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A tool to analyze robust stability for model predictive controllers

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Abstract

A strategy based on Nonlinear Programming (NLP) sensitivity is developed to establish stability bounds on the plant/model mismatch for a class of optimization-based Model Predictive Control (MPC) algorithms. By extending well-known nominal stability properties for these controllers, we derive a sufficient condition for robust stability of these controllers. This condition can also be used to assess the extent of model mismatch that can be tolerated to guarantee robust stability. In this derivation we deal with MPC controllers with final time constraints or infinite time horizons. Also for this initial study we concentrate only on discrete time systems and unconstrained state feedback control laws with all of the states measured. To illustrate this approach we give two examples: a linear first-order dynamic system and a nonlinear SISO system involving a first order reaction.

Keywords: Model predictive controls; Nonlinear programming; Sensitivity; Robustness

1. Introduction

Model predictive control (MPC) for both linear and nonlinear systems has seen considerable research over the past decade. Widely used linear predictive formulations such as Quadratic Dynamic Matrix Control (QDMC), or Generalized Predictive Control (GPC) [1] are common for industrial applications. These controllers have their greatest advantage for MIMO processes as they provide a direct way for coordinating and balancing interactions among inputs and outputs. Qin and Badgwell [2] provide an overview of the industrial application of commercial packages of Model Predictive Control (MPC) technology and also point out future developments such as Nonlinear MPC (see also [3]). Nonlinear model predictive controllers have been shown to be advantageous on a variety of processes. Examples where these are essential include processes with sign changes in the gain matrix over desired regions of operation, non-steady state processes where an optimal profile is required and systems that have complex interactions with constraints and nonlinear phenomena.

For both linear and nonlinear systems, a key feature of the control law is the formulation and solution of a nonlinear programming problem. Using optimization-based formulations, a large class of linear and nonlinear controllers has been derived. Several approaches are used to implement MPC frameworks (see [4] and references therein), which include nonlinear programming (NLP) strategies for nonlinear MPC. In addition, a prominent aspect of the research in the Nonlinear Model Predictive Control field is the development of a theoretical analysis framework to study the stability and robustness of the control system to disturbances and plant/model mismatch.

Related to this analysis, several authors cite shortcomings to the *naive* approaches taken in the implementation of MPC with poor or no guaranteed stability properties (e.g. [5–7]). Lee [8] provides a tutorial survey of the recent theoretical developments in MPC. As a result, stability properties of these controllers have been better understood in recent years. This is particularly true for the nominal (so called perfect model) properties of these controllers. In particular, Lyapunov type stability analyses have been developed for model predictive control of discrete time systems and these apply both to infinite time horizon systems and to finite time problems with endpoint constraints on the states. Another important question deserving further investigation and implicitly related to these issues, is the problem of solution existence for the optimal control problem. Here, relevant results for the case of general discrete-time systems have been shown in Keerthi and Gilbert [9], based on previous work by Keerthi and Gilbert [10] and Dolezal [11].

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Moreover, the performance of nonlinear model predictive controllers is largely determined by the formulation and solution of the optimization problem. The nonlinear MPC formulation, initially termed Newton-type control, is based on the work of Economou [12] and was later extended by Li [13] and Oliveira [14]. This formulation has been demonstrated to deal with common difficulties in nonlinear control, such as open-loop instabilities, stability problems introduced by the presence of active hard-constraints, and some amount of ill-conditioning in the predictive model [15–20]. Moreover, with highly nonlinear and open-loop unstable processes, the integration of the model equations becomes numerically unstable, thus deteriorating the performance and/or leading to a failure of the nonlinear controller. To overcome this type of difficulty a boundary value approach to enhance the stability and problem conditioning of the open-loop prediction problem [21] can be used. Here a multiple shooting strategy is adopted to solve the model equations [22], which includes a Successive Quadratic Programming (SQP) formulation to handle terminal state constraints that are derived for controller stability [5,7,9]. Consequently, since the design of nonlinear predictive controllers includes the development and implementation of optimization tools, future research in the field must be done in parallel with the development of new optimization approaches and the interplay between optimization and control in the design of robust nonlinear model predictive controllers [23]. In addition to the above MPC controllers, recent nonlinear control laws include dual mode controllers [7] as well as direct nonlinear extensions of classical LQ strategies and MPC [5,9].

All of these results rely on perfect predictive models and little work has been done in assessing the stability of model predictive controllers in the presence of model mismatch. In this area, Yang and Polak [24] consider a special class of model predictive controllers and investigated these for robust stability. Scokaert and Rawlings [25] also considered the stability of model predictive controllers under perturbations. Finally, Badgwell [26] considered a modification of the Lyapunov stability results for model predictive control in order to deal with model mismatch. This analysis leads to additional constraints that are added to the NLP for the MPC controller. This constraint then enforces robust stability for a specified set of model uncertainties. The complementary approach of Badgwell [26] enforces robustness by enforcing a constraint derived from a set of plants with known uncertainties. Thus the study of Badgwell [26] addresses the question: "Given a known uncertainty, how can we make the algorithm robust for that range of uncertainty?" The robust MPC algorithm is then implemented by solving a constrained NLP online.

In this study we address the complementary question: "Given an MPC algorithm how much mismatch can it tolerate and how can we assess this mismatch for stability of the MPC controller?" This analysis is offline and the NLPs for the MPC controller are guaranteed to be feasible (based on existence results in [10,11]). This approach applies to all optimization based controllers for which nominal stability can be shown with a Lyapunov type analysis. For clarity of presentation, we treat only the unconstrained state feedback discrete-time case in this study. Therefore we assume that at every time index k all the states can be measured. Also to simplify our analysis we do not consider disturbances. Nevertheless, it should be noted that the analysis tools and results are not restricted to these cases and can be extended to more general ones, as shown in the next section.

In Section 2 we provide a general description of the system plant under study and state some assumptions about the plant, the model and the mismatch error. Section 3 then follows with a description of the model predictive control problem formulation, and a characterization of the convergence properties of the control problem without and with model mismatch. From this we consider the influence of the mismatch term on robust stability; characterization of this term through NLP sensitivity in Section 4 leads to a sufficient condition for robust stability, together with a strategy to estimate the resulting bound on this stability condition. We illustrate this property in Section 5 with two simple examples with parametric plant/model mismatch: a first-order linear system and a SISO reactor system with a first order reaction. Finally, Section 6 provides some perspective for this approach as well as directions for future work, including extension of this analysis to constrained controllers and output feedback systems.

2. Description of the system

In the nonlinear MPC framework we assume the dynamics of the plant to be controlled are described by the following nonlinear, continuous-time set of equations:

$$\dot{x} = \mathbf{f}(x, u) \tag{2.1}$$

$$y = \mathbf{g}(x) \tag{2.2}$$

where $x \in \mathbf{R}^{n_s}$ is the vector of states,$u \in \mathbf{R}^{n_i}$ is the vector of inputs, with $\mathbf{f} : \mathbf{R}^{n_s} \times \mathbf{R}^{n_i} \rightarrow \mathbf{R}^{n_s}$ and $y \in \mathbf{R}^{n_o}$ is the vector of outputs, with $\mathbf{g} : \mathbf{R}^{n_s} \rightarrow \mathbf{R}^{n_o}$.

For this study we treat only the state feedback case and assume that at every time index k all the states can be measured accurately. The stationary discrete-time counterpart of Eq. (2.1) is given by

$$x_{k+1} = f_k(x_k, u_k) = f(\Delta t; x_k, u_k) \tag{2.3}$$

where Δt is the sampling period, $x_k \in \mathbf{R}^{n_s}$, $u_k \in \mathbf{R}^{n_i}$, with $f_k : \mathbf{R}^{n_s} \times \mathbf{R}^{n_i} \to \mathbf{R}^{n_s}$. To develop the nonlinear control law, a model with the same dimension as (Eq. (2.3)) is considered, represented by the following nonlinear stationary discrete-time equations:

$$z_{k+1} = \chi_k(z_k, u_k) \tag{2.4}$$

where $z_k \in \mathbf{R}^{n_s}$ is the vector of *nominal* states, u_k is the same vector of inputs as in Eq. (2.3), with $\chi_k : \mathbf{R}^{n_s} \times \mathbf{R}^{n_i} \to \mathbf{R}^{n_s}$. This model may be linear or nonlinear. We consider $(x_k, u_k) = (0, 0)$ the point at which both the plant and the model operate at steady state, such that $\chi_k(0, 0) = f_k(0, 0) = 0$, for all k, $k \geqslant 0$.

From [9] we also apply the definition of a function belonging to class K_∞, along with related assumptions.

Definition 2.1. A function $W(r) : \mathbf{R}_+ \to \mathbf{R}_+$, $r \in \mathbf{R}_+$, is said to belong to class $\mathscr{K}_\infty$ if:

a. it is continuous;
b. W(r) = 0→r = 0;
c. it is nondecreasing;
d. $W(r) \to \infty$ when $r \to \infty$.

Moreover, if W_1 belongs to class $\mathscr{K}_\infty$ then for

$$W_2(r) = \beta_1 W_1(r),\ W_3(r) = \beta_2 W_1(r)$$

with β_1 and β_2 positive constants, and

$$W_4(r) = W_2(r) + W_3(r),\ W_5(r) = W_2(r) \cdot W_3(r) \tag{2.6}$$

it follows that W_2, W_3, W_4 and W_5 are also in class $\mathscr{K}_\infty$.

We define $\|\cdot\|$ as the Euclidean norm and make the following assumptions:

Assumption 2.1. There exists a Lipschitz constant L (independent of k) such that for all pairs (x_k, u_k), $(x_k^{'}, u_k^{'})$,(z_k, u_k) and $(z_k^{'}, u_k^{'}) \in D_k$,$k \geqslant 0$,$D_k \subset \mathbf{R}^{n_s} \times \mathbf{R}^{n_i}$,

$$\| f_k(x_k, u_k) - f_k(x_k', u_k') \| \leqslant L(\| x_k - x_k' \| + \| u_k - u_k' \|) \tag{2.7}$$

$$\| \chi_k(z_k, u_k) - \chi_k(z_k', u_k') \| \leqslant L(\| z_k - z_k' \| + \| u_k - u_k' \|) \tag{2.8}$$

Assumption 2.2. There exists a modeling bound function W_m, that is in class $\mathscr{K}_\infty$, such that for all pairs $(x_k, u_k) \in D_k$, $k \geqslant 0$, $D_k \subset \mathbf{R}^{n_s} \times \mathbf{R}^{n_i}$,

$$\| f_k(x_k, u_k) - \chi_k(x_k, u_k) \| \leqslant W_m(\| x_k \|) \tag{2.9}$$

Assumption 2.3. There exist positive constants K_m and γ such that for all $k \geqslant 0$ the function W_m in Assumption 2.2 is of the form

$$W_m(\| x_k \|) = K_m \| x_k \|^\gamma \tag{2.10}$$

Assumption 2.1 requires that, for every $k \geqslant 0$, the plant and model system set of equations are continuous and have a unique solution in some region D_k about any pair $(x_k^{'}, u_k^{'})$ and $(z_k^{'}, u_k^{'})$ respectively. Assumption 2.2 states that the mismatch error is bounded by a monotonic function that is in class $\mathscr{K}_\infty$. A similar formalism to bound the plant uncertainty can also be found in [24,27]. Because the inputs in the model predictive control problem formulation are given by a state feedback law, hence the control profile is a function of the initial states, the modeling bound in (Eq. (2.9)) can be expressed as a function of the states only. Assumption 2.3 leads to the derivation of a sufficient condition for stability whose formulation permits us to estimate easily a stability bound using the NLP framework.

While Assumptions 2.2 and 2.3 may appear restrictive in the treatment of uncertainty, there are a number of straightforward extensions that can be made which still satisfy these assumptions. For instance, if we consider an additive state disturbance (d_k) that belongs in class $\mathscr{K}_\infty$, it is clear from (2.6) that:

$$\begin{aligned}\| f_k(x_k, u_k) - (\chi_k(x_k, u_k) + d_k) \| &\leqslant \| f_k(x_k, u_k) \\ &- \chi_k(x_k, u_k) \| + \| d_k \| \leqslant W_m(\| x_k \|)\end{aligned} \tag{2.11}$$

and the analysis developed here can be used. This disturbance class also includes asymptotically decaying disturbances considered by Scokaert and Rawlings [25].

Moreover, if the states are not measured perfectly or estimated from a set of outputs, we can apply the same analysis as long as the difference between the imperfectly measured states and the actual states, $(\tilde{x}_k - x_k)$ are in class $\mathscr{K}_\infty$. In this case, we have from Eq. (2.8):

$$\| \chi_k(\tilde{x}_k, u_k) - \chi_k(x_k, u_k) \| \leqslant L(\| \tilde{x}_k - x_k \|) \tag{2.12}$$

and consequently:

$$\begin{aligned}\| f_k(x_k, u_k) - \chi_k(\tilde{x}_k, u_k) \| &\leqslant \| f_k(x_k, u_k) \\ &- \chi_k(x_k, u_k) \| + L \| \tilde{x}_k - x_k \| \leqslant W_m(\| x_k \|).\end{aligned} \tag{2.13}$$

Therefore, certain cases of disturbances and state mismatches can also be treated by the analysis presented in the next sections. Of course, this uncertainty description does not apply to all cases, but for this study we assume that state estimation procedures allow us to invoke the assumptions made above. The more interesting case of output feedback will be treated in a later study.

3. Moving horizon problem solution

In this analysis we consider the formulation of the MPC problem, also named Moving Horizon Control, defined by an objective function:

$$\Psi(x_i, u_i) = \sum_{k=i}^{i+p} h(z_k, v_k) \tag{3.1}$$

with $\Psi : \mathbf{R}^{n_s+n_i} \to \mathbf{R}$ and subject to the discrete time equations Eq. (2.4). Here $h(z, u) : \mathbf{R}^{n_s+n_i} \to \mathbf{R}$ is a nonnegative, continuously differentiable function and is zero if and only if z and u are both zero. This problem is solved at every time index i, $i \geqslant 0$, with the initial condition, $z_i = x_i$. The objective function for this problem is therefore evaluated over a finite time horizon of a given length $p, p \geqslant 1$. The sequences z_k and v_k (with $u_i = v_i$),$k = i, \ldots, i+p$, are the state and input trajectories over the horizon p, respectively. This formulation also allows a shorter input horizon m, with $m \leqslant p$ and $v_k = v_{i+m-1}, k = i+m, \ldots, i+p$. Finally, we impose the constraint $z_{i+p} = 0$, or if we allow $p \to \infty$ then this constraint is automatically satisfied for a finite value of the objective function Eq. (3.1).

A typical example of $h(z_k, v_k)$ is the quadratic function given by

$$h(z_k, v_k) = z_k^T Q_{1k} z_k + v_k^T Q_{2k} v_k \tag{3.2}$$

with $h : \mathbf{R}^{n_s+n^i} \to \mathbf{R}$, $h(0, 0) = 0$. Here the weighting matrix Q_{2k} is positive definite and Q_{1k} is symmetric positive semidefinite. The development below, however, also applies to more general functions.

Solving the moving horizon problem over horizon p using the initial state conditions x_i and the *nominal* input sequence $\{v_k\}$ leads to a state prediction sequence $\{z_k\}$. Here we assume that the state initial conditions z_i are measured such that $z_i = x_i$. Because the problem is initialized with x_i and the optimal u_i is an implicit function of x_i we therefore denote the objective function Eq. (3.1) as $\Psi(x_i)$.

Thus the problem is to solve the open-loop constrained optimal control problem $\mathscr{P}(i)$, given by

$$\min_v \Psi(x_i) \tag{3.3}$$

$$\text{s.t. } z_{k+1} = \chi_k(z_k, v_k), k = i, \ldots, i+p-1 \tag{3.4}$$

$$z_{i+p} = 0 \tag{3.5}$$

$$z_i = x_i \tag{3.6}$$

with $(z_k, v_k) \in D_k$, $D_k \subset \mathbf{R}^{n_s} \times \mathbf{R}^{n_i}$ and optional constraints added for a shorter input horizon, $m \leqslant p$. Solving $\mathscr{P}(i)$ generates the optimal control sequence $\{v_k\}$ which leads to an optimal predicted state profile $\{z_k\}$ (with $z_i = x_i$). At every i, only the first element of this sequence is implemented in the plant, thus $u_i = v_i$, and the entire calculation procedure is repeated at the next time index. The main steps of the control algorithm are the following:

1. match z_i to the plant measurements, $z_i = x_i$.
2. Solve the optimal control problem $\mathscr{P}(i)$ for the predicted inputs v_k and states z_k over the time horizon p.
3. Set $u_i = v_i$, $i = i+1$ and go to 1.

We denote the optimal value of the objective function Eq. (3.1) from solving the problem $\mathscr{P}(i)$ by $\Psi^*(x_i)$.

Existence properties of the solution of $\mathscr{P}(i)$ have been shown in [9], based on previous work by Keerthi and Gilbert [10] and Dolezal [11]. A key requirement for the existence of a solution to $\mathscr{P}(i)$ is that an admissible profile exists for the endpoint constraint. As a result, we do not impose additional constraints in $\mathscr{P}(i)$ but assume that both the states and controls remain in bounded subspaces. To handle any additional constraints, the problem formulation can easily be extended through the use of exact penalty terms in the objective, as developed in [18], but this extension will be deferred to a future study. In addition, we assume a controllability property (termed *Property C* in [9]) where there exists a sufficiently long horizon that insures an admissible trajectory for the terminal constraint (Eq. (3.5)). Unfortunately, for general nonlinear systems, this property can only be checked by trial and error. Nevertheless, we assume that such a horizon exists in the analysis of our controller. Clearly this also allows the imposition of an infinite state horizon.

3.1. Perfect model case

First, we briefly review conditions for which repeated solution of the optimal control problem $\mathscr{P}(i)$ over i converges to the origin. Here we assume there exists a sufficiently long (and possibly infinite) horizon that insures an admissible trajectory to satisfy the terminal state constraint. As a result of solving the problem $\mathscr{P}(i)$ the states and inputs are zeroed such that $(z_k, v_k) = (0, 0)$ for $k \geqslant i+p$, that is

$$\Psi^*(x_i) = \sum_{k=i}^{i+p-1} h(z_k, v_k) + \underbrace{h(z_{i+p}, v_{i+p})}_{=0} \tag{3.7}$$

Moreover the resulting optimal sequence z_k, v_k, $k = i, \ldots, i+p-1$, is a feasible solution for the problem at time index $i+1$, $\mathscr{P}(i+1)$, when using the model equations Eq. (2.4). Thus the difference between $\Psi^*(x_i)$ and the value of the objective function at $i+1$ evaluated with these state and input sequences, $\Psi(x_{i+1})$, gives

$$\Psi^*(x_i) - \Psi(x_{i+1}) = \sum_{k=i}^{i+p-1} h(z_k, v_k) - \sum_{k=i+1}^{i+p} h(z_k, v_k) = h(x_i, u_i) \tag{3.8}$$

In the perfect model case, $z_i = x_i$, and the solution of the problem $\mathscr{P}(i+1)$ cannot be worse than the value at time index i because now the terminal constraint (Eq. (3.5)) has only to be satisfied one interval ahead. Therefore $\Psi^*(x_{i+1}) \leqslant \Psi(x_{i+1})$ and from Eq. (3.7) and (3.8), this leads to the following inequality:

$$\Psi^*(x_i) - \Psi^*(x_{i+1}) \geqslant h(x_i, u_i). \tag{3.9}$$

This relation is satisfied in the finite horizon case by enforcing the terminal state constraints [Eq. (3.5)] and also holds for an infinite horizon problem.

Thus the sequence $\{\Psi^*(x_i)\}$ over N time indices decreases and because the cost function h is bounded from below by zero it converges. Taking the sum of the differences given by Eq. (3.9) over N we obtain

$$\Psi^*(x_1) - \Psi^*(x_{N+1}) = \sum_{i=1}^{N} (\Psi^*(x_i) - \Psi^*(x_{i+1})) \geqslant \sum_{i=1}^{N} h(x_i, u_i) \tag{3.10}$$

Also, because the sequence $\{\Psi^*(x_i)\}$ is decreasing, then as $N \to \infty$, $h(x_i, u_i) \to 0$ and $x_i \to 0$.

3.2. Model mismatch case

We consider now the problem of plant/model mismatch. The natural question that arises is *how much* error can be tolerated by the closed loop system at every time index i, such that the controller drives the system to the setpoints. Therefore, our aim is to determine how large the modeling bounds in Eq. (2.10), K_m, should be for the closed-loop system to remain stable and converge to its desired state.

From the existence and controllability properties in [9] we have a solvable optimal control problem $\mathscr{P}(i)$ at every i. Let $\bar{x}_{i+1} = z_{i+1}|x_i$, i.e. the state prediction given by the solution of $\mathscr{P}(i)$ at time index $i+1$. Because there is model error the state measurements at $i+1$, x_{i+1}, do not match the prediction $\bar{x}_{i+1}$. Thus, using either $\bar{x}_{i+1}$ or x_{i+1} as initial condition to initialize the model [Eq. (2.4)] in the process of solving $\mathscr{P}(i+1)$ leads to two different optimal solutions, i.e. two different state and input optimal sequences over the $p-step$ horizon (Fig. 1). Here we emphasize that in both cases the state prediction over the horizon is performed using the model equations [Eq. (2.4)].

The optimal objective function resulting from the plant measurements at $i+1$ is denoted by $\Psi^*(x_{i+1})$ with corresponding state and input optimal sequences $\{z_k\}_{k=i+1}^{i+p+1}$ (with $z_{i+1} = x_{i+1}$) and $\{v_k\}_{k=i+1}^{i+p+1}$ respectively. On the other hand, for the initial condition $\bar{x}_{i+1}$ and $\Psi^*(\bar{x}_{i+1})$ we use the bar notation to denote the optimal state and input optimal sequences $\{\bar{z}_k\}_{k=i+1}^{i+p+1}$ (with $\bar{z}_{i+1} = \bar{x}_{i+1}$) and $\{\bar{v}_k\}_{k=i+1}^{i+p+1}$ respectively, using x_i.

To account for the existence of mismatch, we consider the difference between $\Psi^*(x_i)$ and $\Psi^*(x_{i+1})$ by adding and subtracting $\Psi^*(\bar{x}_{i+1})$, that is

$$\Psi^*(x_i) - \Psi^*(x_{i+1}) = (\Psi^*(x_i) - \Psi^*(\bar{x}_{i+1}) - (\Psi^*(x_{i+1}) - \Psi^*(\bar{x}_{i+1})) \tag{3.11}$$

The term $\Psi^*(x_i) - \Psi^*(\bar{x}_{i+1})$ represents the difference between the optimal objective functions at time indices i and $i+1$ for the model prediction and satisfies the inequality [Eq. (3.9)]. It follows that

$$\Psi^*(x_i) - \Psi^*(x_{i+1}) \geqslant h(x_i, u_i) - (\Psi^*(x_{i+1}) - \Psi^*(\bar{x}_{i+1})) \tag{3.12}$$

To ensure the sequence $\{\Psi^*(x_i)\}$ is decreasing, we need the quantity [Eq. (3.12)] to be positive and bounded by a positive function $W(\| x_i \|)$ of class $\mathscr{K}_\infty$ such that

$$h(x_i, u_i) - (\Psi^*(x_{i+1}) - \Psi^*(\bar{x}_{i+1})) \geqslant W(\| x_i \|) \tag{3.13}$$

with $W(\| x_i \|) \to 0$ as $\| x_i \| \to 0$, for all i, $i \geqslant 0$.

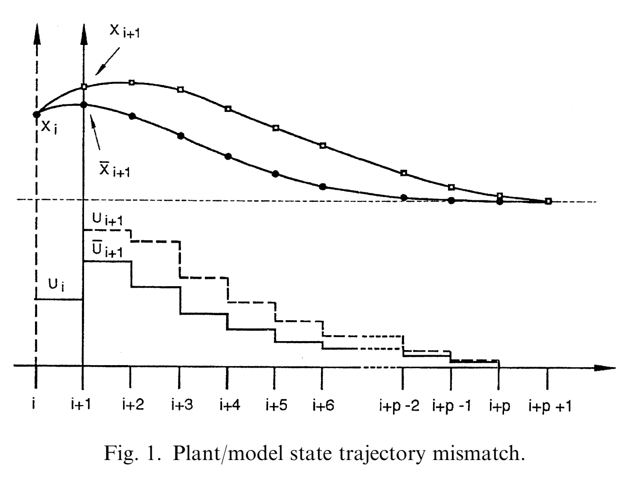

Fig. 1. Plant/model state trajectory mismatch.

Therefore, a sufficient robust stability condition is that, for all i, the mismatch term $\Psi^*(x_{i+1}) - \Psi^*(\bar{x}_{i+1})$ in Eq. (3.13) be smaller than $h(x_i, u_i)$ to satisfy inequality [Eq. (3.12)]. Under this condition, taking the sum of the differences over N time indices, we obtain

$$\begin{aligned}\Psi^*(x_1) - \Psi^*(x_{N+1}) &= \sum_{i=1}^{N}(\Psi^*(x_i) - \Psi^*(x_{i+1})) \\ &\geqslant \sum_{i=1}^{N} W(\| x_i \|)\end{aligned} \tag{3.14}$$

Then as $N \to \infty$, $\| x_N \| \to 0$.

4. Characterizing the mismatch term

To obtain the conditions for which the closed loop system remains stable in the presence of plant/model mismatch, we need to characterize the magnitude of the effect of the mismatch term in the inequality [Eq. (3.13)], expressed in terms of the difference given by

$$\Psi^*(x_{i+1}) - \Psi^*(\bar{x}_{i+1}) = \sum_{k=1}^{p}[h(z_{i+k}, v_{i+k}) - h(\bar{z}_{i+k}, \bar{v}_{i+k})] \tag{4.1}$$

To do this we invoke the mean value theorem:

$$\begin{aligned}\Psi^*(x_{i+1}) - \Psi^*(\bar{x}_{i+1}) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}x}\Psi^*(\bar{x}_{i+1} \\ + \tau(x_{i+1} - \bar{x}_{i+1}))^T(x_{i+1} - \bar{x}_{i+1})\mathrm{d}\tau\end{aligned} \tag{4.2}$$

and rely on concepts from NLP sensitivity analysis to obtain $\mathrm{d}\Psi^*/\mathrm{d}x$. By taking the norm of Eq. (4.2) and using Eqs. (2.3), (2.4), (2.9) or (2.10) to characterize $(x_{i+1} - \bar{x}_{i+1})$ we obtain:

$$\begin{aligned}&|\Psi^*(x_{i+1}) - \Psi^*(\bar{x}_{i+1})| \\ &\leqslant \left\| \int_0^1 \frac{\mathrm{d}}{\mathrm{d}x}\Psi^*(\bar{x}_{i+1} + \tau(x_{i+1} - \bar{x}_{i+1}))^T \mathrm{d}\tau \right\| \cdot \|x_{i+1} - \bar{x}_{i+1}\| \\ &\leqslant \left\| \int_0^1 \frac{\mathrm{d}}{\mathrm{d}x}\Psi^*(\bar{x}_{i+1} + \tau(x_{i+1} - \bar{x}_{i+1}))^T \mathrm{d}\tau \right\| \cdot W_m(\|x_i\|) \\ &= \left\| \int_0^1 \frac{\mathrm{d}}{\mathrm{d}x}\Psi^*(\bar{x}_{i+1} + \tau(x_{i+1} - \bar{x}_{i+1}))^T \mathrm{d}\tau \right\| \cdot K_m\|x_i\|^\gamma\end{aligned} \tag{4.3}$$

We will also see that this approach can also be used to establish a bound on the model errors (e.g. K_m) in order to guarantee stability for the control problem.

To develop these concepts we first simplify the notation and introduce *augmented* vectors as

$$s^*_{i+k} = [z^T_{i+k+1} v^T_{i+k}]^T,\ \bar{s}_{i+k} = [\bar{z}^T_{i+k+1} \bar{v}^T_{i+k}]^T, \text{ and } \varepsilon_{i+k} = [\eta^T_{i+k}, \nu^T_{i+k}]^T,$$

where $\eta_{i+k} = z_{i+k+1} - \bar{z}_{i+k+1}$ and $\nu_{i+k} = v_{i+k} - \bar{v}_{i+k}$, with $k = 1, \ldots, p$. Here the vector s^*_{i+k} corresponds to the optimal state and input vector sequences obtained by solving problem $\mathscr{P}(i)$ with the plant measurement x_i as the initial condition. On the other hand, $\bar{s}_{i+k}$ is the optimal vector of states and inputs obtained from solving $\mathscr{P}(i+1)$ with the initial condition given by $\bar{x}_{i+1}$. The difference between the two solutions, η_{i+k}, provides a measure of the error mismatch.

Since the cost function h is continuously differentiable and its derivative exists over the horizon p and is continuous in $\mathbf{R}^{n_s+n_i}$, then from the mean value theorem it follows that

$$h(s^*_{i+k}) - h(\bar{s}_{i+k}) = \int_0^1 \nabla h(\bar{s}_{i+k} + \tau\varepsilon_{i+k})^T \varepsilon_{i+k}\mathrm{d}\tau \tag{4.4}$$

for any $\bar{s}_{i+k}$, $(\bar{s}_{i+k} + \tau\varepsilon_{i+k}) \in \mathbf{R}^{n_s+n_i}$, $k \geqslant 1$. Substituting Eq. (4.4) in Eq. (4.1) gives

$$\Psi^*(x_{i+1}) - \Psi^*(\bar{x}_{i+1}) = \sum_{k=1}^{p}\int_0^1 \nabla h(\bar{s}_{i+k} + \tau\varepsilon_{i+k})^T \varepsilon_{i+k} d\tau \tag{4.5}$$

To bound Eq. (4.5) we need to determine ε_{i+k}, $k = 1, \ldots, p$, based on the deviation between the two initial conditions for the states available to solve the problem $\mathscr{P}(i+1)$, i.e. the state prediction $\bar{x}_{i+1}$ obtained from the solution of $\mathscr{P}(i)$ and the true state x_{i+1}. As explained at the end of this section, this determination is done by off-line solution of the optimal control problem, using initial conditions x_{i+1} and $\bar{x}_{i+1}$. For this purpose we take the optimality conditions for the optimal control problem $\mathscr{P}(i+1)$ and consider the case with x_{i+1} as the initial state condition. We define the decision vectors

$$s^* \equiv [s^{*T}_{i+1}, s^{*T}_{i+2}, \cdots s^{*T}_{i+p}]^T \tag{4.6}$$

and define $\Psi_{i+1}(s^*) = \Psi(x_{i+1})$. For the equality constrained case, problem $\mathscr{P}(i+1)$ can be rewritten as follows and solved to obtain s^*:

$$\begin{aligned}&\min_s \Psi_{i+1}(s) \\ &\text{s.t. } c(s) = 0\end{aligned} \tag{4.7}$$

where $c(s) = \begin{bmatrix} z_{k+1} - \chi_k(z_k, v_k), k = i+1, \ldots, i+p-1 \\ z_{i+p} \end{bmatrix}$

The Lagrangian for this problem is given by $L(s, \lambda) = \psi_{i+1}(s) + \lambda^T c(s)$, where λ is the Lagrange multiplier vector. The optimality conditions are

$$\begin{bmatrix} \nabla_s \psi_{i+1}(s^*) + \nabla_s c(s^*)^T \lambda^* \\ c(s^*) \end{bmatrix} = 0 \tag{4.8}$$

We also assume that the matrix $\nabla_s c(s^*)$ has full row rank and we define a basis, Z, for the null space of this matrix:

$$\nabla_s c(s^*) Z = 0.$$

By taking the projection of $\nabla_s \psi_{i+1}(s^*) + \nabla_s c(s^*)^T \lambda^*$ on the null space of $\nabla_s c(s^*)$, from (Eq. (4.6)) it follows that

$$\begin{bmatrix} Z^T \nabla_s \Psi_{i+1}(s^*) \\ c(s^*) \end{bmatrix} = 0 \tag{4.9}$$

Consider now the problem of solving $\mathscr{P}(i+1)$ using the *predicted* states $\bar{x}_{i+1}$ as the initial conditions. Here the decision vector is $\bar{s}$ with the same dimension as Eq. (4.6). The Lagrangian is given by $L(\bar{S}, \bar{\lambda}) = \psi_{i+1}(\bar{s}) + \bar{\lambda}^T c(\bar{s})$. Similarly we have that

$$\begin{bmatrix} \bar{Z}^T \nabla_s \psi_{i+1}(\bar{s}) \\ c(\bar{s}) \end{bmatrix} = 0 \tag{4.10}$$

To obtain a bound on Eq. (4.5) we need to determine a bound on ε_{i+k}, i.e. the deviation of $\bar{s}_{i+k}$ from s^*_{i+k}. This can be determined by application of NLP sensitivity and the mean value theorem. For this calculation we solve (Eqs. (4.8)) for (x_{i+1}, s). To simplify the notation we set $\psi^* = \psi_{i+1}(s^*)$ and $c = c(s^*)$.

Now considering the affine approximation to Eq. (4.9) for variations of $\bar{\varphi} = [\bar{x}^T_{i+1}, \bar{s}^T]^T$, such that Eqs. (4.9) and (4.10) are satisfied, it follows that

$$\int_0^1 \cdot \begin{bmatrix} \nabla_{x_{i+1}} Z^T \nabla_s \psi^* + Z^T \nabla_{sx_{i+1}} \psi^* & \nabla_s Z^T \nabla_s \psi^* + Z^T \nabla_{ss} \psi^* \\ \nabla_{x_{i+1}} c & \nabla_s c \end{bmatrix} \Big|_{\varphi=\varphi_\tau} \mathrm{d}\, \mathrm{d}\tau = 0$$

with $\varphi_\tau = [\bar{x}^T_{i+1}, \bar{s}^T]^T + \tau \mathrm{d}$ and $\mathrm{d} = \begin{pmatrix} x_{i+1} - \bar{x}_{i+1} \\ s^* - \bar{s} \end{pmatrix}$.

Solving to get the solution vector of the optimal control problem we obtain

$$s^* - \bar{s} = \left(-\int_0^1 \cdot \begin{bmatrix} \nabla_s Z^T, \nabla_s \psi^* + Z^T, \nabla_{ss} \psi^* \\ \nabla_s c \end{bmatrix} \Big|_{\varphi=\varphi\tau} \mathrm{d}\tau\right)^{-1} \cdot \int_0^1 \cdot \begin{bmatrix} \nabla_{x_{i+1}} Z^T, \nabla_s \psi^* + Z^T, \nabla_{sx_{i+1}} \psi^* \\ \nabla_{x_{i+1}} c \end{bmatrix} \Big|_{\varphi=\varphi\tau} d\tau (x_{i+1} - \bar{x}_{i+1}) \tag{4.11}$$

The solution vector $s^* - \bar{s}$ is well defined and unique and the integral matrix is nonsingular in a neighborhood of the optimal solution, as long as Eq. (4.7) has a strong local minimum and $\nabla_s c(s^*)$ has full row rank [28]. Eq. (4.11) shows how the solution of the NLP changes with errors in the initial conditions $x_{i+1} - \bar{x}_{i+1}$. To bound the integral terms in Eq. (4.11) we define positive constants $B, B_x \in [0, \infty)$, such that

$$\left\| \left(-\int_0^1 \cdot \begin{bmatrix} \nabla_s Z^T \nabla_s \psi^* + Z^T \nabla_{ss} \psi^* \\ \nabla_s c \end{bmatrix} \Big|_{\varphi=\varphi_\tau} \mathrm{d}\tau \right)^{-1} \right\| \leqslant B,$$

$$\left\| \int_0^1 \cdot \begin{bmatrix} \nabla_{x_{i+1}} Z^T \nabla_s \psi^* + Z^T \nabla_{sx_{i+1}} \psi^* \\ \nabla_{x_{i+1}} c \end{bmatrix} \Big|_{\varphi=\varphi_\tau} \mathrm{d}\tau \right\| \leqslant B_x.$$

Using various values of x_{i+1}, these constants can be estimated offline from the plant model and the bound on the model mismatch, and the resulting inequalities are a measure of the sensitivity of the states over the horizon p with respect to the initial state conditions. Substituting in Eq. (4.11) and taking the norm we obtain

$$\| s^* - \bar{s} \| \leqslant BB_x \| x_{i+1} - \bar{x}_{i+1} \| \leqslant \Gamma \| x_{i+1} - \bar{x}_{i+1} \| \tag{4.12}$$

i.e. the solution of the optimal control problem Eq. (4.7) is bounded by the deviation on the initial state conditions. Thus, for all i, $i \geqslant 0$, $k \geqslant i$, we establish a bound on the difference vector ε_{i+k} such that,

$$\|\varepsilon_{i+k}\| \leqslant \|s^* - \bar{s}\| \tag{4.13}$$

These quantities can be evaluated directly from the solution of the NLP. Note also that if the model is linear and the objective is quadratic, then the integrands are constant and B and B_x can be determined once and for all, from a single NLP solution.

We can now relate ε directly to the model mismatch. From Eq. (2.9) it follows that:

$$\|x_{i+1} - \bar{x}_{i+1}\| = \|f_k(x_i, u_i) - \chi_k(x_i, u_i)\| \leqslant W_m(\|x_i\|) \tag{4.14}$$

Combining Eq. (4.12) and (4.14) with Eq. (4.5) leads to

$$|\Psi^*(x_{i+1}) - \Psi^*(\bar{x}_{i+1})| \leqslant \sum_{k=1}^{p} \| \int_0^1 \nabla h^T \mathrm{d}\tau \varepsilon_{i+k} \| \leqslant \Gamma \sum_{k=1}^{p} \| \int_0^1 \nabla h \mathrm{d}\tau \| W_m(\|x_i\|) \tag{4.15}$$

This expression can be applied to general receding horizon control problems and, by using appropriate norms, can even be applied to infinite dimensional problems. On the other hand, if we consider a finite prediction horizon, p, substitute the quadratic function Eq. (3.2) and apply Eq. (4.5) we obtain a sharper bound:

$$\Psi^*(x_{i+1}) - \Psi^*(\bar{x}_{i+1}) = \sum_{k=1}^{p} (\bar{s}_{i+k} + \varepsilon_{i+k})^T Q_{i+k} \varepsilon_{i+k} \tag{4.16}$$

where $Q_{i+k} = \mathrm{diag}\{Q_{1,i+k},\ Q_{2,i+k}\}$ is bounded in norm by Q over the horizon p. Taking the norm of Eq. (4.16), and from Eqs. (4.12) and (4.13), we have the change in the objective function due to error mismatch given by

$$|\Psi^*(x_{i+1}) - \Psi^*(\bar{x}_{i+1})| \leqslant Q \sum_{k=1}^{p} (\| \bar{s}_{i+k} \| + \Gamma \| x_{i+1} - \bar{x}_{i+1} \|) \Gamma \| x_{i+1} - \bar{x}_{i+1} \|. \tag{4.17}$$

Using Eq. (4.14), gives:

$$|\Psi^*(x_{i+1}) - \Psi^*(\bar{x}_{i+1})| \leqslant Q \sum_{k=1}^{p} (\|\bar{s}_{i+k}\| + \Gamma W_m(\| x_i \|)) \Gamma W_m(\| x_i \|) \tag{4.18}$$

Now since $\| \bar{s}_{i+k} \| = \| \bar{z}_{i+k}, \bar{v}_{i+k} \|$, $k \geqslant 1$, depends on the initial state x_i, then from the feedback law and the NLP problem we can establish

$$\| \bar{s}_{i+k} \| \leqslant K \| x_i \| \tag{4.19}$$

Substituting in Eq. (4.18) it follows that

$$|\Psi^*(x_{i+1}) - \Psi^*(\bar{x}_{i+1})| \leqslant Qp\Gamma W_m(\| x_i \|)(\hat{K} \| x_i \| + \Gamma W_m(\| x_i \|)) \tag{4.20}$$

From definition Eq. (2.1) and from Eq. (2.5) and (2.6), the right hand side of Eq. (4.20) is a function that belongs to class $\mathscr{K}_\infty$ and therefore we can write that the mismatch term is bounded by a generic function as follows:

$$|\Psi^*(x_{i+1}) - \Psi^*(\bar{x}_{i+1})| \leqslant W_B(\| x_i \|) \tag{4.21}$$

By Assumption 2.3, substituting in Eq. (4.20) with $\gamma = 1$ it follows that

$$|\Psi^*(x_{i+1}) - \Psi^*(\bar{x}_{i+1})| \leqslant Qp\Gamma K_m(\hat{K} + \Gamma K_m) \| x_i \|^2 = K_B \| x_i \|^2 \tag{4.22}$$

where K_B can be determined off-line from the solution of Eq. (4.7).

In Eq. (3.13), because u_i is an implicit function from x_i we can replace $h(x_i, u_i)$ by $h(x_i)$. Thus, from Eq. (3.13) and (4.22) the following condition is sufficient to ensure stability:

$$h(x_i) - |\Psi^*(x_{i+1}) - \Psi^*(\bar{x}_{i+1})|, \geqslant h(x_i) - K_B \| x_i \|^2 = W(\| x_i \|) \tag{4.23}$$

For general problems, a simpler way to estimate values for K_B we note that Γ can be estimated from information at the solution of Eq. (4.7). Note that since Eq. (4.7) is not specialized to a finite p, it can be applied to infinite time horizons as well. In addition, the value of $\hat{K}$ can be estimated off-line by comparing the solutions of Eq. (4.7) with initial conditions at x_i, and with initial states at zero. Moreover, a tighter value for K_B can be obtained directly from:

$$K_B \geqslant \max_{x_i} \frac{|\Psi^*(x_{i+1}) - \Psi^*(\bar{x}_{i+1})|}{\| x_i \|^2} \tag{4.24}$$

This calculation can be made off-line, according to the following cycle:

1. For a given x_i, $i \geqslant 0$, perform the following steps:
2. Solve the optimal control problem $\mathscr{P}(i)$ for the predicted inputs v_k and states z_k over the time horizon p; save the state prediction for $i+1$, $\bar{x}_{i+1} = z_{i+1}|x_i$.
3. Set $u_i = v_i$ and $i = i + 1$.
4. Solve the optimal control problem $\mathscr{P}(i+1)$ using as initial condition the state measurement from the plant, x_{i+1}; obtain $\Psi^*(x_{i+1})$.
5. Solve the optimal control problem $\mathscr{P}(i+1)$ using as initial condition the state prediction obtained from the solution of the problem $\mathscr{P}(i)$, $\bar{x}_{i+1}$; obtain $\Psi^*(\bar{x}_{i+1})$.
6. Go to 1 and repeat steps with new values of x_i.

Therefore for a nonzero x_i we can compute a lower bound for K_B from Eq. (4.24). In addition, from this calculation we estimate the value of K_m in Eq. (2.10) (with $\gamma = 1$) which is needed to maintain robust stability.

In the next section we give two examples which illustrate this approach.

5. Examples

In this section we apply the procedure of the previous section to assess the robust stability properties of both linear and nonlinear model predictive controllers. In both cases we use the objective function Eq. (3.2) and apply Eq. (4.23) as the sufficient robust stability condition. Moreover, the constant K_B is estimated using Eq. (4.24) and the bounding procedure described above.

5.1. First-order linear system

Consider an open-loop stable first-order system described by the following continuous-time model

$$\dot{x} = -\frac{1}{\tau_p}x + \frac{\kappa_p}{\tau_p}u \tag{5.1}$$

where x is the state variable, u the control variable and τ_p and κ_p are the process time constant and the gain respectively. The operating steady-state point is the origin $(x, u) = (0, 0)$. For this example we set $\kappa_p = 3$ and $\tau_p = 5$. We introduce parametric mismatch by considering different values of the plant gain (κ_p) from the model (κ_m), and we set $\tau_m = \tau_p$. For the evaluation of K_B we note that for linear systems, the matrices in Eq. (4.11) are independent of x_i and these constants can be evaluated relatively cheaply. In this example, this constant was estimated for several values of model mismatch.

Fig. 2(a) and (b) represent the evolution of states and the respective control action from the MPC for various negative and positive values of the model gain, respectively. For the MPC, these profiles are obtained using output horizon lengths (p) and input horizon lengths (m) of $(p, m) = (15, 1)$. The weighting matrices in the cost function Eq. (3.2) are $Q_{1k} = 1$ and $Q_{2k} = 0$ and the sampling time is $\Delta t = 1$. Consequently, from Eq. (4.23) we require that $K_B < 1$ for robust stability.

With values of κ_m below 0.05578 and above -0.055975 the feedback system for the MPC controller becomes unstable. Also, in this range of κ_m the constant K_B is greater than one and the sufficient condition for robust stability [Eq. (4.23)] is not satisfied. The corresponding values of K_B and of the modeling bound constant in Eq. (2.10), K_m, are presented in Table 1. Figs. 3–5 show how K_B and K_m change with the parametric mismatch on the gain and illustrate the effect of the horizon lengths as well. In Fig. 3, notice that in the range of positive values of κ_m the discontinuity of K_B versus κ_m coincides with appearance of the ringing phenomenon in curves (f) and (e) in Fig. 2(b).

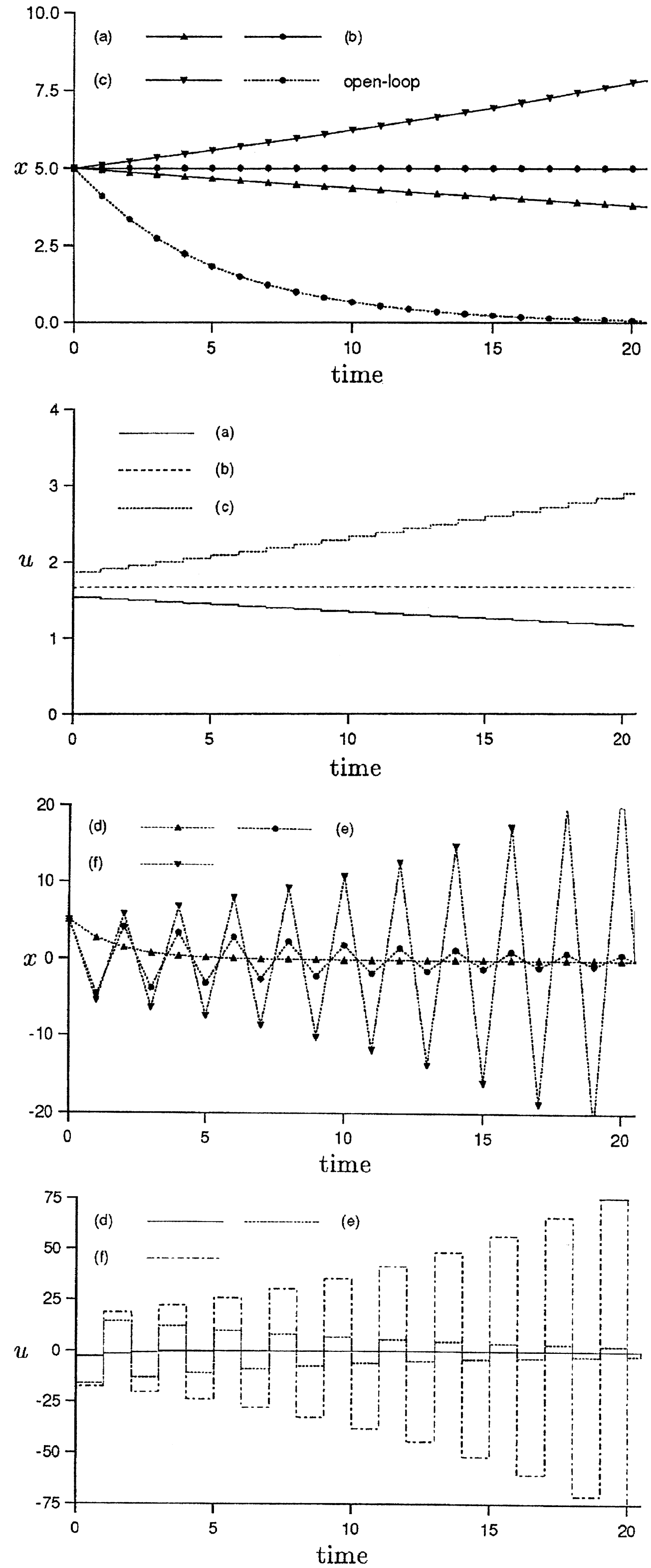

Fig. 2. (a)Plant open-loop and closed-loop responses for initial state $x_0 = 5$ and control moves for various values of the model gain,with $(p, m) = (15, 1)$: (a) $\kappa_m = -0.17$; (b) $\kappa_m = -0.157189$; (c) $\kappa_m = -0.14$. (b) Plant closed-loop response for initial state $x_0 = 5$ and control moves for various values of the model gain, with $(p, m) = (15, 1)$: (d) $\kappa_m = 0.1$; (e) $\kappa_m = 0.0165$; (f) $\kappa_m = 0.015$.

Table 1
Values of the sufficient condition stability [Eq. (4.23)] constant, K_B, and of the modeling bound constant in Eq. (2.10), K_m, for different parametric mismatch cases and with $(p, m) = (15, 1)$

Curve	(a)	(b)	(c)	(d)	(e)	(f)
κ_m	−0.17	−0.157189	−0.14	0.1	0.0165	0.015
K_m	0.17710	0.19076	0.21302	0.27543	1.71736	1.89005
K_B	0.89491	0.97128	1.09794	1.04101	0.47796	1.44454

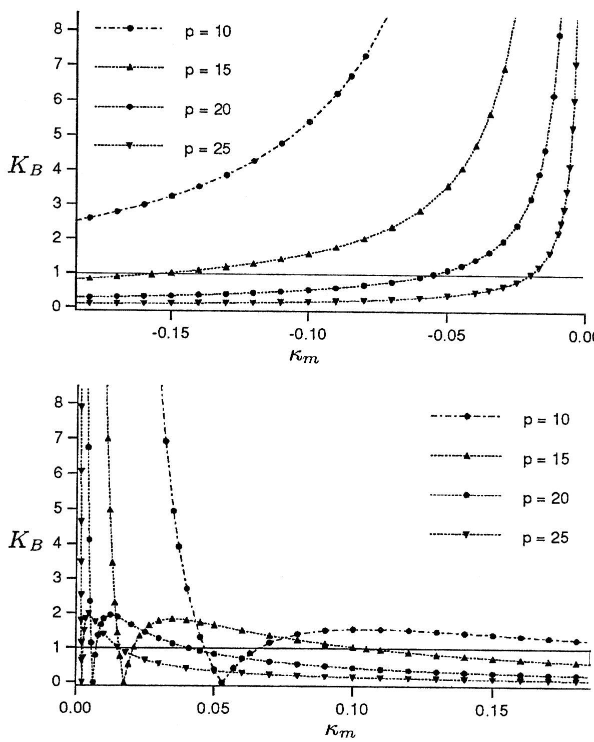

Fig. 3. Variation of the constant bound in Eq. (4.23), K_B, in function of the model gain (κ_m) for various output horizons, with $m = 1$.

We observe that the robustness of the closed-loop system increases with the increase of p. Thus $K_B > 1$ is reached only for a bigger difference of κ_m from κ_p and therefore the MPC controller is robust to a higher degree of parametric mismatch (Fig. 3). On the other hand, with a fixed $p = 15$, the robustness of the system deteriorates when the input horizon is increased from one to two intervals (Fig. 5). Since the controller has more degrees of freedom with $m = 2$, the control action is more aggressive. With $m > 2$ the control profiles are very similar and the estimated K_B over various values of the model gain are identical to the $m = 2$ case.

The total user CPU time to obtain the profiles of 20 samples in Fig. 1 and 2 varies from 9s [case (d)] to 18s [case (a)], on a Sun SPARCstation 10 to solve all of the NLPs related to Eq. (4.24). In particular, to estimate the constant K_B for a given x_i, the total user CPU time is 2.7s for the case with $(p, m) = (15, 1)$ and 4.2s for the case with $(p, m) = (25, 1)$. These times are quite modest especially as offline calculations.

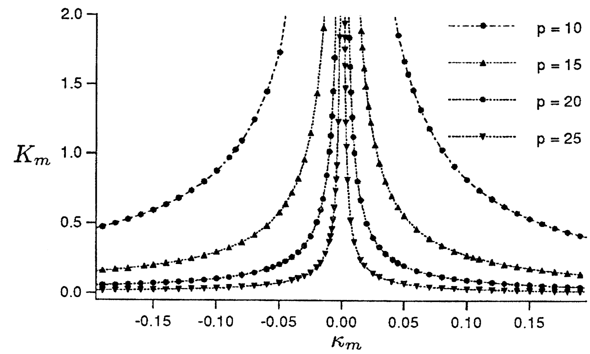

Fig. 4. Variation of the modeling bound in Eq. (2.10), K_m, in function of the model gain (κ_m) for various output horizons, with $m = 1$.

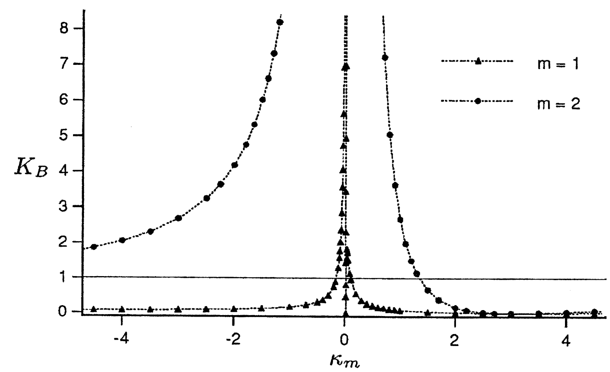

Fig. 5. Variation of the constant bound in Eq. (4.23), K_B, in function of the model gain (κ_m) for various input horizons, with $p = 15$.

5.1.1. *Stable model/unstable plant and vise-versa*

We consider now the case in which either the plant or the model are open-loop unstable by switching the sign on the first term of the right hand side of Eq. (5.1). The results obtained are given in Table 2. Fig. 6 shows closed-loop response of the cases given in Table 2, with horizons $(p, m) = (25, 1)$. From the four configurations only the first one is closed-loop unstable, and therefore $K_B > 1$. Cases III and IV are *perfect model* cases, thus $K_B = 0$. Although for the case II K_B is also greater than one because the model is open-loop unstable, the resulting feedback system is still stable. This is consistent with the fact that the condition $K_B < 1$ is a *sufficient* but not necessary condition for stability. Here the control action is initially much stronger than in case IV, thus driving the state to the origin faster. Also, since the model is unstable, increasing p leads to higher values of K_B.

To obtain the profiles of twenty samples in Fig. 6, the total user CPU time varies from 20s (case III) to 26s

Table 2
Values of the sufficient condition stability [Eq. (4.23)] constant, K_B, and of the modeling bound constant in Eq. (2.10), K_m, for different pairs unstable/stable plant/model with horizons $(p, m) = (15, 1), (25, 1), (35, 1)and(45, 1)$

Case	Open-loop response		K_B for different p with $m = 1$					Closed-loop feedback-system
	Model	Plant	$p=15$	$p=25$	$p=35$	$p=45$	$p=100$	
I	Stable	Unstable	2.28	2.46	2.49	2.49	2.49	Unstable
II	Unstable	Stable	5.45	10.84	16.65	22.56	55.21	Stable
III	Unstable	Unstable	0.00	0.00	0.00	0.00	0.00	Stable
IV	Stable	Stable	0.00	0.00	0.00	0.00	0.00	Stable

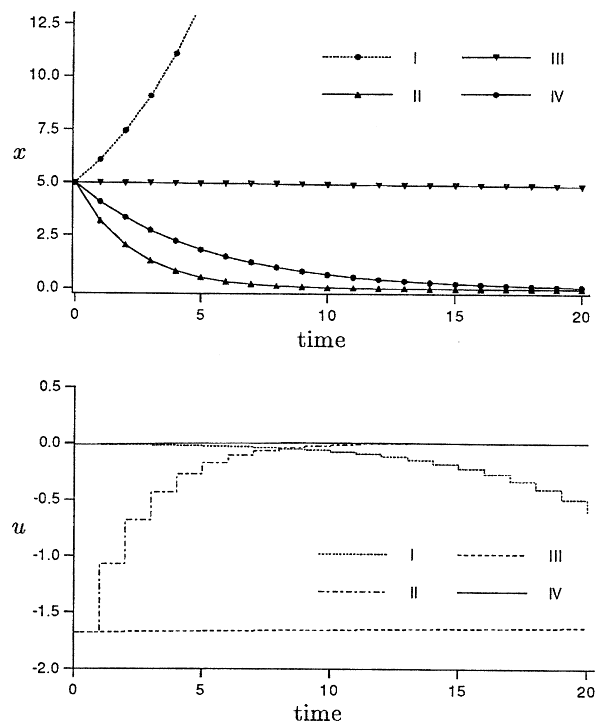

Fig. 6. Plant closed-loop response for initial state $x_0 = 5$ with $(p, m) = (25, 1)$ and for the following cases: (I) stable model and unstable plant; (II) unstable model and stable plant; (III) unstable model and plant; IV) stable model and plant.

(case *I*). Table 3 shows total user CPU time to estimate K_B for a given x_i with horizons $(p, m) = (25, 1)$ and $(100, 1)$.

5.2. SISO nonlinear system

We now consider an open-loop stable example to illustrate the calculation of the modeling bounds K_m and to verify the sufficient stability condition Eq. (4.23). The example is a SISO nonlinear system modeling an ideal CSTR with a first order reaction $A \rightarrow B$, and with temperature and volume constants. The behavior of the system is described by a nonlinear equation derived from the component A mass balance, given by:

Table 3
Total user CPU time to estimate K_B for a given x_i

Case	Open-loop response		CPU (s) with $m = 1$	
	Model	Plant	$p=25$	$p=100$
I	Stable	Unstable	4.3	13.5
II	Unstable	Stable	3.5	22
III	Unstable	Unstable	3.3	21
IV	Stable	Stable	4	12

$$\frac{\mathrm{d}C_A}{\mathrm{d}t} = \frac{F_i}{V}(C_{Ai} - C_A) - k_0 \mathrm{e}^{\frac{-E_a}{RT}} C_A \tag{5.2}$$

The data for the operating conditions and reactor design parameters are taken from a Van de Vusse reaction example cited in several studies (e.g. [29]). The nomenclature and nominal values for the parameters and variables are given in Table 4. The control objective is to keep C_A constant by manipulating the feed rate F_i.

To introduce error mismatch in the model we select different values for the model parameters such that there is no steady-state zero offset. In this example at least two parameters of the model must be changed such that the input and state variables are the same for both the plant and the model. Here we introduce mismatch on the dynamic behavior of the system by selecting different values of V and k_0 for the plant (subscript $_p$) and for the model (subscript $_m$) such that

$$(Vk_0)_m = (Vk_0)_p \tag{5.3}$$

This eliminates the steady state mismatch and satisfies the assumptions made in Section 2. We therefore introduce modeling errors by setting $(k_0)_m = \frac{1}{\alpha}(k_0)_p$, with $\alpha \in (0, \beta]$, $\beta > 0$, and from Eq. (5.3) it follows that $(V)_m = \alpha \cdot (V)_p$.

Fig. 7 shows the open-loop response for various α. Here C_{A_n} is *dimensionless* C_A and $x_0 = C_{A_n} - C_{A_n,sp}$, where $C_{A_{n,sp}=1}$ is the setpoint. Note that $\alpha = 1$ corresponds to a perfect model case. With $\alpha < 1$ the model response is faster than the plant response and for $\alpha > 1$

Table 4
Variables and steady state values of the example model

C_A	Concentration of A	1.3829	mol/l
C_{Ai}	Feed concentration of A	5.1000	mol/l
F_i	Feed rate	0.1883	m^3/h
T	Reactor temperature	407.29	K
V	Reactor volume	0.0100	m^3
E_a/R	Energy of activation $/R$	9758.3	K
k_0	Arrhenius constant	1.287×10^{12}	h^{-1}

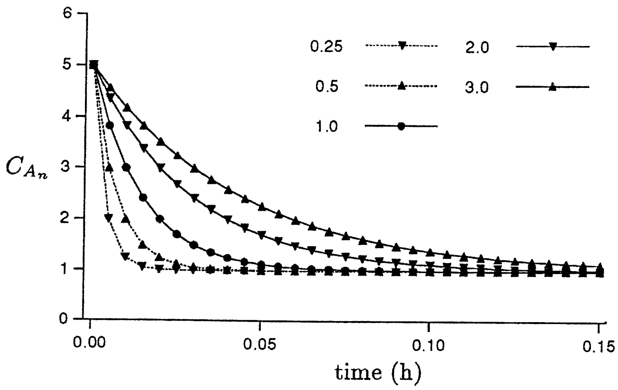

Fig. 7. Open-loop response for various values of α.

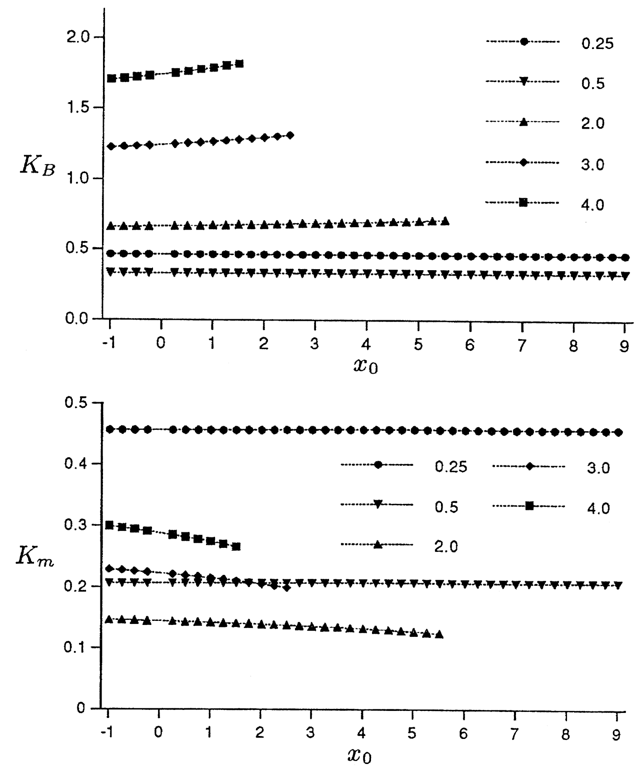

Fig. 8. Variation of K_B and K_m with the initial state x_0 for various α, with $(p, m) = (15, 1)$.

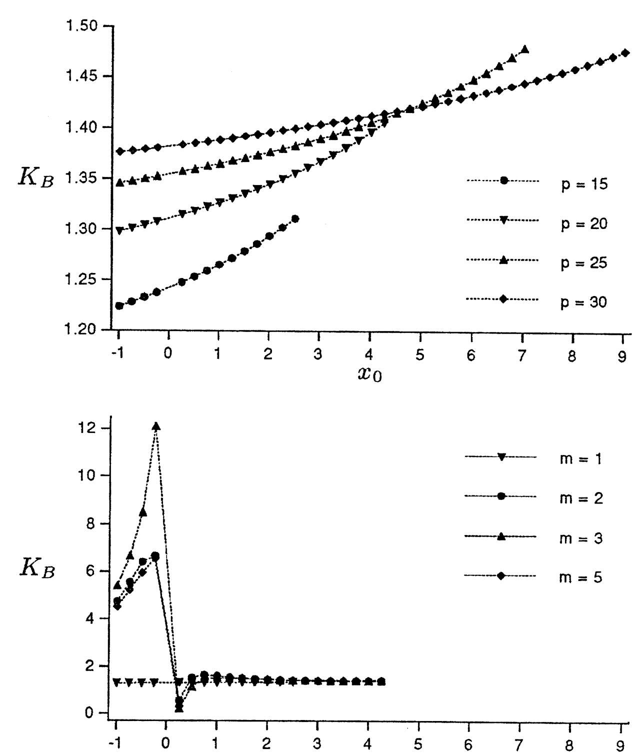

Fig. 9. Variation of K_B with the initial state x_0 for various p, with $m = 1$ (left), and for various m, with $p = 20$ (right), with $\alpha = 3$.

it is slower. The system is always open-loop stable for the range of α values we use in this study, $\alpha \in [0.25, 4]$.

In this simulation the weighting matrices in Eq. (3.2) are $Q_{1,k} = I$ and $Q_{2,k} = 0$ over the horizon p and the sampling interval is $\Delta t = 0.005$ h. Since we do not consider constraints on the variables over the predictive horizon, negative values for the control variable can be obtained from solving the optimization problem. To overcome this difficulty we introduce a simple smoothing technique [30] by adding the following equation to both the plant and model simulation frameworks.

$$F_i = \max\{0, u\} = \frac{(u^2 + \varepsilon^2)^{\frac{1}{2}}}{2} + \frac{u}{2} \tag{7}$$

where u is the solution from the optimization problem and $\varepsilon = 0.001$.

Again, the relation $K_B < 1$ is sufficient for robust stability. To estimate the values of K_m and K_B we solve NLP problems off-line as stated at the end of Section 4, for different values of α (α = 0.25, 0.50, 2.0, 3.0 and 4.0).

We vary x_0 by 0.25 from -1 to 9. In this range of α the MPC controller is always stable either with K_B less or greater than one. This is consistent with our analysis as the condition $K_B < 1$ is sufficient but not necessary for stability. Fig. 8 shows *profiles* of K_B and K_m with $(p, m) = (15, 1)$. We observe that when the model has a slower response than the plant ($\alpha > 1$), K_B increases with x_0. In this case the value of K_B for the given mismatch, is

Table 5
Total user CPU time to estimate K_B (n is the number of x_0 trials)

α	(p, m)	n	CPU (s)
0.25	(15,1)	41	58.2
2.0	(15,1)	27	82.7
3.0	(20,5)	6	22.4
3.0	(30,1)	41	219.8

obtained from (Eq. (4.24)). Also, as α increases the K_B profile terminates for smaller values of x_0. The reason is that as α increases it is more difficult to find a solution that satisfies the terminal constraint [Eq. (3.5)], thus leading to a controller failure.

Fig. 9 shows the effect of p and m when $\alpha = 3$. Here the closed-loop system is also stable. Nevertheless, a solution may not exist if the output horizon is too short, and it becomes more difficult to converge to a solution as we increase the value of m. The total user CPU time to obtain some of the *profiles* of K_B in Fig. 8 and 9 is indicated in Table 5 (the runs were made on a Sun SPARCstation 10). Again, these times are quite modest especially as offline calculations.

6. Conclusions

We develop a strategy based on nonlinear programming sensitivity that determines conditions under which the MPC is robustly stable with respect to modeling errors. Here, a sufficient condition for robust stability is derived and an offline procedure is developed to evaluate constants which determine sufficient conditions for this property. These constants are available from bounds on the model mismatch and from the NLP solution of the receding horizon model. This procedure is applicable to both linear and nonlinear model predictive controllers in discrete time that satisfy nominal stability properties based on Lyapunov arguments. Two small examples, one linear and one nonlinear, are presented to demonstrate the effectiveness of this approach.

As future work, we will demonstrate the application of this approach to more challenging multivariable nonlinear process models and plants. Also considering the fact that the prediction horizon needs to be long enough to insure existence of a feasible trajectory, we will consider a more detailed analysis of the existence problem. Moreover, the extension of this analysis to deal with modeling errors, the effect of disturbances, and the output feedback case is currently under development. Here a more general formulation with plant/model mismatch at the steady state operating point as indicated in Mayne [27] would be taken in consideration.

Finally, this research will be extended to the incorporation of input and output constraints in the formulation of $\mathscr{P}(i)$. Robust stability of MPC with hard constraints was considered by Zafiriou [16] and Zafiriou and Marchal [17]; properties were developed for linear systems using contraction arguments instead of a Lyapunov approach. In future work, we will apply penalty function formulations instead that extend the analysis of this paper in a straightforward way. In particular, the use of exact penalty terms in the objective function as in Oliveira and Biegler [18] will allow us to evaluate the robust stability of constrained MPC controllers directly. Hard constraints will also be considered as needed.

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