$|\Psi\rangle_{in}$ →   → $|\Psi\rangle_{out}$

E.c.e.M.

ΔSE

Entanglement (creation and enhacement) Machine

Ricardo André Faleiro Miguel

A STUDY OF THE ENTANGLEMENT'S ENTROPY VARIATION IN ELASTIC SCATTERING PROCESSES FOR A Φ^4 COMPLEX MODEL

Dissertation presented to the Physics Department at University of Coimbra to obtain the Master's degree in Physics

July 2016

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UNIVERSIDADE DE COIMBRA

A Study of the Entanglement's Entropy Variation in Elastic Scattering Processes for a ϕ^4 Complex Model.

A dissertation presented by

Ricardo André Faleiro Miguel

under the supervision of

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to the

Departamento de Física da Universidade de Coimbra

In the fulfillment of the requirements
for the Master's degree in Physics.

Universidade de Coimbra
July 2016

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Abstract

This thesis aims to evaluate the legitimacy of the role attributed to an elastic scattering procedure as an hypothetical "Entanglement (creation and enhancement) Machine" (EceM). We study such a collision in the framework of Q.F.T (Quantum Field Theory), where the interaction is given by the coupling between two scalar fields, Alice's (A) and Bob's (B) with a ϕ^4 interaction.

Using the Von Neumann entropy as a measure of entanglement, which we call the "Entropy of Entanglement" (S_E), we calculate $(S_E)_{in}$ from a state of momenta before the collision ("in-state") and $(S_E)_{out}$ from the state of momenta after the collision ("out-state") to determine,

$$(\Delta S_E) \equiv (S_E)_{out} - (S_E)_{in}.$$

Thus, we can determine the entanglement between subsystems A and B in the degrees of freedom of momentum before and after the states "enter" the EceM and judge how well the collision behaves as one, knowing that $(\Delta S_E) > 0$ is the condition that must be met.

We separate the calculation into two distinct case studies:

-When the "in-state" is separable;

-When the "in-state" already has some degree of entanglement;

As such we can analyse if the scattering procedure behaves differently when it has to create entanglement from scratch or enhance the already existing one.

(ΔS_E) is shown to have an explicit dependence on the velocities of the CM (center of momentum) frame which corresponds to the momenta of the initial state, and on the coupling of the interaction (λ). These variables could be interpreted as "parameters" of the machine, which we can regulate to create the best possible configuration. We analyse the collision diagrammatically, computing up to 1-loop contributions when the "in-state" is separable. Thus, we can compare how this changes the "performance" of the machine, and how the parameters for the best possible outcome vary.

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Acknowledgements

There are a great number of people without whom, I'm sure, this thesis wouldn't have been made possible to come to fruition the way it has, if at all.

First of all, I would like to thank my thesis adviser, Brigitte Hiller, for guiding me throughout this path in a way that I believe few could and/or would want, that is, in a subject which wasn't known territory with a naive student who wanted to work on "something related to entanglement". This off the wall approach, I believe, would have thrown off most people. I thank her for not being one, and willingly taking time out of a busy schedule to delve into uncharted waters to provide much appreciated guidance. Her intellectual support and nurturing character were crucial in crafting important and much needed traits into my personal approach on physics, definitely traits that I'll try to carry as a physicist and person throughout my life.

I would also like to thank Alex Blin, whose counseling always provided a big push forward when dealing with those bumps inevitable when studying such un-intuitive subjects. To Marcos Sampaio and his team, for setting us on the right path at the beginning when there was no direction defined, and our continuing sharing of new ideas which has provided to be very fruitful.

Also to my colleagues and friends for accompanying me trough this journey, out of whom there are two whose influence in this work whether directly or indirectly was substantial:

Rafael Pavão I thank him for his friendship, comradery and for the countless discussions in what was a daily back and forward of almost two years, which undoubtedly without I would have found my task a lot more difficult both psychologically and intellectually.

Jorge Morais for his friendship and counselling, our discussions were always the last testament of how well I decided I knew something. Of course, this at times would mean that I was made aware of the fact that I didn't understand something all that well.

To my better half, Marta, for her love and support which provided me with the mental stability and diligent method of work that allowed me to accomplish this task. Our daily conversations managed to keep most of my insanity at bay. The wonderful cover art was also product of her enviable talent.

Finally but surely not least, to my mother, father and brother. For giving me the emotional support to carry on in those dire moments when all seemed lost. My achievements are yours.

"Our imagination is stretched to the utmost, not, as in fiction, to imagine things which are not really there, but just to comprehend those things which are there. (...) I think I can safely say that nobody understands quantum mechanics."

-Richard Feynman in *The Quantum Mechanical View of Nature.*

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Chapter 1

Introduction

Note: In this work natural units ($\hbar = c = k_B = 1$) are used unless stated otherwise

Quantum Information can be traced back as far as 1935, to the original paper of Einstein-Podoslky-Rosen (EPR) [1] in which they proposed a gedankenexperiment which has, for more tan eighty years now, challenged physicists to reevaluate the most intricate properties of Quantum theory. They concluded that either single particle entanglement was impossible or the quantum mechanical description of reality was incomplete, which in turn was refuted by Bohr. Thirty years after the EPR paper, J. Bell [2] established an inequality whose violation excludes local realistic theories and validates a spooky action at a distance. Such mathematical formulation has paved the way for Bell test experiments which settle the quantum theory debate between Einstein and Bohr. Experiments carried out at Orsay in 1982 by Aspect, Grangier, Roger and Dalibard [3] showed a violation of Bell's inequalities using calcium atoms excited to a particular state, from which the atoms decay by emitting two photons in opposite directions entangled in polarization. They concluded that such a pair of entangled photons should be considered as a global, inseparable quantum system.

Whilst quantum information was originally formulated in terms of non-relativistic quantum mechanics, recent years have seen increasing research interest in placing quantum information within the more fundamental framework of quantum field theory. Relativistic quantum information aims to understand the relationship between special and general relativity and quantum information theory. Quantum entanglement bits (e-bits) are key resources in quantum communication and quantum computation. Relativistic quantum information plays a key role in studying important current issues like,

quantum cryptography, quantum teleportation, quantum computation and quantum metrology [4] both in inertial and noninertial frames. For instance, in [5] it was pointed out that gravity or noinertial motion may serve to enhance quantum information protocols. Questions such as how different partitions of momentum/spin entanglement of relativistic particles or Bell inequalities behave under Lorentz transformations have become important [6]. Quantum entanglement also serves as a tool to cosmology, since in the early universe, the energy content was largely dominated by highly entangled quantum field background [7]. Even though experimental evidence show that primordial perturbations have undergone quantum-to-classical transition by some decoherence mechanism, some quantum correlations could in principle linger, in the case of weakly interacting fields, and encode information about the evolution of the universe [8, 9]. The appropriate theoretical framework to study all of this phenomena is quantum field theory, as such, having an understanding how entanglement is described within such a context is of prime importance.

There is also a particular interest in the study of relativistic scattering when one has access to a subset of states in the context of quantum information theory. It is rigorously formulated in the framework of Dyson's S-matrix in relativistic quantum field theory [10]. Therefore, entanglement's creation/enhancement in particle decays and collisions can be derived in a complete quantum relativistic framework.

There is an abundance of applications of scattering and entanglement creation/enhancement. For instance, the degree of entanglement created in fermionic scattering within quantum electrodynamics was analysed in [11] and a study of entropy variation between initial and final states asymptotic states to leading order in perturbation theory appears in [12]. In [13] they concluded that for a low energy regime that the differential cross section can be written as a function of the degree of entanglement of the incoming photons. An enhancement of the cross section was observed for photons prepared in a symmetric Bell state in their polarizations as compared with the factorized state.

Then it is the aim of this thesis to add a contribution to the already established works in this field, by providing a detailed analysis of entanglement creation/enhancement by calculating the entropy variation in the scattering of interacting scalar particles in a fully quantum field theoretical framework.

Using the S-matrix formalism, we derive the final state to 1-loop order in perturbation theory and express the entropy variation of the reduced state

as a function of the degree of entanglement of the initial states. We show in Chapter5 that, such correlations between the parties show explicit dependence on the speed (energy) between the colliding particles.

1.1   Structure

The thesis is organized as follows:

In Chap.2 we present a broad and self-contained overview of entanglement, which we expand into Chap.3, where we present the notion of Entanglement Monotones and their applicability's. In Chap.4 we define the necessary formalism used to study entanglement in the framework of quantum field theory. In Chap.5 we present the "bulk" of the calculations separated in two different case studies, one considering and initial separable state and one considering an initial entangled state. Finally, in Chap.6 we present our conclusions and proposals for possible future work.

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Chapter 2

Introduction to Quantum Entanglement

There are mainly two approaches to address the subject of quantum entanglement in an introductory fashion. The first is an historical approach, which starts from the paradoxes that the "Spooky action at a distance" causes [1], and builds from there a chronological exposition of the pertinent insights people had about the subject, culminating in Bell's inequalities. These open the door to study the crucial impact that non-local correlations have on the data pertaining to specially arranged quantum states [2]. Since the existence of non-local correlations is at the core of quantum entanglement, the next step is the need to set up the mathematical framework to study these, which is presently still a subject in active development [14, 15, 16, 17].

The other way starts almost exactly at this point, ignoring the deep conceptual introduction and possible philosophical implications. It is assumed the reader is convinced that quantum entanglement is alive and well and is not a product of inaccuracies in our description of reality (meaning quantum theory). Adopting this route, one should focus on a didactic way to address the basic concepts needed to introduce the subject of entanglement as clearly as possible in a rigorous mathematical way. Although interesting, the first route would lead to a lengthy exposition and wouldn't add any particular insight to the scope of this work, since we treat entanglement in a very pragmatic way.

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We take Entanglement to be a property that quantum systems may exhibit, which can be calculated and "measured" [1], and we want to see how it behaves in some specific situations. Thus, we shall choose the second way of introducing the subject and show the reader "the basic rules of the game" with examples, without going deep into the philosophy of it.

2.1 Presenting Alice and Bob: It takes two to Tango, at least.

The study of entanglement is to a large extent the general study of the influence that quantum systems have on one another (this will become more apparent in the subsequent sections). In physics we are aware that when figuring out how systems are made of smaller components, that sometimes the whole might be greater that the sum of its parts, or maybe less. For instance, the mass of an atomic nucleus is not just the sum of the mass of it's nucleons, one has to take in account the energy that binds them together among other things in order to have an accurate description of the system's mass. Figuring how systems combine to produce larger systems is essential in physics, and such a thing is more subtle than what it might appear at first, specially in quantum mechanics. Then if we are trying to find what influence quantum systems have on one another, we should have at least two of these, they are typically called Alice's and Bob's system. The systems in principle have no restrictions on what they could be, one could just be a particle and the other a black-hole or even the rest of the universe, but for now we'll start with something easier and suppose that both Alice and Bob are trying to describe one electron each. Before jumping into the description of the general composite quantum system of Alice and Bob it's better to define some basic characteristics of Alice's and Bob's sub-systems independently.

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[1]One has to be careful when using such terms loosely, of course entanglement is not an observable and cannot be measured in the same way as, for instance, energies can. We say Entanglement is "measured" in a more broad sense of the word, entanglement is not something which a system possesses but is instead a "behaviourism" of the system, it can be displayed in varying degrees and witnessed by appropriate measurement of the system's observables, in that sense we "measure" Entanglement.

2.1.1 Mathematical Interlude:

Operators, vectors and the Hilbert space

A pure[2] quantum state can be characterized as a vector of an N-dimensional Hilbert space ($\mathcal{H}$), hence, just like any kind of vector the state can be constructed as superposition of N orthogonal base vectors that span the space (a basis). It can be shown that the eigenvectors of any given Hermitean operator in $\mathcal{H}$ can be used to construct such a basis of the space. Let $\hat{\mathcal{A}}$ be any operator of the kind in Alice's space $\mathcal{H}_A$, with the collection of eigenvectors $\forall_i \{|a_i\rangle\}$, which can also be called the *base states* of the quantum state due to their fundamental nature. We then have the eigenvalue equation,

$$\hat{\mathcal{A}} |a_i\rangle = A_i |a_i\rangle, \tag{2.1}$$

where A_i is the eigenvalue associated with state $|a_i\rangle$. By using this collection of eigen-states, we can write down Alice's general description of a quantum state,

$$|\psi\rangle_A = \sum_i \alpha_i |a_i\rangle, \tag{2.2}$$

where α_i is the projection coefficient of the state into the base state $|a_i\rangle$, i.e $\alpha_i = \langle a_i|\psi\rangle_A$; $|\alpha_i|^2$ is interpreted as the probability of $|\psi\rangle$ being in that said base state. Since the state $|psi\rangle$ exists, the probability to find it in a given base state is related to the fact that if it is "somewhere" and one looks "everywhere", it must be found for sure; this is stated mathematically by the normalization condition $\sum_i |\alpha_i|^2 = 1$.

Hermitean operators are intimately related to observables, they are their mathematical representations in Hilbert space, for instance $\hat{\mathcal{A}}$ relates to the observable $\mathbf{A}$ like [18],

$$\langle \mathbf{A} \rangle_{\text{av}} = {}_A\langle \psi| \hat{\mathcal{A}} |\psi\rangle_A. \tag{2.3}$$

$\langle \mathbf{A} \rangle_{\text{av}}$ is the average value of the observable $\mathbf{A}$ (also know as expectation value) if the observable should be measured when the quantum system is in the state $|\psi\rangle_A$.

---

[2] The "pure" is to distinguish from other type of quantum states called "mixed" quantum states, which have their own special properties which we will study later on. For now when we say a "quantum state" it is implied that we are talking about a pure quantum state.

It is clear that if the system happens to be in a base state $|a_i\rangle$ of $\hat{\mathcal{A}}$, then that the measurement of the observable will give for sure A_i, otherwise it will give a weighed average over all possible eigenvalues of the base states of the system.

Evidently all these steps can be taken for Bob as well. The equivalent results in his system are compared to Alice's in the following table,

	Alice	Bob
Space	$\mathcal{H}_A$	$\mathcal{H}_B$
Operator	$\hat{\mathcal{A}}$	$\hat{\mathcal{B}}$
Basis	$\forall_i \{|a_i\rangle\}$	$\forall_i \{|b_i\rangle\}$
State	$|\psi\rangle_A = \sum_i \alpha_i |a_i\rangle$	$|\psi\rangle_B = \sum_i \beta_i |b_i\rangle$
Normalization	$\sum_i |\alpha_i|^2 = 1$	$\sum_i |\beta_i|^2 = 1$
$\langle$Observable$\rangle$	$\langle\mathbf{A}\rangle = {}_A\langle\psi| \hat{\mathcal{A}} |\psi\rangle_A$	$\langle\mathbf{B}\rangle = {}_B\langle\psi| \hat{\mathcal{B}} |\psi\rangle_B$

Table 2.1: Alice Vs. Bob

Until this point we have kept things quite general since we haven't yet specified the nature of the quantum systems they aim to study. Each of their own space's $\mathcal{H}_A$ and $\mathcal{H}_B$ have a general number of dimensions denoted by N_A and N_B. Likewise, the operators $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ could be any operator they wish.

Now that Alice and Bob have all the necessary tools to study their own independent quantum systems we can proceed to combine their systems into some other large composite system.

Combining Systems

How do we combine quantum systems to make larger ones? Intuitively one has an idea how to create such a composite space when combining classical systems. For instance, say Alice wants do describe a regular coin toss, where the only possibilities are either heads (H) or tails (T). Such classical systems can be described by means of a set where the possible outcomes are its elements, in this case the set is $\{H_A, T_A\}$. She could then describe her system schematically in a simple way, imagine that she decided to represent the elements of the set in this box like manner,

H_A	T_A

Table 2.2: Heads or Tails: Table of the possible outcomes of Alice's coin toss. Each possible state is given a probability $P(H_A)$, $P(T_A)$.

Now suppose Bob also wants decides to do the same,

H_B	T_B

Table 2.3: Heads or Tails: Table of the possible outcomes of Bob's coin toss. Each possible state is given a probability $P(H_B)$, $P(T_B)$.

It's not so difficult to imagine what the possibilities would be if they decided to study their coin tosses as a whole system (we denote it as AB) instead of separately. It would look something like,

Alice \\ Bob	H_B	T_B
H_A	$H_A; H_B$	$H_A; T_B$
T_A	$T_A; H_B$	$T_A; T_B$

Table 2.4: Heads or Tails: Table of the possible outcomes of AB's coin tosses. Each possible state is given a probability $P(H_A; H_B)$, $P(H_A; T_B)$, $P(T_A; H_B)$, $P(T_A; T_B)$. Since the processes are independent the probabilities factorize i.e $P(H_A; H_B) = P(H_A)P(H_B)$.

Each box now represents a specific configuration of the possible outcomes the joint system AB could produce.

This sort of operation we performed informally can be generalized in a mathematical rigorous way to any number of subsystems and to an arbitrary number of dimensions of each subsystem. The operation is called a *tensor product*, in this case between two systems, A and B, which is usually denoted as $A \otimes B$.

When we talk about the number of dimensions of a system we mean the minimum "blocks of information" necessary to have a complete description of the system, in a set this means the elements that compose the entire set, and in a vector space the number of vectors necessary to form an orthogonal basis. For instance, in one coin toss we have a full description of the possible outcomes with just two elements $\{H, T\}$ with two coin tosses we need to have four $\{HH, HT, TH, TT\}$, this is because when "multiplying" the systems with one another we also multiplied the number of dimensions of both systems. Then the number of dimensions of a general composite AB system (N_{AB}), is given by $N_{AB} = N_A \times N_B$. This is not so difficult to realize, since the tensor product is a generalized product, we should expect that the regular numbers (like the number of dimensions) attached to the product would just come out in a regular multiplication operation. If we had more than two sub-systems we would just need to multiply them as well, but the composite system is only composed of two sub-systems (Alice's and Bob's), we usually call this kind of system *Bi-partite*.

As it turns out this operation also holds true when combining quantum systems, the main difference is that the systems being "multiplied" aren't just sets but vectors spaces, and consequently the "blocks of information" aren't elements of sets but base vectors of a vector space, this will later on prove to be a source of peculiar characteristics in the composite system.

In the framework of quantum mechanics we have the composite system of Alice and Bob given by, $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, where again, $N_{\mathcal{H}_{AB}} = N_{\mathcal{H}_A} \times N_{\mathcal{H}_B}$. We can easily construct a state $|\psi\rangle_{AB} \in \mathcal{H}_{AB}$ which is given by,

$$|\psi\rangle_{AB} = |\psi\rangle_A \otimes |\psi\rangle_B \, ; \tag{2.4}$$

Such a state is aptly called , *product state* or *separable state*. The general description of such a state is,

$$|\psi\rangle_{AB} = \sum_{i,j} \alpha_i \beta_j \, |a_i \, b_j\rangle \, ; \tag{2.5}$$

$|a_i\, b_j\rangle$ is a short notation for $|a_i\rangle \otimes |b_j\rangle$. We can immediately observe that the tensor product of the two basis of the sub-systems also forms a basis in the composite system. Then, even though $|a_i\, b_j\rangle$ has two labels it only represents one state in the composite system.

Product States and Observables

We shall comment on some characteristics of these *product states*. First, it can be shown that if $|\psi\rangle_A$ and $|\psi\rangle_B$ are both normalized, $|\psi\rangle_{AB}$ is immediately normalized with the same set of normalization conditions (see Appendix).

Now we suppose Alice wants to measure an observable of her choice, when the quantum state in question is a *product state* like in (2.5) but Bob doesn't, this is described in terms of operators as,

$$\hat{\mathcal{A}} \otimes \mathbf{1}_B,$$

$\mathbf{1}_B$ is the identity operator in Bob's space, it's meant to show that he doesn't want to do anything to the system. We want to find,

$$_{AB}\langle\psi|\, \mathcal{A} \hat{\otimes} \mathbf{1}_B\, |\psi\rangle_{AB}. \tag{2.6}$$

Since each operator only exists in their own respective space, $\mathcal{A}$ only acts on states of $\mathcal{H}_A$, likewise $\mathcal{B}$ only acts on states of $\mathcal{H}_B$. Then it can be easily seen that the previous expression reduces to,

$$_A\langle\psi|\, \hat{\mathcal{A}}\, |\psi\rangle_A \; _B\langle\psi|\, \mathbf{1}_B\, |\psi\rangle_B = \langle\mathbf{A}\rangle_{\mathrm{av}}; \tag{2.7}$$

Likewise, in the case when Bob wants to measure something, $\mathbf{1}_A \otimes \hat{\mathcal{B}}$,

$$_A\langle\psi|\, \mathbf{1}_A\, |\psi\rangle_A \; _B\langle\psi|\, \hat{\mathcal{B}}\, |\psi\rangle_B = \langle\mathbf{B}\rangle_{\mathrm{av}}; \tag{2.8}$$

This is to say, Alice and Bob will get exactly the same expected value of the observable whether the system is in the *product state* or their own personal states $|\psi\rangle_A$ (for Alice), and $|\psi\rangle_B$ (for Bob). There's really no surprise here, this is the quantum mechanical counterpart to the previous classical example of the coin toss. To illustrate this let us suppose we assign value (V) to each face of the coin, for example,

$$V(T) = +1;\ V(H) = -1;$$

If we pick a configuration out of Table 2.4 like $(H_A; T_B)$, Alice "measures" $V(H_A) = -1$ in certainly the same way as if she had studied her own coin toss and the outcome had been (H_A) from Table 2.2, the same is true for Bob.

Now imagine both Alice and Bob want to measure their respective observables, the operator which translates this is, $\hat{\mathcal{A}} \otimes \hat{\mathcal{B}}$, or just $\hat{\mathcal{A}}\hat{\mathcal{B}}$. At this point we aren't surprised to see that the result will be,

$$\langle \mathbf{AB} \rangle_{\text{av}} = {}_A\langle \psi | \hat{\mathcal{A}} | \psi \rangle_A \; {}_B\langle \psi | \hat{\mathcal{B}} | \psi \rangle_B = \langle \mathbf{A} \rangle_{\text{av}} \langle \mathbf{B} \rangle_{\text{av}} \tag{2.9}$$

We can write the previous equation as,

$$\langle \mathbf{AB} \rangle_{\text{av}} - \langle \mathbf{A} \rangle_{\text{av}} \langle \mathbf{B} \rangle_{\text{av}} = 0, \tag{2.10}$$

But, the right hand side of the previous equation isn't always zero. It's usually called the *Correlation* between the observables $\mathbf{A}$ and $\mathbf{B}$, and it's only zero when the values factorize. The reason the values factorize in such a way stems from the fact that the probability distributions which describe such observables also factorize, we have already seen this example in the coin toss, and in fact we can confirm this easily. Suppose, $\langle V_A \rangle_{\text{av}}$ and $\langle V_B \rangle_{\text{av}}$ are the averages values that Alice and Bob get, respectively, in a N number of coin tosses, if N is high enough both would be zero since there's one-half probability of getting either heads or tails - What would the value $\langle V_A V_B \rangle_{\text{av}}$ be in the same circumstances ? Since we know that the coin tosses are independent, the probabilities factorize and we get exactly the same probabilities for the outcomes of coin A and coin B which implies the average value will also be zero, thus having

$$\langle V_A V_B \rangle_{\text{av}} = \langle V_A \rangle_{\text{av}} \langle V_B \rangle_{\text{av}} = 0;$$

So just like we've seen in this classical example it seems that when a quantum system is in a *product state* both observers measure un-correlated observables, the reason is obviously the same as in the classical example since this stems from the fact that the "function of probability"[3] $|\psi\rangle_{AB}$ factorizes into $|\psi_A\rangle |\psi_B\rangle$.

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[3]Although in this form the state doesn't have the explicit form of an analytic function, with an appropriate choice of basis we could describe the state as a function, $\psi(x)$ or $\psi(p)$

Classical Correlations

We shall take (2.10) and generalize it into,

$$\langle \mathbf{AB} \rangle_{\text{av}} - \langle \mathbf{A} \rangle_{\text{av}} \langle \mathbf{B} \rangle_{\text{av}} = C(\mathbf{A}, \mathbf{B}), \qquad (2.11)$$

where $C(\mathbf{A}, \mathbf{B})$ is the *Correlation* between observable $\mathbf{A}$ and $\mathbf{B}$.

In what scenario could $C(\mathbf{A}, \mathbf{B})$ be different from zero? Imagine that Alice and Bob instead of coin tossing are interested in studying those power ball machines, each of them puts a ball inside one of these machines with a $+1$ printed on one and a -1 printed on the other. The machine randomly selects one ball which Alice takes and Bob takes the remaining one. On average $\langle V_A \rangle_{\text{av}}$ and $\langle V_B \rangle_{\text{av}}$ will be zero, just like the coin tossing case. But what about $\langle V_A V_B \rangle_{\text{av}}$? This time it won't be zero, because regardless of who gets what, the product of both values will always be -1, so,

$$\langle V_A V_B \rangle_{\text{av}} - \langle V_A \rangle_{\text{av}} \langle V_B \rangle_{\text{av}} = -1;$$

Thus one can appreciate that unlike the observables in the coin tossing example these are perfectly correlated. One wonders in what situation we might find such a thing in the quantum case.

2.1.2 Spin-$\frac{1}{2}$ Systems

At this point we're going to commit ourselves to a specific quantum system to analyse, as it should be. We can only take things in a general way so far, eventually we have to make up our minds on what physical system we're actually talking about.

A spin-$\frac{1}{2}$ system has two dimensions, since it only takes a collection of two base states to characterize all possible states in which the system could be. These two base states are usually denoted as the up-state, $|u\rangle$, and the down-state $|d\rangle$. As we stated before, such base states are eigenvectors of a Hermitean operator which relates to some observable, the observable in question is the particle's spin component along a given axis. If we say without loss of generality that such an axis is the $\hat{z}$ axis, then the operator in question is the known $\hat{\sigma}_z$ spin operator, and the base states are states which correspond to eigenvalue $+1$ in the positive direction of $\hat{z}$, $|u\rangle$, and -1 in the negative direction, $|d\rangle$. Even though $\hat{\sigma}_z$ is an operator which relates to a certain observable component of spin which we want to measure(along

the $\hat{z}$ axis), and also whose eigenvectors serve as a basis, a particle's spin is mathematically a 3-vector and it isn't fully described if we are only equipped to measure one of it's components, we should also have operators which relate to the other observable components along the vectors which form the basis of a 3-vector spin. Since one operator is already defined to be the component along the $\hat{z}$ axis, the other two are obviously going to be along the $\hat{x}$ axis $(\hat{\sigma}_x)$ and the $\hat{y}$ axis $(\hat{\sigma}_y)$axis. We should mention that $\hat{\sigma}_z$ isn't special, we could just as well have used the eigenvectors of the other two operators as base states for the spin system, but usually picking $\hat{\sigma}_z$ is a preferred convention when defining the base states in which we measure the 3 different components of spin. Generally we say the state is "quantized along the $\hat{z}$-axis", this choice implies that we can't measure all 3 components of the system simultaneously, because only one set of base states is available for each operator, in this case the $\hat{z}$-axis base states.

The specifics of the spin-$\frac{1}{2}$ system in which Alice and Bob want measure their particle's spin is present below in a table similar to Table.2.1,

	Alice	Bob
Space	$\mathcal{H}_A, N_A = 2$	$\mathcal{H}_B, N_B = 2$
Operators	$\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z$	$\hat{\tau}_x, \hat{\tau}_y, \hat{\tau}_z$
Basis	$\{|u\rangle_A, |d\rangle_A\}$	$\{|u\rangle_B, |d\rangle_B\}$
State	$|\psi\rangle_A = \alpha_u |u\rangle_A + \alpha_d |d\rangle_A$	$|\psi\rangle_B = \beta_u |u\rangle_B + \beta_d |d\rangle_B$
Normalization	$|\alpha_u|^2 + |\alpha_d|^2 = 1$	$|\beta_u|^2 + |\beta_d|^2 = 1$
$\langle\text{Observable}\rangle$	$\langle\sigma_i\rangle_{\text{av}} = {}_A\langle\psi| \hat{\sigma}_i |\psi\rangle_A \,\,_{|i=x,y,z}$	$\langle\tau_i\rangle_{\text{av}} = {}_B\langle\psi| \hat{\tau}_i |\psi\rangle_B \,\,_{|i=x,y,z}$

Table 2.5: Alice spin-$\frac{1}{2}$ system Vs. Bob spin-$\frac{1}{2}$ system

We should comment that $\hat{\tau}_i$ is very much the same operator as $\hat{\sigma}_i$, but in order to distinguish them between Alice's and Bob's spin operator we decided to represent them with different Greek letters, an A or B subscript would've done the job but the notation would have become too over-encumbered.

Pauli Matrices and Spin observables

$\hat{\sigma}_x$, $\hat{\sigma}_y$ and $\hat{\sigma}_z$ (or $\hat{\tau}_x$, $\hat{\tau}_y$ and $\hat{\tau}_z$) are operators that Alice (Bob) uses to describe the observables that she (he) may want to measure, such operators have a

matrix representation form. The matrices which represent these operators
are widely known as the Pauli matrices which are,

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \ \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \qquad (2.12)$$

If we adopt the column vector representation of the base states,

$$|u\rangle_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \ |d\rangle_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix};^4 \qquad (2.13)$$

We can easily deduce the following relationships between the base states and
the operators,

$$\hat{\sigma}_z |u\rangle_A = |u\rangle_A ; \hat{\sigma}_z |d\rangle_A = -|d\rangle_A ;$$

$$\hat{\sigma}_x |u\rangle_A = |d\rangle_A ; \hat{\sigma}_x |d\rangle_A = |u\rangle_A ; \qquad (2.14)$$

$$\hat{\sigma}_y |u\rangle_A = i |d\rangle_A ; \hat{\sigma}_y |d\rangle_A = -i |u\rangle_A ;$$

The relationships are all equivalent for $(\hat{\tau}_x, \hat{\tau}_y, \hat{\tau}_z)$ and $|u\rangle_B$, $|d\rangle_B$. Now we can
find the general form of the average values of the operators,

$$\langle \sigma_i \rangle_{\text{av}} = {}_A\langle \psi| \hat{\sigma}_i |\psi\rangle_A {}_{|i=x,y,z} ; \quad \langle \tau_i \rangle_{\text{av}} = {}_A\langle \psi| \hat{\tau}_i |\psi\rangle_A {}_{|i=x,y,z} ; \qquad (2.15)$$

We find them to be,

$$\langle \sigma_x \rangle_{\text{av}} = \alpha_u^* \alpha_d + \alpha_d^* \alpha_u;$$

$$\langle \sigma_y \rangle_{\text{av}} = i(\alpha_d^* \alpha_u - \alpha_u^* \alpha_d); \qquad (2.16)$$

$$\langle \sigma_z \rangle_{\text{av}} = |\alpha_u|^2 - |\alpha_d|^2;$$

$\langle \tau_x \rangle$, $\langle \tau_y \rangle$ and $\langle \tau_z \rangle$ have the exact same form, we just have to change the
coefficients to the appropriate coefficients of Bob's space (β's instead of α's).
The previous set of equations give the form of the average values for the
different spin components when the system is in a general state $|\psi\rangle_A$ prepared
along the $\hat{z}$ axis. Now, if we take the observables expectation values, square
them and add them all up we realize, that regardless of what values we have

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^{4}Notice that $\langle u|d\rangle = \langle d|u\rangle = 0$, as it should be since they're orthogonal.

for the coefficients the end result will always be 1, due to the normalization conditions. i.e

$$\langle \sigma_x \rangle_{\mathrm{av}}^2 + \langle \sigma_y \rangle_{\mathrm{av}}^2 + \langle \sigma_z \rangle_{\mathrm{av}}^2 = 1. \tag{2.17}$$

This relationship has exactly the same form as the quantum angular momentum conservation in terms of the Pauli matrices (except for some constants). Spin is a quantum number which is very intimately related to angular momentum since the mathematics that describe them both are the same, this relationship could be thought as a "Spin Conservation Principle" which translates the fact that for each specific combination of coefficients we might choose, there is an orientation $\hat{n}$ for which our state behaves as an eigenvector of the spin component along that vector. For instance if $\alpha_u = \alpha_d = \frac{1}{\sqrt{2}}$, we get that $\langle \sigma_z \rangle_{\mathrm{av}} = \langle \sigma_y \rangle_{\mathrm{av}} = 0$ and $\langle \sigma_x \rangle_{\mathrm{av}} = 1$. This means if the state is prepared with the said coefficients it's equally probable to measure the spin's component along the $\hat{z}$ and $\hat{y}$ as either "up" or "down", but if we measure it along the $\hat{x}$ axis it will be for sure "up". This is true because

$$\hat{\sigma}_x |\psi\rangle_A = |\psi\rangle_A \,; \qquad \text{For,} \ \ |\psi\rangle_A = \frac{1}{\sqrt{2}}(|u\rangle_A + |d\rangle_A);$$

Thus $|\psi\rangle_A$ is an eigenvector of $\hat{\sigma}_x$, and as such the observable's expectation value will be the eigenvalue of the state (i.e +1).

Entangled Spin States

Let us suppose we construct a *product state* between $|\psi\rangle_A$ and $|\psi\rangle_B$,

$$|\psi\rangle_A \otimes |\psi\rangle_B = \{\alpha_u |u\rangle_A + \alpha_d |d\rangle_A\} \otimes \{\beta_u |u\rangle_B + \beta_d |d\rangle_B\},$$

we get,

$$|\psi\rangle_{AB} = \alpha_u \beta_u |u\ u\rangle + \alpha_u \beta_d |u\ d\rangle + \alpha_d \beta_u |d\ u\rangle + \alpha_d \beta_d |d\ d\rangle \,; \tag{2.18}$$

Which clearly has the form of (2.5). This expression gives the general form of a product state in AB for a system of two spins (one for Alice and one for Bob). From this expression we can produce any number of states we would like, for instance, say we know that $\alpha_d = 0$ for a given state, we call it $|\phi_1\rangle_{AB}$. The state is,

$$|\phi_1\rangle_{AB} = \alpha_u \beta_u |u\ u\rangle + \alpha_u \beta_d |u\ d\rangle \,; \tag{2.19}$$

Imagine that $\beta_d = 0$ for another state, let's call it $|\phi_2\rangle_{AB}$,

$$|\phi_2\rangle_{AB} = \alpha_u \beta_u |u\ u\rangle + \alpha_d \beta_u |d\ u\rangle \ ; \tag{2.20}$$

These are two perfectly acceptable *product states* that exist in AB.

Remembering our classical example in the **Combining Systems** subsection, when we figured out how to study the two coin tosses as a whole system; We had a table of all the necessary elements in order to have a complete description of the combined system, there were no "blocks of information" missing. We can appreciate this because the underlying mathematical structure of the theory we used when describing the system doesn't provide us with any operations to expand the number of states inhabiting the composite system, by adding to the collection of states we already had directly from the tensor product. To clarify: Imagine we take two members of the set from the composite system, say, (H_A, T_B) and (H_A, H_B). Could we make something out of these two elements with any kind of operation such that the end result is a different but also viable element in the composite ? No, there isn't;[5]

But in the quantum case, is this also true ? Most certainly not. Since we know quantum states are vectors in the Hilbert space and the algebra of vectors allows us to perform operations that produce a vector in the same space, then we could very well take any two vectors like the ones we calculated, $|\phi_1\rangle_{AB}, |\phi_2\rangle_{AB}$ and subtract them, such that we have,

$$|\phi_1\rangle_{AB} - |\phi_2\rangle_{AB} = \alpha_u \beta_d |u\ d\rangle - \alpha_d \beta_u |d\ u\rangle \ ; \tag{2.21}$$

This vector rightfully belongs in $\mathcal{H}_{AB}$, but the curious thing about it is that, there is no available choice of coefficients such that you can create the state (2.21) out of (2.18), so this state clearly isn't a *product state* but something else, we call this an *entangled state*.

An *entangled state*, by definition, is any state of a composite system which isn't a *product state*. This means that it cannot be decomposed as a tensor product of the sub-systems states alone, this is something which is manifestly different from what we were used to seeing until this point. We said that the superposition of vectors would produce curious results and here it is, the entanglement of states; A phenomenon of quantum mechanics with no

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[5] Imagine we add them up like this, $(H_A + H_A, T_B + H_B)$; This doesn't have any meaning in terms of our set, it's probably an element of a different set pertinent to a different system, but not to our own.

classical counter-part[6], we now realize that in the coin tossing example(and in fact, any classical system) every state is a *product state*.

We define (2.21) as,

$$\left|E_0^-\right\rangle = \alpha_u \beta_d \left|u\ d\right\rangle - \alpha_d \beta_u \left|d\ u\right\rangle; \qquad (2.22)$$

Let's go back to (2.18) and try other combinations of coefficients to see what we get. If for instance we say that now the "up" coefficients (α_u, β_u) are the ones which are null, for $\alpha_u = 0$ we have the state $\left|\phi_3\right\rangle_{AB}$,

$$\left|\phi_3\right\rangle_{AB} = \alpha_d \beta_u \left|d\ u\right\rangle + \alpha_d \beta_d \left|d\ d\right\rangle; \qquad (2.23)$$

And for $\beta_u = 0$ the state $\left|\phi_4\right\rangle_{AB}$,

$$\left|\phi_4\right\rangle_{AB} = \alpha_u \beta_d \left|u\ d\right\rangle + \alpha_d \beta_d \left|d\ d\right\rangle; \qquad (2.24)$$

Subtracting them like we did before we get,

$$\left|\phi_3\right\rangle_{AB} - \left|\phi_4\right\rangle_{AB} = -\left|\mathrm{E}_0^-\right\rangle,$$

so we have the same result with a minus sign. Changing the coefficients which are null from "down" to "up" only adds a global negative factor to the sate. We can try more combinations of adding and subtracting these states to find more entangled states, as long as we make sure we are using states which have a mixed restriction on the coefficients (one α and β), because otherwise this would just retrieve us a formula of the type of (2.18). We can try then,

$$\left|\phi_3\right\rangle_{AB} - \left|\phi_2\right\rangle_{AB} = \alpha_u \beta_u \left|u\ u\right\rangle - \alpha_d \beta_d \left|d\ d\right\rangle := \left|E_1^-\right\rangle; $$

Which is another entangled state. If we change the phase between the base sates in $\left|E_0^-\right\rangle$, from a minus to plus sign, the resulting state would be the symmetric equivalent of state $\left|E_0^-\right\rangle$ and also an entangled state, we call it

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[6]Not unlike entanglement, the principle of "quantum superposition" also stems from the vectorial like nature of the quantum sates. In fact, just as a state $\left|\psi\right\rangle$ is given by a superposition of its base states, so can *entangled state* can also be seen as a superposition of *separable states* (2.21). The major difference is that in a regular superposition, of a given quantum sate, the interference is only between the eigenstates that form a basis of one operator, and in entanglement there needs to be at least two distinct bases of eigenstates in the superposition, which correspond to the observables of each observer in the subsystems.

$\left|E_0^+\right\rangle$. Likewise, if we change the phase in $\left|E_1^-\right\rangle$, we would get an entangled state $\left|E_1^+\right\rangle$. Until this point we have the following collection of *entangled states* in the AB-spin system, and they are,

$$\left|E_0^-\right\rangle = \alpha_u\beta_d \left|u\ d\right\rangle - \alpha_d\beta_u \left|d\ u\right\rangle ;$$

$$\left|E_0^+\right\rangle = \alpha_u\beta_d \left|u\ d\right\rangle + \alpha_d\beta_u \left|d\ u\right\rangle ;$$

$$\left|E_1^-\right\rangle = \alpha_u\beta_u \left|u\ u\right\rangle - \alpha_d\beta_d \left|d\ d\right\rangle ; \qquad (2.25)$$

$$\left|E_1^+\right\rangle = \alpha_u\beta_u \left|u\ u\right\rangle + \alpha_d\beta_d \left|d\ d\right\rangle ;$$

If we say that,

$$\alpha_u = \beta_u = \alpha_d = \beta_d = \left(\frac{1}{\sqrt{2}}\right)^{\left(\frac{1}{2}\right)} , \qquad (2.26)$$

the states in (2.25) become maximally entangled states, known as *Bell States*[7].

The states in (2.25) also form a complete set, [8] this means any other entangled or separable state we could concoct can be described as a combination of these four states. For instance, imagine we add $\left|\phi_1\right\rangle_{AB}$ and $\left|\phi_2\right\rangle_{AB}$, with the choice of coefficients of (2.26), we get,

$$\frac{2}{\sqrt{2}} \left|u\ u\right\rangle + \frac{1}{\sqrt{2}} \left|u\ d\right\rangle + \frac{1}{\sqrt{2}} \left|d\ u\right\rangle ,$$

which is also an entangled state, albeit, with a different structure from the ones in (2.25), because we can factorize at least part of it,

$$\frac{1}{\sqrt{2}} \left|u\right\rangle_A \otimes (2\left|u\right\rangle_B + \left|d\right\rangle_B) + \frac{1}{\sqrt{2}} \left|d\ u\right\rangle ,$$

so they cant ever be perfectly correlated (i.e maximally entangled). Such a state can be written as combination of the Bell states as, $\left|E_0^+\right\rangle + \left|E_1^-\right\rangle + \left|E_1^+\right\rangle$,

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[7]The name is attributed to them because they violate Bell's theory maximally. Entanglement is not a on/off type of thing, depending on the values of coefficients, some states might be "more" or "less" entangled. As it turns out these states are maximum entangled when all their coefficients are $\frac{1}{\sqrt{2}}$.

[8]This is true because the sates are linearly independent from each other and we could construct a orthogonal basis out of them to describe any vector.

which means then that a combination of maximally entangled states won't be maximally entangled if the result isn't of the form of the states in (2.25).

But why can't these *entangled states* be represented in the same way as separable states ? An entangled state is, for all purposes, just a quantum state, which happens to exists in a composite system of various subsystems. To see why this is so, let us write the most general state in a given composite system,

$$|\Gamma\rangle = \sum_i \gamma_i |i\rangle \tag{2.27}$$

Where $\sum_i |\gamma_i|^2 = 1$, if it's normalized. When studying the previous composite spin system $\mathcal{H}_{AB}$, $i \in (1,4)$. The state becomes,

$$|\Gamma\rangle_{AB} = \gamma_1 |1\rangle + \gamma_2 |2\rangle + \gamma_3 |3\rangle + \gamma_4 |4\rangle \tag{2.28}$$

Labels are just labels, we can very well use two labels for each state reminiscent to what we had for the spin system before, such that,

$$|\Gamma\rangle_{AB} = \gamma_{uu} |u\ u\rangle + \gamma_{ud} |u\ d\rangle + \gamma_{du} |d\ u\rangle + \gamma_{dd} |d\ d\rangle \ ; \tag{2.29}$$

The normalization condition becomes,

$$|\gamma_{uu}|^2 + |\gamma_{ud}|^2 + |\gamma_{du}|^2 + |\gamma_{dd}|^2 = 1.$$

Comparing with the general form of a *product state* (2.18), which is completely normalized by

$$|\alpha_u|^2 + |\alpha_d|^2 = |\beta_u|^2 + |\beta_d|^2 = 1,$$

we notice that the general state $|\Gamma\rangle_{AB}$ in the composite system needs two more parameters to be fully described than $|\psi\rangle_{AB}$. It seems Alice and Bob aren't able to create mathematically the most general state in the composite system only from the information necessary to create the most general state in their respective subsystems, indeed it seems the sum of the parts does not make the whole. This is something which will be reinforced later on when we try to compute observables of systems in *entangled states*.

From (2.29) with an appropriate choice of coefficients we can reproduce any of the *entangled states* we calculated before. If we set

$$\gamma_{uu} = \gamma_{dd} = 0; \ \gamma_{ud} = \alpha_u \beta_d; \ \gamma_{du} = -\alpha_d \beta_u$$

we get $\left|E_0^-\right\rangle$, and so forth, but there is no choice one can make of α's and β's to get such a state. So the reason entangled states can't be represented in the same way as *product states*, is that such states need more information to be fully described, information which is out of Alice's and Bob's reach.

Observables and Quantum Correlations

Imagine a quantum system of two electrons in a *entangled state*, like $\left|E_0^-\right\rangle$ when $\alpha_u\beta_d = \alpha_d\beta_u = \frac{1}{\sqrt{2}}$; which is definitely a possible state for the electrons to be at[9], then it stands to reason that if the electrons are in this state, Alice(Bob) could try to measure an observable with her(his) apparatus, like a spin component of one of the electrons along a given axis. We use the same form of operator as before, $\hat{\sigma}_z \otimes \mathbf{1}_B$, and calculate the expectation value of one of the Alice's spin components along the $\hat{z}$ axis,

$$\langle\sigma_z\rangle_{\mathrm{av}} = \left\langle E_0^-\right| \hat{\sigma}_z \otimes \mathbf{1}_B \left|E_0^-\right\rangle \tag{2.30}$$

$$= \frac{1}{2}\Big(\langle u\,d| - \langle d\,u|\Big)\Big(|u\,d\rangle + |d\,u\rangle\Big) = 0.$$

It appears that it's as likely for the spin along the $\hat{z}$ axis to be up or down. What about, $\langle\sigma_x\rangle_{\mathrm{av}}$ and $\langle\sigma_y\rangle_{\mathrm{av}}$?

$$\langle\sigma_x\rangle_{\mathrm{av}} = \left\langle E_0^-\right| \hat{\sigma}_x \otimes \mathbf{1}_B \left|E_0^-\right\rangle \tag{2.31}$$

$$= \frac{1}{2}\Big(\langle u\,d| - \langle d\,u|\Big)\Big(|u\,d\rangle - |u\,u\rangle\Big) = 0.$$

$$\langle\sigma_y\rangle_{\mathrm{av}} = \left\langle E_0^-\right| \hat{\sigma}_y \otimes \mathbf{1}_B \left|E_0^-\right\rangle \tag{2.32}$$

$$= \frac{1}{2}\Big(\langle u\,d| - \langle d\,u|\Big)\Big(i\,|d\,d\rangle + i\,|u\,u\rangle\Big) = 0.$$

Then, $\langle\sigma_x\rangle_{\mathrm{av}} = \langle\sigma_y\rangle_{\mathrm{av}} = \langle\sigma_z\rangle_{\mathrm{av}} = 0$, which is something quite surprising because the observables don't agree with the "Spin Conservation Principle" (2.17). It appears that if the system is in such a state then Alice can't predict what the spin component will be no matter the direction she tries to measure it. This state has no predictive powers whatsoever in regard to Alice's (or Bob's) observables in their own subsystems, which is quite odd because according to the principles of quantum mechanics the system's quantum state is the most one can know about it. At this point we are witnessing first hand the counter intuitive nature of entanglement, just like the saying goes, "one can know everything there is to know about a system and know nothing about its parts".

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[9]For instance, in the quantum description of the interaction between a proton and an electron in the Hydrogen atom, which explains the Hyperfine Splitting phenomenon, an electron is in a state of this kind. We will explore this example in more detail, later on.

Now imagine both Alice and Bob want to measure some component of spin along the same axis, let us say the $\hat{z}$ axis, the operator will be $\hat{\sigma}_z\hat{\tau}_z$, and we have,

$$\langle \sigma_z\tau_z\rangle_{\text{av}} = \left\langle E_0^-\left| \hat{\sigma}_z\hat{\tau}_z \right|E_0^-\right\rangle \tag{2.33}$$

$$= \frac{1}{2}\Big(\langle u\,d| - \langle d\,u|\Big)\Big(-|u\,d\rangle + |d\,u\rangle\Big) = -1.$$

The same is also true for $\langle\sigma_y\tau_y\rangle_{\text{av}}$ and $\langle\sigma_x\tau_x\rangle_{\text{av}}$. If we look back at (2.11), we notice that in this case,

$$\langle\sigma_z\tau_z\rangle_{\text{av}} - \langle\sigma_z\rangle_{\text{av}}\langle\tau_z\rangle_{\text{av}} = C(\sigma_z, \tau_z) \tag{2.34}$$

$$C(\sigma_z, \tau_z) = -1$$

All pairs of observables are perfectly correlated. As we can see the state gives us information about the observables $((\sigma_z\tau_z)$, etc), unfortunately for Alice and Bob all the state provides us with, is the information that whenever Alice measure spin up, Bob measure spin down for sure, (and vice-versa) but there's no predictive information on what only one of the spins will be if they measure it by itself, because the outcome will bear this intrinsic random behaviour in its statistical distribution. It seems Alice and Bob need each other in order to calculate this kind of composite observable by comparing their own measurements, there's no other way to make sense out of it. This is the nature of quantum entanglement, it's this strange behaviour that emerges when we try to measure observables when the system is in such a state. It's as if the "true" quantum system exists by default in the higher dimensional Hilbert space, and we get a glimpse of the "projection" of the state in the lower dimensions that are accessible to the specific experimental configurations of the sub-systems, and only by comparing the said "projections" do we get the full picture. Sometimes we can recreate the picture by ourselves (separable system), but other times it won't be enough and we'll need more information (entangled system).

Before, we said the composite observables can't be measured by Alice or Bob alone. But is this really true ? We know that in some type of systems like two neighboring electrons in an Ising chain or even the Hydrogen atom, which for all purposes is an electron/proton system, a part of the Hamiltonian operator is proportional to the product of the the spin operators for each of the particles of the subsystems. In the Hydrogen atom example, suppose

Alice's subsystem is the electron and Bob's the proton. If the atom is the energy ground state (E_0) and in some entangled spin state $|\psi\rangle_{AB}$, with the interaction part of Hamiltonian given by ,

$$\hat{H} - \hat{H}_0 \propto \hat{\sigma}_e \cdot \hat{\tau}_p; \qquad (2.35)$$

One might wonder if Alice could by her own means (without Bob) determine what the composite observables of spin are, by measuring the variation of the energy of the system, because

$$E - E_0 = \langle\psi| \hat{H} - \hat{H}_0 |\psi\rangle_{AB} \propto \langle\sigma_e \cdot \tau_p\rangle_{\text{av}}.$$

And even though this is an entangled system in terms of the spin states, it's also just a regular atom whose energy transitions can be easily measured. But knowing $\langle\sigma \cdot \tau\rangle_{\text{av}}$ doesn't relate in any way to being more close in knowing the components separately ($\langle\sigma_x\tau_x\rangle_{\text{av}}, \langle\sigma_y\tau_y\rangle_{\text{av}}, \langle\sigma_z\tau_z\rangle_{\text{av}}$), which can only be described by the appropriate measurement by both Alice and Bob, as we said. There's no other way around, like it or not.

Mixed States and Density operators

As it has already been mentioned throughout, there's a lack of information that hinders the measurement of the sub-systems observables if the composite system happens to be in an *entangled state*. But how does this manifests itself in terms of the states ? We already know that we can't measure accurately the observables, but the question now is; is there a way to describe the entangled state, even if incompletely in a sub-system ? It can't be described as a "regular" quantum state because we know that by definition,

$$|\Gamma\rangle_{AB} \neq |\psi\rangle_A \otimes |\psi\rangle_B \qquad (2.36)$$

where $|\Gamma\rangle_{AB}$ is an *entangled state*, so it definitely can't be a vector in the subspace. Fortunately there is a different kind of formalism used to describe states which aren't "regular", such states are called mixed states, as opposed to the "regular states" which are the already mentioned pure states.

To understand what a mixed state is imagine the following: Alice prepares a system, but she is sloppy and she isn't too sure the if system will be in the state $|\psi_1\rangle_A$ like she intended, because there's also a chance that it could be in some other state $|\psi_2\rangle_A$, due to an error in the preparation. How can she describe this "mixing", if she weighs in the fact that there's a $\frac{2}{3}$ probability

that the system is in $|\psi_1\rangle_A$, and $\frac{1}{3}$ in $|\psi_2\rangle_A$? Imagine she would try to describe the system as,

$$\frac{2}{3}|\psi_1\rangle_A + \frac{1}{3}|\psi_2\rangle_A$$

Well, this doesn't cut it at all. A sum of this kind will produce just another vector representing a different state based on the principle of quantum superposition, and not a combination of the two possible states in which the system could be. We need to find another approach. Luckily, there is a kind of operator which "encodes" classical probabilities on the different states in which the system could be. Such operators are called *density operators* and they are given by the external product of the state vector with it's dual,

$$|\psi_1\rangle_A \rightarrow {}_A|\psi_1\rangle\langle\psi_1|_A \,; |\psi_2\rangle_A \rightarrow {}_A|\psi_2\rangle\langle\psi_2|_A\,; \tag{2.37}$$

Thus the general density operator (ρ) of the mixed system is given by,

$$\rho = \frac{2}{3}\,{}_A|\psi_1\rangle\langle\psi_1|_A + \frac{1}{3}\,{}_A|\psi_2\rangle\langle\psi_2|_A \tag{2.38}$$

This seems adequate. It says the system is in either state $|\psi_1\rangle_A$ with probability $\frac{2}{3}$ or state $|\psi_2\rangle_A$ with probability $\frac{1}{3}$, but now the states are described by their density operators instead of vectors so they don't add up. We can generalize this for any distribution of probability for each state and for any number of states,

$$\rho = \sum_i P_i |\psi_i\rangle\langle\psi_i|\,, \tag{2.39}$$

where P_i is the classical probability of the system being in the $|\psi_i\rangle$ state, which should add up to unity summing over all the states. Since we have a new way of writing states we could define the condition of separability (2.4) in terms of these operators as well. A state is separable if its density operator in the composite space (ρ_{AB}) can be written like,

$$\rho_{AB} = \rho_A \otimes \rho_B. \tag{2.40}$$

Why is it useful to have this depiction of states in the context of entanglement ? It seems this is a good way to describe systems whose complete information is lacking, and we are already well assured that lack of information is very important in entanglement, as we'll see "the forced projection" of an entangled state into the lower dimensions of the subsystem is represented by a mixed state and as such must be described by a density operator.

What is this "forced projection" we've been talking about ? Imagine, we have the state,

$$\left| E_0^- \right\rangle = \frac{1}{\sqrt{2}} \big(\left| u\, d \right\rangle - \left| d\, u \right\rangle \big), \tag{2.41}$$

and we want to describe such a state as it is seen by Alice or Bob, we know the state doesn't factor out like a product state, so we must forcefully rid ourselves from the sub-subsystem we don't want. What we do is to trace out the system that we want to make disappear, similarly as when one integrates over a variable of some function and the function ceases to have that explicit dependence. The process is as follows, first we calculate the density operator of the entangled state, which in this case is,

$$\rho_{AB} := \left| E_0^- \right\rangle \left\langle E_0^- \right| = \frac{1}{2} \big(\left| u\, d \right\rangle \left\langle u\, d \right| - \left| u\, d \right\rangle \left\langle d\, u \right| - \left| d\, u \right\rangle \left\langle u\, d \right| + \left| d\, u \right\rangle \left\langle d\, u \right| \big); \tag{2.42}$$

Now we create the tracing out operator, which is similar to a normal trace but only acts on one of the bases of the composite system, just like the operators defined in the subsystems (e.g σ_z for Alice and τ_z for Bob). Let's say we create the operator such that it traces over Bob's space, then whatever we're left with will be something that only exists in Alice space. The operation gives,

$$Tr_B(\rho_{AB}) = \sum_n {}_B \left\langle n \right| \rho_{AB} \left| n \right\rangle_B = \frac{1}{2} \big({}_A \left| u \right\rangle \left\langle u \right|_A + {}_A \left| d \right\rangle \left\langle d \right|_A \big), \tag{2.43}$$

the crossed terms vanish due to the orthogonality conditions. This obviously looks just like (2.38), and has the form of a mixed state. This is usually called the reduced density operator, but we'll just call it Alice's density operator. So we have $Tr_B(\rho_{AB}) = \rho_A$, and not surprisingly, $Tr_A(\rho_{AB}) = \rho_B$, and this is the way we can represent some entangled state by means of states of the sub-system alone.

At this point we're going to present some properties of density operators, which are meaningful to use later on. A density operator ρ defined for a pure state $\left| \psi \right\rangle$ is,

$$\rho = \left| \psi \right\rangle \left\langle \psi \right|,$$

as we've stated. It can be easily seen that $\rho^2 = \rho$, for a pure state. For a mixed state this is not true, we have instead $\rho^2 < \rho$. The condition of normalization,

$$\left\langle \psi | \psi \right\rangle = 1 \ \text{ becomes, } \ Tr(\rho) = 1,$$

as a direct implication of $\rho^2 = \rho$ we also get that $Tr(\rho^2) = 1$ for a pure state. From $\rho^2 < \rho$ we get $Tr(\rho^2) < 1$ for a mixed state. It can be shown that, if ρ_{AB} is the density operator of the entangled state $\left| E_0^- \right\rangle$, defined as $\rho_{AB} = \left| E_0^- \right\rangle \left\langle E_0^- \right|$ then,

$$\langle \mathbf{A} \rangle_{\text{av}} = \left\langle E_0^- \right| \hat{\mathcal{A}} \otimes I_B \left| E_0^- \right\rangle = Tr\left(\rho_{AB}(\hat{\mathcal{A}} \otimes I_B) \right) = Tr(\rho_A \hat{\mathcal{A}}), \qquad (2.44)$$

which is a pretty useful formula for finding the average values of observables. Obvious the same is also true for Bob,

$$\langle \mathbf{B} \rangle_{\text{av}} = \left\langle E_0^- \right| I_A \otimes \hat{\mathcal{B}} \left| E_0^- \right\rangle = Tr\left(\rho_{AB}(I_A \otimes \hat{\mathcal{B}}) \right) = Tr(\rho_B \hat{\mathcal{B}}), \qquad (2.45)$$

This process of tracing out is actually a clever way to know if some state which *a priori* we aren't sure about, is in fact entangled or not. It might seem cumbersome at this point that such a procedure is needed, because all the states we have showed and worked with were simple enough to check immediately if they are entangled or not. But in reality, when they become more complex either by increasing the number of dimensions and/or sub-systems of the composite system, figuring out if a state is entangled becomes something hard to do, impossible , actually, without a more formal mathematical way to do so. The process is quite straightforward, conceptually speaking, you do this operation and if the density operator (ρ) that comes out is one which corresponds to a pure state ($\rho^2 = \rho$) then it means the state in the composite space was a *product state*, and if it correspond to a mixed state ($\rho^2 < \rho$) it means the state was an *entangled state*.

It's easy to see why, imagine you have a state $\left| \psi \right\rangle_{AB}$ in a bi-partite space and you don't know if it's entangled or not, but you know if it is separable there must be some decomposition like $\left| \psi_1 \right\rangle_A \otimes \left| \psi_2 \right\rangle_B$, you just can't seem to find the states, but you know they exist. If you construct the operator of such a state, this gives,

$$\rho_{AB} = \rho_{1_A} \otimes \rho_{2_B}, \qquad (2.46)$$

which is just the separability condition in terms of density operators, again you don't know how this decomposition takes place. But surely, if you trace out ρ_{AB},

$$Tr_B(\rho_{AB}) = \sum_n \rho_{1_A} \left\langle n \right| \rho_{2_B} \left| n \right\rangle \qquad (2.47)$$

if the operators are normalized we get ρ_{1_A}, which is defined as $_A \left| \psi_1 \right\rangle \left\langle \psi_1 \right|_A$, thus, is a pure state and we can check this by the properties of the pure state

operators. If ρ_{1_A} is mixed, it means the operator couldn't be written like
(2.46). The generalization of this idea, of testing to see if a certain state
is entangled or not will be expanded in the next chapter where we present
different ways to "measure" entanglement. Such a criterion is very important
because it gives us tools to test the existence of entanglement and even to
calculate to what degree the states exhibit this behaviour. We until this point
have always picked states with the appropriate coefficients so that they are
maximally entangled to show the most drastic examples of this behaviour,
but it is not true that for a state to be considered an entangled state that
it has to be "maximally entangled". For instance if we write the state $\left|E_0^-\right\rangle$
with coefficients, $\alpha_u \beta_d = \sqrt{\frac{2}{3}}, \alpha_d \beta_u = \sqrt{\frac{1}{3}}$

$$\left|E_0^-\right\rangle = \sqrt{\frac{2}{3}}\left|u\,d\right\rangle - \sqrt{\frac{1}{3}}\left|d\,u\right\rangle ; \tag{2.48}$$

it will also be an entangled state, but in this instance the observables (σ_z, τ_z)
won't have a perfect correlation, we can calculate it to be, $C(\sigma_z, \tau_z) = -\frac{8}{9}$.
The knowledge of the subsystems observables is not completely lacking as
well, from (2.16) we find $\langle\sigma_x\rangle_{\mathrm{av}} = \langle\tau_x\rangle_{\mathrm{av}} = \langle\sigma_y\rangle_{\mathrm{av}} = \langle\tau_y\rangle_{\mathrm{av}} = 0$, but $\langle\sigma_z\rangle_{\mathrm{av}} = \frac{1}{3}$
and $\langle\tau_z\rangle_{\mathrm{av}} = -\frac{1}{3}$. If we sum the squares of the expected values to check the
"Spin Conservation Principle" we see that it gives for both Alice and Bob $\frac{1}{9}$,
so it isn't 1 but it also isn't 0.

As we can see states can be "more" or "less" entangled, finding methods
to find which states are which will be important.

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Chapter 3

A deeper look at Entanglement

3.1 Entanglement Monotones

Entanglement monotone is the general name used to refer to a specific criterion used in either testing or measuring entanglement. The name derives from the fact that such mathematical functions are constructed to remain unchanged (monotonous) under operations which are understood to preserve the entanglement of a system.

There are quite a fair amount of different monotones, but they could be separated in two groups: The ones which are qualitative and behave like an on/off type of functions, testing if there is or there isn't entanglement, but offering no way to differentiate between the degree of entanglement; And the ones which retrieve a number and measure the entanglement in the system by quantifying "how much" entangled the state is. We have already encountered in the previous chapter examples of these two kinds, applicable to isolated systems (explained in the footnotes). The qualitative one was the process of checking if the reduced density operator was a pure state or not, if it was mixed we knew that the state in composite system was entangled, but we didn't know how much it was[1]. The other one was the function of correlations between any two observables $C(\mathbf{A}, \mathbf{B})$, which behaves like a monotone for entangled states if the composite systems aren't interacting[2],

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[1] Of course this is only true if there isn't an environment causing "noise" in one of the subsystems, which could "mix" the pure state. In such a situation we wouldn't be able to tell if the state in the composite system was entangled or not.

[2] This is because the correlation function measures all kinds of correlations, either classical (local) or quantum (non-local), but if the systems aren't interacting then we can

as we saw

$$C(\sigma_z, \tau_z) = -1 \text{ when the state is: } \frac{1}{\sqrt{2}} \left| u\, d \right\rangle - \frac{1}{\sqrt{2}} \left| d\, u \right\rangle;$$

and,

$$C(\sigma_z, \tau_z) = -8/9 \text{ when the state is: } \sqrt{\frac{2}{3}} \left| u\, d \right\rangle - \sqrt{\frac{1}{3}} \left| d\, u \right\rangle;$$

This means the last state was less entangled than the previous one, which was actually perfectly entangled.

Obviously quantifiable monotones provide us with more information, but it's worth mentioning that even from the perspective of a quantifiable monotone that it doesn't matter in what state a system might be if the state is entangled to the same degree as some other state; e.g. $\left| E_0^- \right\rangle$ and $\left| E_0^+ \right\rangle$ are different states but both are maximally entangled for $(\alpha_u \beta_d = \alpha_d \beta_u = \frac{1}{\sqrt{2}})$, then, according to any monotone these states are indistinguishable. If we used a monotone which could quantify the degree of entanglement it would accuse the same value for both states, in this case the maximum value it could register for that specific system, and the other kind would just say that it is indeed entangled.

We should also mention that, depending how certain monotones may be constructed, they don't necessarily have to agree with one another 100% on how they order the states in terms of the degree of entanglement, they should agree on the condition which gives the maximum entanglement and the condition of no entanglement (separability) though, if they are to be consistent with one another [3]. Thus it's important that if we want to compare measurements of entanglement, that these should be done with the same monotone, or monotones which we know have a 1 to 1 correspondence.

The research field of these criteria is still very much pertinent nowadays, and we're still a long way from finding a fail-proof general monotone which can be applied to any sort of system without failing. There is an abundance of monotones which behave appropriately when dealing with systems of low

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safely assume that local correlation aren't an issue.

[3]But in fact even this isn't guaranteed. For big enough dimensions some measurements don't even agree on what is or isn't entangled, creating special kinds of entanglement only detected for specific monotones (like PPT entanglement (ref.)), but this doesn't concerns us though because we are not going to be in that regime.

complexity (i.e low dimensions and few subsystems like $2\otimes2$, $2\otimes3$, or $2\otimes2\otimes2$) but have issues with higher number of degrees of freedom which have greater complexity, [19]. They either cease to be sufficient and only become necessary conditions, (i.e They can prove that there is entanglement but they can't prove that there isn't) or could stop working altogether. On the other hand, the ones which are mathematically sound in a more broad range (usually of the qualitative kind) are virtually useless since they're stated in a very abstruse mathematical manner and aren't applicable.

We choose to present a "standard" quantifiable monotone which is the one we're going to use afterwards in the scope of this work, known as the Von Neumann Entropy of Entanglement.

The Von Neumann Entropy: Entropy of Entanglement

The *Entropy of entanglement*[4] (S_E), is one of most well known monotones and a "go to" measurement due to it's characteristics: A fairly easy applicability, conceptual similarities with the regular thermodynamic entropy, and being the quantum analog of the classical Shannon information entropy. (ref.) Also it's specially easy to compute for bi-partite systems, although one still has to be careful in this scenario if the number of dimensions of the subsystems are too big because, as we will see, the entropy diverges logarithmically with the number of dimensions of a system. The Von Neumann entropy is generally defined as follows,

$$S_V = -Tr(\rho \log \rho). \tag{3.1}$$

Then, for a given composite bi-partite system whose density operator is ρ_{AB}, an unequivocal measure of entanglement of this system is given by

$$S_E = -Tr(\rho_A \log \rho_A), \tag{3.2}$$

where ρ_A stands for the reduced density operator defined earlier as $Tr_B(\rho_{AB})$, and log is the natural base logarithm. This is called the entropy of Entanglement.

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[4]Although there are other kinds of monotones with "Entropy" in the name, like the " Relative entropy of entanglement", or "Renyi entropy of entanglement" etc. which could rightfully also be denoted as "entropy of entanglement", in the context of this work since no other monotone is going to be used, it's implicit that when we say *Entropy of entanglement* that we are talking about Von Neumann's Entropy of Entanglement.

The entropy is defined in terms of Alice's reduced density operator, but as is usual we could have chosen to have it defined with Bob's reduced density operator. The previous expression of the entropy can still be written in a more low-level form, which is the one typically used when doing actual calculations. One knows from linear algebra that, if some given matrix is diagonal then the diagonal entries are the eigenvalues of the said matrix. As long as rho_A is a diagonalizable square matrix, it can be expressed in terms of its eigenvectors by,

$$\rho_A = \sum_i A_i \left| a_i \right\rangle \left\langle a_i \right|, \tag{3.3}$$

where A_i is the eigenvalue in the ith entry of the matrix, interpreted as the probability for Alice's subsystem to be in $\left| a_i \right\rangle \left\langle a_i \right|$, as can be seen directly if we compare (3.3) to (2.39).

Since the log of a diagonal matrix is equal to the matrix of the log of the elements, we can easily see that (3.2) becomes,

$$S_E = - \sum_i (A_i \log A_i); \tag{3.4}$$

If ρ_A is diagonal, A_i is the ith entry of the matrix. But it doesn't need to be, because if we know that it's diagonalizable we only need it to be an eigenvalue of the matrix.

There are some properties which can be stated immediately about the entropy. Like all measures of entanglement this monotone should be zero when the system is in a state which is separable, and have its maximum defined when the state is maximally entangled. The entropy of entanglement is a quantifiable extension of the qualitative measure we mentioned before in the previous chapter, when we checked to see if the reduced density operator was mixed or pure, but now we can compute the degree of "mixedness" with this measure, which equates to the degree of entanglement of the composite system.

As argued in last section of Chapter2, if ρ_A is pure than the state is not entangled, which should mean that $S_E = 0$. Let's see if (3.4) respects this: Looking at (3.3), we know that if the operator is to be a pure state it needs to take the form,

$$\rho_A = A_i \left| a_i \right\rangle \left\langle a_i \right|,$$

where $\left| a_i \right\rangle$ can be any eigenvector. In matrix representation, all the entries are null except the ith entry which is the eigenvalue A_i. If the operator

is normalized then from the condition $Tr(\rho_A) = 1$ comes out directly that, $A_i = 1$. Then (3.4) takes the form,

$$S_E = -1 \log 1 - (n-1) \cdot \lim_{x \to 0}(x \log x) \tag{3.5}$$

The first term obviously is zero. The second term, where n is the total number of entries of the matrix, which are all null except for one, is also zero since the limit converges to zero. Then, like we expected $S_E = 0$ when the reduced density operator is a pure state, which means the composite system is separable.

Now we proceed to obtain the expression for the maximum of the entropy of entanglement.

As we saw previously, a maximally entangled state (Bell state) in a composite system of 2×2 dimensions is given by the collection of states in (2.25) for $(\alpha_u \beta_d = \alpha_d \beta_u = \frac{1}{\sqrt{2}})$. We can generalize the collection of states as,

$$\frac{1}{\sqrt{2}}\big(\,|1\rangle + |2\rangle\,\big); \tag{3.6}$$

(we're only considering the symmetric states, to get the antisymmetric case we just have to substitute the plus with a minus sign). Here $|1\rangle$ and $|2\rangle$ are any two base states which cannot be factorized simultaneously. One could guess that a maximally entangled state in a system of 3×3 dimensions should be given by,

$$\frac{1}{\sqrt{3}}\big(\,|1\rangle\, + |2\rangle + |3\rangle\,\big); \tag{3.7}$$

and this is happens to be true. In fact for a given general composite bi-partite system, of $N_{AB} = N_A \times N_B$ dimensions, a general Bell state is given by [20],

$$|B\rangle = \left(\frac{1}{\sqrt{N_{AB}}}\right)^{\frac{1}{2}} \sum_{i=1}^{i=\sqrt{N_{AB}}} |i\rangle\,; \tag{3.8}$$

If we are in the special case where $N_A = N_B$ we could just could call it N, and $N_{AB} = N^2$, which sets,

$$|B\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{i=N} |i\rangle\,; \tag{3.9}$$

The corresponding operator is given by,

$$|B\rangle\langle B| = \frac{1}{N} \sum_{i=1}^{i=N} |i\rangle \sum_{j=1}^{j=N} \langle j| \, ; \tag{3.10}$$

which can be written as,

$$|B\rangle\langle B| = \frac{1}{N} \sum_{i=1}^{i=N} |i\rangle\langle i| + \text{(crossed terms)}; \tag{3.11}$$

If at this point we apply the tracing operator over Bob's space, we get that Alice's reduced density operator is,

$$\rho_A = \frac{1}{N} \sum_{i=1}^{i=N} |a_i\rangle\langle a_i| \, ; \tag{3.12}$$

As we expected the crossed terms vanish when we calculate the trace. The state $|a_i\rangle$ corresponds to Alice's eigenvector, which was contained in the eigenvector $|i\rangle$ of the composite space. As we can see such operator is a $N \times N$ diagonal matrix, whose entries are $\frac{1}{N}$ for all N diagonal elements. If we apply (3.4) we get that,

$$S_E = -N \times \left(\frac{1}{N} \log \frac{1}{N}\right) = \log N; \tag{3.13}$$

This is the maximum value of the entropy for a given maximally entangled system. Which shows that for a system of the kind $n \otimes n$ when $n \to \infty$, S_E diverges.

Usually the entropy is redefined, such that for a closed quantum system of dimension N its maximum value should be 1 [5] for a maximally entangled state, (i.e it has the same upper and lower bound as the function of correlations). This is easily obtained by taking the N based logarithm. Such that $\log_N N = 1$. Although if we are going to compare results of the entropy where the system isn't closed for all degrees of freedom we should stick with the "universal" natural based logarithm.

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[5]Actually, it's the absolute value of the entropy that should have a maximum value of 1, because the entropy could be negative depending on the convention).

Let's see if this checks out for a simple example. Choosing the already familiar $\left| E_0^- \right\rangle$ state with, $(\alpha_u \beta_d = \alpha_d \beta_u = \frac{1}{\sqrt{2}})$,

$$\left| E_0^- \right\rangle = \frac{1}{\sqrt{2}} \big(\left| u\,d \right\rangle - \left| d\,u \right\rangle \big), \tag{3.14}$$

where it's reduced density operator was already calculated in the previous chapter as,

$$\rho_A = \frac{1}{2} \big({}_A \left| u \right\rangle \left\langle u \right|_A + {}_A \left| d \right\rangle \left\langle d \right|_A \big)$$

This in matrix form is given by,

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}; \tag{3.15}$$

Then evaluating the entropy in a 2 based logarithm we have the expression,

$$S_E = -\frac{1}{2} \log_2 \frac{1}{2} - \frac{1}{2} \log_2 \frac{1}{2}$$

$$= \frac{1}{2} \log_2 2 + \frac{1}{2} \log_2 2 = 1,$$

as it should. The value registered for this monotone is the same value as the correlation between two given observables measured in a system which is maximally entangled, actually the correlation was negative but we could have defined it the other way around, so it doesn't make any difference. But don't get yourself fooled, this doesn't mean that for any other state that the two measures will produce the same value, in fact if we calculate the entropy for the state,

$$\left| E_0^- \right\rangle = \sqrt{\frac{2}{3}} \left| u\,d \right\rangle - \sqrt{\frac{1}{3}} \left| d\,u \right\rangle, \tag{3.16}$$

we get $S_E \sim 0.92$ which is not the same as the value for the correlation which we calculated in the last chapter to be $\frac{8}{9}$, in module.

Actually we can calculate a simple plot of the general function of entropy for a 2 dimensional system in terms of it's eigenvalues (probability). We know that every reduced operator written in terms of it's eigenvectors is going to be of the form,

$$\rho_A = P_u \left| u \right\rangle \left\langle u \right| + (1 - P_u) \left| d \right\rangle \left\langle d \right|,$$

if the operator is to be normalized, because the eigenvalues represent a probability, so they must sum to unity. This implies then that the general form for the entropy is given by,

$$S_E = P_u \log_2 P_u + (1 - P_u) \log_2 (1 - P_u)$$
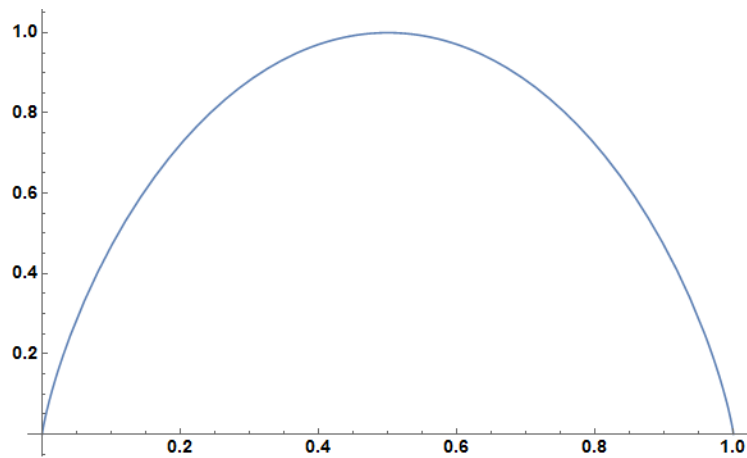
If we plot the function $S_E(P_u)$ we have,

Figure 3.1: $S_E(P_u)$
Entropy as a function of the probability P_u; P_d is given by $(1\text{-}P_u)$

As we can see the maximum entanglement is when $P_u = P_d = 0.5$ like we calculated, and the distribution is symmetric because, when we go to the right of the middle point, the values of P_d are equal to the value P_u to the left because $P_d = (1 - P_u)$.

In the case of the other state $P_u = \frac{2}{3}$, which if you look at the plot it seems to agree with the value we calculated, $S_E \sim 0.92$. Everything check's out.

3.2 Entanglement and Decoherence

Thermodynamic Entropy Vs. Entropy of Entanglement

In the beginning of this chapter we stated that the thermodynamic entropy is closely related to the entropy of entanglement without any explanation. Here will see what's similar between them and perhaps more importantly what is different.

If we consider a classical system in the context of statistical mechanics, we have a collection of states [6] which are also called micro-states, in the sense that they could have various "micro-configurations" capable of reproducing the same thermodynamical observable of a "macro-state", like the average energy of a gas. A given micro-state i of probability p_i is associated to a given energy E_i. The Gibbs Thermodynamic Entropy of such a system is defined as(natural units),

$$S_G = -\sum_i p_i \log p_i \qquad (3.17)$$

Well, this has exactly the same form as (3.4). Then, it's easy to understand why the function (3.4), and consequently (3.2) are also called "entropy". But in what way are they different ? The differences arise from the distinct mechanisms associated with the calculation of the probability in either case. In the thermodynamic entropy, due the high number of states inhabiting the system, the values of p_i are associated with a distribution of probabilities (Boltzmann's for classical examples), such that the probability for a given micro-state i is defined to be,

$$p_i \propto e^{-E_i/T};$$

Where T is the thermodynamic temperature of the "macro-state".

The probability P_i from (3.4) should in principle respect the same properties as $p_i \propto e^{-E_i/T}$ from (3.17), but we don't have an enough number of states to apply any sort of statistical treatment in the quantum example because we can't define the necessary thermodynamical variables (i.e temperature) to compute the probability in this way. We have to find a different mathematical formalism altogether, for this case. This formalism gets it's

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[6]These states are not the same states of quantum mechanics since they don't represent vectors of any sort but only an element of a given set, just like the coin example.

bearings in the vector like nature of quantum states and is of course the definition of a density operator for a given quantum state, whose eigenvalues are the probabilities to be in some specific base state. So you see, although the mathematical function is the same, the context in which we find ourselves and the restrictions imposed by the nature of the systems we're trying to describe provides a very different approach to the same expression.

In summary: When ρ in (3.1) represents a microcanonical or canonical ensemble, the von Neumann entropy gives the thermodynamic entropy. When it represents the reduced density matrix describing a subsystem A of the composite system AB, the von Neumann entropy quantifies the entanglement between A and B, and is called the entropy of entanglement (3.2) [21].

Coherent states and Decoherence

There is another important aspect that we should address. Until this point we have always been in the idealised framework of "noiseless quantum theory"[20].

The "noise" is in respect to the local interactions that the environment has with the entangled system. Such local interactions are detrimental to entanglement because it causes *decoherence* on the quantum states. Until this point we have never mentioned this said "decoherence"; Because we have made throughout the implicit idealization that the quantum systems only interact to become entangled, and then they go on their way without interacting ever again, either with themselves or with anything else, so they are "noiseless".

Although decoherence doesn't play a huge part in the scope of this work, it is something worth mentioning in order to better understand the nature of our results further on. We aren't going explore this concept in much detail, we only add a brief description of the way it works, why it is detrimental to quantum entanglement and, as such, why it's existence provides an intrinsic difference to the behaviour of the entropy of entanglement as compared to thermodynamic entropy.

"Decoherence" is just a coined term that translates to the phenomenon of coherence loss. Then, to better understand what is decoherence in the context of quantum mechanics, one has to know what properties characterize a coherent quantum state, and see how such a state ceases to have those properties.

Suppose we have a state $|\psi\rangle$. This state is said to be coherent if, for some amount of time δt, we let the it evolve and it always has the same "interference pattern" for all times in δt. This occurs if and oly if, the relative phase between the base states remain the same throughout all times. Thus, it's a direct implication that any stationary state is a coherent state.[ref]

The process of decoherence is the loss of such an interference pattern by offsetting the relative phase of wave-functions between themselves. This destroys the character of quantum superposition, and consequently of entanglement.

Like entanglement the process of decoherence can be gradual, it doesn't have to be an all or nothing proposition. This is explained due to the fact that the process of decoherence can be thought as the "unreachable" information that gets entangled with the environment [22].

Let's expand on this; Although entanglement is essentially non-local it emerges from the the inner works that occur when things interact locally on the quantum level[7], so in the same sense that Alice's and Bob's electrons might become entangled between themselves if they interact, so can they become entangled with the environment by interacting with it. What is the difference ? One can book-keep (more or less) the entanglement between various system components if the degrees of freedom are limited and as such, write down a quantum state accordingly and compute the entanglement of the system. This isn't possible in the infinite amount of the degrees of freedom that the surrounding environment has, where the information gets "scrambled" exponentially fast [23] and cannot be used.

The "environment" is a relative thing, as far as entanglement goes everything which can't be mathematically included in the general description of the pure entangled state is "environment".

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[7]Local doesn't mean that it has to be exclusively with one another. For instance, the phenomenon of superconductivity which is explained by the BSC theory uses the concept of a "Cooper pair" which is basically an entangled pair of electrons, such a pair doesn't become entangled by direct interaction of the electrons but rather by a mediated interaction of the lattice phonon's. But it's also local nonetheless [24].

Imagine we have a pair of spin-$\frac{1}{2}$ particles, A and B, which are maximally entangled. Sometimes A and B are also called qubit's (it derives from quantum-bit and it's a general term used to refer to any 2D quantum system), and the maximally entangled pair of qubit is is called and "ebit" (from entangled bit). Suppose that ebit unbeknownst to us interacts very briefly with another spin-$\frac{1}{2}$ particle (qubit), and that particle becomes entangled with the e-bit (this in reality creates a triple of entanglement between the 3 particles)but the way entanglement is shared is not completely at random, there are inequalities that must be fulfilled; For instance, "the principle of entanglement's monogamy" which is stated as: "If two qubits A and B are maximally quantum correlated they cannot be correlated at all with a third qubit C" [25]; which is to say the ebit cannot become entangled with another particle and still continue to be an ebit. This entails that if we go measure again the entangled pair that the entropy of entanglement won't be a maximum, so the particle that interacted caused decoherence by becoming entangled with the pair and disappearing.

After the interaction the "complete quantum system" of spins, couldn't be described in a $2 \otimes 2$ but only in $2 \otimes 2 \otimes 2$ Hilbert space. In this sense the role of the "environment" was played by a single particle.

Chapter 4

Entanglement in Quantum Field Theory

4.1 Brief introduction

The framework in which we aim to study entanglement in this thesis is Quantum Field Theory (QFT), and this chapter serves to introduce the formalism needed in this context. We aren't going to introduce the subject in such a self contained manner as we did previously with quantum entanglement, for various reasons, mainly that, this subject is more widely known and is easier to find didactic and comprehensive literature which relates well to the scope of this work. We recommend further reading for those which might be unfamiliar with the theme, like Peskin and Schroeder [26] which provides an introductory and broad overview of the subject. A deeper analysis on ϕ^4 theories is found in [27], which was heavily used for revision throughout this thesis.

QFT provides us with a fruitful formalism when dealing with the field–particle correspondence on the quantum level. This is made possible by quantizing the fields which are associated with the particles they aim to represent, thus we are able to treat a particle or collection of particles as discrete excitations of their respective fields. There is a particular useful tool in QFT, which we are very interested in, the $\hat{S}$ matrix formalism used to study scattering phenomena between particles. We're going to focus mainly on this formalism for obvious reasons, since our goal in this work is to study entanglement of particles submitted to a certain type of collisions.

4.1.1 Laying down the foundations

Suppose we have two general fields ϕ_A and ϕ_B, which are the fields used to represent the particles associated with Alice's and Bob's subsystems, respectively (the blue and red particles in the previous chapter's example).

ϕ_A and ϕ_B don't have to be equal since they don't need to represent the same kind of particles. But let's assume they do represent the same type of particles, which are only distinguished for the fact that they exist in different subsystems with different observers studying their behaviour, like we did in Chapter2.

How can Alice (or Bob) use this abstract field notion to represent the particles in their subsystems? Burrowing from the mathematics of ladder operators in regular quantum mechanics, they can define a creation $(c_{\vec{k}}^{\dagger})$ and annihilation operator $(c_{\vec{l}})$ for their fields, with the commutation relationship

$$[c_{\vec{l}}, c_{\vec{k}}^{\dagger}] = 2E_{\vec{k}}(2\pi)^3\delta^{(3)}(\vec{k} - \vec{l}). \tag{4.1}$$

Thus, we can define mathematically a given state of an n number of particles with momenta $\vec{k}$ as,

$$\left|n_{\vec{k}}\right\rangle = (c_{\vec{k}}^{\dagger})^n \left|0\right\rangle$$

The operator $(c_{\vec{k}}^{\dagger})$ is applied an n number of times, and for each time it raises from the vacuum-state a particle of momentum $\vec{k}$.

But we are specially interested in the case when both Alice and Bob only have a particle each in their respective subsystem, a $1 \oplus 1$ Fock space as it's usually called. Then from the previous expression we for $n = 1$,

$$\left|1_{\vec{k}}\right\rangle = (c_{\vec{k}}^{\dagger}) \left|0\right\rangle ;$$

When the state that gives us the occupation number for a given momentum $\vec{k}$ is only occupied by one particle, then that single excitation mode can be interpreted as representing the quantum state of that specific momentum $\left(\text{i.e.} \left|1_{\vec{k}}\right\rangle = |\vec{k}\,\rangle\right)$. So, we define a general state of momentum in Alice's subspace as,

$$\left|\vec{p_1}\right\rangle_A = \sqrt{E_{\vec{p_1}}} c_{\vec{p_1}}^{\dagger} \left|0\right\rangle_A ;\, [1]. \tag{4.2}$$

---

[1]The energy factor is only introduced in the definition of the state to be in accordance with Lorentz invariant factor in the definition of the field of (4.3)

Where $c_{\vec{p_1}}^{\dagger}$ is the creation operator for mode $\vec{p_1}$, defined by means of Alice's field which we introduce at this point as a spinless (scalar) field[2]

$$\hat{\phi}_A(x) = \int \frac{d^3 \vec{p_1}}{(2\pi)^3} \frac{1}{2E_{\vec{p_1}}} \left(c_{\vec{p_1}} e^{-i\vec{p_1}\cdot \vec{x}} + c_{\vec{p_1}}^{\dagger} e^{i\vec{p_1}\cdot \vec{x}} \right); \qquad (4.3)$$

We get the same for Bob's description, if we set $A \to B$ and $\vec{p_1} \to \vec{p_2}$. The state is,

$$|\vec{p_2}\rangle_B = \sqrt{E_{\vec{p_2}}} c_{\vec{p_2}}^{\dagger} |0\rangle_B; \qquad (4.4)$$

With the field,

$$\hat{\phi}_B(x) = \int \frac{d^3 \vec{p_2}}{(2\pi)^3} \frac{1}{2E_{\vec{p_2}}} \left(c_{\vec{p_2}} e^{-i\vec{p_2}\cdot \vec{x}} + c_{\vec{p_2}}^{\dagger} e^{i\vec{p_2}\cdot \vec{x}} \right); \qquad (4.5)$$

The overlap (inner product) between any two given states (defined in the previous manner) can be calculated using (4.1) to be,

$$\langle \vec{q} \, | \vec{p} \, \rangle = 2E_{\vec{p}}(2\pi)^3 \delta^{(3)}(\vec{q} - \vec{p}) \qquad (4.6)$$

4.1.2 The Scattering Procedure

The formalism of the $\hat{S}$ matrix is quite straightforward to understand in a conceptual manner. We have the two particles (one for Alice and one for Bob) "going in", the said particles are in some quantum state usually called the in-state, defined as $|\Psi\rangle_{in}$. After the interaction a certain number of particles "go out"[3] in the out-state defined as $|\Psi\rangle_{out}$, which is related to the in-state

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[2] At this point we diverge from the examples of exclusively entangled spin states which we introduced in the previous chapters. Now the particles that Alice and Bob are describing don't have spins, but there's no need for worrying, entanglement is something which can happen for all types of degrees of freedom associated with the descriptions of the quantum systems under study. In this case it will be momentum states which could become entangled.

[3]The "going in" and "going out" relates to the fact that the particles are arriving or leaving a given region of space-time where there is a specific event happening, in this case a collision. Different events(interactions) might have different needs as to how "close" they need to be in order for them to take place, and how "far" they need to be for the interaction to stop, this depends for instance on the couplings strengths on the interactions. Thus to avoid those difficulties, such states are defined as asymptotic states, which are defined infinitely far away in space and time before the interaction (in-state) and infinitely far away in space and time after the interaction (out-state), in this way whatever happens in the middle is nothing which we should concern ourselves with.

as such,

$$|\Psi\rangle_{out} = \hat{S}\,|\Psi\rangle_{in}\,.$$

We are specially interested in elastic scattering so we have an extra restriction to impose upon the character of the collision; The same type and number of particles that go in should come out, such that the out-state $|\Psi\rangle_{out}$ can also be described by the single modes of the Alice's and Bob's fields.
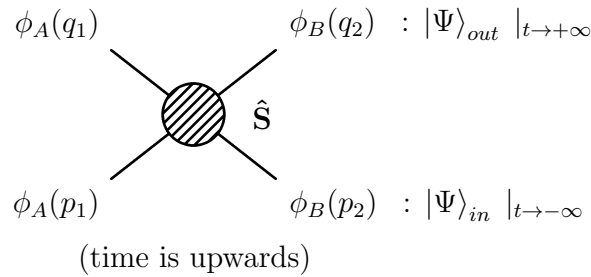
We can represent this diagrammatically by,

$$\phi_A(q_1) \qquad\qquad \phi_B(q_2)\ :\ |\Psi\rangle_{out}\ |_{t\to+\infty}$$

$$\hat{S}$$

$$\phi_A(p_1) \qquad\qquad \phi_B(p_2)\ :\ |\Psi\rangle_{in}\ |_{t\to-\infty}$$

(time is upwards)

Figure 4.1: Scattering diagram.

$\phi_A(p_1)$, $\phi_B(p_2)$ are the excitation modes of 4-momenta p_1 and p_2, for Alice's and Bob's fields, respectively, which correspond to their incoming particles described by the asymptotic state $|\Psi\rangle_{in}$. $\phi_A(q_1)$, $\phi_B(q_2)$, correspond to the modes of the fields which relate to the outgoing particles which are described by the asymptotic state $|\Psi\rangle_{out}$. The "blob" is supposed to represent a black box which encodes the ignorance at this point, about the inner workings of the collision.

We should comment why only the channel,

$$\phi_A\phi_B \to \phi_A\phi_B,$$

is being taken into account in the study of this collision. One might ask, why not collisions also of the form,

$$\phi_A\phi_A \to \phi_B\phi_B, \quad \text{or} \quad \phi_B\phi_B \to \phi_A\phi_A\ ?$$

Although these collisions are completely respectable in terms of elastic scattering they aren't pertinent to the study of entanglement, because entanglement is enabled by the "mixing" of the degrees of freedom in the different

subsystems, for that to happen they must interact. So we must have a collision of the type

$$\phi_A \phi_B \rightarrow \phi_A \phi_B,$$

because this is the one which corresponds to the interaction of Alice's and Bob's subsystem. We have already restricted our collision to some extent by specifying the characteristics of the fields/particles in the asymptotic states, but we are still completely unsure about the nature of the interactions that take place inside the "blob", (i.e the nature of $\hat{S}$).

Scattering matrix and the ϕ^4 interaction

The Scattering matrix ($\hat{S}$) is defined as,

$$\hat{S} = \mathbf{1} + i\hat{T}. \tag{4.7}$$

Where ($i\hat{T}$) is the transfer matrix,

$$\langle \, \vec{q_1} \, \vec{q_2} \, | \, i\hat{T} \, | \, \vec{p_1} \, \vec{p_2} \, \rangle = (2\pi)^4 \delta^{(4)}(p_1 + p_2 - q_1 - q_2) i\mathcal{M}_{(p_1 p_2 \rightarrow q_1 q_2)}, \tag{4.8}$$

$i\mathcal{M}$ is the part which represents the algebra of the collision's dynamics, but to calculate it we have to know what interactions are taking place inside the "blob" such that we could produce this collision with the desired asymptotic states.

For the kind of fields we're dealing with, there's usually a type of interaction called the ϕ^4 *interaction* (or *quartic interaction*), which conceptually pertains a number of interesting phenomena in scalar field theory. Thus, the model we are going to consider is the already well known complex ϕ^4 model, whose action is given by,

$$A = -\int d^4x \left(\frac{1}{2}\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi - \frac{\lambda}{4}(\phi^*\phi)^2 \right). \tag{4.9}$$

Where λ is the coupling constant of the field, which gives the strength of the interaction.

The choice of a complex model is appropriate because we have two real fields ϕ_A, ϕ_B, and we can define the complex field as, $\phi = \phi_A + i\phi_B$. In terms of Alice's and Bob's real fields the action (4.9) becomes,

$$A = -\int d^4x \frac{1}{2}\partial_\mu \phi_A \partial^\mu \phi_A + \frac{1}{2}\partial_\mu \phi_B \partial^\mu \phi_B - m^2(\phi_A^2 + \phi_B^2) - \frac{\lambda}{4}(\phi_A^2 + \phi_B^2)^2. \tag{4.10}$$

Where the interaction term is given by,

$$(\phi^*\phi)^2 = (\phi_A^2 + \phi_B^2)^2 = \phi_A^4 + \phi_B^4 + 2\phi_A^2\phi_B^2.$$

Any coupling of the previous form can occur between the fields inside the "blob", so to produce the collision we desire. But there are an infinite number of ways in which the fields could couple according to the said interaction. How do we proceed do calculate such a thing ?

Diagrammatic Expansion

There's an approach based on the fact that we can treat $\hat{S}$ as an infinite sum of terms around powers of the coupling constant λ , like,

$$\hat{S} = 1 + f^{(1)}\lambda + f^{(2)}\lambda^2 + (...) \tag{4.11}$$

Where $f^{(n)}$ is the complex function for all the possible contributions of λ^n interactions. If we compare the previous series with the definitions of $\hat{S}$ (4.7) and $i\hat{T}$ (4.8), we notice the series is of the form,

$$\langle\, \vec{q_1}\, \vec{q_2}\,|\, \hat{S}\,|\, \vec{p_1}\, \vec{p_2}\,\rangle = 1 + (2\pi)^4\delta^{(4)}(p_1+p_2-q_1-q_2)[i\mathcal{M}_{(\text{tree})}+i\mathcal{M}_{(\text{1-loop})}+\mathcal{O}(\lambda^3)] \tag{4.12}$$

Where we are are ignoring the terms of $\mathcal{O}(\lambda^3)$ and higher. Each of these terms can be represented diagrammatically by means of Feynman diagrams, and by using Feynman's rules we find what the $i\mathcal{M}$ amplitudes are. Bearing in mind the restrictions on the interactions we get the following diagrams: Tree-level term proportional to λ,

$\phi_A(q_1)$ $\phi_B(q_2)$

$\phi_A(p_1)$ $\phi_B(p_2)$

(time is upwards)
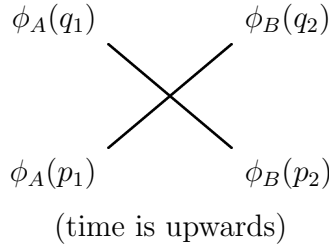
Figure 4.2: Tree-level contribution.

This is the simplest way the particles can interact. The diagram is proportional to λ since it only has one inner vertex.

The next term in the expansion correspond to all the possible[4] 1-loops diagrams which are proportional to λ^2. They are represented as,
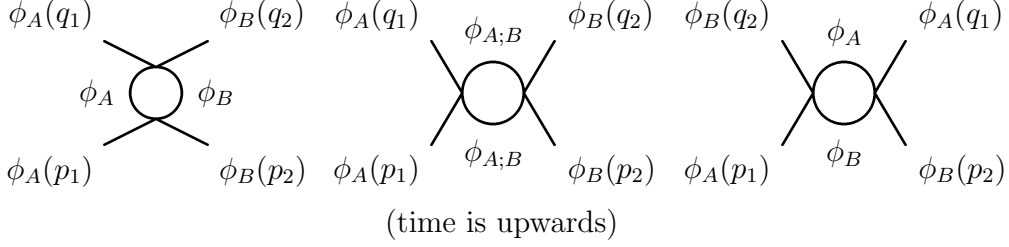

(time is upwards)

Figure 4.3: 1-loop contributions.
These are the only three topologically different 1-loop diagrams that respect the interactions terms. They are known as the s-channel, t-channel and u-channel, respectively. They are proportional to λ^2 since they have two inner vertices.

We will stop here because higher loop diagrams would be proportional to powers of λ which we are ignoring. We can apply the Feynman rules for ϕ^4 theory, and calculate $i\mathcal{M}$ for both cases (See Appendix);
In Tree-level we have,

$$i\mathcal{M}_{(\text{tree})} = -2i\lambda_R + \mathcal{O}(\lambda_R^2); \tag{4.13}$$

And in 1-loop,

$$i\mathcal{M}_{(\text{1-loop})} = -4i(\frac{\lambda_R}{4\pi})^2 \cdot (3G(t) + 2G(u) + G(s) + 2) + \mathcal{O}(\lambda_R^3), \tag{4.14}$$

The coupling has a subscript R to evidentiate that it's redefined by the process of renormalization. But from here on out, we'll only use λ. The arguments of the G function, u,t and s are the Mandelstam variables defined as,

$$t = (p_1 - q_1)^2, u = (p_1 - q_2)^2, s = (p_1 + p_2)^2$$

---

[4]There is actually another legitimate type of diagram, called the "tadpole" diagram which is a self-interacting diagram on the external legs. But these diagrams aren't going to affect the overall collision since they disappear in the process of renormalization.

The function is given by the expression,

$$G(x) = -2 + \sqrt{1 - \frac{4m^2}{x}} \cdot \log\left(\frac{\sqrt{1 - \frac{4m^2}{x}} + 1}{\sqrt{1 - \frac{4m^2}{x}} - 1}\right), \qquad (4.15)$$

$$\text{for } x \in \{s, t, u\}.$$

If we set,

$$\mathcal{F}(s, t, u) \equiv 4 \cdot (3G(t) + 2G(u) + G(s) + 2), \qquad (4.16)$$

then we have a more compact form of (4.14), we sum it to (4.13) and get,

$$i\mathcal{M}_{(\text{tree+1-loop})} = -2i\lambda - i\left(\frac{\lambda}{4\pi}\right)^2 \cdot \mathcal{F}(s, t, u) + \mathcal{O}(\lambda^3). \qquad (4.17)$$

Which gives that the expansion for the scattering matrix in (4.12) is now given by,

$$\hat{S} = 1 + (2\pi)^4 \delta^{(4)}(p_1 + p_2 - q_1 - q_2)[-2i\lambda - i\left(\frac{\lambda}{4\pi}\right)^2 \cdot \mathcal{F}(s, t, u)] \qquad (4.18)$$

So now, we have our scattering matrix and consequently our collision completely defined.

Comment on Unitarity

An important feature in QFT is the existence on the vacuum state. This state interacts in a dynamical manner with the existing particle states, which provides a volatile stage of action where particles can be absorbed and emitted by the vacuum in all sort of manners, this volatility creates possible problems in the preservation of the normalization of systems, a must if one is to give any physical interpretation to the results. Thus, in order to impose a legitimate physical interpretation on any phenomena in QFT, one should be careful as to see if the probability at a given value of energy is properly normalized at all times, since generally the number of particles aren't conserved.

For example, in regular quantum mechanics if we have a normalized and closed quantum system it will remain normalized throughout time, because

the time evolution operator is unitary (a property which preserves the normalization of the state), and since the state is also closed there will be no change in the particles content.

When dealing with scattering (in QFT), unitarity is also a preferred feature of the $\hat{S}$ matrix. Since we can't book-keep the intermediary "zoo" of particles during the collisions, we need to be assured that such interactions don't affect the end points so that the probabilities of the "in" and "out" states are consistent with each other, even if they have a different number of particles (i.e are normalized to all the possible outcomes that the collision could produce).

The condition of unitarity for a given matrix is that it's hermitean conjugate is also it's own inverse. Then if $\hat{S}$ is unitary $\hat{S}^{\dagger}\hat{S} = 1$ is true. Let's imagine then that $\hat{S}$ is indeed unitary, and we write the matrix as the series,

$$\hat{S} = 1 + f^{(1)}\lambda + f^{(2)}\lambda^2 + (...); \tag{4.19}$$

The fact is that although $\hat{S}$ is unitary, only the infinite sum of the terms amounts necessarily to an unitary operator. It is not generally true that if we stop at any give term of the expansion that the approximation needs to be unitary as well.

So we can't assume that the approximation (4.18) is unitary. How does this affect the results ? Fortunately, we are restricting ourselves to the specific scenario where the space in which the initial and final states exist is a (1+1)-particle Fock space, such that the particle content is bound to be the same before and after the collision. As long as we're in a regime of the interaction which respects our restriction, that no particles are created or absorbed in the asymptotic states and we normalize them to the same values of probability, we'll have an ad-hoc recovered "unitarity" on such states, which would legitimize the physical interpretation of the results [12].

There exist precise methods which we'll not pursue in this work, where we could evaluate for what values of λ the regime of "weak-coupling" is valid. Such methods are derived from the Optical Theorem [26]. A rough estimate for the allowed values of lambda is obtained in Chapter5.

The Road-map for the Calculation

We have already determined the nature of the scattering matrix and this point we should reflect on what we want to find, and how to calculate it, in order do devise a road-map of the steps we need to take to get there.

  We want to calculate the variation of the entanglement's entropy (ΔS_E) during an elastic scattering process described by the said $\hat{S}$ matrix.

  From the previous chapters one knows that, if we have a given state $|\Psi\rangle_{AB}$ we can find the entropy by calculating it's density operator ρ_{AB} and tracing it out with respect to one of the subsystems to get the reduced density operator (e.g ρ_A) to input it into the entropy formula (3.2). Thus we need to proceed in this fashion for both the "in-state" $(|\Psi\rangle_{in})$ and the "out-state" $|\Psi\rangle_{out}$), and subtract the values we get for the entropy to compute the entropy's variation (ΔS_E) caused due the scattering process. Then the entropy will be given by,

$$\Delta S_E = S_{E \ (out)} - S_{E \ (in)} \qquad (4.20)$$

  We synthesize the necessary steps of the process in an visual manner bellow,

$$|\Psi\rangle_{in} \rightarrow \rho_{AB}^{(in)} \rightarrow \rho_{A}^{(in)} \rightarrow S_{E \ (in)}$$

$$\hat{S} \ \Big\downarrow$$

$$|\Psi\rangle_{out} \rightarrow \rho_{AB}^{(out)} \rightarrow \rho_{A}^{(out)} \rightarrow S_{E \ (out)}$$
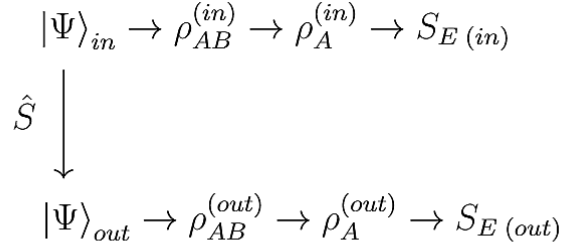
Figure 4.4: Road-Map for the Calculation.
The necessary steps that we have to take in order to get to the desired result of computing the variation of the entanglement's entropy (ΔS_E).

Mathematical Interlude: QFT considerations

We already performed these operations before, but there's some aspects in which this case differs from the examples of the previous chapters (like the one in (2.43)); We are now dealing in a continuous basis of momenta as opposed to a discrete basis of spins. This difference must be taken into account in our calculations. For instance, the tracing operator was previously given by,

$$Tr_B(\rho_{AB}) = \sum_n {}_B\langle n| \rho_{AB} |n\rangle_B . \tag{4.21}$$

If we adopt the usual convention in QFT of changing the summation into an integral over the infinite amount of degrees of freedom,

$$\sum_n \to \int_{-\infty}^{+\infty} \frac{d^3\vec{n}}{(2\pi)^3}, \tag{4.22}$$

we get,

$$\rho_A = Tr_B(\rho_{AB}) = \int_{-\infty}^{+\infty} \frac{d^3\vec{n}}{(2\pi)^3} \frac{{}_B\langle \vec{n}|\rho_{AB} |\vec{n}\rangle_B}{2E_{\vec{n}}}. \tag{4.23}$$

The additional energy factor was introduced to cancel the factor in the state definition (4.2). This factor preserves the covariant nature of QFT because $d^3\vec{n}/(2\pi)^3 2E_{\vec{n}}$ is a Lorentz invariant integration variable [12].

Attributing the covariant properties to the states themselves, we take the inner product,

$$\langle \vec{q} | \vec{p} \rangle = 2E_{\vec{p}}(2\pi)^3 \delta^{(3)}(\vec{q} - \vec{p}) \tag{4.24}$$

This is equivalent to say,

$$\langle \vec{q} | \vec{p} \rangle = \sqrt{2E_{\vec{p}}(2\pi)^3}\sqrt{2E_{\vec{q}}(2\pi)^3}\delta^{(3)}(\vec{q} - \vec{p}) \tag{4.25}$$

If we define,

$$|\vec{p}\rangle^\mu := \frac{|\vec{p}\rangle}{\sqrt{2E_{\vec{p}}(2\pi)^3}}, \tag{4.26}$$

we have,

$${}^\mu\langle \vec{q} | \vec{p} \rangle^\mu = \delta^{(3)}(\vec{q} - \vec{p}). \tag{4.27}$$

This relationship is much more similar to the typical one in regular quantum mechanics $\langle i|j\rangle = \delta_{ij}$. The trace would also be given by,

$$Tr_B(\rho_{AB}) = \int_{-\infty}^{+\infty} d^3\vec{n} \ {}_B^\mu\langle \vec{n}|\rho_{AB} |\vec{n}\rangle_B^\mu, \tag{4.28}$$

which also bears a resemblance to (4.21), where the summation is substituted only by the integral. So, if one considers that the states themselves are defined in such a way the operations don't have to account for covariance.

We will derive some results using this notation, but afterwards we'll revert to using the regular states as they were defined in (4.21).

If we try to calculate the inner product of $|\vec{p}\rangle^{\mu}$ with itself, and assuming that the state is normalized to unity, we should get 1. But we can easily see that ,

$$^{\mu}\langle\vec{p}\,|\vec{p}\,\rangle^{\mu} = \delta^{(3)}(0), \tag{4.29}$$

which is a divergence. To countour this issue, we introduce the definition of the phase space volume defined as,

$$L^3 = (2\pi)^3\delta^3(0). \tag{4.30}$$

From the definition of (4.26), one gets that the inner product is,

$$\frac{\langle\vec{p}\,|\vec{p}\,\rangle}{2E_{\vec{p}}} = L^3 \Leftrightarrow \frac{\langle\vec{p}\,|\vec{p}\,\rangle}{L^3 2E_{\vec{p}}} = 1 \tag{4.31}$$

which can still be written as,

$$\frac{\langle\vec{p}|}{\sqrt{L^3 2E_{\vec{p}}}} \cdot \frac{|\vec{p}\rangle}{\sqrt{L^3 2E_{\vec{p}}}} = 1. \tag{4.32}$$

So, generally one should use this expression whenever the need arises for calculating any kind of inner product between two equal vectors. For instance, the nth diagonal element of the density operator ρ in matrix form would be given by,

$$\rho_n = \frac{\langle\,\vec{n}\,|\,\rho\,|\,\vec{n}\,\rangle}{\sqrt{2E_{\vec{n}}L^3}\sqrt{2E_{\vec{n}}L^3}}. \tag{4.33}$$

This is an expression we'll use quite often to calculate the elements of the matrix to plug into the entropy expression.

If we are dealing with the originally defined states then the operators should bear the burden of a covariant description. As such this translates into the integration over a given momentum to be defined as,

$$\int_{-\infty}^{+\infty} \frac{d^3\vec{n}}{(2\pi)^3}\frac{1}{2E_{\vec{n}}}. \tag{4.34}$$

Then the projection operator changes like,

$$\hat{P} = \sum_n |n\rangle \langle n| \rightarrow \int_{-\infty}^{+\infty} \frac{d^3\vec{n}}{(2\pi)^3} \frac{|\vec{n}\rangle \langle \vec{n}|}{2E_{\vec{n}}}, \tag{4.35}$$

which means that the general description of $|\Psi\rangle_{out} = \hat{S} |\Psi\rangle_{in}$, when projecting the state into the a two labeled momentum basis, will be of the form,

$$|\Psi\rangle_{out} = \int_{-\infty}^{+\infty} \frac{d^3\vec{q_1}}{(2\pi)^3} \frac{1}{2E_{\vec{q_1}}} \frac{d^3\vec{q_2}}{(2\pi)^3} \frac{1}{2E_{\vec{q_2}}} |\vec{q_1}, \vec{q_2}\rangle \langle \vec{q_1}, \vec{q_2} | \hat{S} |\Psi\rangle_{in}. \tag{4.36}$$

This will be the starting point for the calculation of the entropy of the final state.

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Chapter 5

Entanglement's Entropy: The Calculation

As we said in the previous chapter, we're going to measure the state in a basis of momenta, a two labeled orthogonal infinite basis. The general description of such a projection for the state after the collision (out-state) is given by (4.36). At this point we should define exactly the state $|\Psi\rangle_{in}$. We are going to separate this calculation in two distinct case studies: first we'll assume that the initial state is a product state; afterwards we'll assume the initial state to be an entangled state. We're motivated to do this because intuitively we expect that there might be some differences in the processes of the overall phenomena, since we've found throughout this work that entangled states are of an intrinsically different nature than product/separable states. This approach will allow us to separate the scattering process into the two regimes of the EceM, i.e. creating entanglement versus enhancing the already existing one.

5.1 $\quad |\Psi\rangle_{in}$ is separable

Looking back at our initial definition of a product state, (2.4), we can accordingly to these properties construct a given state in the composite system,

$$|\vec{p}_1 \, \vec{p}_2\rangle_{AB} = |\vec{p}_1\rangle_A \otimes |\vec{p}_2\rangle_B \ , \tag{5.1}$$

where the subsystem states were defined in the previous chapter.

Then we can input the state $|\Psi\rangle_{in}$ defined as (5.1) into (4.36), and get,

$$|\Psi\rangle_{out} = \int_{-\infty}^{+\infty} \frac{d^3\vec{q_1}}{(2\pi)^3} \frac{1}{2E_{\vec{q_1}}} \frac{d^3\vec{q_2}}{(2\pi)^3} \frac{1}{2E_{\vec{q_2}}} |\vec{q_1}\ \vec{q_2}\rangle\langle\vec{q_1}\ \vec{q_2}\,|\,\hat{S}\,|\,\vec{p_1}\ \vec{p_2}\rangle. \qquad (5.2)$$

We will need to solve this last expression to compute the entropy of the out-state, but before tackling this somewhat strenuous task we will compute the initial entropy, which we'll need in order to calculate the variation. Recalling a general result of Chapter2, we found that the entropy of entanglement of any separable state is given by (3.5).Then it's a direct consequence that $S_{E\ (in)} = 0$ and, as such, $\Delta S_E = S_{E\ (out)}$.

We revert our attention to the expression (5.2), which is quite over-encumbered. We will rearrange it so to become less eye strenuous without having to write everything explicitly. This integration spans various domains pertaining to specific states configurations, there are 4 different contributions for each of the different cases:

- One for when the new state has the same labels as the state being projected i.e: $|\vec{q_1}\ \vec{q_2}\rangle\Big|_{\vec{q_1}=\vec{p_1},\ \vec{q_2}=\vec{p_2};}$;

-Two for when the new state has one of the labels equal but the other one different from the state being projected,
i.e $\ |\vec{q_1}\ \vec{q_2}\rangle\Big|_{\vec{q_1}\neq\vec{p_1},\ \vec{q_2}=\vec{p_2};}$ and $|\vec{q_1}\ \vec{q_2}\rangle\Big|_{\vec{q_1}=\vec{p_1},\ \vec{q_2}\neq\vec{p_2};}$;

-One for when the new state has both labels different than the state being projected, i.e $\ |\vec{q_1}\ \vec{q_2}\rangle\Big|_{\vec{q_1}\neq\vec{p_1},\ \vec{q_2}\neq\vec{p_2};}$;

We then might think in separating (5.2) into 4 parts, but a more careful look into the results reveals that the contributions which have all except one momenta equal, vanish. This is due to the nature of $\hat{S}$,

$$\langle\vec{q_1}\ \vec{q_2}\,|\,\hat{S}\,|\,\vec{p_1}\ \vec{p_2}\rangle \propto \delta^{(3)}(\vec{q_1}+\vec{q_2}-\vec{p_1}-\vec{p_2}),$$

with, $\vec{q_1} = \vec{p_1}$ and $\vec{q_2} \neq \vec{p_2}$ yields,

$$\delta^{(3)}(\vec{q_2} - \vec{p_2}),$$

which gives zero (if $\vec{q_1} \neq \vec{p_1}$ and $\vec{q_2} = \vec{p_2}$ the result is the same). Then we actually just have to separate the integral into 2 parts. A term stemming from the projection onto a state with equal labels to the state being projected and one being projected to a different labeled state,

$$|\Psi\rangle_{out} = |\vec{p_1}\,\vec{p_2}\,\rangle + \int\limits_{\vec{q_1}\neq\vec{p_1};\vec{q_2}\neq\vec{p_2}} \frac{d^3\vec{q_1}}{(2\pi)^3}\frac{d^3\vec{q_2}}{(2\pi)^3} \frac{|\vec{q_1}\,\vec{q_2}\,\rangle\langle\,\vec{q_1}\,\vec{q_2}\,|\hat{S}\,|\vec{p_1}\,\vec{p_2}\,\rangle}{2E_{\vec{q_1}}2E_{\vec{q_2}}}. \tag{5.3}$$

The first term doesn't have an explicit integration anymore, because we already integrated the inner product with the operator $\hat{S} = 1 + i\hat{T}$ inside, for which only the identity term contributes.

If in the second term of (5.3) we write $\hat{S} = 1 + i\hat{T}$, we notice that $i\hat{T}$ is the one which produces a non-null result. This is because the identity operator leaves the state unchanged and in this scenario the new state was made so to be different from the old one. The second term becomes,

$$\int\limits_{\vec{q_1}\neq\vec{p_1};\vec{q_2}\neq\vec{p_2}} \frac{d^3\vec{q_1}}{(2\pi)^3}\frac{d^3\vec{q_2}}{(2\pi)^3} \frac{|\vec{q_1}\,\vec{q_2}\,\rangle\langle\,\vec{q_1}\,\vec{q_2}\,|i\hat{T}\,|\vec{p_1}\,\vec{p_2}\,\rangle}{2E_{\vec{q_1}}2E_{\vec{q_2}}} \tag{5.4}$$

$$= \int\limits_{\vec{q_1}\neq\vec{p_1};\vec{q_2}\neq\vec{p_2}} \frac{d^3\vec{q_1}}{(2\pi)^3}\frac{d^3\vec{q_2}}{(2\pi)^3} \frac{|\vec{q_1}\,\vec{q_2}\,\rangle(2\pi)^4\delta^{(4)}(p_1+p_2-q_1-q_2)i\mathcal{M}_{(p_1p_2\to q_1q_2)}}{2E_{\vec{q_1}}2E_{\vec{q_2}}}.$$

Then (5.3) becomes,

$$|\Psi\rangle_{out} = |\vec{p_1}\,\vec{p_2}\,\rangle + \int\limits_{\vec{q_1}\neq\vec{p_1};\vec{q_2}\neq\vec{p_2}} \frac{d^3\vec{q_1}}{(2\pi)^3}\frac{d^3\vec{q_2}}{(2\pi)^3} \frac{|\vec{q_1}\,\vec{q_2}\,\rangle(2\pi)^4\delta^{(4)}(p_1+p_2-q_1-q_2)i\mathcal{M}_{(p_1p_2\to q_1q_2)}}{2E_{\vec{q_1}}2E_{\vec{q_2}}}. \tag{5.5}$$

We separate the $\delta^{(4)}$ into

$$\delta^{(3)}(\vec{p_1}+\vec{p_2}-\vec{q_1}-\vec{q_2})\delta(E_{\vec{p_1}}+E_{\vec{p_2}}-E_{\vec{q_1}}-E_{\vec{q_2}})$$

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in order to integrate over

$$\int \frac{d^3\vec{q_2}}{(2\pi)^3} \delta^{(3)}(\vec{p_1} + \vec{p_2} - \vec{q_1} - \vec{q_2}),$$

which by the properties of the delta function, $\vec{q_2} \to \vec{p_1} + \vec{p_2} - \vec{q_1}$. After the integration we have,

$$\int\limits_{\vec{q_1} \neq \vec{p_1}} \frac{d^3\vec{q_1}}{(2\pi)^3} \frac{|\vec{q_1}, \; \vec{p_1} + \vec{p_2} - \vec{q_1} \, \rangle \, [(2\pi)\delta'_E] i\mathcal{M}_{(p_1 p_2 \to q_1 q'_2)}}{2E_{\vec{q_1}} 2E_{\vec{p_1}+\vec{p_2}-\vec{q_1}}}, \tag{5.6}$$

where,

$$q'_2 := (E_{\vec{p_1}+\vec{p_2}-\vec{q_1}}, \vec{p_1} + \vec{p_2} - \vec{q_1})$$

is a 4-vector modified by the integration.

We also use the short notation δ'_E for $\delta(E_{\vec{p_1}} + E_{\vec{p_2}} - E_{\vec{q_1}} - E_{\vec{p_1}+\vec{p_2}-\vec{q_1}})$. Adding the two terms we get,

$$|\Psi\rangle_{after} = |\vec{p_1}\,\vec{p_2}\,\rangle + \int\limits_{\vec{q_1} \neq \vec{p_1}} \frac{d^3\vec{q_1}}{(2\pi)^3} \frac{|\vec{q_1}, \; \vec{p_1} + \vec{p_2} - \vec{q_1} \, \rangle \, [(2\pi)\delta'_E] i\mathcal{M}'}{2E_{\vec{q_1}} 2E_{\vec{p_1}+\vec{p_2}-\vec{q_1}}}. \tag{5.7}$$

5.1.1 The Center of Momenta (CM) frame

Although at this point it's not explicitly necessary to turn to the CM frame in order to evaluate the integrals, we will soon encounter unnecessary difficulties if we keep things as general as they are. We make the decision to go to the CM frame to facilitate an easier approach to solving the integrals. Then, all the dynamical variables associated with momenta and energy will be calculated in the CM, and hence all aspects of the results will be pertinent only to the CM frame. In the CM frame,

$$\vec{p} := \vec{p_1} = -\vec{p_2} \Rightarrow |\vec{p_1}| = |\vec{p_2}| = |\vec{p}|; \tag{5.8}$$

$$\vec{q} := \vec{q_1} = -\vec{q_2} \Rightarrow |\vec{q_1}| = |\vec{q_2}| = |\vec{q}|;$$

Which means that the initial state is,

$$|\Psi\rangle_{in} = |\vec{p}, -\vec{p}\rangle. \tag{5.9}$$

And also from fact the particles are all of the same mass,

$$E_{\vec{p_1}} = E_{\vec{p_2}} := E_{\vec{p}}; \tag{5.10}$$

$$E_{\vec{q_1}} = E_{\vec{q_2}} := E_{\vec{q}};$$

From now on we should remember that, "Our systems is being studied in the CM frame". Then if we evaluate (5.7) in the CM we get,

$$|\Psi\rangle_{out} = |\vec{p}, \, -\vec{p}\,\rangle \; + i\int\limits_{\vec{q}\neq\vec{p};} \frac{d^3\vec{q}}{(2\pi)^3} \frac{|\vec{q}, \, -\vec{q}\,\rangle \, (2\pi)\delta_E \, \mathcal{M}}{(2E_{\vec{q}})^2}, \tag{5.11}$$

where $\mathcal{M} \equiv \mathcal{M}_{(p,-p\rightarrow q,-q)}$. Because,

$$q_2' := (E_{\vec{p_1}+\vec{p_2}-\vec{q_1}}, \vec{p_1} + \vec{p_2} - \vec{q_1}) \xrightarrow{CM} (E_{-\vec{q}}, -\vec{q}) \equiv -q.$$

Also the delta function of energy becomes,

$$\delta_E' \xrightarrow{CM} \delta(2E_{\vec{p}} - 2E_{\vec{q}}) := \delta_E$$

So, (5.11) is the general description of the out-state projected into a basis from the perspective of the CM frame. This is the expression which we will use to carry on the calculation.

Following the next step in fig.(4.4) we need, $\rho_{out} = |\Psi\rangle_{out}\,{}_{out}\langle\Psi|$. In order to do that we defined the dual bra vector of (5.11),

$$\langle\Psi|_{out} = \langle\vec{p}, \, -\vec{p}\,| - i\int\limits_{\vec{l}\neq\vec{p}} \frac{d^3\vec{l}}{(2\pi)^3} \frac{\langle\vec{l}, \, -\vec{l}\,|(2\pi)\delta_E^{(l)}\mathcal{M}_{(l)}^*}{(2E_{\vec{l}})^2}. \tag{5.12}$$

We explicitly put a different variable $\vec{l}$ integrating over the space, so that we don't get confused on what terms belong to what integration. The operator is given explicitly as,

$$\rho_{out} = |\vec{p}, \; -\vec{p}\,\rangle \, \langle \vec{p}, \; -\vec{p}\,| + \int\limits_{\vec{q} \neq \vec{p}; \; \vec{l} \neq \vec{p}} \frac{d^3\vec{q}}{(2\pi)^3} \frac{d^3\vec{l}}{(2\pi)^3} \frac{(2\pi)^2 \big(\delta_E \delta_E^{(l)} \; \mathcal{M}\mathcal{M}_{(l)}^*\big)}{(2E_{\vec{q}})^2 (2E_{\vec{l}})^2} |\vec{q} - \vec{q}\,\rangle \, \langle \vec{l}, \; -\vec{l}\,|$$

<div align="right">(5.13)</div>

$$+ (\text{Crossed Terms}).$$

We didn't write the crossed terms explicitly because they disappear in the process of calculating the reduced density operator. This is easily noticeable because the two crossed terms will be proportional to either to

$$\delta^{(3)}(\vec{q} - \vec{p}), \;\; \text{or} \;\; \delta^{(3)}(\vec{l} - \vec{p}),$$

and since we would integrate over domains, where $\vec{q} \neq \vec{p}; \;\; \vec{l} \neq \vec{p}$, this would obviously give zero.

We now shall use the previous expression derived for the trace in QFT (4.23), and apply it to the operator (5.13),

$$(\rho_A)_{out} = Tr_B(\rho_{out}) = \int_{-\infty}^{+\infty} \frac{d^3\vec{n}}{(2\pi)^3} \frac{{}_B\langle \vec{n}|\rho_{out}\,|\vec{n}\rangle_B}{2E_{\vec{n}}}.$$

<div align="right">(5.14)</div>

We will have two terms. The first one is given by,

$$\int_{-\infty}^{+\infty} \frac{d^3\vec{n}}{(2\pi)^3} \frac{{}_B\langle \vec{n}\,|\vec{p}, \; -\vec{p}\,\rangle \, \langle \vec{p}, \; -\vec{p}\,|\vec{n}\rangle_B}{2E_{\vec{n}}} = \int_{-\infty}^{+\infty} \frac{d^3\vec{n}}{(2\pi)^3} \frac{\langle \vec{n}\,| \; -\vec{p}\,\rangle \, \langle -\vec{p}\,| \; -\vec{n}\,\rangle}{2E_{\vec{n}}} |\vec{p}\,\rangle \, \langle \vec{p}\,| \; .$$

<div align="right">(5.15)</div>

Using the inner product, $\langle \vec{q}\,|\vec{p}\,\rangle = 2E_{\vec{p}}(2\pi)^3 \delta^{(3)}(\vec{q} - \vec{p}\,)$,

$$= \int_{-\infty}^{+\infty} d^3\vec{n} \; 2E_{\vec{n}}(2\pi)^3 \big(\delta^{(3)}(\vec{n} - \vec{p}\,)\big)^2 |\vec{p}\,\rangle \, \langle \vec{p}\,| \; .$$

The integration over one delta function sets $\vec{n} \rightarrow \vec{p}$, but we still have a remaining $\delta^{(3)}(0)$, and since it's multiplied by $(2\pi)^3$, the result for the first term will be, $2E_{\vec{p}}L^3 |\vec{p}\,\rangle \, \langle \vec{p}\,|$. The second term will take a bit more of work. Explicitly we have,

$$\int_{-\infty}^{+\infty} \frac{d^3\vec{n}}{(2\pi)^3} \frac{1}{2E_{\vec{n}}} \int\limits_{\vec{q} \neq \vec{p}; \; \vec{l} \neq \vec{p}} \frac{d^3\vec{q}}{(2\pi)^3} \frac{d^3\vec{l}}{(2\pi)^3} \frac{(2\pi)^2 \big(\delta_E \delta_E^{(l)} \; \mathcal{M}\mathcal{M}_{(l)}^*\big)}{(2E_{\vec{q}})^2 (2E_{\vec{l}})^2} \; \langle \vec{n}\,| {-}\vec{q}\,\rangle \, \langle {-}\vec{l}\,| \, \vec{n}\,\rangle \; |\vec{q}\rangle \langle \vec{l}\,| \; .$$

<div align="right">(5.16)</div>

We integrate first over the l space using the delta $\delta^{(3)}(\vec{n} + \vec{l})$ that comes from the inner product $\langle -\vec{l} \mid \vec{n} \rangle = 2E_{\vec{n}}(2\pi)^3\delta^{(3)}(\vec{n} + \vec{l})$. The integration sets $\vec{l} \to -\vec{n}$ which accordingly gives,

$$\int_{-\infty}^{+\infty} \frac{d^3\vec{n}}{(2E_{\vec{n}})^2} \int_{\vec{q}\neq\vec{p}} \frac{d^3\vec{q}}{(2\pi)^3} \frac{(2\pi)^2\left(\delta_E \, \delta_E^{(n)} \, \mathcal{M} \, \mathcal{M}_{(n)}^*\right)}{(2E_{\vec{q}})^2} \langle \vec{n} \mid -\vec{q} \rangle \, |\vec{q}\rangle \, \langle -\vec{n} \, .|$$

We can switch the order of integration and integrate over the space of n using the delta $\delta^{(3)}(\vec{n} + \vec{q})$ which stems from the inner product $\langle \vec{n} \mid -\vec{q} \rangle = 2E_{\vec{q}}(2\pi)^3\delta^{(3)}(\vec{n} + \vec{q})$.

The integration sets $-\vec{n} \to \vec{q}$ and we have,

$$\int_{\vec{q}\neq\vec{p}} \frac{d^3\vec{q}}{(2\pi)^3} \frac{\left(2\pi\delta_E\right)^2 |\mathcal{M}|^2}{(2E_{\vec{q}})^3} |\vec{q}\rangle \langle \vec{q}|,$$

since it turns out that, $\vec{q} = \vec{l}$, and as such $\delta_E^{(l)} = \delta_E$ and $\mathcal{M}_{(l)} = \mathcal{M}$.

By adding this term to the first term, we get that Alice's reduced density operator is,

$$(\rho_A)_{out} = 2E_{\vec{p}}L^3 \, |\vec{p}\rangle \, \langle \vec{p}| + \int_{\vec{q}\neq\vec{p}} \frac{d^3\vec{q}}{(2\pi)^3} \frac{\left(2\pi\delta_E\right)^2 |\mathcal{M}|^2}{(2E_{\vec{q}})^3} |\vec{q}\rangle \langle \vec{q}|; \qquad (5.17)$$

As one can appreciate $|\mathcal{M}|^2$ is given by $\mathcal{M}\mathcal{M}^*$, which implies that $\mathcal{M}$ is generally a complex function. And in fact, the expression for $\mathcal{M}$ in 1-loop (4.14) is complex, because $\mathcal{F}(u, t, s)$ defined in chapter 4, (4.16), is a complex function. Although obviously in tree-level $\mathcal{M}$ is real since it's just a constant. Having this general expression for the reduced density operator, now we will plug in the values of $\mathcal{M}$ for both the tree-level case and the tree-level plus 1-loop corrections.

5.1.2 Calculating $(S_E)_{out}$ in Tree-level

We want to perform this calculation for the Tree-level approximation. In tree-level we have,

$$i\mathcal{M}_{(\text{tree})} = -2i\lambda,$$

and as such
$$|\mathcal{M}|^2 = 4\lambda^2.$$

We input this into the expression and get,

$$(\rho_A)_{out} = 2E_{\vec{p}}L^3 \, |\, \vec{p}\, \rangle \, \langle\, \vec{p}\, | + \int\limits_{\vec{q}\neq\vec{p}} \frac{d^3\vec{q}}{(2\pi)^3} \frac{\left(2\pi\delta_E\right)^2 \, |\vec{q}\, \rangle \, \langle\vec{q}\, |}{(2E_{\vec{q}})^3}\left(4\lambda^2\right); \qquad (5.18)$$

The operator is not normalized, and we stressed previously the importance for it to be. Then we can write the normalized operator in compact notation as,

$$(\rho_A)_{out} = \mathcal{N}\left(\, |\, \vec{p}\, \rangle \, \langle\, \vec{p}\, |\, (2E_{\vec{p}}L^3) + \mathcal{I}_2\right). \qquad (5.19)$$

Where $\mathcal{N}$ is the normalization constant and $\mathcal{I}_2$,

$$\mathcal{I}_2 \equiv \int\limits_{\vec{q}\neq\vec{p}} \frac{d^3\vec{q}}{(2\pi)^3} \frac{\left(2\pi\delta_E\right)^2 \, |\vec{q}\, \rangle \, \langle\vec{q}\, |}{(2E_{\vec{q}})^3}\left(4\lambda^2\right), \qquad (5.20)$$

the subscript in the integral notation is equal to the power of λ, inside the integral.

We need to find the value for $\mathcal{N}$, by using the normalizing condition we introduced in Chapter2, we know that $Tr(\rho) = 1$ for a normalized ρ. Then

$$\mathcal{N} = \frac{1}{Tr(\rho_A)_{out}}.$$

If we trace the operator (5.19) we have two terms. The first one is,

$$Tr(\rho_A)_{out} = 2E_{\vec{p}}L^3 \int_{-\infty}^{+\infty} \frac{d^3\vec{n}}{(2\pi)^3} \frac{\langle \vec{n}|\, \vec{p}\, \rangle \, \langle\, \vec{p}\, |\vec{n}\, \rangle}{2E_{\vec{n}}} = \qquad (5.21)$$

$$2E_{\vec{p}}L^3 \int_{-\infty}^{+\infty} d^3\vec{n}(2\pi)^3 2E_{\vec{n}}(\delta^{(3)}(\vec{n}-\vec{p}))^2.$$

The integration over one of the delta's sets $\vec{n} \to \vec{p}$, and we get afterwards the factor $(2\pi)^3\delta^{(3)}(0)$ so that the end result is, $(2E_{\vec{p}}L^3)^2$. Thus, we find that

$$\mathcal{N} = \frac{1}{(2E_{\vec{p_2}}L^3)^2 + \langle\mathcal{I}_2\rangle}.$$

Here we have defined the implicit trace over $\mathcal{I}_2$ to be,

$$\int_{-\infty}^{+\infty} \frac{d^3\vec{n}}{(2\pi)^3} \frac{\langle \vec{n}| \, \mathcal{I}_2 \, |\vec{n}\rangle}{2E_{\vec{n}}} \equiv \langle \mathcal{I}_2 \rangle \tag{5.22}$$

Explicitly the normalized operator is,

$$(\rho_A)_{out}^{(n)} = \frac{|\vec{p}\rangle \langle \vec{p}| \, (2E_{\vec{p}}L^3) + \mathcal{I}_2}{(2E_{\vec{p_2}}L^3)^2 + \langle \mathcal{I}_2 \rangle}, \tag{5.23}$$

where we've put the superscript (n) in the operator to indicate that it's already normalized.

If we factor out $(2E_{\vec{p_2}}L^3)^2$ in the denominator, we can write the expression for the operator still in the form,

$$(\rho_A)_{out}^{(n)} = \frac{|\vec{p}\rangle \langle \vec{p}|}{(2E_{\vec{p}}L^3)(1 + \mathcal{A}_{tree})} + \frac{\mathcal{I}_2}{(2E_{\vec{p}}L^3)^2(1 + \mathcal{A}_{tree})}, \tag{5.24}$$

with,

$$\mathcal{A}_{tree} \equiv \frac{\langle \mathcal{I}_2 \rangle}{(2E_{\vec{p}}L^3)^2}. \tag{5.25}$$

We shall proceed to calculate $\langle \mathcal{I}_2 \rangle$, we write (5.22) explicitly,

$$\int_{-\infty}^{+\infty} \frac{d^3\vec{n}}{(2\pi)^3} \frac{1}{2E_{\vec{n}}} \int\limits_{\vec{q}\neq\vec{p}} \frac{d^3\vec{q}}{(2\pi)^3} \frac{\left(2\pi\delta_E\right)^2 \langle \vec{n}|\vec{q}\rangle \langle \vec{q}|\vec{n}\rangle}{(2E_{\vec{q}})^3} \left(4\lambda^2\right) = \tag{5.26}$$

$$\int\limits_{\vec{q}\neq\vec{p}} \frac{d^3\vec{q}}{(2\pi)^3} \int_{-\infty}^{+\infty} d^3\vec{n} \frac{\left(2\pi\delta_E\right)^2(2\pi)^3\left(\delta^{(3)}(\vec{n}-\vec{q})\right)^2}{(2E_{\vec{n}})(2E_{\vec{q}})} \left(4\lambda^2\right); \tag{5.27}$$

Again we integrate $d^3\vec{n}$ over one delta such that, $\vec{n} \to \vec{q}$ and have a term $(2\pi)^3\delta^{(3)}(0)$. The result is,

$$\langle \mathcal{I}_2 \rangle = \frac{L^3}{(2\pi)} \int\limits_{\vec{q}\neq\vec{p}} d^3\vec{q} \, \frac{\delta^2(2E_{\vec{q}} - 2E_{\vec{p}})}{(2E_{\vec{q}})^2} \left(4\lambda^2\right). \tag{5.28}$$

We need to calculate this integral, but we have a delta function whose arguments are energies and we are integrating over momenta, we need to take this into consideration. First we go to spherical coordinates,

$$\int_{-\infty}^{+\infty} d^3\vec{q} \to \int_0^{+\infty} d|\vec{q}| \cdot q^2 \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \ .$$

Since there is no dependency in θ and ϕ in the expression for $\langle \mathcal{I}_2 \rangle$, the integration over the angles is isotropic and yields 4π.

We use the relativistic relationship between momentum and energy $q^2 = E_{\vec{q}}^2 - m^2$, and notice that,

$$|\vec{q}\,|d|\vec{q}\,| = E_{\vec{q}} \, dE_{\vec{q}}.$$

So using this identity the integration becomes,

$$4\pi \int_m^{+\infty} dE_{\vec{q}} \ (E_{\vec{q}}\sqrt{E_{\vec{q}}^2 - m^2}).$$

If we substitute this in (5.28) we get the integral,

$$\langle \mathcal{I}_2 \rangle = (2\lambda^2)L^3 \int_m^{+\infty} dE_{\vec{q}} \ \sqrt{E_{\vec{q}}^2 - m^2} \frac{\delta^2(2E_{\vec{q}} - 2E_{\vec{p}})}{(E_{\vec{q}})}. \tag{5.29}$$

We can now integrate the delta, which sets $E_{\vec{q}} \to E_{\vec{p}}$ with an additional factor of $\frac{1}{2}$, due the following property of the delta function,

$$\int_{-\infty}^{+\infty} \delta[c(x - x_0)]f(x)dx = \frac{f(x_0)}{c};$$

Then we have the following result,

$$\langle \mathcal{I}_2 \rangle = \frac{L^4\lambda^2}{2\pi} \frac{|\vec{p}|}{E_{\vec{p}}} = \frac{L^4\lambda^2}{2\pi} |v_{\vec{p}}|. \tag{5.30}$$

The extra factor of L came from the $2\pi\delta(0)$, and we have used the fact that,

$$\frac{|\vec{p}|}{E_{\vec{p}}} = \frac{m\gamma_{\vec{p}}|v_{\vec{p}}|}{m\gamma_{\vec{p}}} = |v_{\vec{p}}|,$$

where $\gamma_{\vec{p}}$ is the Lorentz factor associated with momentum $\vec{p}$.

If we substitute (5.30) into (5.25) we find that,

$$\mathcal{A}_{tree} = \frac{\lambda^2}{8\pi} \frac{|v_{\vec{p}}|}{(\bar{E}_{\vec{p}})^2}; \tag{5.31}$$

Where we have defined, $\bar{E}_{\vec{p}} \equiv E_{\vec{p}}L = \gamma_{\vec{p}}\bar{m}$, if $\bar{m} = mL$, in order to absorb the divergences.

Entropy in Tree-level

Alice's out-state reduced operator is completely defined at this point, with (5.24) and (5.31). We are endowed with all the information needed to calculate the entropy. Let's look at the expression (5.24) and write $\mathcal{I}_2$ explicitly,

$$(\rho_A)_{out} = \frac{|\vec{p}\rangle\langle\vec{p}|}{(2E_{\vec{p}}L^3)(1+\mathcal{A}_{tree})} \tag{5.32}$$

$$+\frac{1}{(2E_{\vec{p}}L^3)^2(1+\mathcal{A}_{tree})}\int\limits_{\vec{q}\neq\vec{p}}\frac{d^3\vec{q}}{(2\pi)^3}\frac{\left(2\pi\delta_E\right)^2|\vec{q}\rangle\langle\vec{q}|}{(2E_{\vec{q}})^3}\left(4\lambda^2\right);$$

Clearly the operator in matrix form is a diagonal operator, and as such, we can use the expression for the entropy (3.4), which in this context will be given by,

$$S_E = -\sum_i (\rho_i \log \rho_i) = -\rho_p \log \rho_p - \frac{L^3}{(2\pi)^3}\int_{-\infty}^{+\infty} d\vec{k}\ (\rho_k \log \rho_k); \text{ With, } k \neq p\,. \tag{5.33}$$

Here the elements of the operator are given by (4.33),

$$\rho_n = \frac{\langle\vec{n}|\rho|\vec{n}\rangle}{\sqrt{2E_{\vec{n}}L^3}\sqrt{2E_{\vec{n}}L^3}}, \tag{5.34}$$

such that,

$$(\rho_p)_{out} = \frac{\langle\vec{p}|(\rho_A)_{out}|\vec{p}\rangle}{\sqrt{2E_{\vec{p}}L^3}\sqrt{2E_{\vec{p}}L^3}} = \frac{\langle\vec{p}|\vec{p}\rangle\langle\vec{p}|\vec{p}\rangle}{(2E_{\vec{p}}L^3)^2(1+\mathcal{A}_{tree})} = \frac{1}{1+\mathcal{A}_{tree}}; \tag{5.35}$$

The first term of the entropy is calculated to be,

$$-\rho_p \log \rho_p = -\frac{1}{1+\mathcal{A}_{tree}}\log\left(\frac{1}{1+\mathcal{A}_{tree}}\right) = \frac{\log(1+\mathcal{A}_{tree})}{1+\mathcal{A}_{tree}}. \tag{5.36}$$

Now for,

$$(\rho_k)_{out} = \frac{\langle\vec{p}|(\rho_A)_{out}|\vec{p}\rangle}{\sqrt{2E_{\vec{p}}L^3}\sqrt{2E_{\vec{p}}L^3}} \tag{5.37}$$

with, $\vec{k}\neq\vec{p}$, we get,

$$\frac{4\lambda^2}{(2E_{\vec{k}}L^3)(2E_{\vec{p}}L^3)^2(1+\mathcal{A}_{tree})}\int\limits_{\vec{q}\neq\vec{p}}\frac{d^3\vec{q}}{(2\pi)^3}\frac{\left(2\pi\delta_E\right)^2}{(2E_{\vec{q}})^3}\langle\vec{k}|\vec{q}\rangle\langle\vec{q}|\vec{k}\rangle;$$

As usual we integrate $\vec{q}$ over one of the deltas, such that $\vec{q} \to \vec{k}$. And we get,

$$(\rho_k)_{out} = \left(\frac{2\pi\delta(2E_{\vec{k}} - 2E_{\vec{p}})}{2E_{\vec{k}}2E_{\vec{p}}L^3}\right)^2 \frac{4\lambda^2}{1 + \mathcal{A}_{tree}}, \qquad (5.38)$$

which means the second term of the entropy is given by,

$$-\frac{L^3}{(2\pi)^3} \int_{\vec{k} \neq \vec{p}} d\vec{k} \left(\frac{2\pi\delta(2E_{\vec{k}} - 2E_{\vec{p}})}{2E_{\vec{k}}2E_{\vec{p}}L^3}\right)^2 \frac{4\lambda^2}{1 + \mathcal{A}_{tree}} \log\left[\left(\frac{2\pi\delta(2E_{\vec{k}} - 2E_{\vec{p}})}{2E_{\vec{k}}2E_{\vec{p}}L^3}\right)^2 \frac{4\lambda^2}{1 + \mathcal{A}_{tree}}\right];$$

$$= -\frac{L^{-3}\lambda^2}{8\pi E_{\vec{p}}(1 + \mathcal{A}_{tree})} \int_{\vec{k} \neq \vec{p}} d\vec{k} \left(\frac{\delta(2E_{\vec{k}} - 2E_{\vec{p}})}{E_{\vec{k}}}\right)^2 \log\left[\left(\frac{2\pi\delta(2E_{\vec{k}} - 2E_{\vec{p}})}{2E_{\vec{k}}2E_{\vec{p}}L^3}\right)^2 \frac{4\lambda^2}{1 + \mathcal{A}_{tree}}\right];$$

We solve this integration in the same manner as before, when calculating $\langle \mathcal{I}_2 \rangle$ by changing the integration over the momenta to an integration over the energies,

$$-\frac{L^{-3}\lambda^2}{2E_{\vec{p}}(1 + \mathcal{A}_{tree})} \int_m^{+\infty} dE_{\vec{q}} \, \frac{\sqrt{E_{\vec{q}}^2 - m^2}}{E_{\vec{q}}} \delta^2(2E_{\vec{k}} - 2E_{\vec{p}}) \log\left[\left(\frac{2\pi\delta(2E_{\vec{k}} - 2E_{\vec{p}})}{2E_{\vec{k}}2E_{\vec{p}}L^3}\right)^2 \frac{4\lambda^2}{1 + \mathcal{A}_{tree}}\right];$$

Integrating over the delta of the energy, we set $E_{\vec{q}} \to E_{\vec{k}}$ with the $\frac{1}{2}$ factor. After multiplying with the appropriate (2π) constants we have the following expression,

$$\lambda^2 |v_{\vec{p}}| \frac{\log\left((4\bar{E}_{\vec{p}}^4(1 + \mathcal{A}_{tree}))/\lambda^2\right)}{8\pi \bar{E}_{\vec{p}}^2(1 + \mathcal{A}_{tree})},$$

Adding to the first term of the entropy we finally have,

$$\Delta S_E = (S_E)_{out} = \frac{\log(1 + \mathcal{A}_{tree})}{1 + \mathcal{A}_{tree}} + \lambda^2 |v_{\vec{p}}| \frac{\log\left((4\bar{E}_{\vec{p}}^4(1 + \mathcal{A}_{tree}))/\lambda^2\right)}{8\pi \bar{E}_{\vec{p}}^2(1 + \mathcal{A}_{tree})}. \quad (5.39)$$

5.1.3 Calculating $(S_E)_{out}$ with 1-loop Corrections

Now we will proceed to add 1-loop corrections to see how this changes the results. The result of $i\mathcal{M}$ from the 1-loop diagrams was calculated in (4.14) to be,

$$i\mathcal{M}_{\text{(1-loop)}} = -i(\frac{\lambda}{4\pi})^2 \cdot \mathcal{F}(s,t,u).$$

Then the amplitude with 1-loop corrections is,

$$i\mathcal{M}_{\text{(tree+1-loop)}} = -2i\lambda - i(\frac{\lambda}{4\pi})^2 \cdot \mathcal{F}(s,t,u) \tag{5.40}$$

where we defined $\mathcal{F}(s,t,u)$ in (4.16) to be,

$$4 \cdot (3G(t) + 2G(u) + G(s) + 2)$$

with,

$$t = (p_1 - q_1)^2, u = (p_1 - q_2)^2, s = (p_1 + p_2)^2.$$

And the function,

$$G(x) = -2 + \sqrt{1 - \frac{4m^2}{x}} \cdot \log\left(\frac{\sqrt{1 - \frac{4m^2}{x}} + 1}{\sqrt{1 - \frac{4m^2}{x}} - 1}\right),$$

$$\text{for } x \in \{s,t,u\}.$$

But we are in the CM frame, and we should compute the variables accordingly.

Mandelstam variables in the CM

Bearing in mind the CM considerations,

$$\vec{p} := \vec{p_1} = -\vec{p_2} \Rightarrow |\vec{p_1}| = |\vec{p_2}| = |\vec{p}|;$$

$$\vec{q} := \vec{q_1} = -\vec{q_2} \Rightarrow |\vec{q_1}| = |\vec{q_2}| = |\vec{q}|;$$

we find that,

$$s = p_1^2 + p_2^2 + 2p_1 \cdot p_2 = 2m^2 + 2(E_{\vec{p}}^2 + |\vec{p}\,|^2) = 4(m^2 + |\vec{p}\,|^2),$$

$$u = p_1^2 + q_2^2 - 2p_1 \cdot q_2 = 2m^2 - 2\big(E_{\vec{p}}E_{\vec{q}} + |\vec{p}\,||\vec{q}\,|\cos(\theta)\big) = 2m^2 - 2\big(E_{\vec{p}}E_{\vec{q}} + |\vec{p}|\sqrt{E_{\vec{q}}^2 - m^2}\cos(\theta)\big),$$

$$t = p_1^2 + q_1^2 - 2p_1 \cdot q_1 = 2m^2 - 2\big(E_{\vec{p}}E_{\vec{q}} - |\vec{p}\,||\vec{q}\,|\cos(\theta)\big) = 2m^2 - 2\big(E_{\vec{p}}E_{\vec{q}} - |\vec{p}|\sqrt{E_{\vec{q}}^2 - m^2}\cos(\theta)\big).$$

So, the Mandelstam variables are actually functions of θ and $E_{\vec{q}}$, where θ is the scattering angle between $\vec{p}$ and $\vec{q}$. This means that, $\mathcal{F}(s,t,u)$ can be defined as the functional,

$$\mathcal{F}(s,t,u) \equiv \mathcal{F}[s, t(\theta, E_{\vec{q}}), u(\theta, E_{\vec{q}})] \tag{5.41}$$

But, as we have seen previously in the Tree-level calculation, when we integrate over a given delta of energy $\delta(E_{\vec{q}} - E_{\vec{p}})$, as we'll do further on, we set $E_{\vec{q}} \to E_{\vec{p}}$. This will eliminate the explicit dependence of the Mandelstam variables in the energy $E_{\vec{q}}$, and the functional becomes,

$$\mathcal{F}[s, t(\theta), u(\theta)]. \tag{5.42}$$

Thus, we will evaluate explicitly this functional, since when the need arises in the upcoming calculations for an explicit substitution, this will be the form it takes.

What are the functions $t(\theta)$, and $u(\theta)$? We take the expressions we calculated before for u and t, and substitute $E_{\vec{q}} \to E_{\vec{p}}$ which give,

$$u = -2|\vec{p}\,|^2(1 + \cos\theta) \equiv u(\theta);$$

$$t = -2|\vec{p}\,|^2(1 - \cos\theta) := t(\theta);$$

$$s = 4(m^2 + |\vec{p}\,|^2);$$

The s stays unchanged, but the other two expressions are significantly simpler.

What is the expression for $\mathcal{F}[s, t(\theta), u(\theta)]$? Writing them out in terms of the $G(x)$ function we have, that,

$$\mathcal{F}[s, t(\theta), u(\theta)] = 12G[t(\theta)] + 8G[u(\theta)] + 4G(s) + 8$$

We can easily check that substituting the functions and inputting them into the functional we have,

$$G[t(\theta)] = -2 + \sqrt{\frac{2/|v_{\vec{p}}|^2 - (1 + \cos\theta)}{1 - \cos\theta}} \log \frac{\sqrt{\frac{2/|v_{\vec{p}}|^2 - (1+\cos\theta)}{1-\cos\theta}} + 1}{\sqrt{\frac{2/|v_{\vec{p}}|^2 - (1+\cos\theta)}{1-\cos\theta}} - 1}; \qquad (5.43)$$

$$G[u(\theta)] = -2 + \sqrt{\frac{2/|v_{\vec{p}}|^2 + (-1 + \cos\theta)}{1 + \cos\theta}} \log \frac{\sqrt{\frac{2/|v_{\vec{p}}|^2 + (-1+\cos\theta)}{1+\cos\theta}} + 1}{\sqrt{\frac{2/|v_{\vec{p}}|^2 + (-1+\cos\theta)}{1+\cos\theta}} - 1};$$

And also,

$$G(s) = -2 + |v_{\vec{p}}| \log\left(\frac{|v_{\vec{p}}| + 1}{|v_{\vec{p}}| - 1}\right).$$

So we have $\mathcal{F}$ calculated explicitly at this point. We mentioned earlier in the beginning of the section that $\mathcal{F}$ is complex. This is seen as follows. An inspection on both expressions, $G[t(\theta)]$ and $G[u(\theta)]$, reveal that they're real for all possible combination of values in the domain of the variable θ an the parameter $|v_{\vec{p}}|^2$. We can see a specific graphical representation for a specific value of the parameter,
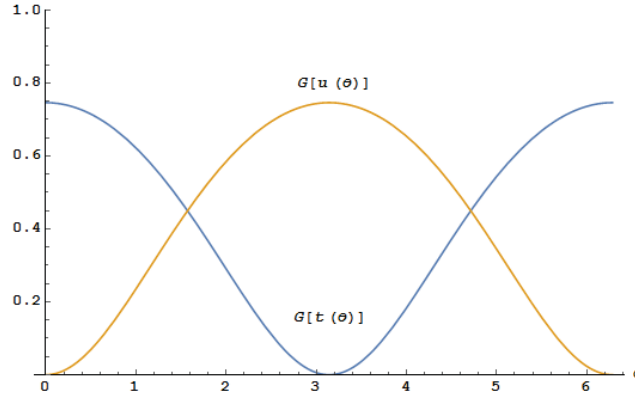
Figure 5.1: Evaluation of $G[t(\theta)]$, $G[u(\theta)]$ for all values of their domain in θ in the specific case where $|v_{\vec{p}}| = 0.8$;

But what about $G(s)$? Looking at the expression,

$$G(s) = -2 + |v_{\vec{p}}| \log\left(\frac{|v_{\vec{p}}| + 1}{|v_{\vec{p}}| - 1}\right)$$

it's easy to see that the logarithm is a complex number, since its argument is negative, as $|v_{\vec{p}}| < 1.$.

We shall treat this inconvenience by separating the domains for which the function produces imaginary and real parts. The logarithm can be re-written as,

$$\log\left(-1 \cdot \frac{1 + |v_{\vec{p}}|}{1 - |v_{\vec{p}}|}\right) = \log\left(e^{i\pi}\right) + \log\frac{1 + |v_{\vec{p}}|}{1 - |v_{\vec{p}}|} = i\pi + \log\frac{1 + |v_{\vec{p}}|}{1 - |v_{\vec{p}}|}.$$

Then we have that,

$$G(s) = -2 + |v_{\vec{p}}| \log\frac{1 + |v_{\vec{p}}|}{1 - |v_{\vec{p}}|} + i\pi|v_{\vec{p}}|, \tag{5.44}$$

such that,

$$\mathrm{Re}\left\{G(s)\right\} = -2 + |v_{\vec{p}}| \log\frac{1 + |v_{\vec{p}}|}{1 - |v_{\vec{p}}|}, \quad \text{and,} \quad \mathrm{Im}\left\{G(s)\right\} = \pi|v_{\vec{p}}|.$$

This implies that,

$$\mathrm{Re}\left\{\mathcal{F}[s, t(\theta), u(\theta)]\right\} = 12G[t(\theta)] + 8G[u(\theta)] + 4\mathrm{Re}\left\{G(s)\right\} + 8 \tag{5.45}$$

$$= 12G[t(\theta)] + 8G[u(\theta)] + 4|v_{\vec{p}}| \log\frac{1 + |v_{\vec{p}}|}{1 - |v_{\vec{p}}|},$$

and,

$$\mathrm{Im}\left\{\mathcal{F}[s, t(\theta), u(\theta)]\right\} = 4\pi|v_{\vec{p}}|. \tag{5.46}$$

We have $\mathcal{F} = \mathrm{Re}\left\{\mathcal{F}\right\} + i\mathrm{Im}\left\{\mathcal{F}\right\}$ fully described, to be used in the calculation of $|\mathcal{M}_{(\text{tree}+1\text{-loop})}|^2$.

$(\rho_A)_{out}$ with 1-loop corrections

From

$$i\mathcal{M}_{(\text{tree+1-loop})} = -2i\lambda - i(\frac{\lambda}{4\pi})^2\mathcal{F}$$

we calculate $|\mathcal{M}|^2 = \mathcal{M}_{(\text{tree+1-loop})}\mathcal{M}^*_{(\text{tree+1-loop})}$,

$$= \left(-2\lambda - (\frac{\lambda}{4\pi})^2\mathcal{F}\right)\left(-2\lambda - (\frac{\lambda}{4\pi})^2\mathcal{F}^*\right) = 4\lambda^2 + 2\lambda(\frac{\lambda}{4\pi})^2(\mathcal{F}+\mathcal{F}^*) + (\frac{\lambda}{4\pi})^4|\mathcal{F}|^2 \tag{5.47}$$

$$= 4\lambda^2 + 4\lambda(\frac{\lambda}{4\pi})^2 \operatorname{Re}\{\mathcal{F}\} + (\frac{\lambda}{4\pi})^4(\operatorname{Re}^2\{\mathcal{F}\} + \operatorname{Im}^2\{\mathcal{F}\}).$$

Take notice though that at this point the functions inside the functional $\mathcal{F}$ are still dependent explicitly on $E_{\vec{q}}$, so we can't substitute it with the expressions we calculated before when under the assumption that $E_{\vec{q}} \to E_{\vec{p}}$. But since we know the functional will take the form of the previously calculated expressions further on we will be able to substitute them after integrating the delta which sets $E_{\vec{q}} \to E_{\vec{p}}$.

From the expression for the reduced density operator,

$$(\rho_A)_{out} = 2E_{\vec{p}}L^3 |\vec{p}\,\rangle\langle\vec{p}\,| + \int\limits_{\vec{q}\neq\vec{p}} \frac{d^3\vec{q}}{(2\pi)^3} \frac{(2\pi\delta_E)^2 |\mathcal{M}|^2}{(2E_{\vec{q}})^3} |\vec{q}\,\rangle\langle\vec{q}\,|, \tag{5.48}$$

where we use 5.47.

We have the normalized operator,

$$(\rho_A)_{out} = \mathcal{N}(|\vec{p}\,\rangle\langle\vec{p}\,|(2E_{\vec{p}}L^3) + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4). \tag{5.49}$$

Where $\mathcal{I}_2$ is defined in the same way as before and ,

$$\mathcal{I}_3 \equiv \int\limits_{\vec{q}\neq\vec{p}} \frac{d^3\vec{q}}{(2\pi)^3} \frac{|\vec{q}\,\rangle\langle\vec{q}\,|[(2\pi)\delta(E)]^2}{(2E_{\vec{q}})^3} \left(4\lambda(\frac{\lambda}{4\pi})^2\operatorname{Re}\{\mathcal{F}\}\right);$$

$$\mathcal{I}_4 \equiv \int\limits_{\vec{q}\neq\vec{p}} \frac{d^3\vec{q}}{(2\pi)^3} \frac{|\vec{q}\,\rangle\langle\vec{q}\,|[(2\pi)\delta(E)]^2}{(2E_{\vec{q}})^3} \left(\frac{\lambda}{4\pi}\right)^4 (\operatorname{Re}^2\{\mathcal{F}\} + \operatorname{Im}^2\{\mathcal{F}\});$$

Following the same procedure as in the Tree-level calculation, by applying the normalization condition to find $\mathcal{N}$ we get,

$$\rho_{Aout} = \frac{|\,\vec{p}\,\rangle\,\langle\,\vec{p}\,|\,(2E_{\vec{p}}L^3) + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4}{(2E_{\vec{p}}L^3)^2 + \langle\mathcal{I}_2\rangle + \langle\mathcal{I}_3\rangle + \langle\mathcal{I}_4\rangle}. \tag{5.50}$$

Here similarly to $\langle\mathcal{I}_2\rangle$ in (5.28), we have,

$$\langle\mathcal{I}_3\rangle \equiv \frac{L^3}{(2\pi)} \int\limits_{\vec{q}\neq\vec{p}} d^3\vec{q}\;\frac{\delta^2(2E_{\vec{q}} - 2E_{\vec{p}})}{(2E_{\vec{q}})^2}\,\big(4\lambda(\frac{\lambda}{4\pi})^2\,\mathrm{Re}\,\{\mathcal{F}\}\,\big); \tag{5.51}$$

$$\langle\mathcal{I}_4\rangle \equiv \frac{L^3}{(2\pi)} \int\limits_{\vec{q}\neq\vec{p}} d^3\vec{q}\;\frac{\delta^2(2E_{\vec{q}} - 2E_{\vec{p}})}{(2E_{\vec{q}})^2}\,(\frac{\lambda}{4\pi})^4\big(\mathrm{Re}^2\,\{\mathcal{F}\} + \mathrm{Im}^2\,\{\mathcal{F}\}\,\big); \tag{5.52}$$

Take notice that, $\langle\mathcal{I}_2\rangle$, $\langle\mathcal{I}_3\rangle$ and $\langle\mathcal{I}_4\rangle$ are exactly the same integral aside the algebraic part pertinent to the dynamics of the collision.

We can still take (5.50) and rearrange it in a similar manner as in the Tree-level case by factoring the energy in the denominator and separating the terms,

$$\rho_{Aout} = \frac{|\,\vec{p}\,\rangle\,\langle\,\vec{p}\,|}{(2E_{\vec{p}}L^3)(1 + \mathcal{A}_{1loop})} + \tag{5.53}$$

$$\frac{1}{(1 + \mathcal{A}_{1loop})}\left(\frac{\mathcal{I}_2}{(2E_{\vec{p}}L^3)^2} + \frac{\mathcal{I}_3}{(2E_{\vec{p}}L^3)^2} + \frac{\mathcal{I}_4}{(2E_{\vec{p}}L^3)^2}\right),$$

where,

$$\mathcal{A}_{1loop} \equiv \frac{\langle\mathcal{I}_2\rangle}{(2E_{\vec{p}}L^3)^2} + \frac{\langle\mathcal{I}_3\rangle}{(2E_{\vec{p}}L^3)^2} + \frac{\langle\mathcal{I}_4\rangle}{(2E_{\vec{p}}L^3)^2}.$$

We already have $\langle\mathcal{I}_2\rangle$ calculated, now we need to calculate $\langle\mathcal{I}_3\rangle$ and $\langle\mathcal{I}_4\rangle$ in order to have $\mathcal{A}_{1loop}$ completely defined.

Calculation of $\langle \mathcal{I}_3 \rangle$

Remembering that,

$$\mathcal{F}(s,t,u) \equiv \mathcal{F}[s, t(\theta, E_{\vec{q}}), u(\theta, E_{\vec{q}})]; \tag{5.54}$$

$$\langle \mathcal{I}_3 \rangle = 4\lambda (\frac{\lambda}{4\pi})^2 \frac{L^3}{(2\pi)} \int\limits_{\vec{q} \neq \vec{p}} d^3 \vec{q} \ \frac{\delta^2(2E_{\vec{q}} - 2E_{\vec{p}})}{(2E_{\vec{q}})^2} \ \text{Re} \left\{ \mathcal{F}[s, t(\theta, E_{\vec{q}}), u(\theta, E_{\vec{q}})] \right\}; \tag{5.55}$$

We change the integration over momenta to energies,

$$\lambda (\frac{\lambda}{4\pi})^2 L^3 \int_0^\pi d\theta \sin\theta \int_m^{+\infty} dE_{\vec{q}} \ \frac{\sqrt{E_{\vec{q}}^2 - m^2}}{E_{\vec{q}}} \ \delta^2(2E_{\vec{q}} - 2E_{\vec{p}}) \ \text{Re} \left\{ \mathcal{F}[s, t(\theta, E_{\vec{q}}), u(\theta, E_{\vec{q}})] \right\}; \tag{5.56}$$

We can't integrate the solid angle isotropically because there's a θ dependency inside the real part of $\mathcal{F}$. We integrate over one energy delta and as expected set $E_{\vec{q}} \to E_{\vec{p}}$. Such that we have,

$$\langle \mathcal{I}_3 \rangle = \lambda (\frac{\lambda}{4\pi})^2 \frac{L^4}{4\pi} |v_{\vec{p}}| \int_0^\pi d\theta \sin\theta \ \text{Re} \left\{ \mathcal{F}[s, t(\theta), u(\theta)] \right\}; \tag{5.57}$$

And at this point the real part of the functional is indeed given by the previous expression we calculated in full,

$$\text{Re} \left\{ \mathcal{F}[s, t(\theta), u(\theta)] \right\} = 12 G[t(\theta)] + 8 G[u(\theta)] + 4|v_{\vec{p}}| \log \frac{1 + |v_{\vec{p}}|}{1 - |v_{\vec{p}}|}.$$

Thus, the integration over the angle θ of $\mathcal{F}$ is given by,

$$\int_0^\pi d\theta \sin\theta \ \text{Re} \left\{ \mathcal{F}[s, t(\theta), u(\theta)] \right\} = 12 \int_0^\pi d\theta \sin\theta G[t(\theta)] + 8 \int_0^\pi d\theta \sin\theta G[u(\theta)] \tag{5.58}$$

$$+ 4 \int_0^\pi d\theta \sin\theta |v_{\vec{p}}| \log \frac{1 + |v_{\vec{p}}|}{1 - |v_{\vec{p}}|}$$

The last term has no dependency in θ so the integration just produces a factor of 2.

The first term could be rewritten as,

$$I_{(u,t)} \equiv \int_0^\pi d\theta \sin\theta \big(12G[t(\theta)] + 8G[u(\theta)] \big)$$

Where the functions are given, as we know,

$$G[t(\theta)] = -2 + \sqrt{\frac{2/|v_{\vec{p}}|^2 - (1+\cos\theta)}{1-\cos\theta}} \log \frac{\sqrt{\frac{2/|v_{\vec{p}}|^2-(1+\cos\theta)}{1-\cos\theta}}+1}{\sqrt{\frac{2/|v_{\vec{p}}|^2-(1+\cos\theta)}{1-\cos\theta}}-1}; \qquad (5.59)$$

$$G[u(\theta)] = -2 + \sqrt{\frac{2/|v_{\vec{p}}|^2 + (-1+\cos\theta)}{1+\cos\theta}} \log \frac{\sqrt{\frac{2/|v_{\vec{p}}|^2+(-1+\cos\theta)}{1+\cos\theta}}+1}{\sqrt{\frac{2/|v_{\vec{p}}|^2+(-1+\cos\theta)}{1+\cos\theta}}-1};$$

And also,

$$G(s) = -2 + |v_{\vec{p}}| \log\left(\frac{|v_{\vec{p}}|+1}{|v_{\vec{p}}|-1} \right).$$

We perform this calculation using Mathematica by inputting the functions explicitly and integrating over θ. Which produces,

$$\mathcal{I}_{(u,t)} = 6\left(-10 + \log\frac{1/|v_{\vec{p}}|+1}{1/|v_{\vec{p}}|-1}\left(\frac{4}{|v_{\vec{p}}|}\right) + (\frac{1}{v_{\vec{p}}^2}-1)\left(\log\frac{1/|v_{\vec{p}}|+1}{1/|v_{\vec{p}}|-1}\right)^2 \right)$$

$$-4\left(15 - \log\frac{1/|v_{\vec{p}}|+1}{1/|v_{\vec{p}}|-1}\left(\frac{-4}{|v_{\vec{p}}|}\right) - (\frac{1}{v_{\vec{p}}^2}-1)\left(\log\frac{1/|v_{\vec{p}}|+1}{1/|v_{\vec{p}}|-1}\right)^2 \right)$$

$$= (-120 + \log\frac{1/|v_{\vec{p}}|+1}{1/|v_{\vec{p}}|-1}\left(\frac{40}{|v_{\vec{p}}|}\right)(\frac{10}{v_{\vec{p}}^2}-10)\left(\log\frac{1/|v_{\vec{p}}|+1}{1/|v_{\vec{p}}|-1}\right)^2 + 8|v_{\vec{p}}|\log\frac{1+|v_{\vec{p}}|}{1-|v_{\vec{p}}|}).$$

Then we calculate,

$$\langle\mathcal{I}_3\rangle = \frac{L^4}{4\pi}\lambda(\frac{\lambda}{4\pi})^2|v_{\vec{p}}|(-120 + \log\frac{1/|v_{\vec{p}}|+1}{1/|v_{\vec{p}}|-1}\left(\frac{40}{|v_{\vec{p}}|}\right) + \qquad (5.60)$$

$$+ (\frac{10}{v_{\vec{p}}^2}-10)\left(\log\frac{1/|v_{\vec{p}}|+1}{1/|v_{\vec{p}}|-1}\right)^2 + 8|v_{\vec{p}}|\log\frac{1+|v_{\vec{p}}|}{1-|v_{\vec{p}}|}).$$

If we call part of the expression with the explicit dependence on $|v_{\vec{p}}|$, $f(|v_{\vec{p}}|)$, then we have in compact notation,

$$\langle \mathcal{I}_3 \rangle = \frac{L^4}{4\pi} \lambda (\frac{\lambda}{4\pi})^2 f(|v_{\vec{p}}|), \tag{5.61}$$

If we compare with the result for $\langle \mathcal{I}_2 \rangle$, which we calculated to be

$$\langle \mathcal{I}_2 \rangle = \frac{L^4 \lambda^2}{2\pi} |v_{\vec{p}}|. \tag{5.62}$$

We realize that both expressions have the same form: dimensionally they're the same and have the appropriate powers of λ corresponding to their subscripts, multiplied by an explicit function of $|v_{\vec{p}}|$, which in the case of $\langle \mathcal{I}_3 \rangle$ is $f(|v_{\vec{p}}|)$ and $\langle \mathcal{I}_2 \rangle$ just $|v_{\vec{p}}|$ itself.

We have calculated $\langle \mathcal{I}_2 \rangle$, $\langle \mathcal{I}_3 \rangle$, we're just missing $\langle \mathcal{I}_4 \rangle$.

Calculation of $\langle \mathcal{I}_4 \rangle$

The first part of the calculation will be inherently the same as with $\langle \mathcal{I}_3 \rangle$, changing the variable of integration to the energy ($E_{\vec{p}}$) and integrating over one of the delta's such that $E_{\vec{q}} \to E_{\vec{p}}$. We aren't going to write all the intermediary steps again, instead we start from the equivalent of (5.57) for this case,

$$\langle \mathcal{I}_4 \rangle = (\frac{\lambda}{4\pi})^4 \frac{L^4}{16\pi} |v_{\vec{p}}| \int_0^\pi d\theta \sin\theta \ \left(\text{Re}^2 \left\{ \mathcal{F}[s, t(\theta), u(\theta)] \right\} + \text{Im}^2 \left\{ \mathcal{F}[s, t(\theta), u(\theta)] \right\} \right); \tag{5.63}$$

We know the real part squared is,

$$\text{Re}^2 \left\{ \mathcal{F}[s, t(\theta), u(\theta)] \right\} = \left(12 G[t(\theta)] + 8 G[u(\theta)] + 4|v_{\vec{p}}| \log \frac{1 + |v_{\vec{p}}|}{1 - |v_{\vec{p}}|} \right)^2,$$

and the imaginary part is,

$$\text{Im}^2 \left\{ \mathcal{F}[s, t(\theta), u(\theta)] \right\} = 16\pi^2 |v_{\vec{p}}|^2;$$

The imaginary part squared is not dependent of θ so the integral of that part is going produce a factor of 2. The expression is given by,

$$\int_0^\pi d\theta \sin\theta \ \left(\text{Re}^2 \left\{ \mathcal{F}[s, t(\theta), u(\theta)] \right\} \right) + 32\pi^2 |v_{\vec{p}}|^2.$$

What about the first term ? It's easy to see that the expression for $\mathrm{Re}^2 \{\mathcal{F}[s, t(\theta), u(\theta)]\}$ will become quite complicated, and unfortunately no analytical solution of the integral can be found. We will have to compute numerically the integration on θ, but obviously the result also depends on the parameter $|v_{\vec{p}}|$, so we will have to carry on the calculations with the implicit integration until it's time to extract numerical results for specific values of the velocity parameter. We define a compact notation so that we don't have to keep writing the expression in it's integral form,

$$I_{|v_{\vec{p}}|} \equiv \int_0^\pi d\theta \sin\theta \ \left(\mathrm{Re}^2 \{\mathcal{F}[s, t(\theta), u(\theta)]\} \right); \qquad (5.64)$$

We have $\langle \mathcal{I}_4 \rangle$ calculated as,

$$\langle \mathcal{I}_4 \rangle = (\frac{\lambda}{4\pi})^4 \frac{L^4}{16\pi} |v_{\vec{p}}| \left(I_{|v_{\vec{p}}|} + 32\pi^2 |v_{\vec{p}}|^2 \right) \qquad (5.65)$$

Although the expression still has an implicit integration, we know the result will surely yield an expression dependent only on the velocity parameter. Then if we compare the previously calculated expressions for $\langle \mathcal{I}_2 \rangle$ and $\langle \mathcal{I}_3 \rangle$, we realize that $\langle \mathcal{I}_4 \rangle$ has the same structure as well.

We have all the necessary integrals calculated in order to characterize fully the expression for Alice's reduced density operator, (5.53).

We can determine $\mathcal{A}_{1loop}$, which was previously defined as,

$$\mathcal{A}_{1loop} = \frac{\langle \mathcal{I}_2 \rangle}{(2E_{\vec{p}}L^3)^2} + \frac{\langle \mathcal{I}_3 \rangle}{(2E_{\vec{p}}L^3)^2} + \frac{\langle \mathcal{I}_4 \rangle}{(2E_{\vec{p}}L^3)^2}.$$

Substituting into the previous expression,

$$\langle \mathcal{I}_2 \rangle = \frac{L^4 \lambda^2}{2\pi} |v_{\vec{p}}|.$$

$$\langle \mathcal{I}_3 \rangle = \frac{L^4}{4\pi} \lambda (\frac{\lambda}{4\pi})^2 f(|v_{\vec{p}}|),$$

$$\langle \mathcal{I}_4 \rangle = (\frac{\lambda}{4\pi})^4 \frac{L^4}{16\pi} |v_{\vec{p}}| \left(I_{|v_{\vec{p}}|} + 32\pi^2 |v_{\vec{p}}|^2 \right)$$

We have,

$$\mathcal{A}_{1loop} = \frac{\lambda^2}{8\pi} \frac{|v_{\vec{p}}|}{\bar{E}_{\vec{p}}} + \frac{\lambda}{16\pi} \left(\frac{\lambda}{4\pi} \right)^2 \frac{f(|v_{\vec{p}}|)}{\bar{E}_{\vec{p}}^2} + \left(\frac{\lambda}{4\pi} \right)^4 \frac{|v_{\vec{p}}| \left(I_{v_{\vec{p}}} + 32\pi^2 v_{\vec{p}}^2 \right)}{64\pi \bar{E}_{\vec{p}}^2}. \quad (5.66)$$

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As such we have the operator completely defined,

$$(\rho_A)_{out} = \frac{|\vec{p}\rangle\langle\vec{p}|}{(2E_{\vec{p}}L^3)(1 + \mathcal{A}_{1loop})} + \qquad (5.67)$$

$$+ \frac{1}{(1 + \mathcal{A}_{1loop})}\left(\frac{\mathcal{I}_2}{(2E_{\vec{p}}L^3)^2} + \frac{\mathcal{I}_3}{(2E_{\vec{p}}L^3)^2} + \frac{\mathcal{I}_4}{(2E_{\vec{p}}L^3)^2}\right);$$

Writing the integrals explicitly we have,

$$(\rho_A)_{out} = \frac{|\vec{p}\rangle\langle\vec{p}|}{(2E_{\vec{p}}L^3)(1 + \mathcal{A}_{1loop})} + \qquad (5.68)$$

$$+ \frac{1}{(2E_{\vec{p}}L^3)^2(1 + \mathcal{A}_{1loop})}\int\limits_{\vec{q}\neq\vec{p}}\frac{d^3\vec{q}}{(2\pi)^3}\frac{\left(2\pi\delta_E\right)^2|\vec{q}\rangle\langle\vec{q}|}{(2E_{\vec{q}})^3}\times$$

$$\times\left(4\lambda^2 + 4\lambda(\frac{\lambda}{4\pi})^2\mathrm{Re}\left\{\mathcal{F}\right\} + (\frac{\lambda}{4\pi})^4(\mathrm{Re}^2\left\{\mathcal{F}\right\} + \mathrm{Im}^2\left\{\mathcal{F}\right\})\right);$$

The real and imaginary parts inside the integral can't be substituted at this point by any of the expressions we calculated before, because they are still explicitly dependent on the energy, as such in order to simplify the notation we define the following expression,

$$\left(4\lambda^2 + 4\lambda(\frac{\lambda}{4\pi})^2\mathrm{Re}\left\{\mathcal{F}[s, t(\theta', E_{\vec{q}}), u(\theta', E_{\vec{q}})]\right\} + \qquad (5.69)\right.$$

$$+ (\frac{\lambda}{4\pi})^4(\mathrm{Re}^2\left\{\mathcal{F}[s, t(\theta', E_{\vec{q}}), u(\theta', E_{\vec{q}})]\right\} +$$

$$\left. + \mathrm{Im}^2\left\{\mathcal{F}[s, t(\theta', E_{\vec{q}}), u(\theta', E_{\vec{q}})]\right\}\right) \equiv \Omega[s, t(\theta', E_{\vec{q}}), u(\theta', E_{\vec{q}})]$$

We explicitly denote this angle θ', so that there's no confusion with the angle θ still present in the implicit integration $I_{|v_{\vec{p}}|}$.

The operator can be written more succinctly as,

$$(\rho_A)_{out} = \frac{|\vec{p}\rangle\langle\vec{p}|}{(2E_{\vec{p}}L^3)(1 + \mathcal{A}_{1loop})} + \qquad (5.70)$$

$$+\frac{1}{(2E_{\vec{p}}L^3)^2(1 + \mathcal{A}_{1loop})}\int\limits_{\vec{q}\neq\vec{p}}\frac{d^3\vec{q}}{(2\pi)^3}\frac{\left(2\pi\delta_E\right)^2|\vec{q}\rangle\langle\vec{q}|}{(2E_{\vec{q}})^3}\left(\Omega[s, t(\theta', E_{\vec{q}}), u(\theta', E_{\vec{q}})]\right);$$

And now we can calculate the entropy associated with this operator.

Entropy with 1-loop corrections

Alice's reduced operator with 1-loop corrections is completely defined and we shall proceed to calculate the entropy associated to it. The same principles apply as with the Tree-level calculation, the density operator is obviously diagonal. As such, the entropy will still be given by,

$$S_E = -\sum_i (\rho_i \log \rho_i) = -\rho_p \log \rho_p - \frac{L^3}{(2\pi)^3} \int_{-\infty}^{+\infty} d\vec{k} \ (\rho_k \log \rho_k); \ \text{With,} \ k \neq p. \tag{5.71}$$

Where the elements are given by (4.33),

$$\rho_n = \frac{\langle \vec{n} \mid \rho \mid \vec{n} \rangle}{\sqrt{2E_{\vec{n}}L^3}\sqrt{2E_{\vec{n}}L^3}}. \tag{5.72}$$

If we look back at the operator in Tree-level (5.32) and compare it with the operator with 1-loop corrections (5.70), we realize that the first term is exactly the same if $\mathcal{A}_{tree} \to \mathcal{A}_{1loop}$. So we can deduce that the corresponding element of the matrix will be also of the same form, and consequently the entropy term derived from it as well. Let's see if this is true.

Using (5.70) as the expression for $(\rho_p)_{out}$ (i.e to 1-loop),

$$(\rho_p) = \frac{\langle \vec{p} \mid (\rho_A)_{out} \mid \vec{p} \rangle}{\sqrt{2E_{\vec{p}}L^3}\sqrt{2E_{\vec{p}}L^3}} = \frac{\langle \vec{p} \mid \vec{p} \rangle \langle \vec{p} \mid \vec{p} \rangle}{(2E_{\vec{p}}L^3)^2(1 + \mathcal{A}_{1loop})} = \frac{1}{1 + \mathcal{A}_{1loop}}; \tag{5.73}$$

As such we calculate the first term of the entropy to be,

$$-\rho_p \log \rho_p = -\frac{1}{1 + \mathcal{A}_{1loop}} \log \left(\frac{1}{1 + \mathcal{A}_{1loop}} \right) = \frac{\log(1 + \mathcal{A}_{1loop})}{1 + \mathcal{A}_{1loop}}. \tag{5.74}$$

This is exactly the same term as the entropy in the Tree-level case, but now with $\mathcal{A}_{1loop}$ instead of $\mathcal{A}_{tree}$.

The second term of the operator (5.70) seems to be more complicated than the one from (5.32). In fact the term is indeed more cumbersome algebraically but the expression for the element associated with it will be of the same form as in the Tree-level case. We get that (ρ_k) is given by,

$$\frac{1}{(2E_{\vec{k}}L^3)(2E_{\vec{p}}L^3)^2(1 + \mathcal{A}_{tree})} \int\limits_{\vec{q} \neq \vec{p}} \frac{d^3\vec{q}}{(2\pi)^3} \frac{\left(2\pi\delta_E\right)^2 \ \langle \vec{k} \mid \vec{q} \rangle \langle \vec{q} \mid \vec{k} \rangle}{(2E_{\vec{q}})^3} \Omega[s, t(\theta', E_{\vec{q}}), u(\theta', E_{\vec{q}})];$$

We integrate $\vec{q}$ over one of the deltas, such that $\vec{q} \to \vec{k}$. And we get,

$$(\rho_k) = \left(\frac{2\pi\delta(2E_{\vec{k}} - 2E_{\vec{p}})}{2E_{\vec{k}}2E_{\vec{p}}L^3}\right)^2 \frac{\Omega[s, t(\theta', E_{\vec{k}}), u(\theta', E_{\vec{k}})]}{1 + \mathcal{A}_{1loop}}. \qquad (5.75)$$

If we are to compare this formula with the one calculated earlier for the same element in the Tree-level calculation, we realize that it's exactly the same if $4\lambda^2 \to \Omega[s, t(\theta', E_{\vec{k}}), u(\theta', E_{\vec{k}})]$.

Using this expression and inputting it into the pertinent part of (5.71) we calculate the second term of the entropy to be,

$$-\frac{L^3}{(2\pi)^3} \int_{\vec{k} \neq \vec{p}} d\vec{k} \ \left(\frac{2\pi\delta(2E_{\vec{k}} - 2E_{\vec{p}})}{2E_{\vec{k}}2E_{\vec{p}}L^3}\right)^2 \frac{\Omega[s, t(\theta', E_{\vec{k}}), u(\theta', E_{\vec{k}})]}{1 + \mathcal{A}_{1loop}} \times$$

$$\times \log\left[\left(\frac{2\pi\delta(2E_{\vec{k}} - 2E_{\vec{p}})}{2E_{\vec{k}}2E_{\vec{p}}L^3}\right)^2 \frac{\Omega[s, t(\theta', E_{\vec{k}}), u(\theta', E_{\vec{k}})]}{1 + \mathcal{A}_{1loop}}\right];$$

We calculate the integral in the same manner as before. First we change integration into spherical coordinates and then change the variable of momenta to energy,

$$-\frac{L^{-3}}{16E_{\vec{p}}(1 + \mathcal{A}_{1loop})} \int_0^\pi d\theta' \sin\theta' \int_m^{+\infty} dE_{\vec{q}} \ \frac{\sqrt{E_{\vec{q}}^2 - m^2}}{E_{\vec{q}}} \delta^2(2E_{\vec{k}} - 2E_{\vec{p}}) \times$$

$$\times \Omega[s, t(\theta', E_{\vec{q}}), u(\theta', E_{\vec{q}})] \times \log\left[\left(\frac{2\pi\delta(2E_{\vec{k}} - 2E_{\vec{p}})}{2E_{\vec{k}}2E_{\vec{p}}L^3}\right)^2 \frac{\Omega[s, t(\theta', E_{\vec{q}}), u(\theta', E_{\vec{q}})]}{1 + \mathcal{A}_{1loop}}\right],$$

which we then integrate over the delta of energy to set $E_{\vec{k}} \to E_{\vec{p}}$. This eliminates the explicit dependence on the energy inside the Mandelstam functions of the functional, which produce

$$\Omega[s, t(\theta'), u(\theta')] = \left(4\lambda^2 + 4\lambda(\frac{\lambda}{4\pi})^2 \text{Re}\left\{\mathcal{F}[s, t(\theta'), u(\theta')]\right\} + \qquad (5.76)$$

$$+ (\frac{\lambda}{4\pi})^4 (\text{Re}^2\left\{\mathcal{F}[s, t(\theta'), u(\theta')]\right\} + \text{Im}^2\left\{\mathcal{F}[s, t(\theta'), u(\theta')]\right\}\right),$$

this expression can be explicitly calculated by using the algebraic relationships we found previously. One major difference in this calculation from the Tree-level one is that, when changing over to spherical we couldn't integrate the solid angle isotropically, because there is an explicit dependence on θ'.

So there's an additional integration that we need to solve, and once again as it were the case when trying to calculate A_{1loop} an analytical solution won't be possible. We present this term of the entropy in it's integral form,

$$\frac{|v_{\vec{p}}|/64\pi}{(\bar{E}_{\vec{p}})^2(1+\mathcal{A}_{1loop})}\int_0^\pi \sin\theta'\,\Omega[s,t(\theta'),u(\theta')]\log\Big[\frac{16(\bar{E}_{\vec{p}})^4(1+\mathcal{A}_{1loop})}{\Omega[s,t(\theta'),u(\theta')]}\Big]d\theta'.$$

Adding to the first term, we have

$$\Delta S_{E\ 1loop}=S_{E\ (out)}=\frac{\log\,(1+\mathcal{A}_{1loop})}{(1+\mathcal{A}_{1loop})}+ \tag{5.77}$$

$$\frac{|v_{\vec{p}}|/64\pi}{(\bar{E}_{\vec{p}})^2(1+\mathcal{A}_{1loop})}\int_0^\pi \sin\theta'\,\Omega[s,t(\theta'),u(\theta')]\log\Big[\frac{16(\bar{E}_{\vec{p}})^4(1+\mathcal{A}_{1loop})}{\Omega[s,t(\theta'),u(\theta')]}\Big]d\theta'.$$

This is the variation of the entropy of entanglement for an initial separable state to 1-loop. It is completely defined knowing that, $\mathcal{A}_{1loop}$ is given by the expression (5.66) and $\Omega[s,t(\theta'),u(\theta')]$ by (5.76), with imaginary and real parts determined in (5.45) and (5.46).

It's interesting to note that if we take the limiting case where the 1-loop contributions are zero, which is equivalent to say, $\Omega[s,t(\theta'),u(\theta')])\to 4\lambda^2$ and $\mathcal{A}_{1loop}\to\mathcal{A}_{tree}$ then we reproduce the previous result of the entropy variation for Tree-level,

$$\Delta S_{E\ tree}=(S_E)_{out}=\frac{\log\,(1+\mathcal{A}_{tree})}{1+\mathcal{A}_{tree}}+\lambda^2|v_{\vec{p}}|\frac{\log\,\big((4\bar{E}_{\vec{p}}^4(1+\mathcal{A}_{tree}))/\lambda^2\big)}{8\pi\bar{E}_{\vec{p}}^2(1+\mathcal{A}_{tree})},$$
$$\tag{5.78}$$

which gives positive reinforcement that the calculations are coherent.

Cross Sections

We have reasons to believe there might be some relationship between the variation of entropy and the differential cross-section, since both of these entities are proportional themselves to $|\mathcal{M}|^2$. We will try to find this relationship. If we take the expression (5.78) and expand it for the case when $\mathcal{A}$ is small enough, which is up to $(\lambda \sim 2)$. The expression will become,

$$\Delta S_{E\,tree} = \frac{4\lambda^2}{32\pi} \left(\frac{|v_{\vec{p}}|}{\bar{E}^2} \right) (1 - \log 4\lambda^2 + \log 16\bar{E}^4).$$

Knowing that the definition of the differential cross-section in the CM is,

$$\left(\frac{d\sigma}{d\Omega} \right)_{CM} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{\vec{p}}^2}, \tag{5.79}$$

it can be promptly seen that for $|\mathcal{M}|^2 = 4\lambda^2$,

$$\Delta S_{E\,tree} = \left(\frac{d\sigma}{d\Omega} \right)_{CM} \frac{2\pi}{L^2} |v_{\vec{p}}| \log \left(\frac{4\bar{E}^4 e}{\lambda^2} \right).$$

Thus the profile of the variation of the entropy of entanglement has an intimate relationship of proportionality with the collision's differential cross-section. At least for the regime in question of weak coupling, which is respected as long as the collision remains purely elastic (see "**Comment on Unitarity**", Chapter4). This provides an inkling onto a possible mechanism to measure the variation of entropy by experimentally measuring the cross sections of such elastic collisions.

We could integrate it still so that we get it in terms of the total cross-section σ_{CM}, since there's no dependence on any angle we just have to divide by 4π, thus getting,

$$\Delta S_{E\,tree} = \left(\frac{\sigma_{CM}}{2L^2} \right) |v_{\vec{p}}| \log \left(\frac{4\bar{E}^4 e}{\lambda^2} \right) \tag{5.80}$$

If we add the loop corrections to the calculation, $|\mathcal{M}|^2$ will become more complicated and it's not so easy to factor out the terms pertaining to the cross-section to give the entropy in such a form. But we can still find a relationship between these two entities. The total cross section is ,

$$\sigma_{CM} = \frac{1}{64\pi^2 E_{\vec{p}}^2} \int d\phi d\theta \sin\theta |\mathcal{M}|^2 =$$

$$\frac{1}{32\pi E_{\vec{p}}^2} \int_0^\pi d\theta \sin\theta \left(4\lambda^2 + 4\lambda \left(\frac{\lambda}{4\pi}\right)^2 \mathrm{Re}\left\{\mathcal{F}\right\} + \left(\frac{\lambda}{4\pi}\right)^4 \left(\mathrm{Re}^2\left\{\mathcal{F}\right\} + \mathrm{Im}^2\left\{\mathcal{F}\right\}\right)\right).$$

These integrals have been solved before.

$$\sigma_{CM} = \frac{1}{32\pi E_{\vec{p}}^2} \left(8\lambda^2 + 4\lambda \left(\frac{\lambda}{4\pi}\right)^2 f(|v_{\vec{p}}|) + \left(\frac{\lambda}{4\pi}\right)^4 \left(\mathcal{I}_{v_{CM}} + 32\pi^2 |v_{\vec{p}}|^2\right)\right),$$

comparing this expression with the one for $\mathcal{A}$ (5.31) we easily notice that,

$$\mathcal{A}_{tree} = \frac{\sigma_{CM}}{2L^2} |v_{\vec{p}}|.$$

Now, let us take the general form of the entropy with 1-loop corrections (5.77) and expand it for a small $\mathcal{A}$, the formula will be such that when we substitute it with this last equality we get,

$$\Delta S_{1loop} = \frac{\sigma_{CM}}{2L^2} |v_{\vec{p}}| \cdot \log\left(16\bar{E}^4 e\right) - \frac{|v_{\vec{p}}|}{64\pi \bar{E}^2} \int_0^\pi \sin\theta' \ \Omega[s, t(\theta'), u(\theta')] \times \quad (5.81)$$

$$\times \log \Omega[s, t(\theta'), u(\theta')] d\theta' + \mathcal{O}(\mathcal{A}^2).$$

The way in which the variation of the entropy is related to the total cross section is not so evident in this case. It doesn't factor out directly as with the previous case in Tree-level.

For $\Omega[s, t(\theta'), u(\theta')] \to 4\lambda^2$ (Tree-level) there's no θ' dependency and the logarithm can be taken out from the integral, which reduces to,

$$\Delta S_{E \ tree} = \left(\frac{\sigma_{CM}}{2L^2}\right) |v_{\vec{p}}| \log\left(\frac{4\bar{E}^4 e}{\lambda^2}\right). \qquad (5.82)$$

Again reproducing the Tree-level result.

5.1.4 Graphical Solutions

In order for us to be able to get graphical solutions we must choose a prototype, to avoid the divergences in the expressions. We previously defined,

$$\bar{E}_{\vec{p}} = E_{\vec{p}}L = \gamma_{\vec{p}}\bar{m}$$

Where we said that $\bar{m} = mL$. Now we choose the specific value for which, $\bar{m} = 1$ such that,

$$\bar{E}_{\vec{p}} = \gamma_{\vec{p}} = \frac{1}{\sqrt{1 - v_{\vec{p}}^2}}.$$

Then in both expressions of the variation, in Tree-level and to 1-loop,

$$\Delta S_{E\ 1loop} = S_{E\ (out)} = \frac{\log\left(1 + \mathcal{A}_{1loop}\right)}{(1 + \mathcal{A}_{1loop})} +$$

$$\frac{|v_{\vec{p}}|/64\pi}{(\bar{E}_{\vec{p}})^2(1 + \mathcal{A}_{1loop})} \int_0^\pi \sin\theta'\,\Omega[s, t(\theta'), u(\theta')] \log\left[\frac{16(\bar{E}_{\vec{p}})^4(1 + \mathcal{A}_{1loop})}{\Omega[s, t(\theta'), u(\theta')]}\right] d\theta'.$$

$$\Delta S_{E\ tree} = (S_E)_{out} = \frac{\log\left(1 + \mathcal{A}_{tree}\right)}{1 + \mathcal{A}_{tree}} + \lambda^2|v_{\vec{p}}|\frac{\log\left((4\bar{E}_{\vec{p}}^4(1 + \mathcal{A}_{tree}))/\lambda^2\right)}{8\pi\bar{E}_{\vec{p}}^2(1 + \mathcal{A}_{tree})}.$$

We shall set,

$$\bar{E}_{\vec{p}} \to \frac{1}{\sqrt{1 - v_{\vec{p}}^2}};$$

Such that, both entropies will becomes functions of the velocity $\Delta S_{E\ tree}(|v_{\vec{p}}|)$ and $\Delta S_{E\ 1loop}(|v_{\vec{p}}|)$. And our plots will be of the variation as a function of the said velocity, for various values of λ. But to what values of λ is the calculation valid ? We know that lambda should converge since the contributions of the higher order terms need to be increasingly smaller, that is a necessity imposed by the diagrammatic expansion we made on $\hat{S}$.

We have found a conservative [1] upper bounded limit of $\lambda = 2$ by looking at the behavior of the graphical solutions themselves.

All the graphics are of the form (ΔS_E vs $|v_{\vec{p}}|$). The Tree-level contribution is in orange and the "complete" contribution (which is the Tree-level with the higher 1-loop corrections) is in blue. Beneath every graph we're going to present the maximum values for both terms of ΔS_E and the corresponding preferred velocity for which the the maximum occurs ($|v_{\vec{p}}^*|$). The graphics are,
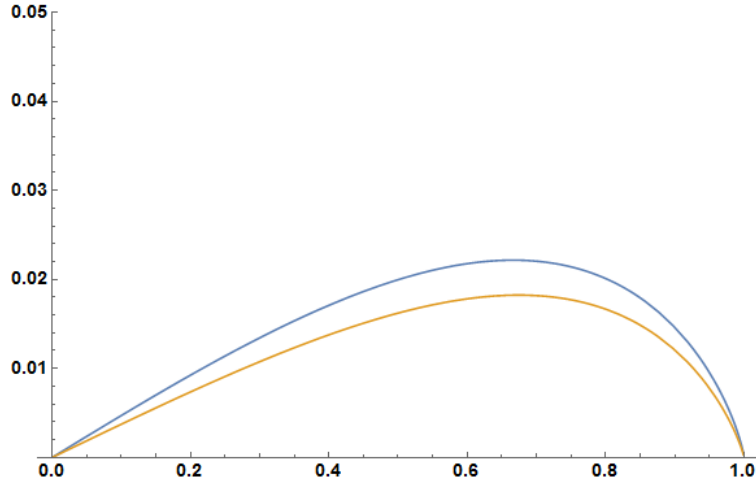
Figure 5.2: $\Delta S_E(|v_{\vec{p}}|),\quad \lambda = 0.5$

$$\text{Máx.}\Delta S_{E\ Tree} = 0.01824;\ |v_{\vec{p}}^*| = 0.67453$$
$$\text{Máx.}\Delta S_{E\ complete} = 0.02216;\ |v_{\vec{p}}^*| = 0.66639$$

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[1] This means that it was estimated in a cautious manner, using the optical theorem [26], bearing in mind the relation between entropy of entanglement and cross section (5.82). With basis on the values of the velocities obtained at the maximum of entaglement for a given coupling, see fig.(5.6), this yields λ 2.
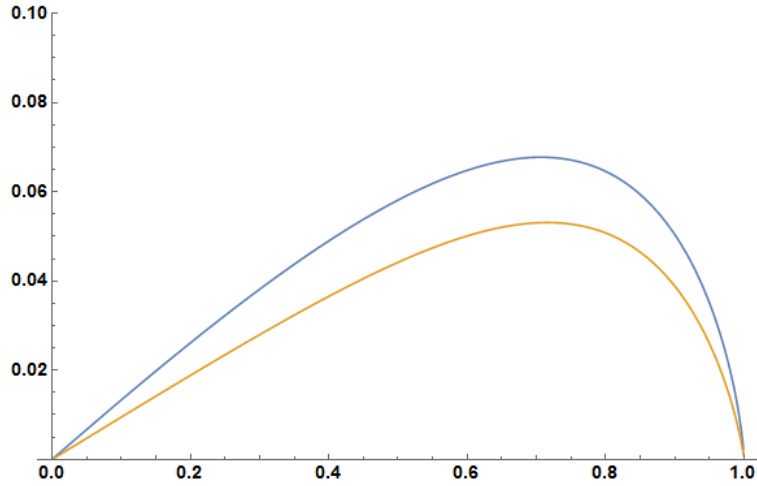
Figure 5.3: $\Delta S_E(|v_{\vec{p}}|)$, $\lambda = 1$

Máx.$\Delta S_{E\ Tree} = 0.05308$; $|v_{\vec{p}}^*| = 0.71553$
Máx.$\Delta S_{E\ complete} = 0.06772$; $|v_{\vec{p}}^*| = 0.70653$
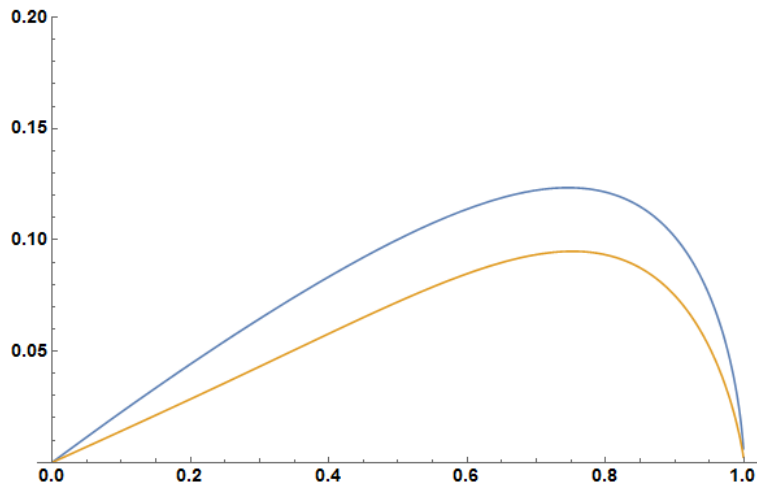
Figure 5.4: $\Delta S_E(|v_{\vec{p}}|)$, $\lambda = 1.5$

Máx.$\Delta S_{E\ Tree} = 0.09484$; $|v_{\vec{p}}^*| = 0.75226$
Máx.$\Delta S_{E\ complete} = 0.12344$; $|v_{\vec{p}}^*| = 0.74568$
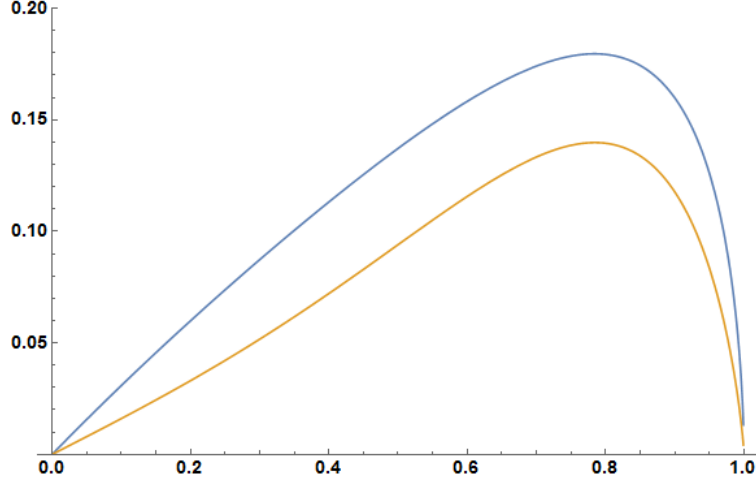
Figure 5.5: $\Delta S_E(|v_{\vec{p}}|), \quad \lambda = 2$

Máx.$\Delta S_{E\,Tree} = 0.13979;\ |v_{\vec{p}}^*| = 0.78434$
Máx.$\Delta S_{E\,complete} = 0.17956;\ |v_{\vec{p}}^*| = 0.78386$

As we can see the 1-loop corrections always increase the variation of the entropy, and this increment gets bigger as λ values increase (as one would expect). It seems that there is always a "preferred" velocity of entanglement for which the variation is maximum ($|v_{\vec{p}}^*|$), and there's an explicit dependence of such a velocity on the coupling constant, it increases with it. The 1-loop corrections also seems to indicate that the preferred velocity is lower than the result produced by the Tree-level contribution.

We are going to analyse the evolution of the preferred velocities ($|v_{\vec{p}}^*|$), associated to the Tree-level and 1-loop contributions, plotted against the coupling constant (λ). We use more values of λ than the ones present in the previous plots of the entropy's variation, in order to better evaluate the way the velocity evolves. The values of $|v_{\vec{p}}^*|$ in Tree-level are represented by discs, and the values with 1-loop corrections by squares,
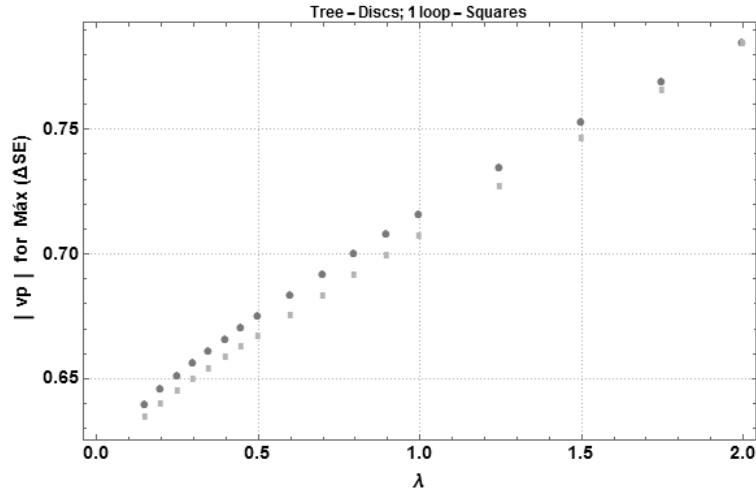
Figure 5.6: $|v_{\vec{p}}^*|$ vs. λ : Tree-level(Discs) and 1-loop(Squares)

By looking at the collection of values for $|v_{\vec{p}}^*|$ in Tree-level and 1-loop, we witness they evolve in pretty much the same way, as it should be, since both calculations aim to describe the same phenomena. But just like we witnessed before, the contributions of 1-loop corrections evidently show that the preferred velocity is always a bit smaller than the values predicted by the Tree level calculation, almost in a constant manner until $\approx \lambda = 1.5$, at that point the gap between the velocities starts to become smaller and when $\lambda = 2$ it's virtually non existent. Actually we should stress that in this region one isn't entirely sure if the "weak coupling" approximation is completely respected, so it might be that this behaviour changes because we are leaving the region for which the unitarity is respected. At least in the regime of "weak coupling", it seems that the profile of the evolution of such velocities in 1-loop could be approximately given by the profile of the Tree-level calculation with a negative "shift" applied to it.

Now we analyse the evolution of the maximum values the entropy's variation plotted against λ.
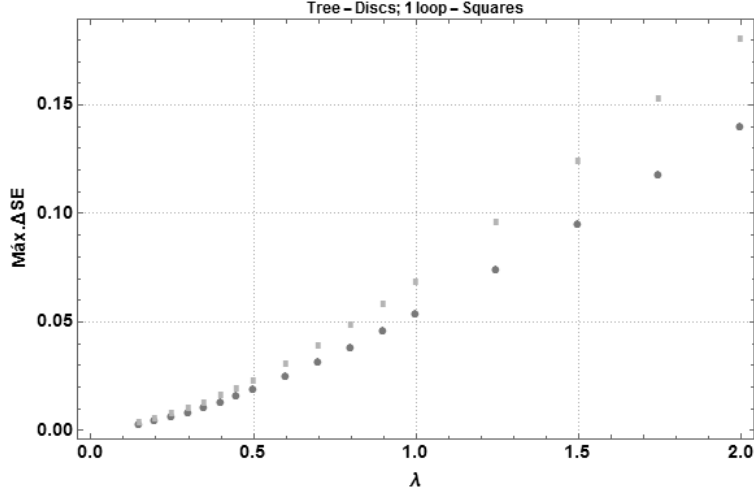
Figure 5.7: Máx(ΔS_E) vs. λ : Tree-level(Discs) and 1-loop(Squares)

The values of Máx(ΔS_E) also increase for small values of λ approximately in the same way, and the values computed for 1-loop are only marginally higher than the Tree-level ones until $\lambda = 0.5$. After $\lambda = 0.5$ the difference in the contributions starts increased with λ and the maximum difference is of order ≈ 0.05.

At this point we realize that the scattering procedure behaves appropriately as an EceM, and it seems the velocity parameters of the "machine" change the optimum configuration for which it produces the best possible outcome, depending on whether one considers only the Tree-level result or take the 1-loop corrections into account. The results seem to indicate that the EceM "works better" when 1-loop corrections are included. Two effects are at work. First one sees from fig.(5.7) that for a fixed value of the coupling, a larger entropy of entanglement is obtained when they are taken into account. Secondly, as indicated in fig.(5.6), their inclusion lowers the CM velocities needed to reach the maximum of entanglement, as compared to the case at Tree-level. This is interesting to know because even if the velocities differ by amounts as low as ≈ 0.01 near $|v_{\vec{p}}| = 0.7$ (in the case of $\lambda = 1$ for example), there would be energies differences of order $\approx 0.025 \times (mass)$ to get the maximally entangled result.

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5.2 $|\Psi\rangle_{in}$ is Entangled

Now we'll proceed with the calculation of the entropy's variation, just like before, with the added difference that the in-state is entangled before the collision as opposed to the separable state used previously. This implies that we have a non null $(S_E)_{in}$, which we have to calculate as well with $(S_E)_{out}$, in order to determine ΔS_E.

What kind of entangled state are we talking about ? We propose the following state,

$$|\Psi\rangle_{in} = \alpha_p \, |\vec{p}_1, \vec{p}_2\rangle + \alpha_k \, |\vec{k}_1, \vec{k}_2\rangle \, , \text{for all} \, \left\{ \vec{p}_1 \neq \vec{k}_1; \vec{p}_2 \neq \vec{k}_2 \right\}; |\alpha_p|^2 + |\alpha_k|^2 = 1.$$
(5.83)

Here the inequality constraints on the momenta won't allow for the factorization of the base states. We shall comment on the choice of the proposed entangled state. This state is akin to the entangled states showed in Chapter2 when we presented entanglement for spin-$\frac{1}{2}$ systems, but there's one major difference to take into account, which is the number of dimensions of the Hilbert spaces.

Such states of Chapter2 existed in a 4-dimensional $(2 \otimes 2)$ Hilbert space and we realized that there were a limited amount of states that had no degree of factorization, they were $\left|E_0^-\right\rangle$, $\left|E_0^+\right\rangle$, $\left|E_1^-\right\rangle$ and $\left|E_1^+\right\rangle$, 4 of them because the system was described by a 4-dimensional Hilbert space.

But what about our system in the space of momenta ? We know this space to be infinite, because unlike spin-$\frac{1}{2}$ which has only two degrees of freedom (i.e up/down) (and hence it only takes two base states to characterize the full space), momentum has an infinite amount of degrees of freedom and as such it stands to reason that there could be an infinite amount of states which could not be factored in any way. This is true and the fact is that the state (5.83) is just one of an infinite amount of entangled states possible. We could keep adding two labeled states forever (since the basis is infinite we wouldn't run out of vectors) and impose the inequality conditions for every label. Then for every new state added we would have a different entangled state and we would end up with an infinite amount of them. Unluckily there is no apparent easy way to describe all the degrees of the most general description of a state like this. Thus we decide to pick simplest possible entangled state. A state

which is by no means invalid since, as we'll see, it can be reduced to known special cases and still produce curious results different than a separable state.

5.2.1   Calculating $(S_E)_{in}$

The first thing we shall do is to go to the CM frame, such that the state becomes,

$$|\Psi\rangle_{in} = \alpha_p \,|\vec{p}, -\vec{p}\rangle + \alpha_k \,|\vec{k}, -\vec{k}\rangle \,, \; \forall \left\{\vec{p} \neq \vec{k}\right\}; |\alpha_p|^2 + |\alpha_k|^2 = 1. \qquad (5.84)$$

Following the recipe (4.4), we now need to calculate the density operator of $|\Psi\rangle_{in}$, to be able to compute it's entropy.

We calculate the density operator ρ_{in},

$$\rho_{in} = |\,\Psi\,\rangle_{in\,in}\langle\,\Psi\,| = \left(\alpha_p \,|\vec{p}, -\vec{p}\rangle + \alpha_k \,|\vec{k}, -\vec{k}\rangle\right)\left(\langle\vec{p}, -\vec{p}|\,\alpha_p^* + \langle\vec{k}, -\vec{k}|\,\alpha_k^*\right)$$

$$\tag{5.85}$$

$$= |\alpha_p|^2 \,|\vec{p}, -\vec{p}\rangle\,\langle\vec{p}, -\vec{p}| + \alpha_p\alpha_k^* \,|\vec{p}, -\vec{p}\rangle\langle\vec{k}, -\vec{k}| +$$

$$+ \alpha_k\alpha_p^* \,|\vec{k}, -\vec{k}\rangle\,\langle\vec{p}, -\vec{p}| + |\alpha_k|^2 \,|\vec{k}, -\vec{k}\rangle\,\langle\vec{k}, -\vec{k}|\,,$$

$$\text{for} \quad \left\{\vec{p} \neq \vec{k};\right\}.$$

Now we trace over to calculate Alice's reduced operator, using the trace operator (4.23). It's noticeable that the crossed terms will be null due the conditioning inequalities. That is, the result of such inner products will be proportional to $\delta^{(3)}(\vec{p} - \vec{k})$, and we know that $\vec{p} \neq \vec{k}$, so they're bound to be zero.

From the first term we get,

$$|\alpha_p|^2 \int_{\infty}^{\infty} \frac{d^3\vec{n}}{(2\pi)^3} \frac{|\vec{p}\rangle\,\langle\vec{p}|}{2E_{\vec{n}}} \left(2E_{\vec{p}}(2\pi)^3\delta^{(3)}(\vec{n} + \vec{p})\right)^2. \qquad (5.86)$$

Integrating we set, $n \to -\vec{p}$, and it yields,

$$|\alpha_p|^2 2E_{\vec{p}}L^3 \,|\vec{p}\rangle\,\langle\vec{p}|.$$

Similarly the second term is,

$$|\alpha_k|^2 2E_{\vec{k}}L^3 \,|\vec{k}\rangle\,\langle\vec{k}|.$$

Then we get that Alice's normalized operator is,

$$(\rho_A)_{in} = \mathcal{N}\left(|\alpha_p|^2 2E_{\vec{p}} L^3 |\vec{p}\rangle \langle \vec{p}| + |\alpha_k|^2 2E_{\vec{k}} L^3 |\vec{k}\rangle \langle \vec{k}| \right). \tag{5.87}$$

Now we calculate the constant $\mathcal{N}$ by using the condition $Tr(\rho_{(A\,in)}) = 1$. The tracing over these states are the same has the ones we calculated in the reduced operator just now, so the factors that come out of it are going to be the same. We then have,

$$\mathcal{N} = \frac{1}{(L^3)^2 \left(|\alpha_p|^2 (2E_{\vec{p}})^2 + |\alpha_k|^2 (2E_{\vec{k}})^2\right)},$$

if we substitute it in the operator's expression we have Alice's operator given by,

$$(\rho_A)_{in} = \frac{|\alpha_p|^2 2E_{\vec{p}} |\vec{p}\rangle \langle \vec{p}| + |\alpha_k|^2 2E_{\vec{k}} |\vec{k}\rangle \langle \vec{k}|}{(L^3) \left(|\alpha_p|^2 (2E_{\vec{p}})^2 + |\alpha_k|^2 (2E_{\vec{k}})^2\right)}; \text{With } \{\vec{p} \neq \vec{k}\}. \tag{5.88}$$

We can rewrite the expression in the form,

$$(\rho_A)_{in} = \frac{|\vec{p}\rangle \langle \vec{p}|}{(L^3)(2E_{\vec{p}})(1 + 1/\mathcal{C})} + \frac{|\vec{k}\rangle \langle \vec{k}|}{(L^3)(2E_{\vec{k}})(1 + \mathcal{C})}. \tag{5.89}$$

Where we defined

$$\mathcal{C} \equiv \frac{|\alpha_p|^2}{|\alpha_k|^2} \left(\frac{E_{\vec{p}}}{E_{\vec{k}}}\right)^2 = \frac{|\alpha_p|^2}{|\alpha_k|^2} \left(\frac{1 - v_{\vec{k}}^2}{1 - v_{\vec{p}}^2}\right). \tag{5.90}$$

With the reduced density operator we are capable of determining the entropy. Note that the terms in this operator are exactly of the same form of the first terms of the operators that we calculated before (see (5.32)), as such it's a direct consequence that the elements of the operator, ρ_p and ρ_k are equivalent to those elements. They are,

$$\rho_p = \frac{1}{1 + 1/\mathcal{C}},$$

and

$$\rho_k = \frac{1}{1 + \mathcal{C}},$$

.

The operator is diagonal so the entropy is defined as,

$$(S_E)_{in} = -\rho_p \log \rho_p - \rho_k \log \rho_k \tag{5.91}$$

Substituting the elements we have,

$$(S_E)_{in} = \frac{\log\left(1 + 1/\mathcal{C}\right)}{1 + 1/\mathcal{C}} + \frac{\log\left(1 + \mathcal{C}\right)}{1 + \mathcal{C}} \tag{5.92}$$

Comment on $\mathcal{C}$

Before plotting the entropy as a function of $\mathcal{C}$, we want to do a qualitative analysis of it's domain to see what values it takes.

By looking at the expression,

$$\mathcal{C} \equiv \frac{|\alpha_p|^2}{|\alpha_k|^2}\left(\frac{E_{\vec{p}}}{E_{\vec{k}}}\right)^2 = \frac{|\alpha_p|^2}{|\alpha_k|^2}\left(\frac{1 - v_{\vec{k}}^2}{1 - v_{\vec{p}}^2}\right).$$

it's a direct consequence that $\mathcal{C}$ must be positive, because $\frac{|\alpha_p|^2}{|\alpha_k|^2}$ is a ratio of probabilities and we know that the velocities are restricted to take values between 0 and 1 (they are in units of c). Is there any upper bound on the values that it can take ? Ignoring $\frac{|\alpha_p|^2}{|\alpha_k|^2}$ which are just constants, we look at the nature of,

$$\left(\frac{1 - v_{\vec{k}}^2}{1 - v_{\vec{p}}^2}\right).$$

We are always free to pick ever increasingly bigger values of $|v_{\vec{p}}|$ up to 1, such that the denominator becomes as small as desirable. And we can pick ever increasingly smaller values of $|v_{\vec{k}}|$ down to 0 so that the numerator becomes as big possible, as such the ratio of the numerator and denominator can take any numbers without any upper bound, which implies $\mathcal{C}$ can take any values approaching 0 and ∞.

Let's suppose that $|\alpha_p|^2|v_{\vec{p}}| \gg |\alpha_k|^2|v_{\vec{k}}|$, since none of these values can be equal to 1, this is the same as saying $|\alpha_k|^2|v_{\vec{k}_{CM}}| \ll 1$, then

$$\mathcal{C} \sim |\alpha_p|^2(1 - v_{\vec{p}}^2),$$

and

$$1/\mathcal{C} \sim 1/|\alpha_p|^2(1 - v_{\vec{p}}^2).$$

It is evident that for increasing values of $|\alpha_p|^2|v_{\vec{p}}|$, $1/\mathcal{C}$ approaches zero and $\mathcal{C}$ approaches ∞. We take the expression for the entropy and substitute it with the limiting case $\mathcal{C} \to \infty$,

$$(S_E)_{in} = \frac{\log(1)}{1} + \frac{\log(1 + \mathcal{C}_{\to\infty})}{1 + \mathcal{C}_{\to\infty}}$$

The first term vanishes immediately and the second one takes the same form as the following limit,

$$\lim_{x\to\infty} \frac{\log x}{x} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

Which implies that $S_{E\ in} \to 0$ when $\mathcal{C}$ approaches infinity. But the argument also holds true if we had said $|\alpha_k|^2|v_{\vec{k}}| \gg |\alpha_p|^2|v_{\vec{p}}|$ which would imply $\mathcal{C}$ approached 0.

Then we expect that the entropy plotted as a function of $\mathcal{C}$ will be such that when $\mathcal{C} \gg 1$ (or $\mathcal{C} \ll 1$) it will approach zero. Which means that, the bigger the difference of the weighed CM velocities of the particles in a quantum superposition of the proposed entangled state, lesser the entanglement witnessed by an observation made in one of the subsystems.

We can also determine the value for which the entropy is maximum by computing the derivative of the expression in order to $\mathcal{C}$ and finding it's the zeros, but that won't be necessary if take a careful look at the expression (in base of $\log_2$),

$$(S_E)_{in} = \frac{\log_2(1 + 1/\mathcal{C})}{1 + 1/\mathcal{C}} + \frac{\log_2(1 + \mathcal{C})}{1 + \mathcal{C}}$$

We know the entropy to be a maximum if $S_{E\ in} = 1$, and there's only one possibility for that to happen, the function has to take the form $\log_2(2)$ which we can easily see only happens if $\mathcal{C} = 1$. So we have that the condition for a maximally entangled state of the form we presented is $C = 1$.

Now we pose the question, is there any choice of values pertinent to the characterization of the entangled sate which might impose a restriction such that this condition can't be met ? This is to say, for any choice of the

velocities(energies) and the probabilities coefficients is it possible to have a maximum entangled state of the form we proposed initially?

One can appreciate that for any choice of non null values of α_p and α_k we can get a maximum entangled state for an appropriate choice of velocities values. This becomes evident if we take the condition for maximum entanglement,

$$\frac{|\alpha_p|^2}{|\alpha_k|^2}\left(\frac{1-v_{\vec{k}}^2}{1-v_{\vec{p}}^2}\right)=1,$$

and rewrite it as,

$$(1-|\alpha_p|^2)(1-v_{\vec{p}}^2)+|\alpha_p|^2(v_{\vec{k}}^2-1)=0,\ |\alpha_p|^2\in(0,1)$$

(we substituted $|\alpha_k|^2=1-|\alpha_p|^2$).

This is the equation of a line segment between two points, one is negative $(v_{\vec{k}}^2-1)$ and the other one is positive$(1-v_{\vec{p}}^2)$, so regardless what their values there is always one value of $|\alpha_p|^2$ where the line passes trough the origin, hence the condition of maximum entangled is satisfied.

The symmetric case is of particular interest because it retrieves a familiar state. If we assume $|\alpha_p|^2=|\alpha_k|^2=0.5$, the point when the line crosses the origin is when $|v_{\vec{p}}|=|v_{\vec{k}}|$, since the two points are at the exact same distance on either side of the origin. But we have to remember this is only allowed if $\vec{p}\neq\vec{k}$.

We know that $|v_{\vec{p}}|$ is the absolute value of velocity associated with the 3-momentum vector present in the subscript, so we can have the condition of maximum entanglement in the symmetric case $|v_{\vec{p}}|=|v_{\vec{k}}|$ respected, if $\vec{p}\neq-\vec{k}$. Which gives a special case of the state (5.84),

$$|\Psi\rangle_{in}=\frac{1}{\sqrt{2}}\,|\vec{p},-\vec{p}\rangle+\frac{1}{\sqrt{2}}\,|-\vec{p},\ \vec{p}\rangle. \tag{5.93}$$

This is quite familiar, in fact, it has exactly the same form as one the Bell sates we encountered in Chapter2,

$$\left|E_0^+\right\rangle=\frac{1}{\sqrt{2}}\,|u\ d\rangle+\frac{1}{\sqrt{2}}\,|d\ u\rangle\ ;$$

But again there is a major difference, the state (5.93) even though normalized to the the two kets which we used to describe it, represents an infinite collection of possible momentum states.

There is also another issue, the state (5.93) is just a special case of all the possible maximally entangled symmetric states. We can appreciate this by looking at the equations that relate the absolute values of momentum to their respective values of the velocity,

$$|\vec{p}| = \gamma m |v_{\vec{p}}|;$$

$$|\vec{k}| = \gamma m |v_{\vec{k}}|,$$

since both masses in the expressions are the same ,

$$|\vec{p}| = |\vec{k}| \Rightarrow |v_{\vec{p}}| = |v_{\vec{k}}|.$$

Thus we can get the general condition for maximum entanglement without breaking the inequality $\vec{p} \neq \vec{k}$, if we consider $|\vec{p}| = |\vec{k}|$ and explicitly $\theta_p \neq \theta_k$, which are the angles of the respective vectors[2].

If we take the state (5.84)

$$|\Psi\rangle_{in} = \frac{1}{\sqrt{2}} |\vec{p}, -\vec{p}\rangle + \frac{1}{\sqrt{2}} |\vec{k}, -\vec{k}\rangle \,, \text{for all } \left\{ \vec{p} \neq \vec{k} \right\}, \qquad (5.94)$$

and write it explicitly in terms of the 3-vector components,

$$|\Psi\rangle_{in} = \frac{1}{\sqrt{2}} |(|\vec{p}|, \theta_p); (|\vec{p}|, \pi - \theta_p)\rangle + \frac{1}{\sqrt{2}} |(|\vec{k}|, \theta_k); (|\vec{k}|, \pi - \theta_k)\rangle \,,$$

$$\text{for all } \left\{ \vec{p} \neq \vec{k} \right\}.$$

If we set $|\vec{p}| = |\vec{k}|$, and say without loss of generality that $\theta_p = 0$ and $\theta_k = \theta$, we get,

$$|\Psi\rangle_{in} = \frac{1}{\sqrt{2}} |(|\vec{p}|, 0); (|\vec{p}|, \pi)\rangle + \frac{1}{\sqrt{2}} |(|\vec{p}|, \theta); (|\vec{p}|, \pi - \theta)\rangle \,,$$

$$\text{for all } \left\{ \theta \in (0, 2\pi) \right\}.$$

So we have a continuous amount of maximum entangled states spanning for all the possible values of θ , and we can appreciate that the sate (5.93) which occurs when $\vec{p} = -\vec{k}$ is just the special case when $\theta = \pi$. But apart

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[2] Even though we are dealing with 3-vectors we assume that $\phi_p = \phi_k = 0$ without loss of generality.

from that one, there is a continuous amount of symmetric states which also produce a maximally entangled state.

We show this in a visually in a figure that represents the superposition of the states,
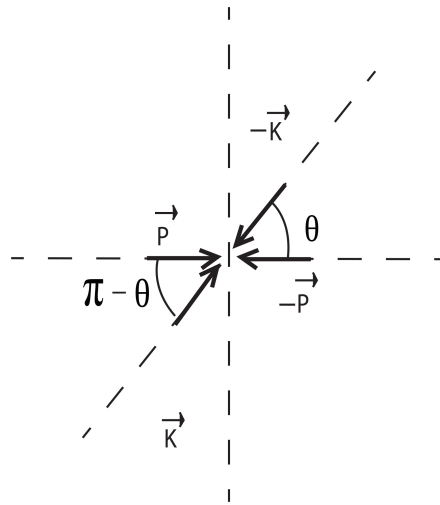
Figure 5.8: Pictorial view of the states superposition in the CM frame. We can see that in the case of $|\vec{p}| = |\vec{k}|$, the superposition corresponds to a maximally entangled state as long as $\theta \neq 0$ or any integer multiple of 2π. If that would happen the vectors would lie on top of each other and there would only be one state ;

If by chance $\theta = 0$, the two states would reduce to a single state because the inequality would not longer be met and thus we would get separable state. But the angles only have to be different in the symmetric case, if the coefficients weren't the same we would need different velocities in order to get $\mathcal{C} = 1$, in that case even if $\theta = 0$ we would have an entangled state because the inequality would be respected since $|\vec{p}| \neq |\vec{k}|$.

If generally we aren't in the symmetric case how can we get the limiting separable state ?

Remember that we commented that the entropy goes to zero if the relative difference of the weighed velocities would become to big, this is because the greater the difference the closer it gets to the limiting case of one of the coefficients being zero which would produce a separable state. And in fact if either $\alpha_p, \alpha_k \to 0$ regardless of what values the velocities have we can see from the expression,

$$\mathcal{C} \equiv \frac{|\alpha_p|^2}{|\alpha_k|^2} \left(\frac{E_{\vec{p}}}{E_{\vec{k}}}\right)^2 = \frac{|\alpha_p|^2}{|\alpha_k|^2} \left(\frac{1 - v_{\vec{k}}^2}{1 - v_{\vec{p}}^2}\right), \tag{5.95}$$

that $\mathcal{C}$ is either going to be 0 or infinite, producing a null entropy which we know to be a sufficient condition for separability. Thus, it's good to know that the limit to get a separable case is to set one of the coefficients to zero.

Graphical Solutions to $(S_E)_{in}$

Firs we plot the expression for the entropy $(S_E)_{in}$

$$\frac{\log\left(1+1/\mathcal{C}\right)}{1+1/\mathcal{C}}+\frac{\log\left(1+\mathcal{C}\right)}{1+\mathcal{C}},$$

as a function of $\mathcal{C}$ we calculate it in logarithmic base of 2 so that the maximum value is 1,
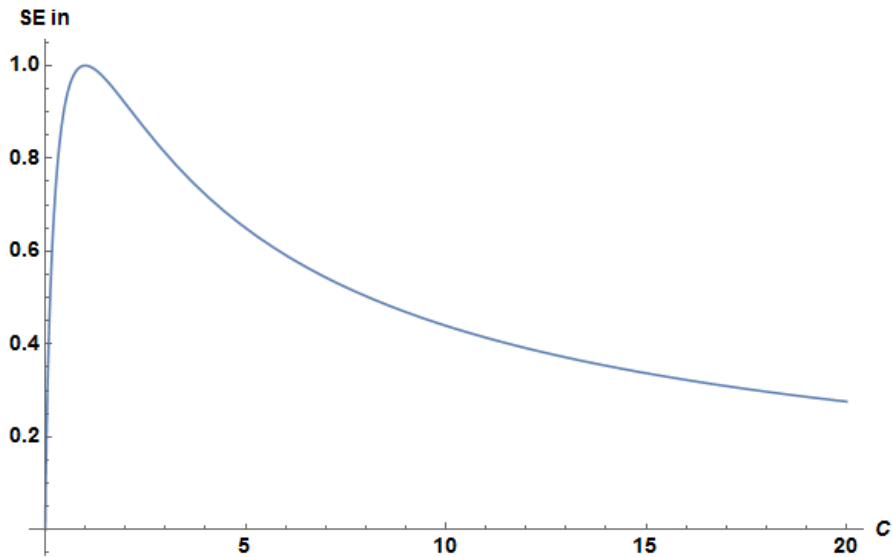
Figure 5.9: $S_{E\,in}(\mathcal{C})$.
It shows it's maximum for $\mathcal{C}=1$, as it was expected. We present values only up to $\mathcal{C}=1$ as it's evident that it's behaviour when $\mathcal{C}\gg 1$ (or $\mathcal{C}\ll 1$) takes the limit zero.

It seems our analysis was correct, the entropy has it's maximum for $\mathcal{C}=1$ and it's limiting cases is also in agreement it our predictions.

It's interesting to compare this graph with the graph we obtained in Chapter3 for the entropy of a given two-dimensional state fig.(3.1). There is an evident difference, the plot of the distribution against the variable $\mathcal{C}$ is not symmetric, this is because the particles can have a continuum of possible momentum's and incidentally those values when used to create the variable $\mathcal{C}$ (which is a probability) produce a greater number of values higher than 1. thus creating an asymmetric distribution. In the other graph of

the spin-$\frac{1}{2}$ system we had symmetric distribution of entropy, because the probabilities were associated with a finite collection of spin eigenvalues which were symmetric around 0.5.

Now we will calculate the symmetric contour plot of the entropy ($|\alpha_p|^2 = |\alpha_p|^2 = 0.5$) as a function of both velocities associated with each state, $|v_{\vec{p}}|$ and $|v_{\vec{k}}|$,
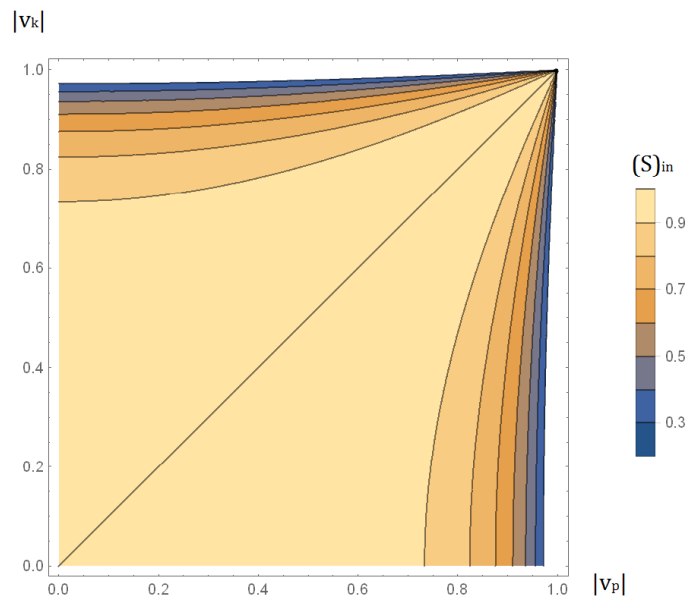
Figure 5.10: $(S_E)_{in}(|v_{\vec{p}}|, |v_{\vec{k}}|)$.
The diagonal line $|v_{\vec{p}}| = |v_{\vec{k}}|$ represents the maximum entropy $(S_E)_{in} = 1$.
This was expected since in the symmetric case $|v_{\vec{p}}| = |v_{\vec{k}}|$ implies $\mathcal{C} = 1$.

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5.2.2 Calculating $(S_E)_{out}$

Determining $(\rho_A)_{out}$

Now that we have a complete description of what the entropy of entanglement is before the collision, we are ready to proceed to the calculation of the entropy after the collision.

From here on out we will perform the calculations for the symmetric case, we assume $|\alpha_p|^2 = |\alpha_p|^2 = 0.5$. This is in order to make the exposition of the intermediary calculations less cumbersome and easier to follow by not having to track explicitly the coefficients. But in the end we will also present the result of $(S_E)_{out}$ with the coefficients explicitly in order to compare the expressions.

The starting point for this calculation is the same as it was before,

$$|\Psi\rangle_{out} = \hat{S} |\Psi\rangle_{in} \tag{5.96}$$

By substituting with the proposed entangled state it follows that,

$$|\Psi\rangle_{out} = \left(\int_{-\infty}^{+\infty} \frac{d^3\vec{q_1}}{(2\pi)^3} \frac{1}{2E_{\vec{q_1}}} \frac{d^3\vec{q_2}}{(2\pi)^3} \frac{1}{2E_{\vec{q_2}}} |\vec{q_1}\ \vec{q_2}\rangle \langle\ \vec{q_1}\ \vec{q_2} | \hat{S} | \vec{p_1}\ \vec{p_2}\rangle \right. \tag{5.97}$$

$$\left. + \int_{-\infty}^{+\infty} \frac{d^3\vec{q_1}}{(2\pi)^3} \frac{1}{2E_{\vec{q_1}}} \frac{d^3\vec{q_2}}{(2\pi)^3} \frac{1}{2E_{\vec{q_2}}} |\vec{q_1}\ \vec{q_2}\rangle \langle\ \vec{q_1}\ \vec{q_2} | \hat{S} | \vec{k_1}\ \vec{k_2}\rangle \right) \frac{1}{2},$$

remembering of course that $\left\{ \vec{p_1} \neq \vec{k_1}; \vec{p_2} \neq \vec{k_2} \right\}$. These two integrals are of the same form the previous case, but now we have two of them. The techniques we used before still apply, hence we can write the expression in the form,

$$|\Psi\rangle_{out} = \left(|\vec{p_1}\ \vec{p_2}\rangle + \int_{\left\{(\vec{q_1} \neq \vec{p_1},\vec{k_1});(\vec{q_2} \neq \vec{p_2},\vec{k_2})\right\}} \frac{d^3\vec{q_1}}{(2\pi)^3} \frac{d^3\vec{q_2}}{(2\pi)^3} \frac{|\vec{q_1}\ \vec{q_2}\rangle \langle\ \vec{q_1}\ \vec{q_2} | i\hat{T} | \vec{p_1}\ \vec{p_2}\rangle}{2E_{\vec{q_1}} 2E_{\vec{q_2}}} \right. \tag{5.98}$$

$$\left. + |\vec{k_1}\ \vec{k_2}\rangle + \int_{\left\{(\vec{q_1} \neq \vec{p_1},\vec{k_1});(\vec{q_2} \neq \vec{p_2},\vec{k_2})\right\}} \frac{d^3\vec{q_1}}{(2\pi)^3} \frac{d^3\vec{q_2}}{(2\pi)^3} \frac{|\vec{q_1}\ \vec{q_2}\rangle \langle\ \vec{q_1}\ \vec{q_2} | i\hat{T} | \vec{k_1}\ \vec{k_2}\rangle}{2E_{\vec{q_1}} 2E_{\vec{q_2}}} \right) \frac{1}{2}.$$

Or in a compact notation,

$$|\Psi\rangle_{out} = \left(|\vec{p_1}\ \vec{p_2}\rangle + |\vec{k_1}\ \vec{k_2}\rangle + \mathcal{I}_p + \mathcal{I}_k \right) \frac{1}{2}. \tag{5.99}$$

Where,

$$\mathcal{I}_p \equiv \int\limits_{\left\{(\vec{q_1}\neq\vec{p_1},\vec{k_1});(\vec{q_2}\neq\vec{p_2},\vec{k_2})\right\}} \frac{d^3\vec{q_1}}{(2\pi)^3}\frac{d^3\vec{q_2}}{(2\pi)^3}\frac{|\vec{q_1}\;\vec{q_2}\,\rangle\langle\,\vec{q_1}\;\vec{q_2}\,|\,i\hat{T}\,|\,\vec{p_1}\;\vec{p_2}\,\rangle}{2E_{\vec{q_1}}2E_{\vec{q_2}}};$$

$$\mathcal{I}_k \equiv \int\limits_{\left\{(\vec{q_1}\neq\vec{p_1},\vec{k_1});(\vec{q_2}\neq\vec{p_2},\vec{k_2})\right\}} \frac{d^3\vec{q_1}}{(2\pi)^3}\frac{d^3\vec{q_2}}{(2\pi)^3}\frac{|\vec{q_1}\;\vec{q_2}\,\rangle\langle\,\vec{q_1}\;\vec{q_2}\,|\,i\hat{T}\,|\,\vec{k_1}\;\vec{k_2}\,\rangle}{2E_{\vec{q_1}}2E_{\vec{q_2}}}.$$

Knowing the strenuous algebra that arouse in the previous calculation when taking 1-loop corrections, we decide to simplify our task and calculate only Tree-Level contributions this time . In Tree-Level, we have ,

$$\langle\,\vec{q_1}\;\vec{q_2}\,|\,i\hat{T}\,|\,\vec{p_1}\;\vec{p_2}\,\rangle = -2i\lambda(2\pi)^4\delta^4\big(p_1+p_2-(q_1+q_2)\big)$$

$$\langle\,\vec{q_1}\;\vec{q_2}\,|\,i\hat{T}\,|\,\vec{k_1}\;\vec{k_2}\,\rangle - 2i\lambda(2\pi)^4\delta^4\big(k_1+k_2-(q_1+q_2)\big).$$

After we integrate the 3-momentum part in both integrals, $\int d^3\vec{q_2}\delta^3(\vec{p_1}+\vec{p_2}-\vec{q_1}-\vec{q_2})$ in $\mathcal{I}_p$, and $\int d^3\vec{q_2}\delta^3(\vec{k_1}+\vec{k_2}-\vec{q_1}-\vec{q_2})$ in $\mathcal{I}_k$ we have,

$$\mathcal{I}_p = \int\limits_{\left\{\vec{q_1}\neq\vec{p_1},\vec{k_1}\right\}} \frac{d^3\vec{q_1}}{(2\pi)^3}\frac{|\vec{q_1}\;\vec{p_1}+\vec{p_2}-\vec{q_1}\,\rangle\,(-2i\lambda 2\pi\delta_{E_p})}{2E_{\vec{q_1}}2E_{\vec{p_1}+\vec{p_2}-\vec{q_1}}};$$

$$\mathcal{I}_k = \int\limits_{\left\{\vec{q_1}\neq\vec{p_1},\vec{k_1}\right\}} \frac{d^3\vec{q_1}}{(2\pi)^3}\frac{|\vec{q_1}\;\vec{k_1}+\vec{k_2}-\vec{q_1}\,\rangle\,(-2i\lambda 2\pi\delta_{E_k})}{2E_{\vec{q_1}}2E_{\vec{k_1}+\vec{k_2}-\vec{q_1}}};$$

where,

$$\delta_{E_k} \equiv \delta(E_{\vec{k_1}}+E_{\vec{k_2}}-E_{\vec{q_1}}-E_{\vec{k_1}+\vec{k_2}-\vec{q_1}}); \quad \delta_{E_p} \equiv \delta(E_{\vec{p_1}}+E_{\vec{p_2}}-E_{\vec{q_1}}-E_{\vec{p_1}+\vec{p_2}-\vec{q_1}})$$

At this point we can drop the subscript in q_1 and add both integrals,

$$\mathcal{I}_{p,k} = -2i\lambda\int\limits_{\left\{\vec{q}\neq\vec{p_1},\vec{k_1}\right\}}\frac{d^3\vec{q}}{(2\pi)^3}\frac{1}{2E_{\vec{q}}}\left(\frac{|\vec{q},\;\vec{k_1}+\vec{k_2}-\vec{q}\,\rangle\,(2\pi\delta_{E_k})}{2E_{\vec{k_1}+\vec{k_2}-\vec{q}}} + \frac{|\vec{q},\;\vec{p_1}+\vec{p_2}-\vec{q}\,\rangle\,(2\pi\delta_{E_p})}{2E_{\vec{p_1}+\vec{p_2}-\vec{q}}}\right);$$

$$(5.100)$$

as such the state (5.99) can be written as,

$$|\Psi\rangle_{out} = \big(\,|\,\vec{p_1}\;\vec{p_2}\,\rangle + |\,\vec{k_1}\;\vec{k_2}\,\rangle + \mathcal{I}_{p,k}\big)\frac{1}{2}. \qquad (5.101)$$

Now we proceed to calculate the density operator, $|\Psi\rangle_{out}\,_{out}\langle\Psi|$, which gives,

$$\rho_{out} = \Big(\, |\vec{p_1}\ \vec{p_2}\rangle\,\langle\vec{p_1}\ \vec{p_2}| + |\vec{k_1}\ \vec{k_2}\,\rangle\,\langle\vec{k_1}\ \vec{k_2}|+ \qquad\qquad (5.102)$$

$$+\mathcal{I}_{p,k}\mathcal{I}_{p,k}^{\dagger}\Big)\frac{1}{4} + \text{(crossed terms)}.$$

Where hermitean conjugate is,

$$\mathcal{I}_{p,k}^{\dagger} = 2i\lambda \int\limits_{\{\vec{q}'\neq\vec{p_1},\vec{k_1}\}} \frac{d^3\vec{q}'}{(2\pi)^3}\frac{1}{2E_{\vec{q}'}} \left(\frac{\langle\vec{q}',\ \vec{k_1}+\vec{k_2}-\vec{q}|(2\pi\delta_{E_k'})}{2E_{\vec{k_1}+\vec{k_2}-\vec{q}'}} + \frac{\langle\vec{q}',\ \vec{p_1}+\vec{p_2}-\vec{q}|(2\pi\delta_{E_p'})}{2E_{\vec{p_1}+\vec{p_2}-\vec{q}'}} \right)$$

Just like before and for the same reasons, the crossed terms are going to disappear when we calculate the reduced density operator. At this point it is convenient to go to the CM frame, such that the operator becomes explicitly,

$$\rho_{out} = \Big(\, |\vec{p}\ -\vec{p}\rangle\,\langle\vec{p}\ -\vec{p}| + |\vec{k}\ -\vec{k}\,\rangle\,\langle\vec{k}\ -\vec{k}|+ \qquad (5.103)$$

$$+4\lambda^2 \int\limits_{\{\vec{q}\neq\vec{p},\vec{k}\}} \frac{d^3\vec{q}}{(2\pi)^3}\frac{1}{(2E_{\vec{q}})^2} \int\limits_{\{\vec{q}'\neq\vec{p},\vec{k}\}} \frac{d^3\vec{q}'}{(2\pi)^3}\frac{|\vec{q}',\ -\vec{q}\rangle\,\langle\vec{q}',\ -\vec{q}|}{(2E_{\vec{q}'})^2}(2\pi)^2\,\delslash_E'\,\delslash_E\Big)\frac{1}{4} +$$

$$+\text{(crossed terms)}.$$

Where,

$$\delslash_E'\,\delslash_E \equiv \big(\delta_{E_k'} + \delta_{E_p'}\big)\big(\delta_{E_k} + \delta_{E_p}\big),$$

with $\delta_{E_k} = \delta(2E_{\vec{q}} - 2E_{\vec{k}})$, and $\delta_{E_p} = \delta(2E_{\vec{q}} - 2E_{\vec{p}})$.

We apply the tracing operator over Bob's space (4.23) in the usual way. The first two terms are exactly of the same form as the first term of the density operator in the separable case, as such the result of tracing out this term will give the equivalent form of the first term in the reduced operator (5.17). They will be,

$$(|\vec{p}\rangle\,\langle\vec{p}|\,(2E_{\vec{p}})L^3 + |\vec{k}\,\rangle\langle\vec{k}|(2E_{\vec{k}})L^3$$

Tracing over the integral term evaluated in the CM is also very similar to the tracing over the integral term in the separable case, in fact the only

difference is that in this case the term has two delta's of energy, but aside from that it is exactly the same, if we look at the second term of (5.17) by comparison we get that,

$$4\lambda^2 \int\limits_{\{\vec{q}\neq\vec{p},\vec{k}\}} \frac{d^3\vec{q}}{(2\pi)^3} \frac{|\vec{q}\rangle\langle\vec{q}|}{(2E_{\vec{q}})^3} (2\pi \ \delta_E)^2 \ ;$$

By adding the two terms we get that the normalized operator is given by,

$$(\rho_A)_{out} = \frac{\mathcal{N}}{4} \Bigg(|\vec{p}\rangle\langle\vec{p}| (2E_{\vec{p}})L^3 + |\vec{k}\rangle\langle\vec{k}|(2E_{\vec{k}})L^3 + \qquad (5.104)$$

$$+4\lambda^2 \int\limits_{\{\vec{q}\neq\vec{p},\vec{k}\}} \frac{d^3\vec{q}}{(2\pi)^3} \frac{|\vec{q}\rangle\langle\vec{q}|}{(2E_{\vec{q}})^3} (2\pi \ \delta_E)^2 \Bigg) ;$$

If we expand
$$(\delta_E)^2 = \delta_{E_p}^2 + \delta_{E_k}^2 + 2\delta_{E_p}\delta_{E_k}$$

We have,

$$(\rho_A)_{out} = \frac{\mathcal{N}}{4} \Bigg(|\vec{p}\rangle\langle\vec{p}| (2E_{\vec{p}}) + |\vec{k}\rangle\langle\vec{k}|(2E_{\vec{k}}) + \qquad (5.105)$$

$$+4\lambda^2 \int\limits_{\{\vec{q}\neq\vec{p},\vec{k}\}} \frac{d^3\vec{q}}{(2\pi)^3} \frac{|\vec{q}\rangle\langle\vec{q}|}{(2E_{\vec{q}})^3} (2\pi\delta(2E_{\vec{q}} - 2E_{\vec{p}}))^2 +$$

$$+4\lambda^2 \int\limits_{\{\vec{q}\neq\vec{p},\vec{k}\}} \frac{d^3\vec{q}}{(2\pi)^3} \frac{|\vec{q}\rangle\langle\vec{q}|}{(2E_{\vec{q}})^3} \left(2\pi\delta(2E_{\vec{q}} - 2E_{\vec{k}})\right)^2 \Bigg)^3 .$$

Now we find $\mathcal{N}$, using the condition for normalization $Tr(\rho_{(A) \ out}) = 1$. Once again, the first two terms are equal to the first term of the reduced density operator in the separable case, and we also notice that the integral terms are as well, such that the tracing out would produce an integration like $\langle\mathcal{I}_2\rangle$.

---

[3]The delta's crossed term is peculiar since it only contributes when velocities are the same, this is called a removable discontinuity. In the present work we will ignore this term because it's existence doesn't affect the general profile of all the other points in the entropy, and we aren't able to give a fair treatment to it, and an appropriate interpretation.

Remembering that the constant of normalization in Tree-level for the separable case was given by,

$$\frac{1}{(2E_{\vec{p_2}}L^3)^2 + \langle \mathcal{I}_2 \rangle}.$$

With,

$$\langle \mathcal{I}_2 \rangle = \frac{L^4\lambda^2}{2\pi}|v_{\vec{p}}|. \tag{5.106}$$

By comparison we get that in this case,

$$\mathcal{N} = \frac{4}{(2E_{\vec{p}}L^3)^2 + (2E_{\vec{k}}L^3)^2 + \frac{\lambda^2}{2\pi}L^4(|v_{\vec{p}}| + |v_{\vec{k}}|)}$$

Now we have Alice's the density operator completely defined, we can compute the entropy.

Calculation of the Entropy

The expression of the entropy will be,

$$(S_E)_{out} = -\sum_{i}^{\infty}(\rho_i) \log \rho_i = -\rho_p \log \rho_p \tag{5.107}$$

$$-\rho_k \log \rho_k - \frac{L^3}{(2\pi)^3} \int_{-\infty}^{+\infty} \rho_n \log \rho_n dn$$

These matrix elements will be calculated in the same fashion as previously, by using (4.33) we calculate ρ_p which is the element correspondent to the first term to be,

$$\rho_p = \frac{1}{1 + \frac{1}{\mathcal{C}} + \frac{\lambda^2}{8\pi E_{\vec{p}}}(|v_{\vec{p}}| + |v_{\vec{k}}|)},$$

and ρ_k (corresponding to the second term),

$$\rho_k = \frac{1}{1 + \mathcal{C} + \frac{\lambda^2}{8\pi E_{\vec{k}}}(|v_{\vec{p}}| + |v_{\vec{k}}|)}.$$

These enter into (5.107) as,

$$\frac{\log\left(1 + \frac{1}{\mathcal{C}} + \frac{\lambda^2}{8\pi\bar{E}_{\vec{p}}^2}(|v_{\vec{p}}| + |v_{\vec{k}}|)\right)}{1 + \frac{1}{\mathcal{C}} + \frac{\lambda^2}{8\pi\bar{E}_{\vec{p}}^2}(|v_{\vec{p}}| + |v_{\vec{k}}|)} \tag{5.108}$$

and

$$\frac{\log\left(1 + \mathcal{C} + \frac{\lambda^2}{8\pi\bar{E}_{\vec{k}}^2}(|v_{\vec{p}}| + |v_{\vec{k}}|)\right)}{1 + \mathcal{C} + \frac{\lambda^2}{8\pi\bar{E}_{\vec{k}}^2}(|v_{\vec{p}}| + |v_{\vec{k}}|)}, \tag{5.109}$$

respectively.

The last term is the rest of the matrix's trace. To find it it we first need to compute the general element ρ_n which we calculate it to be,

$$\rho_n = \frac{(2\pi)^2\left(\delta^2(2E_{\vec{p}} - 2E_{\vec{n}}) + \delta^2(2E_{\vec{k}} - 2E_{\vec{n}})\right)}{L^6(2E_{\vec{n}})^2\left((2E_{\vec{k}})^2 + (2E_{\vec{p}})^2 + \frac{\lambda^2}{2\pi L^2}(|v_{\vec{p}}| + |v_{\vec{k}}|)\right)}$$

For simplicity let's call the denominator $\mathcal{D}(E_{\vec{n}})$. Inputting this into the expression into (5.107) we get,

$$-\frac{L^3}{(2\pi)^3}\int \frac{(2\pi)^2\delta_{E_p}^2 + (2\pi)^2\delta_{E_k}^2}{L^6\mathcal{D}(E_{\vec{n}})} \log\frac{(2\pi)^2\delta_{E_p}^2 + (2\pi)^2\delta_{E_k}^2}{L^6\mathcal{D}(E_{\vec{n}})}d^3\vec{n} \tag{5.110}$$

To solve this integral we separate it into the two different integrals like,

$$-\frac{L^3}{(2\pi)^3}\int \frac{(2\pi)^2\delta_{E_p}^2}{L^6\mathcal{D}(E_{\vec{n}})} \log\frac{(2\pi)^2\delta_{E_p}^2 + (2\pi)^2\delta_{E_k}^2}{L^6\mathcal{D}(E_{\vec{n}})}d^3\vec{n}+$$

$$-\frac{L^3}{(2\pi)^3}\int \frac{(2\pi)^2\delta_{E_k}^2}{L^6\mathcal{D}(E_{\vec{n}})} \log\frac{(2\pi)^2\delta_{E_p}^2 + (2\pi)^2\delta_{E_k}^2}{L^6\mathcal{D}(E_{\vec{n}})}d^3\vec{n}$$

These types of integrals are familiar, we solved them in the exact same situation in the separable case. We won't do it again here because there's nothing new besides the fact that now we have to calculate two instead of just one. The process is the same, we go to spherical coordinates and switch to an integration over energies in order to integrate the delta function of energy, but now we have two different delta's, one for each integral. The solution is given by,

$$\lambda^2 |v_{\vec{p}}| \frac{\log\left((4\bar{E}_{\vec{p}}^4 + 4\bar{E}_{\vec{p}}^2\bar{E}_{\vec{k}}^2 + \bar{E}_{\vec{p}}^2\frac{\lambda^2}{2\pi}(|v_{\vec{p}}| + |v_{\vec{k}}|))/\lambda^2\right)}{2\pi\left(4\bar{E}_{\vec{p}}^2 + 4\bar{E}_{\vec{k}}^2 + \frac{\lambda^2}{2\pi}(|v_{\vec{p}}| + |v_{\vec{k}}|)\right)} +$$

$$+\lambda^2 |v_{\vec{k}}| \frac{\log\left((4\bar{E}_{\vec{k}}^4 + 4\bar{E}_{\vec{p}}^2\bar{E}_{\vec{k}}^2 + \bar{E}_{\vec{k}}^2\frac{\lambda^2}{2\pi}(|v_{\vec{p}}| + |v_{\vec{k}}|))/\lambda^2\right)}{2\pi\left(4\bar{E}_{\vec{p}}^2 + 4\bar{E}_{\vec{k}}^2 + \frac{\lambda^2}{2\pi}(|v_{\vec{p}}| + |v_{\vec{k}}|)\right)}$$

By adding this term to (5.108) and (5.109) we find that,

$$S_{E\,out} = \frac{\log\left(1 + \frac{1}{\mathcal{C}} + \frac{\lambda^2}{8\pi\bar{E}_{\vec{p}}^2}(|v_{\vec{p}}| + |v_{\vec{k}}|)\right)}{1 + \frac{1}{\mathcal{C}} + \frac{\lambda^2}{8\pi\bar{E}_{\vec{p}}^2}(|v_{\vec{p}}| + |v_{\vec{k}}|)} + \tag{5.111}$$

$$+\frac{\log\left(1 + \mathcal{C} + \frac{\lambda^2}{8\pi\bar{E}_{\vec{k}_{CM}}^2}(|v_{\vec{p}}| + |v_{\vec{k}}|)\right)}{1 + \mathcal{C} + \frac{\lambda^2}{8\pi\bar{E}_{\vec{k}}^2}(|v_{\vec{p}}| + |v_{\vec{k}}|)} +$$

$$+\lambda^2 |v_{\vec{p}}| \frac{\log\left((4\bar{E}_{\vec{p}}^4 + 4\bar{E}_{\vec{p}}^2\bar{E}_{\vec{k}}^2 + \bar{E}_{\vec{p}}^2\frac{\lambda^2}{2\pi}(|v_{\vec{p}}| + |v_{\vec{k}}|))/\lambda^2\right)}{2\pi\left(4\bar{E}_{\vec{p}}^2 + 4\bar{E}_{\vec{k}}^2 + \frac{\lambda^2}{2\pi}(|v_{\vec{p}}| + |v_{\vec{k}}|)\right)} +$$

$$+\lambda^2 |v_{\vec{k}}| \frac{\log\left((4\bar{E}_{\vec{k}}^4 + 4\bar{E}_{\vec{p}}^2\bar{E}_{\vec{k}}^2 + \bar{E}_{\vec{k}}^2\frac{\lambda^2}{2\pi}(|v_{\vec{p}}| + |v_{\vec{k}}|))/\lambda^2\right)}{2\pi\left(4\bar{E}_{\vec{p}}^2 + 4\bar{E}_{\vec{k}}^2 + \frac{\lambda^2}{2\pi}(|v_{\vec{p}}| + |v_{\vec{k}}|)\right)}$$

Bear in mind that since we are in the symmetric case, $\mathcal{C} = \left(\frac{1 - v_{\vec{k}}^2}{1 - v_{\vec{p}}^2}\right)$.

We present the general formula for any α_p and α_k,

$$(S_E)_{out} = \frac{\log\left(1 + \frac{1}{\mathcal{C}} + \frac{\lambda^2}{8\pi\bar{E}_{\vec{p}}^2}(|v_{\vec{p}}| + \frac{|\alpha_k|^2}{|\alpha_p|^2}|v_{\vec{k}}|)\right)}{1 + \frac{1}{\mathcal{C}} + \frac{\lambda^2}{8\pi\bar{E}_{\vec{p}}^2}(|v_{\vec{p}}| + \frac{|\alpha_k|^2}{|\alpha_p|^2}|v_{\vec{k}}|)} + \tag{5.112}$$

$$+ \frac{\log\left(1 + \mathcal{C} + \frac{\lambda^2}{8\pi\bar{E}_{\vec{k}}^2}(\frac{|\alpha_k|^2}{|\alpha_p|^2}|v_{\vec{p}}| + |v_{\vec{k}}|)\right)}{1 + \mathcal{C} + \frac{\lambda^2}{8\pi\bar{E}_{\vec{k}}^2}(\frac{|\alpha_k|^2}{|\alpha_p|^2}|v_{\vec{p}}| + |v_{\vec{k}}|)} +$$

$$+ \lambda^2|\alpha_p|^2|v_{\vec{p}}| \frac{\log\left((4\bar{E}_{\vec{p}}^4 + 4\frac{|\alpha_k|^2}{|\alpha_p|^2}\bar{E}_{\vec{p}}^2\bar{E}_{\vec{k}}^2 + \bar{E}_{\vec{p}}^2\frac{\lambda^2}{2\pi}(|v_{\vec{p}}| + \frac{|\alpha_k|^2}{|\alpha_p|^2}|v_{\vec{k}}|))/\lambda^2\right)}{2\pi\left(4|\alpha_p|^2\bar{E}_{\vec{p}}^2 + 4|\alpha_k|^2\bar{E}_{\vec{k}}^2 + \frac{\lambda^2}{2\pi}(|\alpha_p|^2|v_{\vec{p}}| + |\alpha_k|^2|v_{\vec{k}}|)\right)} +$$

$$+ \lambda^2|\alpha_k|^2|v_{\vec{k}}| \frac{\log\left((4\bar{E}_{\vec{k}}^4 + 4\frac{|\alpha_p|^2}{|\alpha_k|^2}\bar{E}_{\vec{p}}^2\bar{E}_{\vec{k}}^2 + \bar{E}_{\vec{k}}^2\frac{\lambda^2}{2\pi}(|v_{\vec{k}}| + \frac{|\alpha_p|^2}{|\alpha_k|^2}|v_{\vec{k}}|))/\lambda^2\right)}{2\pi\left(4|\alpha_p|^2\bar{E}_{\vec{p}}^2 + 4|\alpha_k|^2\bar{E}_{\vec{k}}^2 + \frac{\lambda^2}{2\pi}(|\alpha_p|^2|v_{\vec{p}}| + |\alpha_k|^2|v_{\vec{k}}|)\right)}$$

And now, $\mathcal{C} = \frac{|\alpha_p|^2}{|\alpha_k|^2}\left(\frac{1-v_{\vec{k}}^2}{1-v_{\vec{p}}^2}\right)$.

Obviously if we set $|\alpha_p|^2 = |\alpha_k|^2$ we get the symmetric case expression, but more interestingly if we set any of the two coefficients to zero we get $S_{E\,out}$ of the separable state (computed to Tree level of course). Imagine for instance that we say $|\alpha_k|^2 = 0$ and $|\alpha_p|^2 = 1$. The last term is seen to promptly disappear, also since $|\alpha_k|^2 = 0$ implies that $\mathcal{C} \to \infty$ and as such the second term also goes to zero. We are left with the first and the third term which are,

$$(S_E)_{out} = \frac{\log\left(1 + \frac{\lambda^2}{8\pi\bar{E}_{\vec{p}}^2}(|v_{\vec{p}}|)\right)}{1 + \frac{\lambda^2}{8\pi\bar{E}_{\vec{p}}^2}(|v_{\vec{p}}|)} + \tag{5.113}$$

$$+ \lambda^2|v_{\vec{p}}| \frac{\log\left((4\bar{E}_{\vec{p}}^4 + \bar{E}_{\vec{p}}^2\frac{\lambda^2}{2\pi}(|v_{\vec{p}}|))/\lambda^2\right)}{2\pi\left(4\bar{E}_{\vec{p}}^2 + \frac{\lambda^2}{2\pi}(|v_{\vec{p}}|)\right)}$$

And if we remember that,

$$\mathcal{A}_{tree} = \frac{\lambda^2}{8\pi}\frac{|v_{\vec{p}}|}{\bar{E}_{\vec{p}}},$$

we realize that the surviving terms take the form,

$$(S_E)_{out} = \frac{\log\left(1 + \mathcal{A}_{tree}\right)}{1 + \mathcal{A}_{tree}} + \lambda^2 |v_{\vec{p}}| \frac{\log\left((4\bar{E}_{\vec{p}}^4(1 + \mathcal{A}_{tree}))/\lambda^2\right)}{8\pi \bar{E}_{\vec{p}}^2(1 + \mathcal{A}_{tree})},$$

which is exactly the same as (5.39).

We already have the final entropy, in order to compute the variation we just need to subtract $(S_E)_{in}$, (5.92. Thus getting,

$$\Delta S = (S_E)_{out} - (S_E)_{in} = \frac{\log\left(1 + \frac{1}{\mathcal{C}} + \frac{\lambda^2}{8\pi \bar{E}_{\vec{p}}^2}(|v_{\vec{p}}| + \frac{|\alpha_k|^2}{|\alpha_p|^2}|v_{\vec{k}}|)\right)}{1 + \frac{1}{\mathcal{C}} + \frac{\lambda^2}{8\pi \bar{E}_{\vec{p}}^2}(|v_{\vec{p}}| + \frac{|\alpha_k|^2}{|\alpha_p|^2}|v_{\vec{k}}|)} + \quad (5.114)$$

$$+\frac{\log\left(1 + \mathcal{C} + \frac{\lambda^2}{8\pi \bar{E}_{\vec{k}}^2}(\frac{|\alpha_k|^2}{|\alpha_p|^2}|v_{\vec{p}}| + |v_{\vec{k}}|)\right)}{1 + \mathcal{C} + \frac{\lambda^2}{8\pi \bar{E}_{\vec{k}}^2}(\frac{|\alpha_k|^2}{|\alpha_p|^2}|v_{\vec{p}}| + |v_{\vec{k}}|)} +$$

$$+\lambda^2 |\alpha_p|^2 |v_{\vec{p}}| \frac{\log\left((4\bar{E}_{\vec{p}}^4 + 4\frac{|\alpha_k|^2}{|\alpha_p|^2}\bar{E}_{\vec{p}}^2\bar{E}_{\vec{k}}^2 + \bar{E}_{\vec{p}}^2\frac{\lambda^2}{2\pi}(|v_{\vec{p}}| + \frac{|\alpha_k|^2}{|\alpha_p|^2}|v_{\vec{k}}|))/\lambda^2\right)}{2\pi\left(4|\alpha_p|^2\bar{E}_{\vec{p}}^2 + 4|\alpha_k|^2\bar{E}_{\vec{k}}^2 + \frac{\lambda^2}{2\pi}(|\alpha_p|^2|v_{\vec{p}}| + |\alpha_k|^2|v_{\vec{k}}|)\right)} +$$

$$+\lambda^2 |\alpha_k|^2 |v_{\vec{k}}| \frac{\log\left((4\bar{E}_{\vec{k}}^4 + 4\frac{|\alpha_p|^2}{|\alpha_k|^2}\bar{E}_{\vec{p}}^2\bar{E}_{\vec{k}}^2 + \bar{E}_{\vec{k}}^2\frac{\lambda^2}{2\pi}(|v_{\vec{k}}| + \frac{|\alpha_p|^2}{|\alpha_k|^2}|v_{\vec{k}}|))/\lambda^2\right)}{2\pi\left(4|\alpha_p|^2\bar{E}_{\vec{p}}^2 + 4|\alpha_k|^2\bar{E}_{\vec{k}}^2 + \frac{\lambda^2}{2\pi}(|\alpha_p|^2|v_{\vec{p}}| + |\alpha_k|^2|v_{\vec{k}}|)\right)}$$

$$-\left(\frac{\log\left(1 + 1/\mathcal{C}\right)}{1 + 1/\mathcal{C}} + \frac{\log\left(1 + \mathcal{C}\right)}{1 + \mathcal{C}}\right).$$

With, $\mathcal{C} = \frac{|\alpha_p|^2}{|\alpha_k|^2}\left(\frac{1 - v_{\vec{k}}^2}{1 - v_{\vec{p}}^2}\right).$

5.2.3 Graphical Solutions

Again, in order to be able to get graphical solutions we use the prototype $\bar{m} = 1$. We will present first contour plots for the symmetric case for the same values of lambda as in the previous case with the separable initial state. As we will see changing the parameters won't change the maximums values registered for the entropy's variation for the same λ, so the values the coefficients have won't matter when comparing the maximum values. Then we present some variations of $|\alpha_k|^2$ and $|\alpha_p|^2$ as it's interesting to see that although the entropy's variation maximum value doesn't alter when we

change these parameters, that the domains of the velocities for which the such maxima occur have an explicit dependence on them.
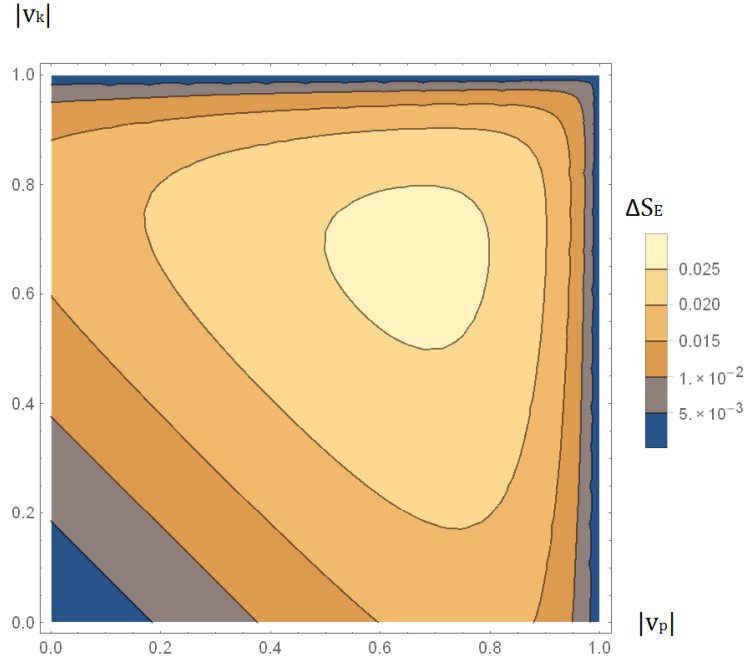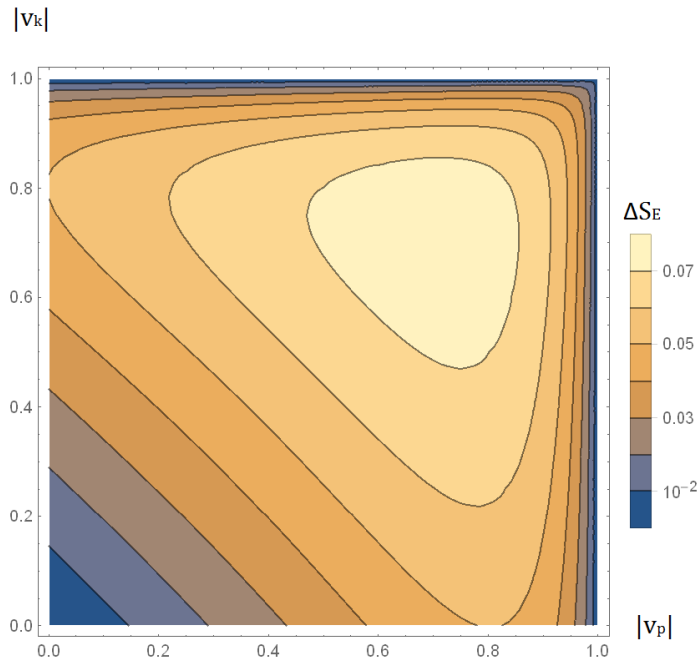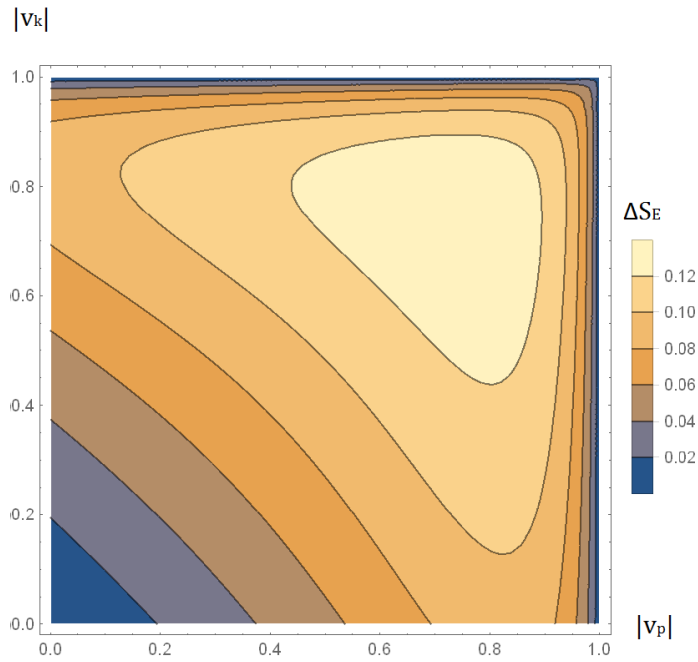

Figure 5.11: $\Delta S_E(|v_{\vec{p}}|, |v_{\vec{k}}|)$ for $\lambda = 0.5$ and $|\alpha_p|^2 = |\alpha_k|^2 = 0.5$.

$$\text{Máx.}\Delta S_E \sim 0.025,$$
$$\text{for} |v_{\vec{p}}| \in (\sim 0.5, \sim 0.75), |v_{\vec{k}}| \in (\sim 0.5, \sim 0.75)$$

That is, there is a domain of certain velocities values which produce the same result of the entropy's variation. The domain pertinent to the maximum value of the variation is symmetric because we are in the symmetric case.

In the case of $\lambda = 1$ we have,

Figure 5.12: $\Delta S_E(|v_{\vec{p}}|, |v_{\vec{k}}|)$ for $\lambda = 1$ and $|\alpha_p|^2 = |\alpha_k|^2 = 0.5$.

$$\text{Máx.}\Delta S_E \sim 0.07,$$
$$\text{for} |v_{\vec{p}}| \in (\sim 0.45, \sim 0.85), |v_{\vec{k}}| \in (\sim 0.45, \sim 0.85)$$

Here the domain's area for the maximum variation got bigger, which means that there is a higher combination of values which produce the maximum value of the entropy's variation.
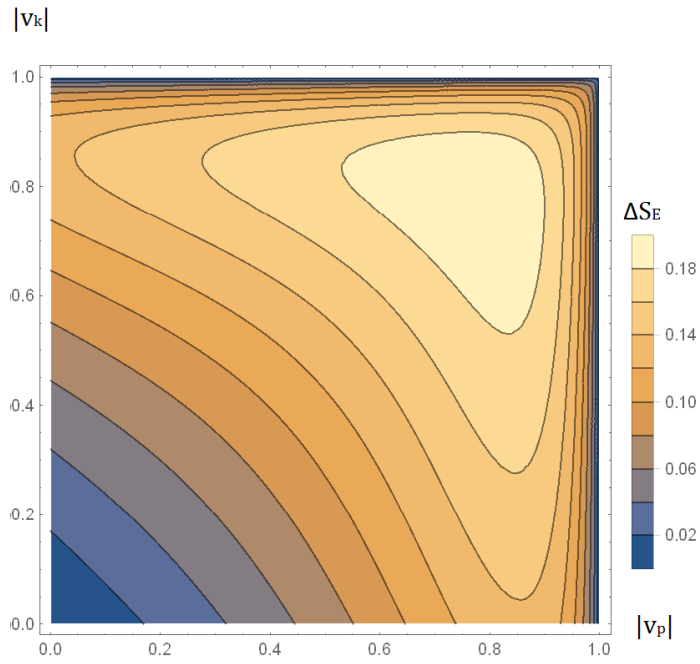
In the case of $\lambda = 1.5$ we have,

Figure 5.13: $\Delta S_E(|v_{\vec{p}}|, |v_{\vec{k}}|)$ for $\lambda = 1.5$ and $|\alpha_p|^2 = |\alpha_k|^2 = 0.5$.

$$\text{Máx.}\Delta S_E \sim 0.012,$$
$$\text{for} |v_{\vec{p}}| \in (\sim 0.4, \sim 0.9), |v_{\vec{k}}| \in (\sim 0.4, \sim 0.9)$$

Here the domain's area for the maximum variation still got bigger, which means that there is even a higher combination of values which produce the maximum value of the entropy's variation.

In the case of $\lambda = 2$ we have,

Figure 5.14: $\Delta S_E(|v_{\vec{p}}|, |v_{\vec{k}}|)$ for $\lambda = 2$ and $|\alpha_p|^2 = |\alpha_k|^2 = 0.5$.

$$\text{Máx.}\Delta S_E \sim 0.012,$$
$$\text{for} |v_{\vec{p}}| \in (\sim 0.5, \sim 0.95), |v_{\vec{k}}| \in (\sim 0.5, \sim 0.95)$$

It appears that in the last value for λ we took, the area for the maximum variation got smaller.

It's interesting to plot a graph similar to (5.7), but instead of comparing the 1-loop and Tree level values for the maximum variation in the separable case, we compare the maximum values for the Tree level in the separable and entangled case. It is as follows,
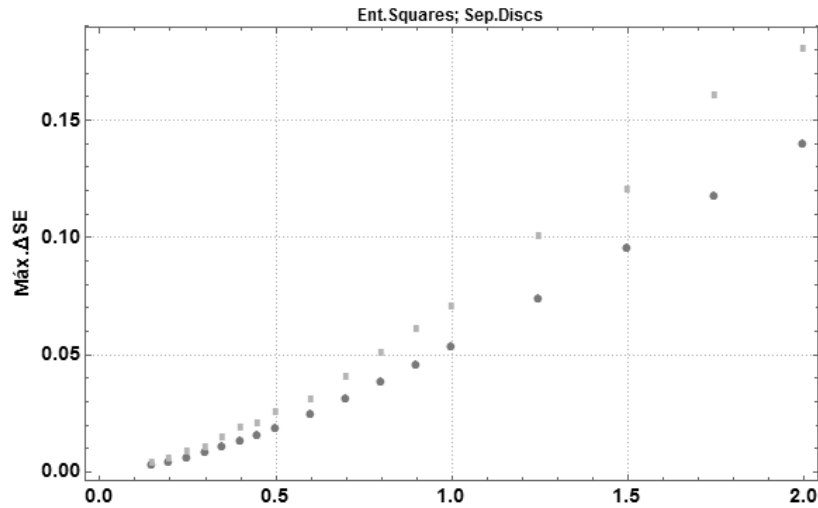
Figure 5.15: Máx(ΔS_E) vs. λ : Separable initial state(Discs) and Entangled initial state (Squares).

There is an undeniable resemblance in this graph and the one in (5.7), the Tree level results of the initial entangled state are almost exactly the same as the 1-loop corrections on the separable state. One wonders if this result is incidental to this particular case or is a consequence of a deeper relationship between the scattering processes to be explored. Unfortunately at this point we can't do much more than guess.

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Now we would like to present an asymmetric case for $\lambda = 1$, and $|\alpha_p|^2 = 0.2$, $|\alpha_k|^2 = 0.8$,
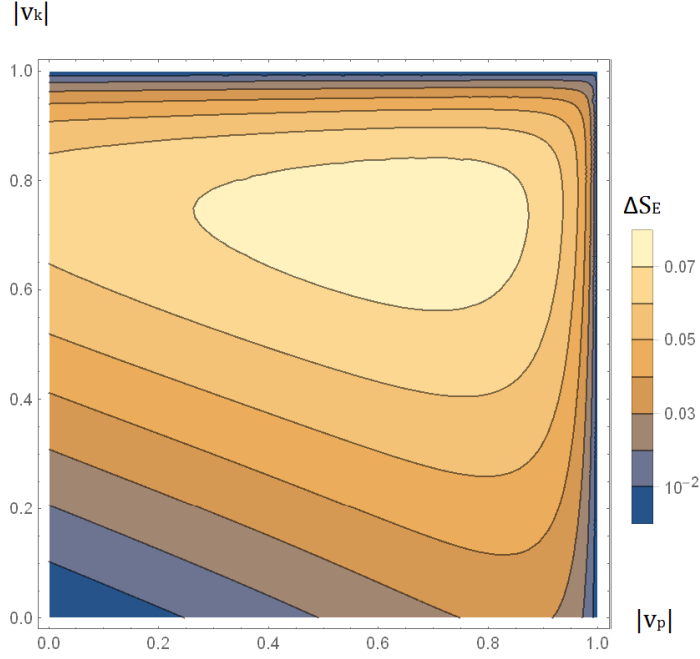
Figure 5.16: $\Delta S_E(|v_{\vec{p}}|, |v_{\vec{k}}|)$ for $\lambda = 1$ and $|\alpha_p|^2 = 0.2$, $|\alpha_k|^2 = 0.8$

$$\text{Máx.}\Delta S_E \sim 0.07,$$
$$\text{for} |v_{\vec{p}}| \in (\sim 0.3, \sim 0.9), |v_{\vec{k}}| \in (\sim 0.6, \sim 0.8)$$

As you can see, the maximum value for the variation still remains the same, as with all of the values pertinent to the different domains. The difference is in the domains themselves, which are no longer symmetric, they are extended to values of the velocity which has the lower weight associated with it, in this case $|v_{\vec{p}}|$. This is because the velocity is "less important", since it's contributions are not going to be of the same order has the contributions of $|v_{\vec{k}}|$, and as such can have a higher range of values without changing the outcome of the variation. On the other hand, the range of values which $|v_{\vec{k}}|$ can take are more restrict.

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If we plot the graph with opposite coefficients $|\alpha_p|^2 = 0.8$, $|\alpha_k|^2 = 0.2$, we get the symmetric distribution for the domains of the entropy,
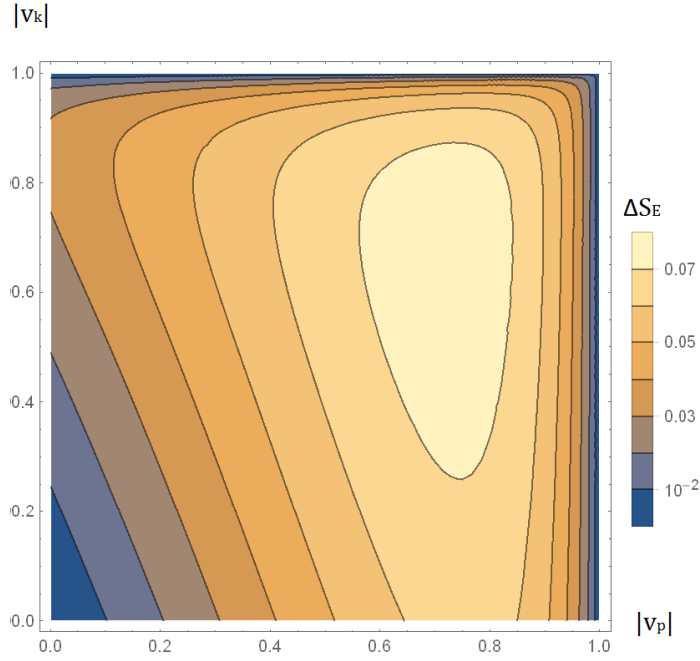

Figure 5.17: $\Delta S_E(|v_{\vec{p}}|, |v_{\vec{k}}|)$ for $\lambda = 1$ and $|\alpha_p|^2 = 0.8$, $|\alpha_k|^2 = 0.2$

$$\text{Máx.}\Delta S_E \sim 0.07,$$
$$\text{for}|v_{\vec{k}}| \in (\sim 0.3, \sim 0.9), |v_{\vec{p}}| \in (\sim 0.6, \sim 0.8)$$

We end here the exposition of the results, and in the next chapter we will present our conclusions.

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Chapter 6

Conclusions

6.1 The Scattering Process as the EceM

We performed a detailed study of the variation of the von Neumann entanglement entropy in the elastic scattering process for a bipartite system comprised of two interacting scalar fields, A and B, to one loop order in perturbation theory. By constructing the final state and the reduced density matrix of the subsystem A via the S-matrix formalism in quantum field theory, we quantified the entanglement generated when we scatter a separable or an entangled momentum initial state. As such we investigate the plausibility of using this phenomenon as an hypothetical "Entanglement creation/enchancement Machine" (EceM), and by looking at the results in the previous chapter it seems that the role although adequate is not optimal, since the maximum of the variation is only ~ 0.15

The correlations between Alice and Bob show explicit dependence on the speed (energy) between the colliding particles in a different manner if the initial state is separable or entangled, which indicates that the configuration of the parameters for the optimum output of the EceM is different if it's creating or enhancing entanglement.

In the case of the separable initial state, the 1-loop corrections indicate that the preferred velocity ($|v_{\vec{p}}^*|$)which creates the maximum entanglement output, is marginally lower than the one predicted by the Tree-level calculation, in the regime of weak coupling.

The maximum of the entropy's variation for same coupling values is approximately 1.5 times higher when the initial state is entangled than when it is separable. This means that the collision has a better performance as the EceM when enhancing already existing entanglement, than creating it from scratch. This difference becomes more accentuated as λ increases.

If the initial state is entangled, the domain of the velocities $|v_{\vec{p}}|, |v_{\vec{k}}|$ for which this peak occurs is strongly influenced by the respective weights associated with the initial states. This entails that there is a greater degree of possible manipulation on the initial entangled state, so that it can reproduce the same outcome of the maximum values of the variation. That is, the machine has more combinations for which we can alter the parameters, such that the output of entanglement is the same.

When the initial state is separable we notice that there is an increase in the values for the variation in 1-loop comparatively to Tree-level. This could be due to the nature of the model we used. Being a self-coupling model there is no background, and as such no apparent means of decoherence imposed on the system. Then much in the same light as thermodynamical entropy, this seems to suggest a kind of 2nd Law of Thermodynamics, for if there is no means for ΔS_E to be negative, then for every interaction the system has it's bound to increase for irreversible operations.

6.2 Further Work

There are a few interesting possible routes which one could pursue to add to the body of this work. We present the most pertinent ones, in no particular order:

-Trying to define a relationship between ΔS_E and the Cross-Section of the interaction for an initial entangled state to determine how the different cross-sections relate to each other. Alternatively, thinking about the initial entangled state as a "bound entangled pair" (e-bit), and trying to find the decay rate of the "particle" in the case where the constituent "qubits" interact weekly in an elastic manner;

-Trying to quantify the decoherence of a collision in this model by studying the example in Chapter3 of an "e-bit" colliding with a "q-bit";

-Applying the same formalism with a non-flat metric, and see how the effects of gravity might influence the "machine";

-Applying this formalism to different kinds of non-self coupling models to see if having different kind of virtual particles mediating the interaction could produce a variation of the entropy with a different kind of profile. Verifying if there are regimes for which the variation could possibly be negative by means of the decoherence;

-Applying this formalism to photons (or some fermionic model) in order to study states which beside the spins are also entangled in momenta, and witness if the different observables starting from the same set of conditions (e.g maximally entangled) behave differently when submitted to the same interaction.

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Appendix A

Derivations

A.1 Calculation of $i\mathcal{M}$ to 1-loop

The diagrammatic expansion to 1-loop is given by the following Dyson series,

(time is upwards)

Figure A.1: Pictorial Dyson Series
Expansion to 1-loop of the diagrams we aim to calculate.

    Such diagrams represent integrals that must be solved. The 1-loop contributions have a factor of $\frac{1}{2}$ which factorizes from the coupling constant. Each diagram has a respective C with a subscript pertaining to it's very channel (the tree-level doesn't have a subscript) which is it's Symmetry Factor, a number that accounts for all the topologically equivalent ways the same diagram can be drawn, and weighs it in in the series.

    We'll pick just a single 1-loop diagram and carry out it's integration explicitly, since the other two are solved exactly in the same manner and the tree-level one is trivial.

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Suppose we solve the t-channel diagram. Ignoring the symmetry factor for the present time we have,
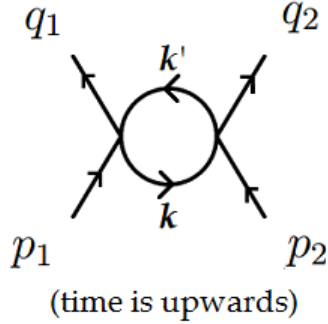

(time is upwards)

Figure A.2: Momentum labeled t-channel.
We put arrows on the external and internal legs, so that we can visualize the ingoing and outgoing momenta.

By using the Feynman rules for a ϕ^4 theory [27],

-For each vertex,

$$(i\lambda)(2\pi)^4\delta^{(4)}(\text{4-momentum conservation on the vertex});$$

-For each propagator with inner momentum k,

$$\frac{i}{k^2 - m^2 + i\epsilon};$$

-Integration over all of the internal momenta;

Applying these rules to the tree-level diagram is quite straightforward. Since there aren't any propagators we just get, $i\mathcal{M}_{(\text{tree})} = -2i\lambda$ which was what we presented in Chapter4 in (4.13).

In the 1-loop diagrams one proceed as follow. We analyze the t-channel in detail, but we'll later see that every calculation can easily generalized to any of the other two 1-loop diagrams. We get that the analytical form of the integral is,

$$\mathcal{I}_t = \int \frac{d^4k}{(2\pi)^4} \frac{d^4k^{'}}{(2\pi)^4} (i\lambda)^2 (2\pi)^4 \delta^{(4)}(p_1 + k^{'} - q_1 - k) \tag{A.1}$$

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$$\times (2\pi)^4 \delta^{(4)}(p_2 + k - q_2 - k^{'}) \frac{(i)^2}{(k^2 - m^2 + i\epsilon)(k^{'\,2} - m^2 + i\epsilon)}$$

After integrating over the space of $k^{'}$ we get,

$$\mathcal{I}_t = (2\pi)^4 \delta^{(4)}(p_1 + p_2 - q_1 - q_2) \times \tag{A.2}$$

$$\int \frac{d^4k}{(2\pi)^4} \frac{(\lambda)^2}{(k^2 - m^2 + i\epsilon)\big((k + q_1 - p_1)^2 - m^2 + i\epsilon\big)}$$

The integral is a Transfer Matrix $(i\hat{T})$ element and as such it has the form,

$$\langle q_1\, q_2|\, i\hat{T}\, |p_1\, p_2\rangle = (2\pi)^4 \delta^{(4)}(p_1 + p_2 - q_1 - q_2) i\mathcal{M}_{p_1\, p_2 \to q_1\, q_2}.$$

Then we can say that,

$$\mathcal{I}_t = (2\pi)^4 \delta^{(4)}(p_1 + p_2 - q_1 - q_2)\mathcal{M}_t, \tag{A.3}$$

where $\mathcal{M}_t$ is the pertinent part of the dynamics for this specific channel given by,

$$\mathcal{M}_t = \int \frac{d^4k}{(2\pi)^4} \frac{(\lambda)^2}{(k^2 - m^2 + i\epsilon)\big((k + q_1 - p_1)^2 - m^2 + i\epsilon\big)} \tag{A.4}$$

The other channels will be of the same form but the integrals will be different. $\mathcal{M}_t$ is the integral we need to calculate.

This integral exists in the complex plane, to avoid complications with the poles when solving it we go to the real plane by applying a Wick rotation. A Wick rotation is a Lorentz invariant transformation from the complex to the real plane, defined for a 4-vector in Minkowski metric as,

$$(ds)^2_M = (dt)^2_M - (d\vec{x})^2 \xrightarrow[rotation]{Wick} (ds)^2_E = (idt)^2_E - (d\vec{x})^2, \tag{A.5}$$

which gives the relationship,

$$(ds)^2_M \xrightarrow[rotation]{Wick} -(ds)^2_E. \tag{A.6}$$

Here the M and E subscripts denote Minkowski and Euclidean space, respectively. The Wick rotation can be generalized for any 4-vector in the same

way. In momentum space, $(p_0)_M \to -i(p_0)_E$. We apply the transformation to the previous propagators inside the integral,

$$\frac{1}{\left((k_0)^2 - (\vec{k})^2 - m^2 + i\epsilon\right)} \xrightarrow[rotation]{Wick} \frac{1}{\left(-(k_0)_E^2 - (\vec{k})^2 - m^2\right)} = -\frac{1}{\left(k_E + m^2\right)}$$

The other propagator also becomes similarly,

$$-\frac{1}{\left(k + (q_1 - p_1)\right)_E^2 + m^2}$$

This means the integral (A.4) in Euclidean space becomes,

$$\mathcal{M}_t = \int \frac{d^4 k_E}{(2\pi)^4} \frac{\lambda^2}{\left(k_E^2 + m^2\right)\left((k + w)_E^2 + m^2\right)}, \quad \text{with, } w = (q_1 - p_1); \quad \text{(A.7)}$$

We'll drop the subscript since it's implicit for all calculations that we are in Euclidean space.

Regularization of the Integral

Even though we are in the real plane, the integral still diverges logarithmically due to the ratio between the powers of momentum in the propagators and the dimension of integration [27]. As such we need to regularize the integral.

This is usually done by either changing the nature of the propagators, for instance, by introducing auxiliary fields (like the Pauli-Villars approach), or changing the dimension of integration (like the dimensional regularization approach). We use the dimensional regularization approach, whose principle is quite simple; Since the ratio between the dimensions of integration and the powers of momentum make the integral diverge, we decide to analyze it in an appropriate chosen dimension of $d = 4 - \epsilon$, because the integral converges for $2 < d < 4$. We'll use the Gamma function (Γ) to isolate the poles stemming from the divergent part, which will be absorbed in the renormalization process (see below). The remaining part converges at d=4, i.e. as $\epsilon \to 0$.

We introduce the Gamma function,

$$\Gamma(p) = \int_0^\infty dx \ e^{-ax} \ x^{p-1}. \quad \text{(A.8)}$$

Used in the following relationship [27],

$$\int_0^\infty \frac{d^d k}{(2\pi)^d} \frac{1}{\left(k^2 + 2kp + m_0^2\right)^n} = \frac{\Gamma(n - \frac{d}{2})}{\Gamma(n)} \frac{(m_0^2 - p^2)^{(d/2 - n)}}{(4\pi)^{d/2}}. \qquad (A.9)$$

We need the previous expression to calculate the integral, it gives a relationship between the kind of expression we have of the integral (A.7), and a computable general expression using the Gamma function.

In order to put the integral (A.7) exactly in the form as it's presented in the left hand side of (A.9), we will use Feynman's trick ,

$$\frac{1}{ab} = \int_0^1 \frac{dx}{\left(ax + b(1 - x)\right)^2}.$$

In our case we set, $a \equiv (k + w)^2 + m^2$ and $b \equiv (k^2 + m^2)$. Expanding the denominator and substituting it into (A.7),

$$\mathcal{M}_t = \int_0^1 dx \int_0^\infty \frac{d^d k}{(2\pi)^d} \frac{\lambda^2}{\left(k^2 + 2kwx + m^2 + w^2 x\right)^2}, \quad \text{with, } w = (q_1 - p_1);$$

$$(A.10)$$

Comparing with (A.9), we see that $p = wx$, $m_0^2 = m^2 + w^2 x$, and $n = 2$. As such we can substitute the expression integrated over k with the right hand side of (A.9), using the appropriate equalities. We get,

$$\mathcal{M}_t = \lambda^2 \int_0^1 dx \frac{\Gamma(2 - \frac{d}{2})}{\Gamma(2)} \frac{(m^2 + w^2 x - (wx)^2)^{(d/2 - 2)}}{(4\pi)^{d/2}}; \qquad (A.11)$$

For $d \to 4 - \epsilon$ and knowing that $\Gamma(2) = 1$, we get the expression,

$$\mathcal{M}_t = \left(\frac{\lambda}{4\pi}\right)^2 \Gamma\left(\frac{\epsilon}{2}\right) \int_0^1 dx \left(\frac{4\pi}{m^2 + w^2 x(1 - x)}\right)^{\epsilon/2}; \qquad (A.12)$$

Where $\Gamma\left(\frac{\epsilon}{2}\right) = \{\ 2/\epsilon + \psi(1) + \mathcal{O}(\epsilon)\ \}$ [27], and $\psi(1)$ is the known Euler–Mascheroni constant.

At this point we shall introduce an arbitrary scale μ by means of a redefinition of the coupling constant. We have

$$\lambda_R = \mu^{d-4} \Leftrightarrow \lambda = \lambda_R\, \mu^\epsilon$$

Then (A.12) becomes,

$$\mathcal{M}_t = \{\, 2/\epsilon + \psi(1) + \mathcal{O}(\epsilon)\, \} \left(\frac{\lambda_R \mu^\epsilon}{4\pi}\right)^2 \int_0^1 dx \left(\frac{4\pi}{m^2 + w^2 x(1-x)}\right)^{\epsilon/2}, \quad (A.13)$$

we can still manipulate the expression in the following way,

$$\mathcal{M}_t = \mu^\epsilon \left(\frac{\lambda_R}{4\pi}\right)^2 \{\, 2/\epsilon + \psi(1) + \mathcal{O}(\epsilon)\, \} \int_0^1 dx \left(\mu^2 \frac{4\pi}{m^2 + w^2 x(1-x)}\right)^{\epsilon/2}. \tag{A.14}$$

If we expand the expression inside the integral in powers of ϵ up to $\mathcal{O}(\epsilon^2)$ we get

$$1 - \frac{\epsilon}{2} \int_0^1 \log\left(\frac{m^2 + w^2 x(1-x)}{4\pi\mu^2}\right) + \mathcal{O}(\epsilon^2),$$

multiplying it by the expression inside the brackets gives,

$$\mathcal{M}_t = \mu^\epsilon \left(\frac{\lambda_R}{4\pi}\right)^2 \{\, 2/\epsilon + \psi(1) - \int_0^1 \log\left(\frac{m^2 + w^2 x(1-x)}{4\pi\mu^2}\right)\} + \mathcal{O}(\epsilon). \quad (A.15)$$

Since we will later set $\epsilon \to 0$, every power higher or equal to ϵ will disappear. The integral of the logarithm function has the form of the following tabulated integral,

$$\int_0^1 dx \log\left(1 + \frac{4}{a} x(1-x)\right) = -2 + \sqrt{1+a}\, \log\frac{\sqrt{1+a}+1}{\sqrt{1+a}-1}.$$

We can write the integral in (A.15) as,

$$\mathcal{M}_t = \log\frac{m^2}{4\pi\mu^2} + \int_0^1 dx \log\left(1 + \frac{4}{a} x(1-x)\right); \quad \text{Where, } a = \frac{4m^2}{w^2}. \quad (A.16)$$

Also realizing that $-w^2 = t \equiv (p_1 - q_1)^2$ (which is also for the other Mandelstam variables in their respective channels), we have,

$$\mathcal{M}(t) = \mu^\epsilon \left(\frac{\lambda_R}{4\pi}\right)^2 \{\, 2/\epsilon + \psi(1) - \left(\log\frac{m^2}{4\pi\mu^2} + G(t)\right)\} \qquad (A.17)$$

Where $G(t)$, is given by the general expression (4.15),

$$G(x) = -2 + \sqrt{1 - \frac{4m^2}{x}} \cdot \log\left(\frac{\sqrt{1 - \frac{4m^2}{x}} + 1}{\sqrt{1 - \frac{4m^2}{x}} - 1}\right), \text{ for } x = t.$$

Since the analytic solution of the integral is a function of the variable t, we dropped the subscript and represented explicitly the dependence.

It can be easily appreciated that all of these steps are exactly the same as if we had picked any of the other two diagrams, the only difference would be that instead of having a dependence on t, it would either be on u or s, depending on the channel.

Renormalization

Although we already have an analytical solution, there are still divergences present in the expressions. But it's noticeable that these expressions are all of the form,

$$\mathcal{M}(t) = \mathcal{M}(t)_D + \mathcal{M}(t)_C.$$

Meaning, we have the sum of both the divergent and the convergent parts discriminated in the expression. In order get the convergent part we have to pick an appropriate point t', such that $\mathcal{M}(t') = \mathcal{M}(t)_D$, in order to calculate the convergent part, as such,

$$\mathcal{M}(t)_C = \mathcal{M}(t) - \mathcal{M}(t').$$

This point is called the "Renormalization point", and we have to pick one also for $\mathcal{M}(s)$ and $\mathcal{M}(u)$. There is no single point that must be necessarily chosen, but a conventional one is $t = u = 0$ and $s = 4m^2$.

We then have,

$$\mathcal{M}(t)_C = \mathcal{M}(t) - \mathcal{M}(t = 0);$$
$$\mathcal{M}(u)_C = \mathcal{M}(u) - \mathcal{M}(u = 0);$$
$$\mathcal{M}(s)_C = \mathcal{M}(s) - \mathcal{M}(s = 4m^2);$$

Substituting the expressions explicitly, and evaluating for each respective point whilst taking the limit $\epsilon \to 0$ we get that,

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$$\mathcal{M}(t)_C = \left(\frac{\lambda}{4\pi}\right)^2 G(t); \; \mathcal{M}(u)_C = \left(\frac{\lambda}{4\pi}\right)^2 G(u); \; \mathcal{M}(s)_C = \left(\frac{\lambda}{4\pi}\right)^2 (2 + G(s));$$

Which means that the algebraic form of the expansion presented in fig.(A.1) is given by,

$$i\mathcal{M}_{(\text{tree }+1\text{loop})} = -iC\lambda - \frac{i}{2}\left(\frac{\lambda}{4\pi}\right)^2 \Big(C_t G(t) + C_u G(u) + C_s(G(s) + 2)\Big) \quad \text{(A.18)}$$

Symmetry Coefficients

The symmetry coefficients are weights attributed to the diagrams in the expansion. We use the interaction picture and calculate them by seeing how many ways we could have constructed the same diagram with all the fields at our disposal. For instance let us take the s-channel diagram,
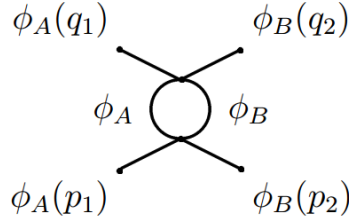

$$\phi_A(q_1) \qquad \phi_B(q_2)$$
$$\phi_A \quad \bigcirc \quad \phi_B$$
$$\phi_A(p_1) \qquad \phi_B(p_2)$$

Figure A.3: s-channel

We take the fields on the external points and the internal points (which we name k and p), and write them all out,

$$\phi_A(p_1)\phi_B(p_2)\phi_A(q_1)\phi_B(q_2) \int d^4p \; : \phi_A^{(1)} : \phi_B^{(1)} : \phi_A^{(2)} : \phi_B^{(2)} : \times 2 \int d^4k \; : \phi_A^{(1)} : \phi_B^{(1)} : \phi_A^{(2)} : \phi_B^{(2)} : \times 2,$$

where the expressions in colons mean normal ordering. Then contractions are performed only among fields pertaining to different momenta.

We have put labels in the fields of the internal points to book keep them. Each term has a factor of 2 which comes from the interaction term in the Lagrangian. The group of fields under d^4p integral depend all on the momentum p, whereas the fields under integral d^4k depend on momentum k.

We contract the internal fields first. Let's take the $\phi_A^{(1)}$ field of the internal point p, which can contract with any other two fields, $\phi_A^{(1)}$ and $\phi_A^{(2)}$ of the internal point k, so we have 2 possibilities. But, suppose it would be $\phi_A^{(2)}$ of the internal point p, that contracted with the fields in point k, in exactly the same manner, this would give a total of 4 combinations. The combinations for the internal fields ϕ_B are the same, so we have a total of (4×4) possible combinations for all the internal fields contractions. After the internal fields are coupled with each other, we couple the external fields. Even though there are all of these possibilities, there are only going to be two couplings in the internal fields, so when we couple the external fields we can choose if the first field couples with some field in point p or point k, so we still have an additional factor of 2

We then have that,
$$C_s = (4 \times 4) \times 2 \times 4,$$

the 4 stems from the two $2''$ from the Lagrangian factor multiplying. We can write it still, $C_s = (2^4 \times 4)(2)$.
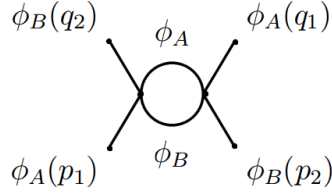
Now the u-channel,

Figure A.4: u-channel

We write it all down, and it's exactly the same as before,

$$\phi_A(p_1)\phi_B(p_2)\phi_A(q_1)\phi_B(q_2) \int d^4p \; : \phi_A^{(1)} : \phi_B^{(1)} : \phi_A^{(2)} : \phi_B^{(2)} : \times 2 \int d^4k \; : \phi_A^{(1)} : \phi_B^{(1)} : \phi_A^{(2)} : \phi_B^{(2)} : \times 2$$

But, in fact, there's one symmetry in this diagram which does not exist in the previous one, which is that if we perform an inversion on all the fields (i.e $\phi_A \leftrightarrow \phi_B$) we would get also a topologically equivalent diagram. This adds an extra factor of 2.

We then have,

$$C_u = (4 \times 4) \times 4 \times 4 = (2^4 \times 4)(2)(2)$$
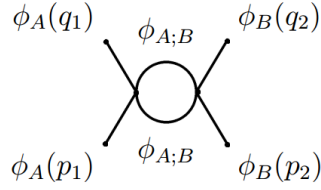
Now the t-channel,

Figure A.5: t-channel

The inner fields in this diagram are of a different nature than the previous ones, for they are entirely composed of either ϕ_A's or ϕ_B's.

We write it all down, in the case of the ϕ_A's,

$$\phi_A(p_1)\phi_B(p_2)\phi_A(q_1)\phi_B(q_2) \int d^4p \; : \phi_A^{(1)} : \phi_A^{(2)} : \phi_A^{(3)} : \phi_A^{(4)} : \int d^4k \; : \phi_A^{(1)} : \phi_B^{(1)} : \phi_A^{(2)} : \phi_B^{(2)} : \times 2$$

Notice that the terms corresponding to the inner point p don't have the factor of 2, because interactions of this type in the Lagrangian don't have any additional factor.

We contract the fields in the internal points, suppose we pick $\phi_A^{(1)}$ of point p, he has two possibilities to couple with the ϕ_A fields in point k, and since we have 4 different ϕ_A fields in point p we have a total of (2×4) combinations, on the first couple and 3 on the second one.

The external ϕ_A fields have two possible choices when coupling with the remainder internal fields, as well as the ϕ_B fields. Then we still have (2×2) more possibilities. This diagram also has an inversion symmetry, so that's another factor of 2, not forgetting the factor do 2 stemming from the interaction term from the Lagrangian, in the internal point k.

We then get,

$$C_t = (2 \times 4) \times 3 \times 2 \times 2 \times \times 4 = (2^4 \times 4)(2)(3)$$

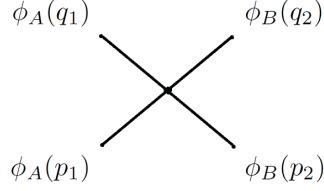We still have the Tree-level contribution,

Figure A.6: Tree level

There are no internal fields, so the fields associated with the internal point are the external ones. We have,

$$\phi_A(p_1)\phi_B(p_2)\phi_A(q_1)\phi_B(q_2)\int d^4k \; : \phi_A^{(1)} : \phi_B^{(1)} : \phi_A^{(2)} : \phi_B^{(2)} : \times 2$$

We have two possibilities for each external field when coupling with the internal field, which sets,

$$C = (2 \times 2 \times 2) = 2^3$$

.

Substituting all of the coefficients in the expression (A.18), we get,

$$i\mathcal{M}_{\text{(tree +1loop)}} = -i2^3\lambda - i\left(\frac{\lambda}{4\pi}\right)^2(2^4 \times 4)\left(3G(t) + 2G(u) + G(s) + 2\right) \quad \text{(A.19)}$$

The Lagrangian was defined initially with a factor of $\frac{1}{4}$ so to cancel with the factorizing part of the coefficients. As such, if we set $\lambda \to \lambda/4$ we get,

$$i\mathcal{M}_{\text{(tree +1loop)}} = -i2\lambda - i\left(\frac{\lambda}{4\pi}\right)^2(4)\left(3G(t) + 2G(u) + G(s) + 2\right) \quad \text{(A.20)}$$

Which is exactly what we have in (4.17).

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