

Short communication

Frequency response of linear systems containing pure capacitance elements

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Abstract

Systems that contain a pure capacitance are not stable and thus the frequency response theorem cannot be applied. Here the response of a pure capacitive element is addressed and erroneous responses in some textbooks are pointed out. Then an easy way to obtain the frequency response of systems consisting of the series a pure capacitance and any stable system is presented.

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1. Introduction

Pure capacitances are elements for which the dynamic behaviour is given by an equation of the form

$$C \frac{dy}{dt} = x(t) \quad (1)$$

where $x(t)$ and $y(t)$ are the input and the output variables, respectively, defined as deviation variables relative to the steady-state. The transfer function for such systems is given by

$$G(s) = \frac{Y(s)}{X(s)} = \frac{1}{Cs} \quad (2)$$

In chemical processes the most common example of a capacitance is a storage tank with either the inlet or the outlet flowrate kept constant. This and other examples of pure capacitances in various systems are shown in Table 1.

This paper first addresses a problem that has been often mistreated in several textbooks on process dy-

namics and control, namely the frequency response of pure capacitances. Then, systems consisting on the series of a pure capacitance and any stable system are analysed. An expedite way of obtaining the frequency response, avoiding the calculation of the inverse Laplace transform, is obtained.

2. Response of pure capacitances

The response of a pure capacitance to a sinusoidal perturbation, $x(t) = a \sin(\omega t)$, can be obtained by direct inversion of

$$Y(s) = \frac{1}{Cs} X(s) = \frac{1}{Cs} \frac{a\omega}{s^2 + \omega^2} \quad (3)$$

resulting in

$$y(t) = \frac{a}{C\omega} (1 - \cos \omega t) \quad (4)$$

Fig. 1 shows the plots of the perturbation $x(t)$ and the response $y(t)$ where it becomes clear that the response is a periodic function that oscillates around $a/C\omega$ in the range $0 \leq y(t) \leq 2a/C\omega$.

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¹ In memoriam.

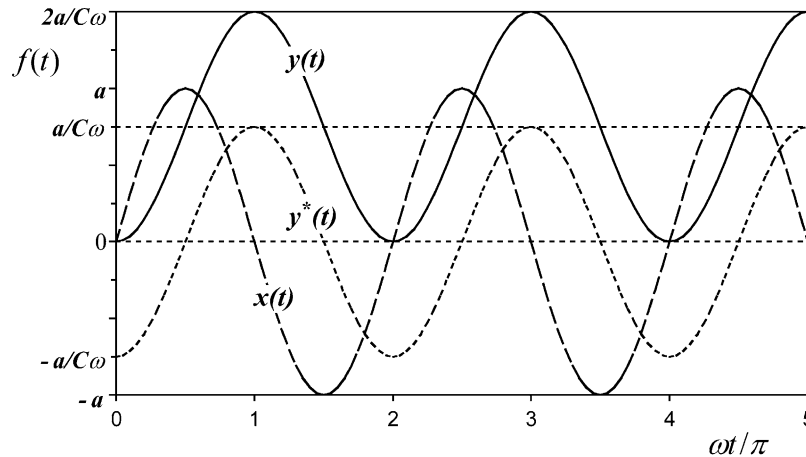


Fig. 1. Response, $y(t)$, of a pure capacitance to a sinusoidal perturbation, $x(t)$, and $y^*(t) = y(t) - a/C\omega$.

3. Frequency response theorem

The fundamental theorem of frequency response analysis states that:

After an initial transient period, the response of a linear, stable system to a sinusoidal input is also a sinusoidal wave with the same frequency; the ratio between the input and the output amplitudes is given by $AR = |G(j\omega)|$ and the phase shift by $\phi = \angle G(j\omega)$.

The blind application of this theorem to the pure capacitance transfer function leads to

$$\text{Amplitude Ratio } AR = |G(j\omega)| = \frac{1}{C\omega} \tag{5a}$$

$$\text{Phase Shift } \phi = \angle G(j\omega) = -\frac{\pi}{2} \tag{5b}$$

Thus, the ultimate sinusoidal response of a pure capacitance could be presented as

$$y^*(t) = \frac{a}{C\omega} \sin\left(\omega t - \frac{\pi}{2}\right) \tag{6}$$

This is clearly not the correct solution since it leads to a sinusoidal response oscillating around zero with amplitude equal to $a/C\omega$. The pure capacitance system is not stable, and therefore, the frequency response theorem cannot be applied.

On the other hand, Eq. (4) may be rewritten in the form

$$y(t) = \frac{a}{C\omega} + \frac{a}{C\omega} \sin\left(\omega t - \frac{\pi}{2}\right) \tag{7}$$

or

$$y^*(t) = y(t) - \frac{a}{C\omega} = \frac{a}{C\omega} \sin\left(\omega t - \frac{\pi}{2}\right) \tag{8}$$

This function, $y^*(t)$, is a sinusoidal wave with

amplitude and phase shift functions equal to the modulus and angle directly obtained by replacing $s = j\omega$ on the pure capacitance transfer function. In this way the frequency response of a pure capacitance, $y(t)$, may be obtained easily from $G(j\omega)$ adding the term $a/C\omega$ to $y^*(t)$.

Arising from the abusive application of the frequency response theorem, erroneous solutions have been presented in textbooks. This is the case in Stephanopoulos [1] (example 17.1, page 323) who writes Eqs. (5a) and (5b) as being applied to the pure capacitance system.

Table 1
Examples of pure capacitance in various systems

System	Scheme	Conservation equation
Hydraulic Storage		$F_i(t) - F_o = A \frac{dh}{dt}$ $F_o = \text{constant}$
Electric Capacitance		$i(t) = C \frac{dv}{dt}$
Thermal Capacitance		$q(t) = mC_p \frac{dT}{dt}$
Mechanic Translational Mass		$F(t) = m \frac{du}{dt}$
Mechanic Rotational Inertia		$\Gamma(t) = J \frac{d\Omega}{dt}$

Ogunnaike and Ray [2] present the sinusoidal response of a capacitance system in the correct form of Eq. (4), and they then rewrite it in the form

$$y(t) = \frac{a}{C\omega} \left[1 + \sin\left(\omega t - \frac{\pi}{2}\right) \right] \quad (9)$$

Based on this formulation they go on and say that the frequency response of a pure capacity system has an amplitude ratio and a phase angle given by Eqs. (5a) and (5b), and show the plot of $y^*(t)$ as the sinusoidal response of a pure capacitance (figure 5.13, page 161).

4. Response of series of a pure capacitance and a stable system

Ogata [3] (page 458) points out that if the system with transfer function $G(s) = Y(s)/X(s)$ is not stable then the response to a sinusoidal perturbation may be obtained by taking the inverse Laplace transform of $Y(s) = G(s)[a\omega/(a^2 + \omega^2)]$. In these cases and if the transfer function is complex the calculation of the inverse may be a difficult and tedious process.

Systems composed of the series of a pure capacitance and a stable system, whose transfer function may be written as

$$G(s) = \frac{1}{C_s} G_1(s) \quad (10)$$

where $G_1(s)$ is the transfer function of any stable system, are also unstable and so the frequency response theorem may not be applied. However, the result of Eq. (8) may be extended to these types of systems in the form

$$y^*(t) = y(t) - \frac{aK}{C\omega} = \frac{aK}{C\omega} \text{AR}_1^N \sin\left(\omega t - \frac{\pi}{2} + \phi_1\right) \quad (11a)$$

$$\text{AR}_1^N = \frac{1}{K} |G_1(j\omega)| \quad \phi_1 = \angle G_1(j\omega) \quad (11b)$$

where K is the gain of the system with transfer function $G_1(s)$.

For example, consider the system composed of a pure capacitance in series with a first order system, that is

$$G(s) = \frac{1}{C_s} \frac{K}{\tau s + 1} \quad (12)$$

Then

$$y^*(t) = \frac{aK}{C\omega} \frac{1}{\sqrt{\omega^2\tau^2 + 1}} \sin\left(\omega t - \frac{\pi}{2} + \phi\right) \quad (13)$$

$$\phi = -\arctan(\omega\tau)$$

and the system response is obtained by

$$y(t) = y^*(t) + \frac{aK}{C\omega} \quad (14)$$

as plotted in Fig. 2. The response at long times is a periodic function oscillating around $aK/C\omega$ with amplitude $\pm aK/(C\omega\sqrt{\omega^2\tau^2 + 1})$.

5. Conclusions

The problem of the frequency response of pure capacitance systems has been addressed, and it was shown that it has been wrongly solved in some textbooks on process dynamics and control. The reason for these errors comes from the inadequate application of the fundamental theorem of frequency response that can only be applied to stable systems, which is not the case with pure capacitance systems. An extension to the frequency response of systems consisting on the series of a pure capacitance and any stable system has been introduced and an easy way to obtain the solution is presented.

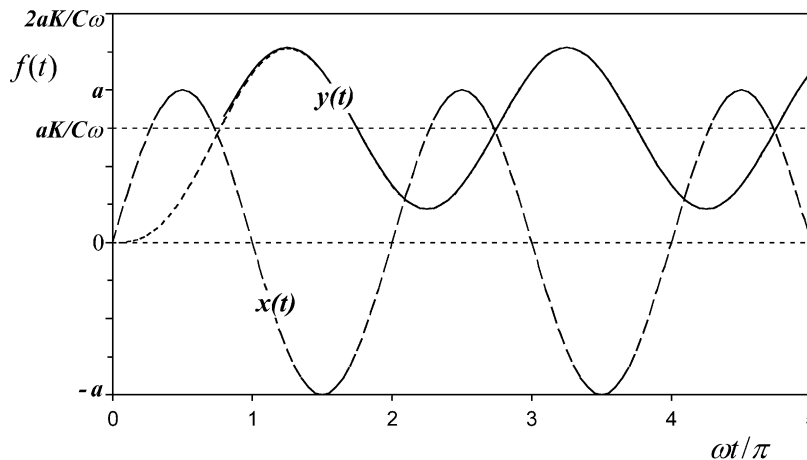


Fig. 2. Response, $y(t)$, of a series of a first-order system and a pure capacitance to a sinusoidal perturbation, $x(t)$.

Appendix A: Nomenclature

A	hydraulic storage tank cross-sectional area (m^2)
a	sinusoidal input wave amplitude
AR	output wave amplitude ratio
C	electric capacitance (F)
C_p	thermal capacitance specific heat ($\text{J}/(\text{kg K})$)
F	mechanic translational force (N)
F_o	hydraulic storage tank outlet flowrate (m^3/s)
F_i	hydraulic storage tank inlet flowrate (m^3/s)
$G(s)$	transfer function
h	liquid level (m)
I	electrical current (A)
J	mechanic rotational moment of inertia (N m s^2)
K	system gain
m	mechanic translational mass; thermal capacitance mass (kg)
q	heat flow rate through thermal capacitance (J/s)
T	thermal capacitance temperature (K)
T	time (s)
s	Laplace domain variable (per s)
u	mechanic translational velocity (m/s)
v	electrical voltage (V)

x	input deviation variable
$X(s)$	input variable transform
y	output deviation variable
y^*	$= y(t) - a/C\omega$
$Y(s)$	Output variable transform

Greek symbols

ϕ	output wave lag or phase-shift
Ω	mechanic rotational angular velocity (rad/s)
Γ	mechanic rotational torque (N m)
τ	system time constant (s)
ω	sinusoidal input wave frequency

References

- [1] G. Stephanopoulos, Chemical Process Control. An Introduction to Theory and Practice, Prentice Hall, 1984.
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