On the regularity of infinity-harmonic functions

João Morais Carreira Pereira

DEPARTAMENTO DE MATEMÁTICA
FACULDADE DE CIÊNCIAS E TECNOLOGIA
UNIVERSIDADE DE COIMBRA

On the regularity of infinity-harmonic functions

João Morais Carreira Pereira

Dissertação para a obtenção do Grau de **Mestre em Matemática**
Área de Especialização em **Análise Aplicada e Matemática Computacional**

Júri

Presidente: José Carlos Soares Petronilho
Orientador: José Miguel Urbano
Vogal: Susana Domingues de Moura

Data: Julho de 2014

Resumo

Neste texto são apresentados os principais resultados da teoria das funções ∞-harmónicas, com ênfase nos resultados recentes relativos à questão da regularidade.

Palavras Chave: Extensões de Lipschitz, Soluções de Viscosidade, Funções ∞-harmónicas, Regularidade

Abstract

In the thesis the main results concerning ∞-harmonic functions are presented, with a focus on the latest results concerning regularity.

Keywords: Lipschitz Extensions, Viscosity Solutions, ∞-harmonic Functions, Regularity

Agradecimentos

Gostava primeiro de agradecer ao meu orientador José Miguel Urbano, não só pelo apoio e disponibilidade que teve para me guiar na realização da dissertação, mas também pela ajuda e envolvimento na minha candidatura ao doutoramento, onde vou crescer como matemático.

Gostava também de agradecer à minha família, pelo apoio incondicional que sempre me deram para perseguir aquilo que gostava, e mais que isso, estarem sempre lá nos momentos importantes e ajudarem sempre que preciso, às vezes mesmo sem eu pedir.

Por fim queria agradecer aos professores do Departamento de Matemática da Universidade de Coimbra, que me ajudaram a crescer como matemático, e aos meus colegas e amigos, que me fizeram crescer como pessoa e contribuíram para que estes últimos 5 anos, em que fui aluno na Universidade de Coimbra, fossem dos melhores da minha vida.

Contents

Chapter 1

Introduction

The theory of ∞-harmonic functions began in 1967, with the paper *Extension of functions satisfying Lipschitz conditions* from Gunnar Aronsson [2]. The problem of minimizing the Lipschitz constant of a function subject to Dirichlet conditions in a bounded set was known to have a smallest and largest solution, called the McShane-Whitney extensions. However, as we will see, this solutions have some 'unpleasant' properties, and in general are distinct, so Aronsson questioned if it would be possible to find a canonical Lipschitz Extension, also solution to the original problem, that would have much nicer properties, and be a unique solution with such properties. Aronsson would end up succeeding, when he introduced the class of absolutely minimizing Lipschitz extensions, and even proved the existence of such extensions. However the uniqueness and stability problems remained open for the next 26 years.

Aronsson also came across the ∞-Laplace equation

$$\Delta_\infty u = \sum_{i,j=1}^{n} u_{x_i} u_{x_j x_i} u_{x_j} = 0,$$

which he discovered by considering the p-Laplace equation. and passing to the limit in p. In fact, for $p > 2$, the equation is equivalent to

$$\frac{1}{p-2}|Du|^2 \Delta u + \Delta_\infty u = 0 \quad \text{in } U$$

and taking $p \to \infty$ we obtain $\Delta_\infty u = 0$. He also proved that if $u \in C^2$, then it is a classical solution to the ∞-Laplacian equation if, and only if, it is absolutely minimizing Lipschitz.

However Aronsson couldn't go further than this, as the mathematical concepts for treating this problem were not developed yet. In fact, the theory of viscosity solutions wouldn't be developed until the 80's, one of the first using such ideas being L. C. Evans. This theory was developed in that decade, where strong results about general viscosity equations, such as Comparison and Existence Theorems were discovered. It was only in 1993, in [9], that R. Jensen would settle the uniqueness and stability of ∞-harmonic functions, using the results about viscosity solutions developed so far.

The regularity question was also on the table. Aronsson had also given the example of the absolutely minimizing Lipschitz function $u(x,y) = x^{\frac{4}{3}} - y^{\frac{4}{3}}$, whose first derivatives are Hölder continuous with exponent $1/3$, so we can not have a better regularity. Jensen had already shown in his work that the ∞-subharmonic functions were Lipschitz continuous, hence differentiable almost everywhere. P. Lindqvist and J. Manfredi [10], considering the limit to ∞ of the p-Laplace equation, proved the sharper Harnack inequality presented in this paper, and a version of the Liouville's Theorem for the ∞-Laplacian.

In 2001, Crandall, Evans and Gariepy, in [6], introduce the notion of comparison with cones and prove its equivalence with ∞-harmonicity. They obtained much simpler proofs for the questions already answered and it became clear at the time that probably the approximation by p-harmonic functions was not the most efficient path to derive properties about ∞-harmonic functions.

In [5], Crandall and Evans obtain a result that would be crucial for proving that ∞-harmonic functions are differentiable everywhere. This result is then explored by Evans and Smart, in [8], to finally settle the everywhere differentiability of ∞-harmonic functions.

About the C^1 and $C^{1,\alpha}$ regularity, we only know them to hold in 2 dimensions, due to Savin in [12] and Evans-Savin in [7]. However the arguments they used are restricted to the two dimensions case, and the generalization to $n > 2$ is not obvious.

Finally, we'll get a flavor of some applications and generalizations of ∞-harmonic functions. There is by now a lot of papers concerning optimization problems with supremum type functionals, and many of this papers are related to the ∞-harmonic theory. Some generalizations of this problem, such as minimizing $\|H(x,u,Du)\|_{L^\infty(U)}$ for some well-chosen functions H, or considering generalizations of comparison with cones, are subject of intense study nowadays. One example of the extent of the applications of this problem is the paper [11] from Peres, Schramm, Sheffield and Wilson, where it was discovered that the value function of a random turn "tug of war" game was given by a ∞-harmonic function.

For more examples of applications, or to get a better insight of the historic background of this problem, I suggest consulting [4], from Crandall.

Chapter 2

Absolutely Minimizing Lipschitz Extensions

2.1. The Lipschitz Extension Problem

The Lipschitz Extension Problem was the motivation to the study of absolutely minimizing Lipschitz extensions. Let's start with the definition of Lipschitz function.

Definition 2.1.1 (Lipschitz Function). *Let $X \subset \mathbb{R}^n$. A function $f : X \to \mathbb{R}$ is Lipschitz continuous on X, or $f \in \mathrm{Lip}(X)$, if there exists a constant $L \in \mathbb{R}_0^+$ such that*

$$|f(x) - f(y)| \leq |x - y|, \quad \forall x, y \in X. \tag{2.1.1}$$

If (2.1.1) holds for $L \in \mathbb{R}_0^+$ then L is called a Lipschitz constant for f in X.
The least constant $L \in \mathbb{R}_0^+$ for which (2.1.1) holds is denoted by $\mathrm{Lip}_f(X)$.
If there is no L for which (2.1.1) holds, we write $\mathrm{Lip}_f(X) = \infty$.

Let $U \subset \mathbb{R}^n$ be open and bounded and denote its boundary with ∂U. The Lipschitz Extension Problem consists in extending one function defined on ∂U to U without increasing its Lipschitz constant.

Problem 2.1.2 (Lipschitz Extension Problem). *Given $f \in \mathrm{Lip}(\partial U)$, find $u \in \mathrm{Lip}(\overline{U})$ such that*

$$u = f \text{ on } \partial U \quad \text{and} \quad \mathrm{Lip}_u(\overline{U}) = \mathrm{Lip}_f(\partial U).$$

We will see that this problem is not well posed. In fact the next theorem will show that this problem has always solution, but in general it's not unique. First we have to define MacShane-Whitney extensions.

Definition 2.1.3. *The **MacShane-Whitney extensions** of $f \in \mathrm{Lip}(\partial U)$ are the real functions defined in $\overline{U}$ by*

$$\mathcal{MW}_*(f)(x) := \sup_{z \in \partial U} \{f(z) - \mathrm{Lip}_f(\partial U)|x - z|\}$$

and

$$\mathcal{MW}^*(f)(x) := \inf_{z\in\partial U}\{f(z) + \mathrm{Lip}_f(\partial U)|x-z|\}.$$

We now state the theorem.

Theorem 2.1.4. *The MacShane-Whitney extensions $\mathcal{MW}_*(f)$ and $\mathcal{MW}^*(f)$ solve the Lipschitz extension problem for $f \in \mathrm{Lip}(\partial U)$ and for any other solution of the problem we have*

$$\mathcal{MW}_*(f) \leq u \leq \mathcal{MW}_*(f) \text{ in } \overline{U}. \tag{2.1.2}$$

Thus the Lipschitz Extension Problem is uniquely solvable if and only if

$$\mathcal{MW}_*(f) = \mathcal{MW}*(f) \text{ in } \overline{U}.$$

Proof. We can see that if u is a solution to the problem 2.1.2 for f defined in ∂U, then $-u$ is a solution to the same problem for $-f$. Taking this into account, and

$$\begin{aligned}
\mathcal{MW}_*(f)(x) &= \sup_{z\in\partial U}\{f(z) - \mathrm{Lip}_f(\partial U)|x-z|\} \\
&= -\inf_{z\in\partial U}\{-f(z) + \mathrm{Lip}_f(\partial U)|x-z|\} \\
&= -\mathcal{MW}^*(-f)(x),
\end{aligned}$$

we conclude it's enough to prove that $\mathcal{MW}_*(f)$ is solution to the problem and the first inequality in (2.1.2).

We will first show the inequality. Suppose u solves 2.1.2 for f defined in ∂U.

Let $z \in \partial U$ and for $x \in \overline{U}$ define

$$F_z(x) = f(z) - \mathrm{Lip}_f(\partial U)|x-z|, \quad x \in \overline{U}.$$

Using (2.1.1) and remembering that $\mathrm{Lip}_u(\overline{U}) = \mathrm{Lip}_f(\partial U)$ and $u(z) = f(z)$, we obtain, through simple calculations

$$u(x) \geq f(z) - \mathrm{Lip}_f(\partial U)|x-z| = F_z(x).$$

Taking now the supremum in ∂U at the right-hand side, we finally obtain

$$u(x) \geq \sup_{z\in\partial U} F_z(x) = \mathcal{MW}_*(f)(x).$$

We will now prove that $\mathcal{MW}_*(f)$ is solution to the problem 2.1.2. First we observe that

$$\begin{aligned}
|F_z(x) - F_z(\tilde{x})| &= \left| f(z) - \mathrm{Lip}_f(\partial U)|x-z| - f(z) + \mathrm{Lip}_f(\partial U)|\tilde{x}-z| \right| \\
&= \mathrm{Lip}_f(\partial U)\left| |\tilde{x}-z| - |x-z| \right| \\
&\leq \mathrm{Lip}_f(\partial U)|\tilde{x}-z-x+z| \\
&= \mathrm{Lip}_f(\partial U)|x-\tilde{x}|.
\end{aligned}$$

Then we use this and take the supremum

$$\begin{aligned}
|\mathcal{MW}_*(f)(x) - \mathcal{MW}_*(f)(\tilde{x})| &= \left| \sup_{z\in\partial U} F_z(x) - \sup_{y\in\partial U} F_y(\tilde{x}) \right| \\
&\leq \sup_{z\in\partial U} |F_z(x) - F_z(\tilde{x})| \\
&\leq \mathrm{Lip}_f(\partial U)|x-\tilde{x}|.
\end{aligned}$$

So $\mathcal{MW}_*(f)$ is Lipschitz continuous with constant no bigger than $\mathrm{Lip}_f(\partial U)$. We next show that $\mathcal{MW}_*(f) = f$ on ∂U, this way we guarantee that the Lipschitz constant is indeed $\mathrm{Lip}_f(\partial U)$. Take $x \in \partial U$. Then

$$\mathcal{MW}_*(f)(x) \geq F_x(x) = f(x) - \mathrm{Lip}_f(\partial U)|x-x| = f(x).$$

On the other hand, since f is Lipschitz continuous, we have for $z \in \partial U$,

$$f(z) - \mathrm{Lip}_f(\partial U)|x-z| \leq f(x),$$

and taking the supremum at left-hand side, we finally obtain

$$\mathcal{MW}_*(f)(x) = \sup_{z\in\partial U} \{f(z) - \mathrm{Lip}_f(\partial U)|x-z|\} \leq f(x).$$

$\square$

We will now see a example where $\mathcal{MW}_*(f) \not\equiv \mathcal{MW}^*(f)$.

Example 2.1.5. *Let $n = 1$ and $U = (-1,1) \cup (1,2)$, so $\partial U = \{-1,1,2\}$. Consider $f : \partial U \to \mathbb{R}$ defined by $f(-1) = f(1) = 0$ and $f(2) = 1$. Then we have* $\mathrm{Lip}_f(\partial U) = 1$,

$$\mathcal{MW}_*(f)(x) = \begin{cases} |x| - 1 & \text{if} \quad -1 \leq x < 1 \\ x - 1 & \text{if} \quad 1 \leq x \leq 2 \end{cases}$$

and

$$\mathcal{MW}^*(f)(x) = \begin{cases} 1 - |x| & \text{if} \quad -1 \leq x < 1 \\ x - 1 & \text{if} \quad 1 \leq x \leq 2 \end{cases}$$

The non-existence of unique solution to problem 2.1.2 is not the only issue, as we will show next, with the help of the example above.

Non Comparison: Consider the function $g : \partial U \to \mathbb{R}$ defined by $g(-1) = 0$, $g(1) = \frac{2}{3}$ and $g(2) = 1$. Then we can see that $\mathrm{Lip}_g(\partial U) = \frac{1}{3}$ and

$$MW_*(g)(x) = MW^*(g)(x) = \frac{x+1}{3}, \quad x \in [0,3]$$

so this is the unique solution. Notice that $f \leq g$ but neither

$$\mathcal{MW}^*(f) \leq \mathcal{MW}^*(g)$$

nor

$$\mathcal{MW}^*(f) \leq \mathcal{MW}^*(g)$$

hold.

Non Stability Let $V = (-\frac{1}{2}, \frac{1}{2})$. Then $\mathcal{MW}^*(f)|_{\partial V} \equiv \frac{1}{2}$ and

$$\mathcal{MW}^*(\mathcal{MW}^*(f)|_{\partial V}) \equiv \frac{1}{2} \not\equiv \mathcal{MW}^*(f)|_{\overline{V}}.$$

Non Locality Choosing the same set V, we have

$$\mathrm{Lip}_{\mathcal{MW}^*(f)}(V) = 1 \neq 0 = \mathrm{Lip}_{\mathcal{MW}^*(f)}(\partial V).$$

So $\mathcal{MW}^*(f)$ isn't a solution to the same Lipschitz Extension problem restricted to V. However the extension defined in $\overline{U}$ by:

$$u(x) = \begin{cases} 0 & \text{if} \quad -1 \leq x \leq 1 \\ x-1 & \text{if} \quad 1 \leq x \leq 2 \end{cases}$$

not only solves the Lipschitz Extension problem presented at 2.1.5, but also verifies this locality property for all set $V \subset\subset U$ (This means V is a open bounded set and $\overline{U} \subset V$).

The following definition will state this property more clearly.

Definition 2.1.6. *A function $u \in C(U)$ is **absolutely minimizing Lipschitz** on u, denoted by $u \in \mathrm{AML}(U)$, if*

$$\mathrm{Lip}_u(V) = \mathrm{Lip}_u(\partial V), \quad \forall V \subset\subset U.$$

Given this definition, we can now try to recast problem 2.1.2 as the following:

Problem 2.1.7. *Given $f \in \mathrm{Lip}(\partial U)$, find $u \in C(\overline{U})$ such that*

$$u \in \mathrm{AML}(U) \quad \textit{and} \quad u = f \ \textit{on}\ \partial U.$$

It can be shown that a solution to this problem satisfies the Lipschitz Extension Problem [3].

2.2. Comparison with cones

In this section we introduce a geometric notion that will be very useful throughout the paper. First we should present some introductory notions.

Definition 2.2.1. *A **cone** is a function of the form*

$$C(x) = a + b|x - x_0|,$$

where $x_0 \in \mathbb{R}^n$ is the vertex of C, $a \in \mathbb{R}$ is its height and $b \in \mathbb{R}$ is its slope. The ray of C through x is the half-line defined by

$$\{x_0 + t(x - x_0),\, t \geq 0\}.$$

Lemma 2.2.2. *If a set V contains two distinct points on the same ray of a cone C with slope b, then*

$$\mathrm{Lip}_C(V) = |b|.$$

Proof. Suppose $C(x) = a + b|x - x_0|$. Then, for any $x, y \in \mathbb{R}^n$,

$$\begin{aligned} \frac{|C(x) - C(y)|}{|x - y|} &= \frac{\big|a + b|x - x_0| - a - b|y - x_0|\big|}{|x - y|} \\ &= |b| \frac{\big||x - x_0| - |y - x_0|\big|}{|x - y|} \\ &\leq |b|. \end{aligned} \tag{2.2.1}$$

So $\mathrm{Lip}_C(V) \leq |b|$. However if we take x, y distinct points in the same ray then the inequality (2.2.1) is in fact an equality, and so we get $\mathrm{Lip}_C(V) \geq |b|$. □

Corollary 2.2.3. *If $V \subset \mathbb{R}^n$ is non-empty and open and C is a cone with slope b, then*

$$\mathrm{Lip}_C(V) = |b|.$$

Moreover, if V is bounded and does not contain the vertex of C, then

$$\mathrm{Lip}_C(\partial V) = |b|.$$

We now present the main concept of this section.

Definition 2.2.4. *A function $u \in C(U)$ enjoys **comparison with cones from above** in U if, for every $V \subset\subset U$ and every cone whose vertex is not in V,*

$$u \leq C \text{ on } \partial V \quad \Rightarrow \quad u \leq C \text{ in } V.$$

*A function u enjoys **comparison with cones from below** if $-u$ enjoys comparison with cones from above.*

*A function u enjoys **comparison with cones** if it enjoys comparison with cones from above and below.*

We give now a equivalent condition for u to satisfy comparison with cones from above.

Lemma 2.2.5. *The function $u \in C(U)$ enjoys comparison with cones from above if, and only if, for every $V \subset\subset U$, $b \in \mathbb{R}$ and $z \notin V$,*

$$u(x) - b|x-z| \leq \max_{w\in\partial V}(u(w) - b|w-z|), \quad \forall x \in V. \tag{2.2.2}$$

Proof. We start with the necessity of the condition. Given $V \subset\subset U$, $b \in \mathbb{R}$ and $z \notin V$ we have

$$u(x) - b|x-z| \leq \max_{w\in\partial V}(u(w) - b|w-z|), \quad \forall x \in \partial V. \tag{2.2.3}$$

This can be rewritten as

$$u(x) \leq C(x) := \max_{w\in\partial V}(u(w) - b|w-z|) + b|x-z|, \quad \forall x \in \partial V. \tag{2.2.4}$$

Since u enjoys comparison with cones from above, (2.2.4) holds for any $x \in V$, and so does (2.2.3).

Now the sufficiency. Let $V \subset\subset U$ and

$$C(x) = a + b|x-z|,$$

with $a, b \in \mathbb{R}$ and $z \notin V$, be a cone such that $u \leq C$ on ∂V, or in another words

$$\max_{w\in\partial V}(u(w) - C(w)) \leq 0.$$

Then using 2.2.2 we conclude, for every $x \in V$

$$\begin{aligned} u(x) - C(x) &= u(x) - a - b|x-z| \\ &\leq \max_{w\in\partial V}(u(w) - a - b|w-z|) \\ &= \max_{w\in\partial V}(u(w) - C(w)) \\ &\leq 0. \end{aligned}$$

□

2.3. Comparison with cones and absolutely minimizing Lipschitz

We now prove the equivalence between absolutely minimizing Lipschitz functions and enjoying comparison with cones.

Theorem 2.3.1. *A function $u \in C(U)$ is absolutely minimizing Lipschitz in U if, and only if, it enjoys comparison with cones in U.*

Proof. We start by proving the sufficiency. Suppose u enjoys comparison with cones in U and let $V \subset\subset U$. We want to show that

$$\mathrm{Lip}_u(V) = \mathrm{Lip}_u(\partial V).$$

Since $u \in C(\overline{V})$, one can prove that $\mathrm{Lip}_u(V) = \mathrm{Lip}_u(\overline{V})$. Then, as $\partial V \subset \overline{V}$, we get that $\mathrm{Lip}_u(V) \geq \mathrm{Lip}_u(\partial V)$ and we only need to prove the other inequality. First, notice that, for any $x \in V$,

$$\mathrm{Lip}_u\left(\partial\big(V\backslash\{x\}\big)\right) = \mathrm{Lip}_u\left(\partial V \cup \{x\}\right) = \mathrm{Lip}_u\left(\partial V\right) \tag{2.3.1}$$

To see the second equation holds, we need only to make sure we have, for any $y \in \partial V$,

$$|u(y) - u(x)| \leq \mathrm{Lip}_u(\partial V)|x - y|.$$

or in another words

$$u(y) - \mathrm{Lip}_u(\partial V)|x - y| \leq u(x) \leq u(y) + \mathrm{Lip}_u(\partial V)|x - y|. \tag{2.3.2}$$

Although we want to prove this for $x \in V$, it's true that this holds for $x \in \partial V$. However, if we focus on the second inequality we can regard the right-hand side as the cone with vertex y, height $u(y)$ and slope $\mathrm{Lip}_u(\partial V)$. Since $y \in \partial V$, $y \notin V$, and because u enjoys comparison with cones from above in U, the second inequality in (2.3.2), which holds for $x \in \partial V$, must also hold for $x \in V$. The first inequality is proved analogously, recurring to comparison with cones from below.

Now let $x, y \in V$. We now use (2.3.1) twice to get

$$\begin{aligned}
\mathrm{Lip}_u\left(\partial V\right) &= \mathrm{Lip}_u\left(\partial\big(V\backslash\{x\}\big)\right) \\
&= \mathrm{Lip}_u\left(\partial\big(V\backslash\{x,y\}\big)\right) \\
&= \mathrm{Lip}_u\left(\partial V \cup \{x,y\}\right).
\end{aligned}$$

So we have

$$|u(x) - u(y)| \leq \mathrm{Lip}_u\left(\partial V \cup \{x, y\}\right)|x - y| \leq \mathrm{Lip}_u\left(\partial V\right)|x - y|,$$

and thus

$$\mathrm{Lip}_u(V) \leq \mathrm{Lip}_u(\partial V).$$

Now the necessity. It is sufficient to prove u satisfies comparison with cones from above, since both u and $-u$ belong to $\mathrm{AML}(U)$. Due to lemma 2.2.5, its sufficient to prove that for every $b \in \mathbb{R}$ and $z \notin V$,

$$u(x) - b|x - z| \leq \max_{w \in \partial V}(u(w) - b|w - z|), \quad \forall x \in V.$$

So we set

$$W = \left\{x \in V \ : \ u(x) - b|x - z| > \max_{w \in \partial V}\left(u(w) - b|w - z|\right)\right\}$$

and prove, by contradiction, that $W = \emptyset$. Consider the cone

$$C(x) := \max_{w \in \partial V}\left(u(w) - b|w - z|\right) + b|x - z|.$$

Then we can write $W = V \cap (u - C)^{-1}\big((0, \infty)\big)$, so it is a open set. Moreover if $W = A \cap B$, with A and B open sets in $\mathbb{R}^n$, then:

$$\begin{aligned}
\partial W &= \overline{W} - W \\
&= \overline{A \cap B} - (A \cap B) \\
&= \overline{A} \cap \overline{B} - (A \cap B) \\
&= (\overline{A} \cap \partial B) \cup (\overline{A} \cap B) - (A \cap B) \\
&= (\overline{A} \cap \partial B) \cup (\partial A \cap B).
\end{aligned}$$

We use this with $A = V$ and $B = (u - C)^{-1}\big((0, \infty)\big)$ and notice that $u \leq C$ on ∂V, so $\overline{A} \cap \partial B = \partial V \cap (u - C)^{-1}\big((0, \infty)\big) = \emptyset$. Now we observe that $\partial B = (u - C)^{-1}(0)$, because $u - C$ is continuous, hence $\partial W = \overline{V} \cap (u - C)^{-1}(0)$, thus

$$u = C \text{ on } \partial W.$$

Then we have, since $u \in \mathrm{AML}(U)$,

$$\mathrm{Lip}_u(W) = \mathrm{Lip}_u(\partial W) = \mathrm{Lip}_C(\partial W) = |b|,$$

due to Corollary 2.2.3, since $z \notin W \subset V$.

Take $x_0 \in W$. The ray of C through x_0

$$\{z + t(x_0 - z, t \geq 0)\}$$

intersects ∂W at least at 2 points x_1 and x_2, the first with $t < 1$ (because $z \notin W$), and the other one with $t > 1$ (because W is bounded and the set above isn't).

Suppose $b \geq 0$, then

$$\mathrm{Lip}_u(W) \geq \frac{|u(x_0) - u(x_1)|}{|x_0 - x_1|} \geq \frac{u(x_0) - u(x_1)}{|x_0 - x_1|} > \frac{C(x_0) - C(x_1)}{|x_0 - x_1|} = b = |b|,$$

because $u(x_0) > C(x_0)$, $u(x_1) = C(x_1)$ and x_0 and x_1 belong to the same ray. For $b < 0$ we argue analogously, using x_2 instead. We observe we obtained a contradiction, since $\mathrm{Lip}_u(W) = |b|$, thus conclude the proof. □

Chapter 3

The ∞-Laplace equation

3.1. The viscosity formulation

We now present the ∞-Laplacian.

Definition 3.1.1. *Let $D\varphi$ and $D^2\varphi$ be the gradient and hessian of φ. The partial differential operator given, on smooth functions φ, by*

$$\Delta_\infty \varphi := \sum_{i,j=1}^{n} \varphi_{x_i}\varphi_{x_i x_j}\varphi_{x_j} = D\varphi^\intercal D^2\varphi D\varphi$$

is called the ***∞-Laplacian****.*

This operator is not in divergence form so we can not (formally) integrate by parts to define a weak solution. We should instead consider the notion of viscosity solution.

Definition 3.1.2. *A function $w \in C(U)$ is a* ***viscosity subsolution of*** *$\Delta_\infty u = 0$ (or a* ***viscosity solution of*** *$\Delta_\infty u \geq 0$ or* ***∞-subharmonic****) in U if, for every $\hat{x} \in U$ and every $\varphi \in C^2(U)$ such that $w - \varphi$ has a local maximum at $\hat{x}$, we have*

$$\Delta_\infty \varphi(\hat{x}) \geq 0.$$

A function $w \in C(U)$ is ***∞-superharmonic*** *in U if $-w$ is ∞-subharmonic in U. A function $w \in C(U)$ is* ***∞-harmonic*** *in U if it is both ∞-subharmonic and ∞-superharmonic in U.*

Lemma 3.1.3. *If $u \in C^2(U)$ then u is ∞-harmonic in U if, and only if, $\Delta_\infty u = 0$ in the pointwise sense.*

Proof. Suppose u is ∞-harmonic, then it is ∞-subharmonic. Take $\varphi = u$ in the definition, then every point $x \in U$ is a local maximum of $\varphi - u \equiv 0$, so $\Delta_\infty u(x) \geq 0$ for every $x \in U$. Since also $-u$ is ∞-subharmonic we have also $\Delta_\infty u(x) \leq 0$, thus $\Delta_\infty u(x) = 0$ in the pointwise sense.

Reciprocally, suppose $\Delta_\infty u(x) = 0$ in the pointwise sense. We will prove that u is ∞-subharmonic, the other case is analogous. Take $\hat{x} \in U$ and $\varphi \in C^2(U)$ so that $w - \varphi$ has a local maximum at $\hat{x}$, we then want to prove $\Delta_\infty \varphi(\hat{x}) \geq 0$. Since $u - \varphi \in C^2(U)$ and $\hat{x} \in U$ is a local maximum, the gradient of $u - \varphi$ is null and its hessian is positive-semidefinite, this is

$$D(u - \varphi)(\hat{x}) = 0 \Leftrightarrow Du(\hat{x}) = D\varphi(\hat{x}). \tag{3.1.1}$$

and, for every $\eta \in \mathbb{R}^n$

$$\eta^\intercal D^2(u - \varphi)(\hat{x})\eta \geq 0 \Leftrightarrow \eta^\intercal D^2 u(\hat{x})\eta \geq \eta^\intercal D^2 \varphi(\hat{x})\eta. \tag{3.1.2}$$

We now use (3.1.1) and (3.1.2) to finally obtain:

$$\begin{aligned} \Delta_\infty \varphi(\hat{x}) &= D\varphi(\hat{x})^\intercal D^2\varphi(\hat{x}) D\varphi(\hat{x}) \\ &\geq D\varphi(\hat{x})^\intercal D^2 u(\hat{x}) D\varphi(\hat{x}) \\ &= Du(\hat{x})^\intercal D^2 u(\hat{x}) Du(\hat{x}) \\ &= \Delta_\infty u(\hat{x}) \\ &= 0. \end{aligned}$$

□

We now give an example on how to use this viscosity formulation to prove the function in $\mathbb{R}^2$

$$u(x,y) = x^{\frac{4}{3}} - y^{\frac{4}{3}}$$

is ∞-harmonic.

Take any point $(x_0, y_0) \in \mathbb{R}^2$ and $\varphi C^2(\mathbb{R}^2)$ such that $u - \varphi$ has a local maximum at (x_0, y_0). We first observe that, since $u \in C^1(\mathbb{R}^2)$,

$$D(u - \varphi)(x_0, y_0) = 0,$$

so we have

$$\varphi_x(x_0, y_0) = u_x(x_0, y_0) = \frac{4}{3} x_0^{\frac{1}{3}} \tag{3.1.3}$$

and

$$\varphi_y(x_0, y_0) = u_y(x_0, y_0) = -\frac{4}{3} y_0^{\frac{1}{3}}. \tag{3.1.4}$$

Suppose $x_0 = 0$. Since $u - \varphi$ has a local maximum at $(0, y_0)$, we have locally $(u - \varphi)(x, y_0) \leq (u - \varphi)(0, y_0)$, so

$$x^{\frac{4}{3}} \leq \varphi(x, y_0) - \varphi(0, y_0). \tag{3.1.5}$$

We will see this can't hold. Set $F(x) = \varphi(x, y_0) - \varphi(0, y_0)$, we then have $F(0) = 0$ and

$$F'(0) = \varphi_x(0, y_0) = 0.$$

Now by Taylor's theorem,

$$\lim_{x\to 0} \frac{F(x)}{x^2} = \lim_{x\to 0} \frac{F''(0)}{2} = \lim_{x\to 0} \frac{\varphi_{xx}(0, y_0)}{2} \leq +\infty.$$

However if (3.1.5) holds

$$\lim_{x\to 0} \frac{F(x)}{x^2} \geq \lim_{x\to 0} \frac{x^{\frac{4}{3}}}{x^2} = \lim_{x\to 0} x^{-\frac{2}{3}} = +\infty,$$

and we obtain a contradiction.

Consider now the case $x_0 \neq 0$ and $y_0 = 0$. If $\varphi \in C^2(\mathbb{R}^2)$ is such that $u - \varphi$ has a local maximum at $(x_0, 0)$, then $(u - \varphi)(x, 0) \leq (u - \varphi)(x_0, 0)$ locally, this is

$$x^{\frac{4}{3}} - \varphi(x, 0) \leq x_0^{\frac{4}{3}} - \varphi(x_0, 0).$$

Set $G(x) = x^{\frac{4}{3}} - \varphi(x, 0)$ then G has a local maximum at x_0. Since it is C^2 in a neighborhood of $x_0 \neq 0$, we have $G'(x_0) = 0$ and

$$G''(x_0) \leq 0 \Leftrightarrow \varphi_{xx}(x_0, 0) \geq \frac{4}{9} x_0^{-\frac{2}{3}} \geq 0. \tag{3.1.6}$$

Now we use (3.1.3), (3.1.4) and (3.1.6) to obtain:

$$\begin{aligned} \Delta_\infty \varphi(x_0, 0) &= \left(\varphi_x^2 \varphi_{xx} + 2\varphi_x \varphi_y \varphi_{xy} + \varphi_y^2 \varphi_{yy}\right)(x_0, 0) \\ &= \varphi_x^2(x_0, 0)\varphi_{xx}(x_0, 0) \\ &\geq 0. \end{aligned}$$

Finally, if both $x_0 \neq 0$ and $y_0 \neq 0$, u is C^2 in a neighbourhood of (x_0, y_0), and we can use Lemma 3.1.3 to reduce the problem to the pointwise calculation of the ∞-Laplacian, which is trivial.

3.2. Comparison with cones and ∞-harmonic functions

In this section we show that a function is ∞-subharmonic if, and only if, it enjoys comparison with cones from above. Thus we can conclude the ∞-harmonic functions are exactly the ones that enjoy comparison with cones or are absolute minimizing Lipschitz. The proof will be split into the next 2 theorems.

Theorem 3.2.1. *If $u \in C(U)$ is ∞-subharmonic then it enjoys comparison with cones from above.*

Proof. Thanks to Lemma 2.2.5, we only need to show that, for $V \subset\subset U$, $b \in \mathbb{R}$ and $z \notin V$, we have

$$u(x) - b|x-z| \leq \max_{w \in \partial V}(u(w) - b|w-z|), \quad \forall x \in V. \tag{3.2.1}$$

Note that if $G : \mathbb{R} \to \mathbb{R}$ is C^2, we can obtain, with some calculations

$$\Delta_\infty G(|x-z|) = G''(|x-z|)G'(|x-z|)^2, \quad x \in V$$

because $z \notin V$. Take now $G(t) = bt - \gamma t^2$, for some $\gamma > 0$, to obtain for all $x \in V$

$$\begin{aligned} \Delta_\infty(b|x-z| - \gamma|x-z|^2) &= \Delta_\infty G(|x-z|) \\ &= G''(|x-z|)G'(|x-z|)^2 \\ &= -\gamma(b - 2\gamma|x-z|)^2 \qquad (3.2.2) \\ &< 0 \end{aligned}$$

if γ is small enough. In fact, because V is bounded, there's always a constant γ_0 for which if we choose $\gamma_0 > \gamma > 0$ we always have $b - 2\gamma|x-z| \neq 0$, hence (3.2.2) is negative.

Now, since u is ∞-subharmonic in $V \subset\subset U$

$$u(x) - (b|x-z| - \gamma|x-z|^2)$$

can't have a local maximum in V. Hence, for all $x \in V$

$$\begin{aligned} u(x) - b|x-z| - \gamma|x-z|^2 &\leq \max_{w \in \partial V}(u(w) - b|w-z| - \gamma|w-z|^2) \\ &\leq \max_{w \in \partial V}(u(w) - b|w-z|) \end{aligned}$$

We now let $\gamma \to 0$ on the left-hand side and finally obtain (3.2.1). □

Theorem 3.2.2. *If $u \in C(U)$ enjoys comparison with cones from above then it is ∞-subharmonic.*

Proof. Take $B_r(y) \subset\subset U$. Then for $x \in \partial(B_r(y) \backslash \{y\}) = \partial B_r(y) \cup \{y\}$, we trivially have

$$u(x) \leq C(x) := u(y) + \max_{w \in \partial B_r(y)} \left(\frac{u(w) - u(y)}{r} \right) |x-y| \tag{3.2.3}$$

However since u enjoys comparison with cones, this inequality must also hold in $B_r(y) \backslash \{y\}$, so (3.2.3) holds for all $x \in B_r(y)$.

We now do some algebraic manipulations

$$\begin{aligned}
u(x) &\leq u(y) + \max_{w\in\partial B_r(y)} \left(\frac{u(w)-u(y)}{r}\right)|x-y| \\
\Leftrightarrow u(x) &\leq u(y) + \max_{w\in\partial B_r(y)} u(w)\frac{|x-y|}{r} - u(y)\frac{|x-y|}{r} \\
\Leftrightarrow u(x) &\leq \left(\frac{r-|x-y|}{r}\right)u(y) + \left(\max_{w\in\partial B_r(y)} u(w)\right)\frac{|x-y|}{r} \\
\Leftrightarrow \left(\frac{r}{r-|x-y|}\right)u(x) &\leq u(y) + \left(\max_{w\in\partial B_r(y)} u(w)\right)\frac{|x-y|}{r-|x-y|} \\
\Leftrightarrow \left(1+\frac{|x-y|}{r-|x-y|}\right)u(x) &\leq u(y) + \left(\max_{w\in\partial B_r(y)} u(w)\right)\frac{|x-y|}{r-|x-y|} \\
\Leftrightarrow u(x)-u(y) &\leq \max_{w\in\partial B_r(y)} \Big(u(w)-u(x)\Big)\frac{|x-y|}{r-|x-y|}. \qquad (3.2.4)
\end{aligned}$$

We first show the result at points of twice differentiability. Let x_0 be one of that points, for the sake of simplicity, consider $x_0 = 0$. Then there is a vector $p \in \mathbb{R}^n$ and a matrix $X \in \mathbb{R}^{n\times n}$ such that, for $z \in U$:

$$u(z) = u(0) + z^\intercal p + \frac{1}{2}z^\intercal Xz + o(|z|^2), \qquad (3.2.5)$$

where $p = Du(0)$ and $X = D^2u(x_0)$. We will show that

$$\Delta_\infty u(0) = p^\intercal Xp \geq 0.$$

The condition is trivially satisfied if $p = 0$, so from now on consider $p \neq 0$. Choose

$$r < \frac{1}{2}\operatorname{dist}(0, \partial U)$$

and λ small enough so that, for $y_0 = -\lambda p$, we have $B_r(y_0) \subset\subset U$ and

$$0 \in B_r(y_0) \Leftrightarrow |y_0| \leq r \Leftrightarrow \lambda \leq \frac{r}{|p|}.$$

Put $z = y_0$ in (3.2.5) to obtain

$$u(y_0) = u(0) - \lambda p^\intercal p + \frac{1}{2}\lambda^2 p^\intercal Xp + o(|\lambda p|^2),$$

or equivalently

$$u(0) - u(y_0) = \lambda|p|^2 - \frac{1}{2}\lambda^2 p^\intercal Xp + o(\lambda^2). \qquad (3.2.6)$$

Then let $w_{r,\lambda} \in \partial B_r(y_0)$ be such that

$$u(w_{r,\lambda}) = \max_{w\in\partial B_r(y_0)} u(w),$$

and put $z = w_{r,\lambda}$ in (3.2.5) to obtain

$$u(w_{r,\lambda}) - u(0) = w_{r,\lambda}^\intercal p + \frac{1}{2}w_{r,\lambda}^\intercal Xw_{r,\lambda} + o((r+\lambda)^2) \qquad (3.2.7)$$

since $|w_{r,\lambda}| = |w_{r,\lambda} - y_0 - \lambda p| \leq r + \lambda |p|$.

We now choose $x = 0$ and $y = y_0$ in (3.2.4) and use equations (3.2.6) and (3.2.7) to get, after dividing by λ,

$$|p|^2 - \frac{1}{2}\lambda p^\intercal X p + o(\lambda) \leq \left(w_{r,\lambda}^\intercal p + \frac{1}{2} w_{r,\lambda}^\intercal X w_{r,\lambda} + o((r+\lambda)^2) \right) \frac{|p|}{r - \lambda |p|}.$$

Then we send $\lambda \downarrow 0$ and divide by $|p|$ to get

$$|p| \leq \left(\frac{w_r}{r}\right)^\intercal p + \frac{1}{2} r \left(\frac{w_r}{r}\right)^\intercal X \left(\frac{w_r}{r}\right) + o(r) \tag{3.2.8}$$

where w_r is any limit point of $w_{r,\lambda}$, thus $w_r \in \partial B_r(x_0)$ and

$$\left|\frac{w_r}{r}\right| = 1.$$

Now using Cauchy-Schwarz inequality

$$|p| \leq |p| + \frac{1}{2} r \left(\frac{w_r}{r}\right)^\intercal X \left(\frac{w_r}{r}\right) + o(r). \tag{3.2.9}$$

We take $r \downarrow 0$ in (3.2.8) and use Cauchy-Schwarz inequality

$$|p| \leq \left(\lim_{r\downarrow 0} \frac{w_r}{r}\right)^\intercal p \leq |p|,$$

so these quantities must be the same. We now see:

$$\left|\lim_{r\downarrow 0} \frac{w_r}{r} - \frac{p}{|p|}\right|^2 = 0 \Rightarrow \lim_{r\downarrow 0} \frac{w_r}{r} = \frac{p}{|p|}.$$

Finally take $r \downarrow 0$ in (3.2.8), after dividing by r, to get

$$\frac{1}{2}\left(\frac{p}{|p|}\right)^\intercal X \left(\frac{p}{|p|}\right) \geq 0 \Leftrightarrow \Delta_\infty u(0) = p^\intercal X p \geq 0.$$

To conclude the proof, consider $\hat{x} \in U$ and $\varphi \in C^2(U)$ such that $u - \varphi$ has a local maximum at $\hat{x}$. Then for y and w close to $\hat{x}$

$$\varphi(\hat{x}) - \varphi(y) \leq u(\hat{x}) - u(y)$$

and

$$u(w) - u(\hat{x}) \leq \varphi(w) - \varphi(\hat{x}).$$

So we have, recalling (3.2.4),

$$\begin{aligned}
\varphi(\hat{x}) - \varphi(y) &\leq u(\hat{x}) - u(y) \\
&\leq \max_{w \in \partial B_r(y)} \Big(u(w) - u(\hat{x}) \Big) \frac{|\hat{x} - y|}{r - |\hat{x} - y|} \\
&\leq \max_{w \in \partial B_r(y)} \Big(\varphi(w) - \varphi(\hat{x}) \Big) \frac{|\hat{x} - y|}{r - |\hat{x} - y|}
\end{aligned}$$

and we obtained the same inequality for $\varphi \in C^2$, so we can use the reasoning used above to finally conclude

$$\Delta_\infty \varphi(\hat{x}) \geq 0.$$

□

3.3. Existence

In this section it is only presented the main result about the existence of solution to the ∞-harmonic functions.

Theorem 3.3.1. *Let $U \subset \mathbb{R}^n$ be open, and $f \in C(\partial U)$. Let $z \in \partial U$ and $A^\pm, B^\pm \in \mathbb{R}$, with $A^+ \geq A^-$, be such that*

$$A^-|x-z| + B^- \leq f(x) \leq A^+|x-z| + B^+ \quad \forall x \in \partial U.$$

Then there exists $u \in C(\overline{U})$ which is ∞-harmonic in U and satisfies $u = f$ on ∂U. Moreover,

$$A^-|x-z| + B^- \leq u(x) \leq A^+|x-z| + B^+ \quad \forall x \in \overline{U}.$$

The proof is not presented here, since the techniques used are not relevant for the rest of the paper, however a intrigued reader can consult the proof in [13] or [3].

Notice this theorem also guarantees the existence of solution to problem 2.1.7, for $f \in \mathrm{Lip}(\partial U)$, thanks to the equivalence between absolutely minimizing Lipschitz and infinity harmonic functions.

3.4. Uniqueness

The question of uniqueness was first solved by Jensen in [9], where he uses techniques from viscosity solutions. However in this paper is presented a far much simpler proof, given by Armstrong and Smart [1], where they explore the equivalence between comparison with cones and ∞-harmonic.

We start with some definitions.

Definition 3.4.1. *Given an open bounded subset $U \subset \mathbb{R}^n$ and $r > 0$, define*

$$U_r := \{x \in U : \overline{B_r}(x) \subset U\}.$$

For $u \in C(U)$ and $x \in U_r$, define

$$u^r(x) := \max_{\overline{B_r}(x)} u \quad \textit{and} \quad u_r(x) := \min_{\overline{B_r}(x)} u$$

and

$$L_r^+ u(x) = \frac{u^r(x) - u(x)}{r} \quad \textit{and} \quad L_r^- u(x) = \frac{u(x) - u_r(x)}{r}.$$

Note that both $L_r^+ u \geq 0$ and $L_r^- u \geq 0$.

Lemma 3.4.2. *Suppose $u, v \in C(U)$ are bounded functions and satisfy*

$$L_r^- u(x) - L_r^+ u(x) \leq 0 \leq L_r^- v(x) - L_r^+ v(x),\ \forall x \in U_r. \tag{3.4.1}$$

Then

$$\sup_U (u - v) = \sup_{U \setminus U_r} (u - v)$$

Proof. We prove this by contradiction. Suppose the theorem is false, then there is a constant a such that

$$\sup_U (u - v) > a > \sup_{u \setminus U_r} (u - v).$$

We then have the set

$$A := \{x \in U : \ (u - v)(x) \geq a\},$$

which is a nonempty, closed and bounded set contained in U_r, because $u - v \in C(u)$, and so the set

$$E := \left\{ x \in U : \ (u - v)(x) = \max_A (u - v) = \sup_U (u - v) \right\}$$

is also nonempty and closed. We can now define

$$F := \left\{ x \in E : \ (u)(x) = \max_E (u) \right\}$$

and note it is also nonempty and closed. Choose a point $x_0 \in \partial F \subset F$. Since $u - v$ attains its maximum at x_0, we have

$$\begin{aligned} (u - v)(x_0) &\geq \max_{w \in \overline{B_r}(x_0)} (u(w) - v(w)) \\ &\geq \max_{w \in \overline{B_r}(x_0)} \left(\min_{\overline{B_r}(x_0)} u - v(w) \right) \\ &= \min_{\overline{B_r}(x_0)} u - \min_{\overline{B_r}(x_0)} v \\ &= u_r(x_0) - v_r(x_0). \end{aligned}$$

Thus

$$L_r^- v(x_0) \leq L_r^- u(x_0). \tag{3.4.2}$$

We have $L_r^+ u(x_0) \geq 0$, so we consider two cases:

1. $L_r^+ u(x_0) = 0$. Then $\max_{\overline{B_r}(x_0)} u = u(x_0)$. From (3.4.1) first inequality, we obtain

$$L_r^- u(x_0) \leq 0 \Rightarrow L_r^- u(x_0) = 0 \Rightarrow \min_{\overline{B_r}(x_0)} u = u(x_0).$$

and now from (3.4.2) we get

$$L_r^- v(x_0) \leq 0 \Rightarrow L_r^- v(x_0) = 0 \Rightarrow \min_{\overline{B_r}(x_0)} v = v(x_0).$$

Then from (3.4.1) second inequality, we obtain

$$L_r^+ v(x_0) \leq 0 \Rightarrow L_r^+ v(x_0) = 0 \Rightarrow \max_{\overline{B_r}(x_0)} v = v(x_0).$$

So we have u and v are constant in $\overline{B_r}(x_0)$, and so is $u - v$. This way we conclude $\overline{B_r}(x_0) \subset E$, and then because u is constant in this set, we have also $\overline{B_r}(x_0) \subset F$, but this is a contradiction since $x_0 \in \partial F$.

2. $L_r^+ u(x_0) > 0$. In this case, since $B_r(x_0)$ is closed and bounded, we can choose z in that set such that $u(z) = u_r(x_0)$, and then:

$$rL_r^+ u(x_0) = u(z) - u(x_0). \tag{3.4.3}$$

If $z \in E$, we would have $u(z) \leq u(x_0)$, because $x_0 \in F$, however this is false according to the last equation, so we must have $z \notin E$. Then we have $(u - v)(z) < (u - v)(x_0)$, because $x_0 \in E$, thus

$$rL_r^+ v(x_0) \geq v(z) - v(x_0) > u(z) - u(x_0) = rL_r^+ u(x_0).$$

We now use this and (3.4.2) to get

$$L_r^- u(x_0) - L_r^+ u(x_0) > L_r^- v(x_0) - L_r^+ v(x_0),$$

which is false by (3.4.1).

□

Lemma 3.4.3. *If $u \in C(U)$ is ∞-subharmonic in U, then*

$$L_r^- u^r(x) - L_r^+ u^r(x) \leq 0, \quad \forall x \in U_{2r}$$

and if $v \in C(U)$ is ∞-superharmonic in U, then

$$L_r^- v_r(x) - L_r^+ v_r(x) \geq 0, \quad \forall x \in U_{2r}.$$

Proof. We only need to prove the first statement; the second one is consequence of the first and the fact that $(-v)^r = -v_r$.

Fix a point x_0 in U_{2r}. Now as in (3.4.3) we can choose $y_0 \in \overline{B_r}(x_0)$ and $z_0 \in \overline{B_{2r}}(x_0)$ such that

$$u(y_0) = u^r(x_0) \quad \text{and} \quad u(y_0) = u^{2r}(x_0).$$

Then

$$\begin{aligned} r\left(L_r^- u^r(x) - L_r^+ u^r(x)\right) &= 2u^r(x_0) - (u^r)_r(x_0) - (u^r)^r(x_0) \\ &\leq 2u(y_0) - u(x_0) - u(z_0), \end{aligned} \tag{3.4.4}$$

since we have

1. $(u^r)^r(x_0) = u(z_0) = u^{2r}(x_0)$. Notice that

   $$(u^r)^r(x_0) = \max_{z \in \overline{B_r}(x_0)} u^r(z) = \max_{z \in \overline{B_r}(x_0)} \max_{y \in \overline{B_r}(z)} u(y).$$

   Then we can choose $x_1 \in \overline{B_r}(x_0)$ and $x_2 \in \overline{B_r}(x_1)$ such that $(u^r)^r(x_0) = u^r(x_1) = u(x_2)$. Now since

   $$|x_2 - x_0| \leq |x_2 - x_1| + |x_1 - x_0| \leq r + r = 2r,$$

   we have $x_2 \in \overline{B_{2r}}(x_0)$, thus $u(z_0) \geq u(x_2) = (u^r)^r(x_0)$.

   Reciprocally, let $z_1 = \frac{z_0 + x_0}{2}$. Since $z_0 \in \overline{B_{2r}}(x_0)$, we have $z_0 \in \overline{B_r(z_1)}$ and $z_1 \in \overline{B_r(x_0)}$, thus

   $$u(z_0) \leq u^r(z_1) \leq (u^r)^r(x_0).$$

2. $(u^r)_r(x_0) \geq u(x_0)$. We have

   $$(u^r)_r(x_0) = \min_{z \in \overline{B_r}(x_0)} u^r(z) = \min_{z \in \overline{B_r}(x_0)} \max_{y \in \overline{B_r}(z)} u(y).$$

   Since $z \in \overline{B_r}(x_0)$, we have $x_0 \in \overline{B_r}(z)$ and

   $$\max_{y \in \overline{B_r}(z)} u(y) \geq u(x_0)$$

   for all $z \in \overline{B_r}(x_0)$, the result follows.

We now recall inequality (3.2.3)

$$u(w) \leq u(x_0) + \frac{u(z_0) - u(x_0)}{2r}|w - x_0| \quad \forall w \in \overline{B_{2r}}(x_0),$$

and take $w = y_0$ to get

$$\begin{aligned} u(y_0) &\leq u(x_0) + \frac{u(z_0) - u(x_0)}{2r}|y_0 - x_0| \\ &\leq u(x_0) + \frac{u(z_0) - u(x_0)}{2} \\ &= \frac{u(z_0) + u(x_0)}{2}, \end{aligned}$$

since $|y_0 - x_0| \leq r$. Thus quantity (3.4.4) is non-positive and the proof is complete. $\square$

Finally we have the theorem

Theorem 3.4.4 (Jensen's Uniqueness Theorem). *Let $u, v \in C(\overline{U})$ be respectively, ∞-subharmonic and ∞-superharmonic. Then*

$$\max_{\overline{U}}(u - v) = \max_{\partial U}(u - v).$$

Proof. Suppose this assertion is false, then we can choose $a \in \mathbb{R}^n$ so that

$$\max_{\overline{U}}(u - v) > a > \max_{\partial U}(u - v).$$

Since $u - v$ is continuous the set

$$A := \{x \in \overline{U} : \, u - v(x) \geq a\}$$

is closed and $R = \min\{|x - y|, \, x \in A, y \in \partial U\}$ is well-defined and positive. We now choose $R/2 > r > 0$ and have, from lemmas 3.4.2 and 3.4.3 that

$$\sup_{U_r}(u^r - v_r) = \sup_{U_r \setminus U_{2r}}(u^r - v_r). \tag{3.4.5}$$

Since $u_r \geq u$ and $v_r \leq v$, we have

$$\sup_{U_r}(u^r - v_r) \geq \sup_{U_r}(u - v) = \max_{\overline{U}}(u - v) > a,$$

since this maximum is achieved in $A \subset U_r$.

On the other hand, we have u^r and v_r converge local uniformly to u and v, so as $r \downarrow 0$, the right-hand side of (3.4.5) becomes $\sup(u - v)$ in some set B that does not contain any point of A. Then

$$u - v \leq a \Rightarrow \sup_{B}(u - v) \leq a,$$

and we reach a contradiction, since the left-hand side is always greater than a. $\square$

Chapter 4

Regularity of ∞-harmonic functions

We will finally present some results about the regularity of ∞-harmonic functions. For an open set $x \in U$ and $x \in U$, denote

$$d(x) := \operatorname{dist}(x, \partial U).$$

4.1. Harnack Inequalities

We start to present a Harnack inequality

Lemma 4.1.1 (Harnack Inequality)**.** *Let $0 \geq u \in C(U)$ satisfy comparison with cones from above. If $z \in U$, $4R < d(z)$ and $x, y \in B_R(z)$, then*

$$\sup_{B_R(z)} u \leq \frac{1}{3} \inf_{B_R(z)} u.$$

Proof. Since u satisfies comparison with cones from above, we recall inequality (3.2.3), to get for $\hat{x} \in B_r(y)$

$$u(\hat{x}) \leq u(y) + \max_{w \in \partial B_r(y)} \left(\frac{u(w) - u(y)}{r} \right) |\hat{x} - y| \leq u(y) \left(1 - \frac{|\hat{x} - y|}{r} \right)$$

since $u(w) \leq 0$. We now take $r \uparrow d(y)$, and note $d(y) \geq 3R$ and $|x - y| \leq 2R$, so for $d(y) > r > 2R$, we have $x \in B_r(y)$ and

$$\begin{aligned} u(x) &\leq u(y) \left(1 - \frac{|x - y|}{d(y)} \right) \qquad (4.1.1) \\ &\leq u(y) \left(1 - \frac{2R}{3R} \right) \\ &= \frac{1}{3} u(y). \end{aligned}$$

Now we take the supremum at the left-hand side and the infimum at the right-hand side to obtain the result. $\square$

We now sharpen this estimate, with a direct proof of the result in [10], while the proof in that paper follows from looking at the ∞-Laplace equation as the limit as $p \to \infty$ of the p-Laplace equation.

Theorem 4.1.2 (The Harnack Inequality of Lindqvist-Manfredi)**.** *Let $0 \geq u \in C(U)$ satisfy comparison with cones from above. If $z \in U$ and $0 < R < d(z)$, then*

$$u(x) \leq \exp\left(-\frac{|x-y|}{d(z)-R}\right) u(y), \quad \forall x, y \in B_R(z). \tag{4.1.2}$$

Proof. Let $x, y \in B_R(z)$, $m \in \mathbb{N}$ and define

$$x_k = x + k\frac{y-x}{m}, \quad k = 0, 1, \ldots, m.$$

Since $x_k \in B_R(z)$ we have $d(x_k) \geq d(z) - R$, and for $k > 0$

$$|x_k - x_{k-1}| = \frac{|x-y|}{m} < d(x_k),$$

if we choose m sufficiently large. We now apply (4.1.1) to get

$$\begin{aligned} u(x_{k-1}) &\leq u(x_k)\left(1 - \frac{|x_k - x_{k-1}|}{d(x_k)}\right) \\ &\leq u(x_k)\left(1 - \frac{|x-y|}{m(d(z)-R)}\right). \end{aligned}$$

We now use this inequality with k varying between 1 and m to obtain

$$u(x) \leq u(y)\left(1 - \frac{|x-y|}{m(d(z)-R)}\right)^m$$

and take the limit $m \to \infty$ to obtain the result □

4.2. Locally Lipschitz Regularity

We now make use of Harnack inequality to show that ∞-harmonic function are locally Lipschitz and hence differentiable almost everywhere.

Theorem 4.2.1. *If $u \in C(U)$ is ∞-harmonic then is locally Lipschitz.*

Proof. Assume first that $u \leq 0$. Since u enjoys comparison with cones from above it satisfies (4.1.1) for $x, y \in B_R(z)$, as long as $z \in U$ and $4R < d(z)$

$$\begin{aligned} u(x) - u(y) &\leq -u(y)\frac{|x-y|}{d(y)} \\ &\leq -\inf_{B_R(z)} u\frac{|x-y|}{3R} \\ &\leq -\sup_{B_R(z)} u\frac{|x-y|}{R}, \end{aligned}$$

the last inequality being a consequence of the Harnack Inequality.

Now for u not non-positive, we see this inequality holds for

$$v = u - \sup_{B_{4R}(z)} u,$$

which is non-positive in $B_{4R}(z)$, and so

$$\begin{aligned} u(x)-u(y) &\leq v(x)-v(y) \\ &\leq -\sup_{B_R(z)} v\frac{|x-y|}{R} \\ &= \left(\sup_{B_{4R}(z)} u - \sup_{B_R(z)} u\right)\frac{|x-y|}{R}. \end{aligned}$$

Now interchanging x and y, we finally obtain the bound to the local Lipschitz constant

$$|u(x)-u(y)| \leq \frac{1}{R}\left(\sup_{B_{4R}(z)} u - \sup_{B_R(z)} u\right)|x-y|.$$

□

4.3. Everywhere differentiability of ∞-harmonic functions

We just proved that ∞-harmonic functions are differentiable almost everywhere, however a recent result, from Evans-Smart [8], asserts they are in fact differentiable at every point.

4.3.1. An intermediate result

In this subsection it's presented a theorem, from [5], that will be crucial to prove the differentiability at every point.

We now recall the definitions (3.4.1):

Remark 4.3.1. *If u enjoys comparison with cones from above in U, then for $z \in U$, $r < d(z)$*

$$u^r(x) = \max_{\overline{B_r}(x)} u = \max_{\partial B_r(x)} u.$$

Proof. Clearly for $y \in \partial B_r(x)$ we have

$$u(y) \leq \max_{\partial B_r(x)} u,$$

and since u enjoys comparison with cones, this holds also in $B_r(x)$ and the result follows. □

Lemma 4.3.2. *Let u enjoy comparison with cones in U. Then $L_r^+u(x)$ and $L_r^-u(x)$ is a non-decreasing and nonnegative function of r. Moreover, we have*

$$\lim_{r\to 0} L_r^+ u \equiv \lim_{r\to 0} L_r^- u. \tag{4.3.1}$$

Proof. We prove the result only for L_r^+, the other case being analogous.

We already noted $L_r^+ u$ is always nonnegative, so we only have to show the monotonicity. Recall inequality (3.2.3) and the last remark to obtain for $z \in U$, $r < d(z)$ and $x \in B_r(u)$

$$\begin{aligned} u(x) &\leq u(z) + \left(\frac{u^r(z) - u(z)}{r}\right)|x - z| \qquad (4.3.2) \\ \Leftrightarrow \frac{u(x) - u(z)}{|x - z|} &\leq \frac{u^r(z) - u(z)}{r}. \end{aligned}$$

We now choose $r_1 < r_2$, then for $x \in \partial B_{r_1}(z) \subset U_{r_2}$

$$\frac{u(x) - u(z)}{r_1} \leq \frac{u^{r_2}(z) - u(z)}{r_2} = L_{r_2}^+ u(z).$$

Now we take the maximum at left-hand side and obtain $L_{r_1}^+ u(z)$ and thus proving the monotonicity result.

We now prove the (4.3.1). Choose $x \in U$, without loss of generality consider $x = 0$. Let the left-hand side limit be L^+ and the right-hand side L^-. Let's see we can't have $L^+ < L^-$, the other case is analogous considering $-u$.

Since $L^+ < L^-$ there must be some $r_0 > 0$ such that for $M = L_{r_0}^+ u(0) < L^-$, then by (4.3.2) for $x \in B_{r_0}(0)$

$$u(x) \leq u(0) + M|x|. \qquad (4.3.3)$$

Now use (4.3.2) centered at x to obtain, for $r < |x|$

$$u(0) \leq u(x) + \max_{z \in \partial B_r(x)} \left(\frac{u(z) - u(x)}{r}\right)|x|.$$

We now use (4.3.3) with $x = z$ and $|z| \geq |x| + r < r_0$ to bound the maximum term in the last inequality, obtaining

$$u(0) \leq u(x) + \frac{u(0) + M(|x| + r) - u(x)}{r}|x|.$$

Then it follows

$$-M\frac{r + |x|}{r - |x|}|x| \leq u(x) - u(0),$$

thus

$$-M\frac{r + \varepsilon}{r - \varepsilon} \leq \min_{x \in \partial B_\varepsilon(0)} \left(\frac{u(x) - u(0)}{\varepsilon}\right) = -L_\varepsilon^- u(0).$$

We now take the limit $\varepsilon \downarrow 0$ to obtain $L^- \leq M < L^-$, hence a contradiction.

□

We can now define the next notion

Definition 4.3.3. *Let $u \in C(\overline{U})$ enjoy comparison with cones in U. Then for $x \in U$*

$$Lu(x) := \lim_{r \to 0} L_r^+ u(x) = \lim_{r \to 0} L_r^- u(x).$$

We can see this notion as a Lipschitz constant defined at every point, in the sense that this constant is the infimum of the values of M which 4.3.3 holds for any $r_0 > 0$. This becomes clear in the next lemma.

Lemma 4.3.4. *Let $u \in C(\overline{U})$ enjoy comparison with cones in U. Then*

$$\sup_{y \in U} Lu(y) = \|Du\|_{L^\infty(U)}.$$

The proof of this lemma is present in [6]. We are now able to present the theorem.

Theorem 4.3.5. *If u enjoys comparison with cones in U, then for $x_0 \in U$ and a sequence of real positive numbers $r_j \downarrow 0$ for which the limit*

$$v(x) = \lim_{j \to \infty} \frac{u(r_j x + x_0) - u(0)}{r_j} \tag{4.3.4}$$

holds locally uniformly in $\mathbb{R}^n$, then exists $z \in \partial B_1(0)$ such that

$$v(x) = Lu(x_0) \langle x, z \rangle .$$

Before starting the proof of this theorem, we will do some considerations. Since u is locally Lipschitz continuous, any sequence $R_k \downarrow 0$ has a subsequence $r_j \downarrow 0$ for which the limit (4.3.4) is defined on $\mathbb{R}^n$ and the convergence is uniform on any bounded subset of $\mathbb{R}^n$.

Since this happens, one might actually believe this result would suffice to prove the differentiability at every point. However this is not true, as the function v can have different behaviour, depending on the sequence chosen. One example of this phenomena in $\mathbb{R}$, provided by D. Priess, is the function: $u(x) = x \sin(\log(|\log(|x|)|))$. This function, in a neighborhood of 0, is differentiable except at $x = 0$ and Lipschitz continuous near $x = 0$. All the limits of the quotients $u(r_j x)/r_j$ as $r_j \downarrow 0$ are linear, but this numbers can be chosen so that the slope of this limit can take any value between -1 and 1.

To simplify the calculations, from now on consider in theorem 4.3.5 $x_0 = 0$. The next lemma will be useful in the proof of the theorem.

Lemma 4.3.6. *Let v be defined as in 4.3.5. Then for $y \in \mathbb{R}^n$, $r > 0$*

$$L_r^+ v(y),\ L_r^- v(y) \leq Lu(0) \tag{4.3.5}$$

and

$$L_r^+ v(0) = L_r^- v(0) = Lu(0) \tag{4.3.6}$$

Proof. Since v is the locally uniform limit of functions enjoying comparison with cones, the function itself enjoys comparison with cones, so we can use some of the results we already used. We only consider the case of the superscript $^+$, the other is obtained considering $-u$. Fix y, and take $z \in \partial B_r(y)$ such that

$$L_r^+ v(y) = \frac{v(z) - v(y)}{r} = \lim_{j\to\infty} \frac{u(r_j z) - u(r_j y)}{r_j r}.$$

We have $r_j z \in \partial B_{r_j r}(r_j y)$, then for $R < d(0)$

$$\frac{u(r_j z) - u(r_j y)}{r_j r} \le L_{r_j r}^+ u(r_j y) \le L_R^+ u(r_j y),$$

if we choose j big enough for $r_j r < R < d(r_j y)$ to hold. We now take the limit at both sides of the inequality

$$L_r^+ v(y) = \lim_{j\to\infty} \frac{u(r_j z) - u(r_j y)}{r_j r} \le \lim_{j\to\infty} L_R^+ u(r_j y) = L_R^+ u(0),$$

since $L_R^+ u$ is continuous. We now take $R \downarrow 0$ to obtain the (4.3.5). We now turn to (4.3.6). For each j choose $z_j \in B_r(0)$ such that

$$u(r_j z_j) = \max_{\partial B_{r_j r}(0)} u,$$

then

$$L_r^+ v(0) \ge \sup_j \frac{v(z_j)}{r} \ge \lim_{j\to\infty} \frac{u(r_j z_j) - u(0)}{r_j r} = \lim_{j\to\infty} L_{r_j r}^+ u(0) = Lu(0).$$

□

We are now ready to prove theorem 4.3.5.

Proof. Take $L_0 = Lu(0)$. The first result of lemma above implies that

$$\mathrm{Lip}_v(\mathbb{R}^n) = L_0, \tag{4.3.7}$$

while the second guarantees there exists $z_r^\pm \in \partial B_r(0)$ such that

$$rL_0 = rL_r^+ v(0) = v(z_r^+) - v(0) = v(z_r^+)$$

and

$$-rL_0 = -rL_r^- v(0) = v(z_r^-) - v(0) = v(z_r^-).$$

Now by (4.3.7)

$$2rL_0 = |v(z_r^+) - v(z_r^-)| \leq L_0|z_r^+ - z_r^-|.$$

Thus $|z_r^+ - z_r^-| \geq 2r$, and since $z_r^\pm \in \partial B_r(0)$, it is required that

$$z_r^- = -z_r^+.$$

Let $z \in B_1(0)$ be such that $rz = z_r^+ = -z_r^-$. Now using (4.3.7) again, we have, for $-r \leq t \leq r$

$$L_0 t = v(rz) - L_0(r-t)|z| \leq v(tz) \leq v(-rz) + L_0(t+r)|z| = L_0 t,$$

hence $v(tz) = L_0 t$. We can now see that z doesn't depend on r. Take $t < r$ then

$$v(tz) = L_0 t = -v(z_t^-).$$

Repeating the reasoning above for tz instead z_t^+, we get

$$tz = -z_t^- = z_t^+,$$

as we wanted. Now taking $r \to \infty$, we get

$$v(tz) = L_0 t \quad \forall t \in \mathbb{R}.$$

We will now prove

$$v(x) = L_0 \langle x, z \rangle.$$

In fact by (4.3.7) we have, for $t > 0$

$$\begin{aligned} [v(x) - v(tz)]^2 &\leq L_0|x + tz|^2 \\ \Leftrightarrow v(x)^2 - 2L_0 t v(x) + L_0^2 t^2 &\leq L_0^2 \left(|x|^2 + 2t\langle x, z \rangle + t^2\right) \end{aligned}$$

and

$$v(x) \geq \frac{v(x)^2 - L_0^2|x|^2}{2L_0 t} + L_0 \langle x, z \rangle.$$

We now take $t \to \infty$ to obtain $v(x) \geq L_0 \langle x, z \rangle$. The other inequality is obtained reasoning analogously for negative t and taking the limit to $-\infty$.

$\square$

4.3.2. The main result

The main contribution of Evans and Smart is proving that the point $z \in \partial B_1(0)$ referred in theorem 4.3.5, is unique for each sequence $r_j \downarrow 0$, thus the full limit

$$\lim_{r \to 0} \frac{u(ry + x_0) - u(x_0)}{r} = Lu(x_0) \langle x, z \rangle$$

exists locally uniformly for each x_0, and $Du(x) = Lu(x_0)z$, $|Du(x)| = Lu(x_0)$.

For obtaining the result it was important the following theorems, regarding smooth functions that approximate u.

Theorem 4.3.7. *There exists a unique smooth solution u^ε to the PDE*

$$\begin{cases} -\Delta_\infty u^\varepsilon - \varepsilon \Delta u^\varepsilon = 0 & in \quad B_3(0) \\ u^\varepsilon = u & on \quad \partial B_3(0) \end{cases}$$

and constants C_1 and C_2, not depending on ε, such that

$$\max_{B_2(0)} |u^\varepsilon| \leq C_1 \quad and \quad \max_{B_2(0)} |Du^\varepsilon| \leq C_2.$$

Furthermore

$$u^\varepsilon \to u \quad uniformly\ on\ U.$$

Theorem 4.3.8. *Consider u^ε as above, and suppose there is some small constant λ such that*

$$\max_{x \in B_2(0)} |u^\varepsilon(x) - \langle x, e_n \rangle| = \lambda.$$

Then the bound

$$|Du^\varepsilon|^2 \leq u^\varepsilon_{x_n} + C_3 \left(\lambda^{\frac{1}{2}} + \frac{\varepsilon^{\frac{1}{2}}}{\lambda^{\frac{1}{2}}} \right) \tag{4.3.8}$$

holds for everypoint in $B_1(0)$, for a constant C_3 that does not depend upon ε.

Although these estimates are very important to establish the main result, the proof, which is in [8], will not be present in this paper, since they are very complex and not relevant to the understanding of the main result.

Another lemma that will be used:

Lemma 4.3.9. *Let $b \in \mathbb{R}^n$, $|b| = 1$. Let v be a smooth function satisfying*

$$\max_{x \in B_1(0)} |v(x) - \langle b, x \rangle| \leq \eta$$

for some constant η. Then there exists a point $x_0 \in B_1(0)$ at which

$$|Dv(x^0) - b| \leq 4\eta. \tag{4.3.9}$$

Proof. Define
$$w := \langle b, x\rangle - 2\eta|x|^2.$$
We have $(v-w)(0) = v(0) \leq \eta$ however for $x \in \partial B_1(0)$
$$(v-w)(x) = v - \langle b, x\rangle + 2\eta \geq \eta.$$
Consequently $v - w$ attains its minimum over $B_1(0)$ at some point x^0, at which
$$Dv(x^0) = Dw(x^0) = b - 4\eta x^0$$
and the result follows □

We now state the main result

Theorem 4.3.10. *If u is ∞-harmonic in U, then u is differentiable at each point in U.*

Proof. Select any point in U, without loss of generality we may assume is 0. Suppose the limit in theorem 4.3.5 doesn't produce a unique tangent plane at 0, this is, there are two sequences $r_j, s_j \downarrow 0$ and vectors $z_1 \neq z_2 \in \partial B_1(0)$ for which
$$\max_{x\in B_{r_j}(0)} \frac{1}{r_j}|u(x) - u(0) - Lu(x_0)\langle z_1, x\rangle| \to 0 \tag{4.3.10}$$
and
$$\max_{x\in B_{s_j}(0)} \frac{1}{s_j}|u(x) - u(0) - Lu(x_0)\langle z_2, x\rangle| \to 0. \tag{4.3.11}$$
Again without loss of generality consider $u(0) = 0$, $Lu(x_0) = 1$ and $z_1 = e_n$. Define
$$\theta := 1 - \langle z_2, e_n\rangle. \tag{4.3.12}$$
Take C_3 the constant from (4.3.8) and choose $\lambda > 0$ so that
$$2C\lambda^{\frac{1}{2}} = \frac{\theta}{4}. \tag{4.3.13}$$
Put
$$\varepsilon_1 = \lambda^2 \tag{4.3.14}$$
Now use (4.3.10) (remember $z_1 = e_n$ and $Lu(x_0) = 1$) to choose a radius $r > 0$ such that
$$\max_{x\in B_r(0)} \frac{1}{r}|u(x) - \langle x, e_n\rangle| \leq \frac{\lambda}{4}.$$
Then we can rescale the function u so that we can take $r = 2$, without changing $Lu(0)$, and get
$$\max_{x\in B_2(0)} |u(x) - \langle x, e_n\rangle| \leq \frac{\lambda}{2}.$$

Now fix $\varepsilon_2 > 0$ so small that

$$\max_{x \in B_2(0)} |u^\varepsilon(x) - \langle x, e_n \rangle| \leq \lambda \tag{4.3.15}$$

for all $0 < \varepsilon \leq \varepsilon_2$. We now introduce one more constant $\eta > 0$, picked so that

$$12\eta = \frac{\theta}{4} \tag{4.3.16}$$

Regarding (4.3.11), we can find a radius $0 < s < 1$ for which

$$\max_{x \in B_s(0)} \frac{1}{s} |u(x) - \langle b, x \rangle| \leq \frac{\eta}{2}.$$

We select $\varepsilon_3 > 0$ so that

$$\max_{x \in B_s(0)} \frac{1}{s} |u^\varepsilon(x) - \langle b, x \rangle| \leq \eta \tag{4.3.17}$$

for all $0 < \varepsilon \leq \varepsilon_3$. Now take

$$\varepsilon := \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}.$$

We can rescale (4.3.17) to the unit ball and apply Lemma 4.3.9 to obtain some point in $B_1(0)$ such that (4.3.9) holds. Now we rescale it back to the original function to obtain $x^0 \in B_s(0) \subset B_1(0)$ at which

$$|Du^\varepsilon(x^0) - b| \leq 4\eta.$$

Then

$$|u^\varepsilon_{x_n} - b_n| \leq 4\eta \Rightarrow u^\varepsilon_{x_n} \leq 4\eta + b_n \tag{4.3.18}$$

and

$$|Du^\varepsilon(x^0)| \geq |b| - |Du^\varepsilon(x^0) - b| \geq 1 - 4\eta. \tag{4.3.19}$$

We now use (4.3.8), and the choices (4.3.13) of λ and (4.3.14) of $\varepsilon_1 \geq \varepsilon$, to deduce

$$|Du^\varepsilon(x^0)|^2 \leq u^\varepsilon_{x_n}(x_0) + C_3(\lambda^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}}/\lambda^{\frac{1}{2}}) \leq u^\varepsilon_{x_n}(x_0) + \frac{\theta}{4}.$$

But (4.3.18) and (4.3.19) imply that

$$(1 - 4\eta)^2 \leq b_n + 4\eta + \frac{\theta}{4},$$

hence

$$\theta = 1 - b_n \leq 12\eta - 16\eta^2 + \frac{\theta}{4} \leq 12\eta + \frac{\theta}{4} = \frac{\theta}{2},$$

by (4.3.16). This is a contradiction, since $\theta > 0$. $\square$

Bibliography

[1] S. Armstrong and C. Smart. *An easy proof of Jensen's theorem on the uniqueness of infinity harmonic functions.* Calc. Var. Partial Diff. Equations 37, 2010, 381–384.

[2] G. Aronsson. *Extension of functions satisfying Lipschitz conditions.* Ark. Mat. 6, 1967, 551–561.

[3] G. Aronsson, M. Crandall, and P. Juutinen. *A tour of the theory absolutely minimizing functions.* Bull. Amer. Math. Soc. (N.S) 41, 2004, 439-505.

[4] M. Crandall. *A visit with the ∞-Laplace equation.* Lecture Notes in Math., 1927, Springer, Berlin, 2013, 75-122.

[5] M. Crandall and L. C. Evans. *A remark on infinity harmonic functions.* Proceedings of the USA-Chile Workshop on Nonlinear Analysis, 2000, 123-129.

[6] M. Crandall, L. C. Evans, and R. Gariepy. *Optimal Lipschitz extensions and the infinity Laplacian.* Calc. Var. Partial Diff. Equations 13, 2001, 123–139.

[7] L. C. Evans and O. Savin. *$C^{1,\alpha}$ regularity for infinity harmonic functions in two dimensions.* Calc. Var. Partial Diff. Equations 32, 2008, 325-347.

[8] L. C. Evans and C. K. Smart. *Everywhere differentiability of infinity harmonic functions.* Calc. Var. Partial Diff. Equations 42, 2011, 289-299.

[9] R. Jensen. *Uniqueness of Lipschitz extensions: minimizing the sup norm of the gradient.* Arch. Rational Mech. Anal. 123, 1993, 51-74.

[10] P. Lindqvist and J. J. Manfredi. *The Harnack inequality for ∞-harmonic functions.* Elec. J. Differential Equations 04, 1995, 1-5.

[11] Y. Peres, O. Schramm, S. Sheffield, and D. Wilson. *Tug-of-war and the infinity Laplacian.* J. Amer. Math. Soc. 22, 2009, 167-210.

[12] O. Savin. *C^1 regularity for infinity harmonic functions in two dimensions.* Arch. Rational Mech. Anl. 176, 2005, 351-361.

[13] J. M. Urbano. *An Introduction to the ∞-Laplacian.* Short Course, 2013.