equivalent post-limit stiffness being defined from equilibrium solution \( n+1 \), yielding

\[
\begin{align*}
k_{e,eq} &= \frac{1}{\sum_{i=1}^{n} k_{ei}} \\
k_{p,eq} &= \frac{1}{\sum_{i=1}^{n} k_{pi}} \\
P_{B,eq} &= P_{B1} 
\end{align*}
\]

(8)
2.3. Equivalent elastic system for elastic-plastic springs in parallel

Analogous to the previous section, it is also required to replace a sequence of components assembled in parallel with a single equivalent spring, as shown in Fig. 8(a). Following [4], the $2n+1$ degree-of-freedom system of Fig. 8(b) is considered, which comprises the following degrees-of-freedom

\[ Q_1 \] — total displacement at the level of force $F$;
\[ Q_{1i} \] — displacement of elastic spring $i$;

Fig. 8. Equivalent transformation for assembly of components in parallel. (a) Single equivalent elastic-plastic spring; (b) Equivalent elastic system transformation.
\( Q_{2_i} \) — rotation of rigid links of length \( L_i \);

yielding the following \( 2^n \) equilibrium paths, reproduced from [4] for the particular case of two components in parallel (\( n=2 \)),

\[ S_1: \text{Fundamental (linear elastic) solution:} \]
\[
\begin{align*}
Q_{2_1} &= 0 \\
Q_{2_2} &= 0 \\
F &= \sum_{i=1}^{2} \frac{h_i^2 k_{ei}}{h^2} Q_i
\end{align*}
\]  \hspace{1cm} (9)

\( S_2 \) to \( S_3 \): Uncoupled solutions in each spring:

\[
F_{(j+1)} = \left[ \frac{h_j^2}{h^2} \frac{1}{k_{ej}} + \frac{h_i^2}{h^2} \frac{1}{k_{ei}} \right] Q_i + \frac{h_j}{2h k_{ej} + k_{pj}} k_{ei} Q_j, \quad 1 \leq j \leq 2, i \neq j
\]  \hspace{1cm} (10)

\( S_4 \): Fully coupled solution:

\[
F = \sum_{i=1}^{2} \frac{h_i^2}{h^2} \frac{1}{k_{ei}} Q_i + \sum_{i=1}^{2} \frac{h_i}{2h k_{ei} + k_{pi}} k_{ei} Q_i
\]  \hspace{1cm} (11)

corresponding to the force-displacement diagram of Fig. 9.

Similarly to the equivalent system for an assembly in series, the equivalent elastic system of Fig. 8(b) is established using an appropriate transformation criterium, yielding the following general expressions for the particular case of three components assembled in parallel, \( m^{(j)} \) and \( b^{(j)} \) being next defined:
\[ k_{e,eq} = \sum_{i=1}^{2} \frac{h_i^2}{h^2} k_{ei} \]
\[ k_{p,eq} = \frac{k_{e,eq} m^{(j)}}{k_{e,eq} - m^{(j)}} \]
\[ P_{eq}^{B} = 2b^{(j)} \frac{k_{e,eq}}{k_{e,eq} - m^{(j)}} \]  

(12)

2.3.1. Transformation criterium 1: equivalent coincident post-buckling stiffness

Matching the equivalent post-buckling path to equilibrium solution \( S_2 \) gives, after [4],
\[ m^{(2)} = \frac{h_i^2}{h^2} \frac{1}{h_i k_{e1} + k_{p1}} + \frac{h_i^2}{h_i k_{e1} + k_{p1}} k_{p1} \]
\[ b^{(2)} = \frac{h_i k_{e1} P_{1}^{B}}{2h k_{e1} + k_{p1}} \]  

(13)

Alternatively, matching the equivalent post-buckling path to equilibrium solution \( S_4 \) gives,
\[ m^{(4)} = \sum_{i=1}^{2} \frac{h_i^2}{h_i^2} \frac{1}{1} k_{e1} k_{p1} \]
\[ b^{(4)} = \sum_{i=1}^{2} \frac{h_i k_{ei}}{h_i k_{ei} + k_{pi}} P_{i}^{B} \]  

(14)

2.3.2. Transformation criterium 2: equivalent secant post-buckling stiffness

In this case, the equivalent post-limit stiffness should be defined as the straight line between the lowest and the 3rd critical loads, with
\[ m_{eq,-Crit.2}^{(eq)} = \frac{h_i k_{ei} k_{e2}}{h k_{ei} + k_{p1}} \left[ \frac{h_1 k_{e1}}{h_1 k_{e1} + k_{p1}} \left( \frac{P_{1}^{B} + h_1 k_{p1} P_{2}^{B}}{h_2 k_{e2}} \right) \frac{h_1 k_{e1}}{h_1 k_{e1} + k_{p1}} \frac{h_1 k_{e1}}{h_1 k_{e1} + k_{p1}} \right] \]
\[ P_{eq}^{B} = 2 \sum_{i=1}^{2} h_i^2 k_{ei} \frac{P_{1}^{B}}{2h k_{ei}} \]  

(15)

(16)

2.3.3. Transformation criterium 3: equivalent lower-bound minimum post-buckling stiffness

Again in this case, the equivalent model should exhibit the same elastic stiffness and the same equivalent pre-compression as the previous case, the equivalent post-limit stiffness being defined from equilibrium solution \( S_4 \), given by
\[ k_{eq} = \sum_{i=1}^{2} \frac{h_i^2}{h_i^2} k_{ei} \]
\[ m_{eq,-Crit.3}^{(eq)} = m^{(4)} \]
\[ P_{eq}^{B} = 2 \sum_{i=1}^{2} h_i^2 k_{ei} \times \frac{P_{1}^{B}}{2h k_{ei}} \]  

(17)
2.4. General non-linear model of a steel connection

Using the equivalent systems derived above, a simplified general model can be proposed that caters for bi-linear component behaviour both in the tensile and compressive zones, illustrated in Fig. 10 for negative (hogging) moment, where \( q_1 \) denotes the total rotation of the joint, \( q_2 \) and \( q_3 \) the rotation of the rigid links, \( q_4 \) the axial displacement of the connection and indexes \( t \) and \( c \) denote, respectively, tension and compression. Introducing, where appropriate, the equivalent properties of Eqs. (5, 6) and (8) for the assemblies in series and Eqs. (12)–(17) for the assemblies in parallel, the following results are obtained [3]:

(i) Fundamental solution

\[
\begin{align*}
M &= \frac{z^2 k_{ec} k_{ct}}{2(k_{ec} + k_{ct})} \sin(2q_1) \\
q_2 &= 0 \\
q_3 &= 0
\end{align*}
\]  

(ii) Non-linear solution in \( q_2 \)

\[
\begin{align*}
M &= \frac{zk_{ec} k_{ct}}{k_{ec} + k_{ct}} \left[ z \sin q_1 - \frac{2z k_{ec} k_{ct} \sin q_1 - P_B^b (k_{ec} + k_{ct})}{2[k_{pc}(k_{ec} + k_{ct}) + k_{ec} k_{ct}]} \right] \cos q_1 \\
1 - \cos q_2 &= \frac{2z k_{ec} k_{ct} \sin q_1 - P_B^b (k_{ec} + k_{ct})}{4L_c [k_{pc}(k_{ec} + k_{ct}) + k_{ec} k_{ct}]} \\
q_3 &= 0
\end{align*}
\]

Fig. 10. General equivalent elastic model.
(iii) Non-linear solution in $q_3$

$$M = \frac{z k_{ec} k_{et}}{k_{ec} + k_{et}} \left[ z \sin q_1 - \frac{2 z k_{ec} k_{et} \sin q_1 - P_b}{2[k_p (k_{ec} + k_{et}) + k_{ec} k_{et}]} \right] \cos q_1$$

$$q_2 = 0$$

$$1 - \cos q_3 = \frac{2 z k_{ec} k_{et} \sin q_1 - P_b}{4L_z[k_p (k_{ec} + k_{et}) + k_{ec} k_{et}]}$$

(iv) Non-linear solution in $q_2$ and $q_3$

$$M = \frac{z k_{ec} k_{et}}{k_{ec} + k_{et}} \left[ z \sin q_1 - 2L_z (1 - \cos q_2) - 2L_z (1 - \cos q_3) \right] \cos q_1$$

$$1 - \cos q_2 = \frac{2 z k_{ec} k_{et} \sin q_1 - P_b}{4L_z[k_p (k_{ec} + k_{et}) + k_{ec} k_{et}]}$$

$$1 - \cos q_3 = \frac{2 z k_{ec} k_{et} k_p \sin q_1 - k_p (P_n - P_c) - P_b}{4L_z[k_{ec} k_{et} (k_p k_{et} + k_{ec} k_{et})]}$$

$$(23)$$

2.5. Application of transformation procedures

The choice of transformation criteria plays an important role in the evaluation of the moment–rotation response of the connection. For an assembly of components in series (corresponding to bolt row $i$), the equivalence procedure is straightforward since it only depends on the components of bolt row $i$. The chosen equivalence criterium should constitute the best fit to the force-displacement envelope of the components in series.

However, for an assembly of components in parallel, the equivalent transformation depends on the position of the centre of rotation $h$ (Fig. 8(b)). Taking the value specified in the revised Annex J of Eurocode 3 [5] as the lever arm $z$,

$$z = \frac{\sum_{i} k_{ei} z_i^2}{\sum_{i} k_{ei} z_i}$$

(22)

(23)

that yields only one meaningful solution for $h$ for the range of distances relevant for a steel connection.
Finally, having obtained the moment–rotation response of the connection, it is required to recover the individual results for all components. Since the general model is composed by one equivalent component in the tensile and compressive zones, geometrical compatibility relations yield the axial displacements, $\Delta^t$ and $\Delta^c$, for the tensile and compressive zones, taken as positive values, respectively,

$$\Delta^t = q_4 - \frac{z}{2} \sin q_1 \quad \Delta^c = q_4 + \frac{z}{2} \sin q_1$$

and the corresponding axial forces, $F_t$, and $F_c$:

$$F_c = F_t = \frac{M}{z \cos q_1}.$$  \hfill (25)

Knowing the total force acting on an equivalent component in series, the displacement of component $i$ is given by

$$\Delta_i = \Delta_{e,i} + \Delta_{p,i}$$ \hfill (26)

where $\Delta_{e,i}$ denotes the elastic displacement and $\Delta_{p,i}$ the corresponding plastic displacement, given by

$$\Delta_{e,i} = Q_{1,i} = \frac{F}{k_{ei}}$$ \hfill (27)

and

$$\Delta_{p,i} = \begin{cases} 0 & F < \frac{P_i^B}{2} \\ \frac{F}{k_{pi}} & F \geq \frac{P_i^B}{2} \end{cases}$$ \hfill (28)

Analogously, for an assembly of components in parallel, the total displacement of each component is given by

$$\Delta_i = \frac{h_i}{h} \Delta$$ \hfill (29)

$\Delta$ denoting the total displacement at the level of the resultant force $F$ (t or c), obtained from Eq. (24).
3. Application to bolted end-plate beam-to-column steel connections

3.1. Flush end-plate bolted beam-to-column connection (Bradran 123.001)

In order to illustrate the application of the equivalent elastic models, one connection configuration was chosen from the literature [6], corresponding to a bolted flush end-plate beam-to-column connection, tested by Bradran at the University of Innsbruck in 1994 and illustrated in Fig. 11(a). In the context of the component method, the spring and rigid link model of Fig. 11(b) is adopted, exhibiting four components in series in the tensile zone, column web in tension (3), column flange in bending (4), end-plate in bending (5) and bolts in tension (10), and two components in the compressive zone, column web in shear (1) and column web in compression (2). Table 1 reproduces the required properties for the various components [2].

Following the procedure defined above and using Eqs. (3) and (4), the minimum envelope for the assemblies in series in the tensile and compressive zones is obtained. Next, adequate choice of the transformation criteria for each case leads to the equiv-

<table>
<thead>
<tr>
<th>Component</th>
<th>Designation</th>
<th>$F^c$ (kN)</th>
<th>$k_e$ (kN/m)</th>
<th>$k_p$ (kN/m)</th>
<th>$\Delta'$ (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Column web in shear</td>
<td>1</td>
<td>327.04</td>
<td>802 220</td>
<td>10 000</td>
<td>0.408</td>
</tr>
<tr>
<td>Column web in compression</td>
<td>2</td>
<td>284.79</td>
<td>1 434 300</td>
<td>10</td>
<td>0.199</td>
</tr>
<tr>
<td>Column web in tension</td>
<td>3</td>
<td>301.05</td>
<td>1 033 200</td>
<td>10</td>
<td>0.291</td>
</tr>
<tr>
<td>Column flange in bending</td>
<td>4</td>
<td>219.47</td>
<td>3 242 400</td>
<td>10 000</td>
<td>0.068</td>
</tr>
<tr>
<td>End-plate in bending</td>
<td>5</td>
<td>229.58</td>
<td>3 263 400</td>
<td>10 000</td>
<td>0.070</td>
</tr>
<tr>
<td>Bolts in tension</td>
<td>10</td>
<td>282.24</td>
<td>1 663 200</td>
<td>10 000</td>
<td>0.170</td>
</tr>
</tbody>
</table>
alent bi-linear springs, as shown in Fig. 12. It is noted that criterium 1 (equivalent coincident post-buckling stiffness with equilibrium path 2) was chosen for the equivalent compressive spring, while criterium 2 (equivalent secant post-buckling stiffness, joining critical points $P_{1,T}$ and $P_{3,T}$) was chosen for the equivalent tensile spring, since, with reference to Fig. 12, critical point $P_{3,T}$ corresponds to failure of the bolts in tension. The resulting values of the equivalent properties, obtained using Eq. (5) for the compressive spring and Eq. (6) for the tensile spring, are summarized in Table 2. Finally, introducing the equivalent properties in the general model of Fig. 10, using Eqs. (18)–(21), yields the results of Fig. 13.

To assess the accuracy of the present model, these results are compared with the numerical results obtained by Silva et al. [2] and the code predictions of the revised annex J of Eurocode 3 [5]. Fig. 13 shows good correlation between the analytical model and the numerical results, while the Eurocode 3 results, only available for initial stiffness and moment resistance, show differences of about 10%. Since the latter does not provide any quantitative guidance on the evaluation of the ductility of the connection, it will be disregarded in further comparisons. The incremental
non-linear analysis performed on the finite element model yields the full moment–rotation response of the connection, allowing, in particular, the identification of the yield points for each component and, consequently, the yielding sequence of the connection: column flange in bending (4), end-plate in bending (5), bolts in tension (10) and, consequently, failure of the connection. The remaining components, column web in shear (1), column web in compression (2) and column web in tension (3), do not yield.

By comparison, the general analytical model, dealing with equivalent components that are restricted to bi-linear behaviour, introduces small errors that are minimised by adequate selection of the transformation criteria, as seen in Fig. 12, where the equivalent post-limit stiffness neglects the bifurcation point $P_{2,T}$. Comparative results for the three procedures are shown in Fig. 13 and summarised in Tables 3 and 4 (values in italic referring to the numerical model). Examination of Table 4 shows the “yield” sequence of the various components and the corresponding levels of ductility: the first component to yield is the column flange in bending, at a yield displacement of 0.068 mm and a total joint rotation of 3.04 mrad (column 3); next, the end-plate in bending reaches yield at 0.070 mm, the total joint rotation reaching Table 3
Bradran 123.001 — Resistance and initial stiffness

<table>
<thead>
<tr>
<th>General non-linear model</th>
<th>Numerical model — Silva et al</th>
<th>Eurocode 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_{j,Rd}=65.5$ kNm</td>
<td>$M_{j,Rd}=65.5$ kNm</td>
<td>$M_{j,Rd}=71.2$ kNm</td>
</tr>
<tr>
<td>$S_{j,ini}=21 550$ kN/m</td>
<td>$S_{j,ini}=21 359$ kN/m</td>
<td>$S_{j,ini}=18 659$ kN/m</td>
</tr>
</tbody>
</table>
### Table 4
Bradran 123.001 — Ductility indexes

<table>
<thead>
<tr>
<th>Component</th>
<th>$\Phi_i$ ($\Delta_i/\Delta_f$)</th>
<th>Absolute displacement $\Delta_i$</th>
<th>Component “yield” sequence</th>
<th>Component ductility index</th>
<th>Failure</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>INF</td>
<td>$-0.274$</td>
<td>$-0.352$</td>
<td>0.671</td>
<td>0.702</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>$-0.153$</td>
<td>$-0.197$</td>
<td>0.771</td>
<td>0.806</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>0.212</td>
<td>0.273</td>
<td>0.729</td>
<td>0.763</td>
</tr>
<tr>
<td>4</td>
<td>INF</td>
<td><strong>0.068</strong></td>
<td>6.364</td>
<td><strong>1.000</strong></td>
<td>15.982</td>
</tr>
<tr>
<td>5</td>
<td>INF</td>
<td><strong>0.067</strong></td>
<td>5.352</td>
<td>0.956</td>
<td><strong>1.000</strong></td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>0.132</td>
<td><strong>0.170</strong></td>
<td>0.778</td>
<td>0.813</td>
</tr>
</tbody>
</table>

**Joint rotation**

<table>
<thead>
<tr>
<th>Absolute rotation (mrad)</th>
<th>Joint ductility index</th>
<th>Failure</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.04</td>
<td>42.62</td>
<td>14.032</td>
</tr>
<tr>
<td>9.44</td>
<td>41.78</td>
<td>13.752</td>
</tr>
</tbody>
</table>

**Bending moment (kNm)**

<table>
<thead>
<tr>
<th>Absolute rotation (mrad)</th>
<th>Joint ductility index</th>
<th>Failure</th>
</tr>
</thead>
<tbody>
<tr>
<td>65.46</td>
<td>84.10</td>
<td>14.032</td>
</tr>
<tr>
<td>68.48</td>
<td>84.18</td>
<td>13.752</td>
</tr>
</tbody>
</table>
9.44 mrad (column 4); finally, the third component to yield (bolts in tension) and the corresponding values for other components are shown in column 5, with a total joint rotation of 42.62 mrad, that corresponds to the maximum rotation capacity of this connection. It is worth noting that the analytical model is able to recover the individual results for each component using Eqs. (26)–(28), as explained in the previous section. The analytical and numerical solutions show a negligible difference, as seen by evaluating the error in the ductility index \([2]\) of the connection,

\[
\varepsilon = \frac{14.03 - 13.75}{13.75} = 0.0204 = 2.04\%
\]  

(30)

3.2. Extended end-plate bolted beam-to-column connection (Humer 109.003)

To illustrate the transformation procedure for assemblies of components in parallel, a second example was chosen from the literature [6], corresponding to an extended end-plate beam-to-column connection, tested by Humer at the University of Innsbruck in 1987 and shown in Fig. 14(a). Starting with the model of Fig. 14(b), the various component properties are listed in Table 5 [2]. Following a similar procedure to the previous example, the components in series are first transformed, as illustrated in Fig. 15. In this case, criterium 1 (equivalent coincident post-buckling stiffness with equilibrium path 2) was adopted for the equivalent compressive spring, while criterium 2 (equivalent secant post-buckling stiffness, joining critical points \(P_{1,T1}\) and \(P_{3,T1}\) and critical points \(P_{1,T2}\) and \(P_{3,T2}\)) was chosen for both the first and second rows of the equivalent springs in tension; the resulting equivalent properties are summarized in Table 6.

Next, application of Eqs. (9)–(11) yields the minimum envelope for an assembly

![Fig. 14. Humer 109.003: model of analysis. (a) Connection geometry; (b) Mechanical model.](image)
Table 5  
Component characterization for extended end-plate beam-to-column connection

<table>
<thead>
<tr>
<th>Component</th>
<th>Designation</th>
<th>$P_c$ (kN)</th>
<th>$k_e$ (kN/m)</th>
<th>$k_p$ (kN/m)</th>
<th>$D_y$ (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Column web in shear</td>
<td>1</td>
<td>262.94</td>
<td>600 600</td>
<td>10 000</td>
<td>0.438</td>
</tr>
<tr>
<td>Column web in compression</td>
<td>2</td>
<td>301.30</td>
<td>1 978 200</td>
<td>10</td>
<td>0.152</td>
</tr>
<tr>
<td>Column web in tension</td>
<td>3.1</td>
<td>277.29</td>
<td>1 213 800</td>
<td>10</td>
<td>0.228</td>
</tr>
<tr>
<td></td>
<td>3.2</td>
<td>277.29</td>
<td>1 213 800</td>
<td>10</td>
<td>0.228</td>
</tr>
<tr>
<td>Column flange in bending</td>
<td>4.1</td>
<td>223.58</td>
<td>2 870 700</td>
<td>10 000</td>
<td>0.078</td>
</tr>
<tr>
<td></td>
<td>4.2</td>
<td>223.58</td>
<td>2 870 700</td>
<td>10 000</td>
<td>0.078</td>
</tr>
<tr>
<td>End-plate in bending</td>
<td>5.1</td>
<td>226.70</td>
<td>9 748 200</td>
<td>10 000</td>
<td>0.023</td>
</tr>
<tr>
<td></td>
<td>5.2</td>
<td>267.41</td>
<td>5 859 000</td>
<td>10 000</td>
<td>0.046</td>
</tr>
<tr>
<td>Bolts in tension</td>
<td>10.1</td>
<td>282.24</td>
<td>1 789 200</td>
<td>10 000</td>
<td>0.158</td>
</tr>
<tr>
<td></td>
<td>10.2</td>
<td>282.24</td>
<td>1 789 200</td>
<td>10 000</td>
<td>0.158</td>
</tr>
</tbody>
</table>

Fig. 15. Humer 109.003 — Transformation criteria (components in series).

Table 6  
Equivalent springs for assemblies in series

<table>
<thead>
<tr>
<th>Equivalent compressive spring</th>
<th>Bolt row 1</th>
<th>Equivalent tensile spring</th>
<th>Bolt row 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_{ec} = 4.607 \times 10^5$ kN/m</td>
<td>$k_{et1} = 5.453 \times 10^5$ kN/m</td>
<td>$k_{et2} = 5.258 \times 10^5$ kN/m</td>
<td></td>
</tr>
<tr>
<td>$k_{pc} = 1.000 \times 10^4$ kN/m</td>
<td>$k_{pt1} = 5.150 \times 10^3$ kN/m</td>
<td>$k_{pt2} = 8.446 \times 10^3$ kN/m</td>
<td></td>
</tr>
<tr>
<td>$P_{ec} = 525.89$ kN</td>
<td>$P_{t1} = 447.16$ kN</td>
<td>$P_{t2} = 447.16$ kN</td>
<td></td>
</tr>
</tbody>
</table>
of two springs in parallel with transformation criterium 3 (equivalent lower-bound minimum post-buckling stiffness) being adopted — Eqs. (12) and (17), giving the resulting bi-linear force-deformation diagram of Fig. 16. Adopting the lever arm given by Eq. (22),

$$z = \frac{k_{e1}z_1^2 + k_{e2}z_2^2}{k_{e1}z_1 + k_{e2}z_2} = 0.2925 \text{m}$$  \hspace{1cm} (31)

and solving Eq. (23), yields the position of the centre of rotation, \( h = 0.0835 \text{ m} \), and thus the corresponding properties of the equivalent springs for the general non-linear model, shown in Table 7.

Finally, as before, introducing the results of Table 7 in Eqs. (18, 20) and (21) yields the moment–rotation results of Fig. 17. Starting by comparing the moment resistance and initial stiffness obtained analytically with the numerical predictions of Silva et al. [2] and Eurocode 3 [5], good agreement is observed, as shown in Table 8. Similarly, restricting the comparison to the analytical and numerical results,
it is observed that this connection fails in the compressive zone, the failure sequence corresponding to yielding of the column web in shear (1), yielding of the column web in compression (2) and failure of the column web in compression, for a predefined component displacement of five times its yield displacement ($\Delta_t^y = 5\Delta_t^y$). In this example, the remaining connection components, column web in tension, column flange in bending, end-plate in bending, and bolts in tension, do not yield before failure. Table 9 and Fig. 17 show good agreement between the analytical and numerical results, as shown by evaluating the relative error in the ductility index of the connection, given by

$$\varepsilon = \frac{6.74 - 6.52}{6.52} = 0.0337 = 3.37\%$$  \hspace{1cm} (32)
Table 9
Humer 109.003 — Ductility indexes

<table>
<thead>
<tr>
<th>Component</th>
<th>$\Psi_i$</th>
<th>Absolute displacement $\Delta_i$ (mm)</th>
<th>Component “yield” sequence</th>
<th>Component ductility index</th>
<th>Failure</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$(\Delta/\Delta_i)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 INF</td>
<td>0.438</td>
<td>-4.274</td>
<td>4.883</td>
<td>1.000</td>
<td>9.762</td>
</tr>
<tr>
<td>2 5</td>
<td>-0.133</td>
<td>-0.152</td>
<td>0.152</td>
<td>0.758</td>
<td>0.873</td>
</tr>
<tr>
<td>3.1 5</td>
<td>0.123</td>
<td>0.141</td>
<td>0.144</td>
<td>0.540</td>
<td>0.619</td>
</tr>
<tr>
<td>3.2 5</td>
<td>0.093</td>
<td>0.107</td>
<td>0.109</td>
<td>0.408</td>
<td>0.467</td>
</tr>
<tr>
<td>4.1 INF</td>
<td>0.052</td>
<td>0.060</td>
<td>0.061</td>
<td>0.670</td>
<td>0.768</td>
</tr>
<tr>
<td>4.2 INF</td>
<td>0.039</td>
<td>0.045</td>
<td>0.046</td>
<td>0.506</td>
<td>0.580</td>
</tr>
<tr>
<td>5.1 INF</td>
<td>0.015</td>
<td>0.018</td>
<td>0.018</td>
<td>0.661</td>
<td>0.757</td>
</tr>
<tr>
<td>5.2 INF</td>
<td>0.019</td>
<td>0.022</td>
<td>0.023</td>
<td>0.423</td>
<td>0.485</td>
</tr>
<tr>
<td>10.1 1</td>
<td>0.084</td>
<td>0.096</td>
<td>0.098</td>
<td>0.531</td>
<td>0.608</td>
</tr>
<tr>
<td>10.2 1</td>
<td>0.084</td>
<td>0.072</td>
<td>0.074</td>
<td>0.531</td>
<td>0.459</td>
</tr>
</tbody>
</table>

Joint rotation

| Absolute rotation (mrad) | 2.73 | 16.59 | 18.41 | Joint ductility index | 6.739 | 6.74 |
| Bending moment (kNm) | 76.92 | 88.13 | 89.92 |

Joint rotation

| Absolute rotation (mrad) | 2.82 | 16.27 | 18.37 | Joint ductility index | 6.519 | 6.52 |
| Bending moment (kNm) | 76.06 | 87.17 | 87.17 |
4. Concluding remarks

The evaluation of the maximum available rotation of a steel connection, essential to enable the safe utilisation of partial-resistant joints in steel construction, is currently not covered by the code specifications of Eurocode 3. There is a consensual opinion in the literature [7] that any general approach to deal with this problem requires characterisation of the various connection components that extends well into the non-linear range, in opposition to the currently available procedures that simply evaluate the component resistance and initial stiffness (the limit force and initial elastic stiffness of Fig. 1).

A recent attempt by Aribert et al. [8] to evaluate the moment–rotation response of a flush end-plate connection using an appropriate component model required an incremental non-linear numerical analysis. The proposed model presents closed-form analytical solutions that, for a given characterisation of the various components, yield the full moment–rotation response of any steel connection loaded in bending currently covered by the revised annex J of Eurocode 3 [5]. It is thus easily translatable into code recommendations with little extra work from the current revised Annex J specifications, with the added advantage of incorporating all required connection properties (resistance, stiffness and ductility) into one single consistent model.

Practical utilisation of this methodology (as, incidentally, any alternative methodology that attempts to predict the non-linear moment–rotation response of a steel connection in the framework of the component method) requires proper characterisation of the non-linear response of each component up to failure, with thorough evaluation of the post-limit stiffness (positive or negative). Given that this task is currently being actively pursued in various european research centres, it is anticipated that this data will become available in the near future, although some initial results are already available for the component “column web in compression”, that exhibits negative post-limit stiffness [9]. Since this model is able to reproduce any non-linear moment–rotation configuration up to failure, comparison with experimental (illustrated in [3] for the simple case of a welded connection) or theoretical (finite element or other) results only depends on adequate characterisation of the components.

Finally, this methodology combined with the component ductility classes and ductility indexes already proposed [2] may lead to simple, deemed-to-satisfy criteria on ductility requirements.

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References