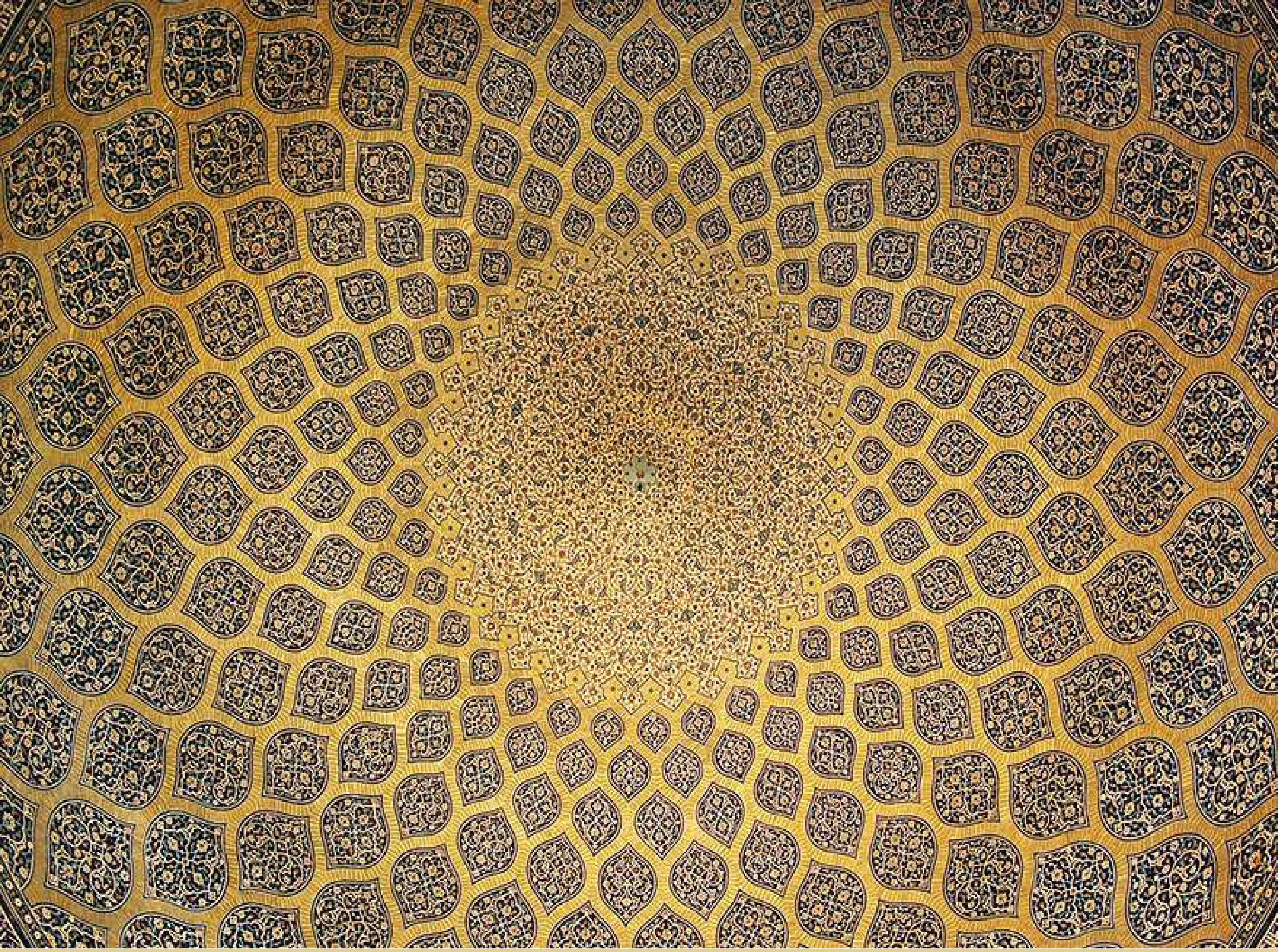

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Numerical Ranges of Linear Pencils

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... to my father and my mother.

Abstract

In recent years, the numerical range of finite matrices and linear operators has been intensively investigated. In this thesis, the concept of numerical range of a linear pencil is discussed, and the geometry of the numerical range is investigated by using techniques of plane algebraic geometry. The classification of all possible boundary generating curves of the numerical range of pencils of two-by-two and three-by-three matrices is explicitly given, when one of the matrices is hermitian.

The numerical range of linear pencils with hermitian coefficients has been studied by some authors. We have characterized the numerical range of self-adjoint linear pencils, pointing out and correcting an error reproduced in the literature.

For the case $n = 2$, the boundary generating curves of numerical range are conics. Geometrical proofs of the Elliptical Range Theorem, Parabolical Range Theorem and Hyperbolical Range Theorem, have been obtained in an unified way. We remark that the two-by-two case is particularly important, since for a pencil of arbitrary dimension the compression to the bidimensional case gives us information on the general n by n case.

For $n = 3$, we obtained the classification of all possible boundary generating curves of the numerical range, distinguishing the case of one of the matrices being positive (negative) definite, semidefinite and indefinite. All the possible boundary generating curves of the numerical range of three-by-three linear pencils can be completely described by using Newton's classification of cubic curves. The obtaining results are illustrated by numerical examples.

Keywords

Numerical range, Linear pencil, Generalized eigenvalue problem, Plane algebraic curve.

Resumo

O contradomínio numérico de matrizes finitas e de operadores lineares tem sido intensivamente investigado. Nesta tese, o conceito de contradomínio numérico de um feixe linear de matrizes e a geometria do contradomínio numérico são estudados usando técnicas de geometria algébrica plana. A classificação de todas as possíveis curvas geradoras de fronteira de feixes de matrizes de ordens dois e três é dada explicitamente, quando uma das matrizes é hermítica.

O contradomínio numérico de feixes lineares com coeficientes hermíticos tem sido objeto de estudo pelos investigadores. Nesta dissertação caracterizámos o contradomínio numérico de feixes lineares auto-adjuntos, apontando e corrigindo um erro reproduzido na literatura.

Para o caso $n = 2$, as curvas geradoras de fronteira são cónicas. As demonstrações do Teorema do Contradomínio Elíptico, Teorema do Contradomínio Parabólico e Teorema do Contradomínio Hiperbólico foram obtidas de modo unificado. O caso dois por dois é particularmente importante, porque um feixe de dimensão arbitrária pode ser reduzido por compressão ao caso bidimensional. Para $n = 3$, e uma das matrizes do feixe hermatica obtivemos a classificação de todas as possíveis curvas geradoras de fronteira do contradomínio numérico, distinguindo-se os casos de matrizes positivas (negativas) definidas, positivas (negativas) semi-definidas e indefinidas. Todas essas curvas são completamente descritas usando a classificação das cúbicas de Newton. Os resultados obtias são ilustrados através de exemplos numéricos.

Keywords

Contradomínio numérico, feixe linear, problemas de valores próprios generalizados, curva algébrica plana.

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Introduction

Studying bounded operators is an important topic in operator theory. The simple sample are matrices which are found in all the fields of mathematics. For instance, in finite dimensions a bounded linear operator may be associated to a corresponding matrix. Matrices were introduced in mathematics and their properties are still studied because they have an important role in mathematics and its applications. This thesis focuses on a well known notion related to matrices called numerical range, in particular numerical range of matrix pencils. Numerical range of a matrix (like spectrum) is the set of complex numbers which naturally depends on a matrix however, the spectrum of matrix is a discrete set also, the numerical range can be a compact and convex set. Since numerical range sits in the complex plane, it is clear that knowledge of its location can be useful.

The notion of numerical range is very extensive and this work tries to study the numerical range of linear pencil. A linear matrix pencil, denoted by (A,B), plays an important role in linear algebra. The problem of finding the eigenvalues of (A,B) is often solved numerically by using the well-known numerical method in [18]. Another approach for exploring the eigenvalues of (A,B) is by way of its numerical range. Matrix pencils are used extensively in studies of control systems with linear descriptors [1]. Generalized eigenvalue problems of matrix pencils have drawn great interest for decades from both mathematicians and engineers.

Study of linear pencils has a rich and long history that goes back to Weierstrass and Kronecker in the nineteenth century, usually in the context of their spectral analysis. In the present work we are particularly interested in studying the geometrical properties of the numerical range of linear pencils. Motivations to investigate this problem come from stability theory and from the study of certain over-damped vibration systems, *e.g.* see [13]. In general, having an accurate plot of the numerical ranges would help one to get deeper insight on the theory of numerical ranges and numerical radii.

Numerical range was introduced for linear operators in complex plane at 1918 by Toeplitz [28]. He associated with any complex $n \times n$ matrix a compact set in the complex plane and Toeplitz-Hausdorff theorem [11] showed that the numerical range is a convex set and also the outer boundary of this set is a convex curve. Hence, to scratch the shape of numerical range we just need boundary points and for this purpose we need support lines. The theorem 10 in [17], all the support lines of the numerical range via the highest eigenvalue of the hermitian part of the matrix $e^{-i\theta}A$, satisfy the characteristic curve of the matrix $P_A(u,v,w) = \det(uH + vK + wI) = 0$, where H,K are the Hermitian part and imaginary part of the matrix respectively.

For general n, the following Kippenhahn's result is useful: For any n-by-n matrix A, consider the homogeneous polynomial $P_A(u,v,w) = \det(uH + vK + wI) = 0$ and the algebraic curve $C(A)$, which is dual to the algebraic curve determined by $P_A(u,v,w) = 0$ in the complex projective plane $\mathbb{CP}^2$. Then the numerical range $W(A)$ is the convex hull of the real points of $C(A)$, which is called Kippenhahn curve or boundary generating curve. In [17] R. Kippenhahn studied the numerical range of three-by-three matrices. He showed that there are four classes of shapes which the numerical range of a three-by-three matrix A can assume. His classification is based on the factorability of the associated characteristic curve $P_A(u,v,w) = \det(uH + vK + wI) = 0$. Similarly, the characteristic polynomial of a linear pencil is $P_{A,B}(u,v,w) = \det(uH + vK + wB) = 0$ and if B is positive definite by an extension of Kippenhahn result, the numerical range $W(A,B)$ is the convex hull of the real points of $C(A,B)$. The case of B indefinite and positive semi definite are also treated (see chapter four). Overall, our studies build on the fact that the numerical range can be reduced under compressions to the bidimensional case. With these approaches we determine the numerical range of self-adjoint linear pencils and a linear pencil with one Hermitian coefficient of small sizes.

In **first chapter** , we focus on the algebraic curves and the dual of algebraic curves that generate the numerical range. Since the characteristic curve of three-by-three matrices are cubic curves, Newton's classification of cubic curves and dual considerations are investigated.

In **second chapter**, we present some properties of the numerical range of linear pencils and the characterization of $W(A,B)$ when A and B are Hermitian, obtained in theorem 4.1 of [19], is revised. We prove this theorem in a different way and we also correct an incorrect statement.

In third and four chapters of the thesis, we study linear pencils (A,B) with one Hermitian coefficient in the two-by-two and the three-by-three cases when B is positive definite, positive semi-definite, and indefinite.

In **third chapter**, we show that for the linear pencil of two-by-two matrices there are three cases which are stated by three theorems, the Elliptical Range Theorem, the Parabolical Range Theorem and the Hyperbolical Range Theorem.

In **fourth chapter**, we present an algorithm for geometric construction of the numerical range of a linear pencil (A,B) of three-by-three matrices using the characteristic polynomial of the linear pencil $P_{A,B}(u,v,w) = \det(uH + vK + wB) = 0$, when B is positive definite, positive semi-definite, indefinite and singular indefinite.

Finally, we give a classification of the generating boundary curves of the numerical range for linear pencils of small size. With our characterization we obtain branches of curves that belong to the boundary of the numerical range and provide the boundary of the numerical range.

Moreover, there are several existing computer programs for plotting the numerical range. In this thesis, we give procedures with Mathematica which allow to draw the numerical range and obtain illustrative examples.

Chapter 1

Preliminaries

1.1 Numerical range

Definition and basic properties

For A a complex square matrix of order n and I the identity matrix of the same size, the equation obtained by equating to zero the determinant of $(A - \lambda I)$, $\lambda \in \mathbb{C}$, is called the **characteristic equation** of A. The roots of this equation are called the **characteristic roots** or **eigenvalues** of A.

If A is Hermitian (i.e, $A = A^*$), it is well known that its characteristic roots are all real. When it is not possible to make any definite statement about the nature of the characteristic roots of a general matrix, several authors have given upper and lower bounds to these roots. The first of these bounds has been obtained by Bendixon in 1900. He obtained upper bounds for the real and imaginary parts of the characteristic roots of a real matrix. In a letter to Bendixon in 1902 [3], Hirsch extended these results to the case of the elements of A being complex numbers. Hirsch obtained an upper bound for the characteristic roots, as well as for their real and imaginary parts. A bound was also obtained by Bromwich in 1904 [4]. These bounds were further refined by Brown and Parker in 1930 and 1937, respectively.

In 1918, in the beautiful paper [28] Toeplitz used the fact that A may be decomposed uniquely in the form

$$A = H(A) + iK(A),$$

where $H(A)$ and $K(A)$ are Hermitian, given by

$$H(A) = Re(A) = \frac{1}{2}(A + A^*), \quad K(A) = Im(A) = \frac{1}{2i}(A - A^*). \tag{1.1}$$

When there is no place for ambiguity, we shall simply write $H = H(A)$, $K = K(A)$. Toeplitz also associated with any complex n-by-n matrix a compact set in the complex plane, which is a containment region for the eigenvalues.

Let $\mathbb{C}^n$ be the standard vector space of complex column vectors with n entries endowed with the scalar product $(x, y) = y^* x$, where y^* denotes the conjugate transpose vector of y, and corresponding norm $\|x\| = \sqrt{(x, x)}$.

Definition 1.1.1. Let $A \in \mathbb{M}_n$. The **field of values** or **numerical range** of A is defined and denoted by

$$W(A) = \{x^*Ax : \ x \in \mathbb{C}^n, \ x^*x = 1\}.$$

Similarly, the numerical range of a linear operator A acting on a Hilbert space $\mathscr{H}$ endowed with an inner product $(.,.)$ is the set of all complex numbers of the form (Af, f), where f varies over all vectors on the unit sphere.

Let $B(\mathscr{H})$ be the algebra of bounded linear operators acting on the Hilbert space $\mathscr{H}$. If dim $\mathscr{H} = n$, we shall identify $B(\mathscr{H})$ with $\mathbb{M}_n$, the algebra of n-by-n complex matrices.

When we try to locate in the complex plane the eigenvalues of complex matrices, we can use the field of values as a containment region for the spectrum of the matrix.

For an eigenvector $x \in \mathbb{C}^n$ of A with corresponding eigenvalue λ, the equation $Ax = \lambda x$ holds. Taking the inner product with x on both sides, we get

$$(Ax, x) = (\lambda x, x)$$

or

$$\frac{(Ax, x)}{(x, x)} = \lambda.$$

The ratio $\frac{(Ax,x)}{(x,x)}$ is well defined for any nonzero vector $x \in \mathbb{C}^n$ and any matrix $A \in \mathbb{M}_n$, and is called the **Rayleigh quotient** of x with respect to A. Thus, the numerical range comprises all the Rayleigh quotients of the matrix.

We present some basic properties of the numerical range, which can be easily verified.

Proposition 1.1.2. *For all linear operators A, B acting on $\mathscr{H}$ and α, $\beta \in \mathbb{C}$, the following holds:*

1) $W(A + \alpha I) = W(A) + \alpha$. *(Translation)*

2) $W(\beta A) = \beta W(A)$. *(Scalar multiplication)*

3) $W(A + B) \subseteq W(A) + W(B)$. *(Subadditivity)*

4) *Let U be a unitary operator, then* $W(U^*AU) = W(A)$. *(Unitary similarity invariance)*

5) $W(A^*) = \{\bar{z}, \ z \in W(A)\}$.

Toeplitz conjectured that $W(A)$ is a convex set, and proved that the outer boundary of $W(A)$ is a convex curve, but did not exclude the existence of interior holes in the set. The following statement is known as the Toeplitz-Hausdorff theorem [28].

Theorem 1.1.3. *The numerical range of an arbitrary linear operator A acting on a Hilbert space is convex.*

The numerical range of an operator is not always closed, not even if the operator is compact [6]. In the finite dimensional case, the numerical range of an operator is the continuous image of a compact set and so is compact.

Proposition 1.1.4. *(Compactness) For any $A \in \mathbb{M}_n$, $W(A)$ is a compact subset of $\mathbb{C}$.*

Further:

Proposition 1.1.5. *(Connectedness) For any linear operator A acting on $\mathscr{H}$, $W(A)$ is a connected subset of $\mathbb{C}$.*

$W(A)$ is a bounded set, but in the infinite dimensional case it is not always closed. The numerical range of the **shift operator** $s : l^2 \longrightarrow l^2$, where l^2 is the linear space of all complex sequences of square summable functions, defined by

$$s(x_0, x_1, ...) = (0, x_0, x_1, ...),$$

is an open set, more precisely, the disk $D = \{z \in \mathbb{C} : |z| < 1\}$.

As it is well known, the spectrum of an operator A consists of those complex numbers λ such that $A - \lambda I$ is not invertible.

Proposition 1.1.6. *(Spectral containment) For all $A \in \mathbb{M}_n$*

$$\sigma(A) \subseteq W(A),$$

where $\sigma(A)$ denotes the spectrum of A.

In infinite dimension, the closure of the numerical range of a bounded linear operator contains its spectrum:

$$\sigma(A) \subseteq \overline{W(A)}.$$

Proof. For a proof see *e.g* [10]. □

Since the diagonal elements of $A \in \mathbb{M}_n$ are the Rayleigh quotients with respect to A of the standard orthogonal basis of $\mathbb{C}^n$ with vectors e_i, that is, $a_{ii} = (Ae_i, e_i)$, the diagonal elements of A belong to the numerical range of A,

$$\{a_{11}, a_{22}, ..., a_{nn}\} \subset W(A).$$

For a normal matrix A, much more can be said. To formulate the precise result, it is necessary to recall the concept of **convex hull** of a set S in the complex plane, in symbols $Co(S)$. By definition, $Co(S)$ is the smallest convex set that includes S.

Proposition 1.1.7. *(Normality) If $A \in \mathbb{M}_n$ is normal, then $W(A)$ is the convex hull of the spectrum of A,*

$$W(A) = Co(\sigma(A)).$$

If A is a normal operator, then $\overline{W(A)} = Co(\sigma(A))$.

Proof. For a proof see *e.g* [10]. □

Corollary 1.1.8. *The numerical range of a Hermitian matrix A is the real interval* $[m, M]$, *where m and M are the minimum and the maximum eigenvalues of A, respectively.*

For a nonnormal matrix A, the knowledge of its spectrum $\sigma(A)$ is not enough to describe the numerical range, and additional information must be taken into account, as will be shown in the sequel.

If the spectrum of an operator lies in the real line, very little can be said about the operator, but if its numerical range is real, then a standard result in Hilbert space theory states that the operator must be self-adjoint.

Proposition 1.1.9. *For any linear operator T acting on $\mathscr{H}$, T is self-adjoint if and only if $W(T)$ is real.*

Proof. For a proof see *e.g* [10]. □

Now we describe the region of the complex plane attained by the values (Ax, x) under the hypothesis that A is a two-by-two complex matrix and $x \in \mathbb{C}^2$ ranges over the unit sphere $x^*x = 1$.

In the sequel, we shall denote by trA the trace of A.

Theorem 1.1.10. *(Elliptical Range Theorem) Let $A \in \mathbb{M}_2$ be given, and set $A_0 = A - (\frac{1}{2}trA)I$. Then*

1) *The numerical range of A is a closed ellipse (with interior).*

2) *The center of the ellipse $W(A)$ is at the point $\frac{1}{2}trA$. The length of the major axis is $(trA_0A_0^* + 2|\det A_0|)^{\frac{1}{2}}$, the length of the minor axis is $(trA_0A_0^* - 2|\det A_0|)^{\frac{1}{2}}$, and the distance of the foci from the center is $|\det A_0|^{\frac{1}{2}}$. The major axis lies on the line passing through the two eigenvalues of A, which are the foci of $W(A)$. These two eigenvalues coincide if and only if the ellipse is a circle (possibly a point).*

3) *$W(A)$ is a closed line segment if and only if A is normal, being the endpoints of the line segment the eigenvalues of A. Furthermore, it is a single point if and only if A is a scalar matrix.*

4) *$W(A)$ is a nondegenerate ellipse (with interior) if and only if A is not normal, and in this event the eigenvalues of A are interior points of $W(A)$.*

Proof. For a proof see *e.g* [14]. □

1.2 Plane algebraic curves

Since the publication of the seminal paper of Kippenhahan [28], many authors have developed the theory of numerical range in several directions. One of these directions is algebraic geometry. The starting point of this approach is the consideration of the plane algebraic curve given by

$$P_A(x, y, z) = \det(xH + yK + zI) = 0,$$

where $H,\ K$ were defined in (1.1), for $A = H + iK$ an n-by-n complex matrix. To describe this approach, we need some terminology and concepts from plane projective geometry, such as homogeneous coordinates, dual curves and foci of a curve. Such concepts belong to the classical heritage of mathematical literature, and a classical treatise on this area is [29]. For a modern treatment see [8].

A point in **nonhomogeneous coordinates** is an ordered pair of complex numbers (u,v). If u and v are real , then (u,v) is called a **real point**.

A point in **homogeneous coordinates** is an ordered triple $[x,y,z]$ of complex numbers $x,\ y,\ z$, which are not all zero.

Definition 1.2.1. Two points $[x,y,z]$ and $[x',y',z']$ are **equivalent** over a field $\mathcal{F},\mathbb{C}$ or $\mathbb{R}$ (notation: $[x,y,z]\sim[x',y',z']$), if $[x,y,z]=\lambda[x',y',z']$ for some nonzero scalar $\lambda\in\mathcal{F}\backslash\{0\}$.

The **complex projective plane** $\mathbb{CP}^2$ is the set of the equivalence classes of all points $[x,y,z]$ under the equivalence relation of the above definition,

$$\mathbb{CP}^2=\{[x,y,z]:(x,y,z)\in\mathbb{C}^3\setminus\{0\}\},$$

or

$$\mathbb{CP}^2\equiv\frac{\mathbb{C}^3\setminus\{0\}}{\sim}.$$

Any point in the projective plane is represented by a triple $[x,y,z]$, called the **homogeneous coordinates** or **projective coordinates** of the point, where $x,\ y$ and z are not simultaneously zero.

The point $[x,y,z]$ in $\mathbb{CP}^2$ with $z\neq 0$ can be identified with the point $(\frac{x}{z},\frac{y}{z})$ in nonhomogeneous coordinates. On the other hand, the point (u,v) becomes $[u,v,1]$ in homogeneous coordinates.

The set of all points $[x,y,z]$ in $\mathbb{CP}^2$ with $z\neq 0$ satisfying a homogeneous equation of degree one, $ax+by+cz=0$, where $a,\ b$ and c are complex numbers not all zero, is a **line**.

If $f(x,y,z)$ is a homogeneous polynomial of degree two,

$$f(x,y,z)=ax^2+bx^2+cz^2+dxy+eyz+fzx,$$

with the coefficients not all zero, then $f(x,y,z)=0$ defines a **conic**, which is an algebraic curve of order two.

More generally, if $f(x,y,z)$ is a homogeneous polynomial of degree d in $x,\ y$ and z, then the set of points $[x,y,z]$ in $\mathbb{CP}^2$ satisfying the equation $f(x,y,z)=0$ is an **algebraic curve** of **order** d.

We observe that any such curve can be dehomogenized to yield the curve $f(x,y,1)=0$ in $\mathbb{C}^2$, and conversely, an algebraic curve $P(x,y)=0$, where $P(x,y)$ is a polynomial in x and y, can be homogenized to a curve in $\mathbb{CP}^2$ with equation obtained by simplifying $P(\frac{x}{z},\frac{y}{z})=0$.

Remark 1.2.2. Let $ax+by+cz=0$ be a fixed line, and let $P(x,y,z)=0$ be an algebraic curve of order d. Setting $z=1$ yields the nonhomogeneous equations $ax+by+c=0$ and $P(x,y,1)=0$. Assume $b\neq 0$. Then

$$y=\frac{-c-ax}{b},$$

and the solutions of

$$P(x, \frac{-c-ax}{b}, 1) = 0,$$

are the x coordinates of the points common to both the line and the curve. Since P has degree d, there are precisely d solutions of the above equation, where multiple roots are counted according to their multiplicities. Thus, a line intersects an algebraic curve of order d in d points, counted according to their multiplicities.

1.2.1 Dual of a plane algebraic curve

The theory of duality plays a central role in projective geometry. In plane projective geometry, duality can be naively introduced:

Any theorem (properly phrased) remains true, by replacing every reference to " point " with " line ", and vice versa. For example, the statement " any two points are incident to an unique line, " corresponds to the dual statement " any two lines are incident to a unique point. " For example, write $x,\ y,\ z$ for homogeneous coordinates in $\mathbb{CP}^2$ and $X,\ Y,\ Z$ for homogeneous coordinates in the dual plane. For fixed $X,\ Y,\ Z$, the line $Xx+Yy+Zz=0$ in $\mathbb{CP}^2$ corresponds to the point $(X:Y:Z)$ in the dual plane, and conversely, for fixed $x,\ y,\ z$ the line $Xx+Yy+Zz=0$ in the dual of $\mathbb{CP}^2$ defines a pencil of lines in $\mathbb{CP}^2$, through the point (x,y,z) in $\mathbb{CP}^2$. In this way, the dual of the dual complex plane is identified with itself.

Coordinate systems are often used to specify the position of a point, but they may also be used to specify the position of other objects, such as lines, planes, circles or spheres. Usually, the type of figure being described is used to distinguish the type of coordinate system, for instance line coordinates are used for any coordinate system that specifies the position of a line. Three coordinates (l,m,n) specify the line for which the equation $lx+my+n=0$ holds. Here l and m may not be simultaneously zero. In this equation, if the coordinates are multiplied by a nonzero scalar then the represented line remains the same. So (l,m,n) is a system of homogeneous coordinates of the line.

Definition 1.2.3. If C is an algebraic curve of order d, given by $f(\frac{x}{z}, \frac{y}{z}) = 0$, then its **dual** is defined and denoted by

$$C^* = \{[X,Y,Z] \in \mathbb{CP}^2 : Xx+Yy+Zz=0 \ \textit{ is a tangent line of } C\ \},$$

and d is called the **class of the dual curve**.

If C is an algebraic curve with real coefficients, C^* is also an algebraic curve and is given by an homogeneous polynomial equation. Since the dual of a dual curve yields the original one, properties of a curve and its dual are indeed dual to each other. In particular, the order of C is the class of C^*. While a line intersects a curve of order d_1 in d_1 points (counting multiplicities), through a point there are d_2 tangent lines to a curve of class d_2. The tangent lines to C^* are exactly those $ax+by+cz=0$ satisfying $f(a,b,c)=0$ or, in other words, C^* is the envelope of the family of lines $ax+by+cz=0$ such that $f(a,b,c)=0$. This gives an alternative description of the dual curve. For example, the point $[a,b,c]$ and the line $ax+by+cz=0$ are dual to each other, and it will be shown below that the dual curve of a conic is again a conic.

1.2.2 Computation of the dual of a plane curve

Let $f(x,y,z) = 0$ be the equation of a plane algebraic curve in homogeneous coordinates. Let $Xx+Yy+Zz=0$ be the equation of a line with line coordinates (X,Y,Z). The condition that the line is tangent to the curve can be expressed in the form of a polynomial equation $F(X,Y,Z)=0$, which is the tangential equation of the curve.

Let (p,q,r) be a point on the curve. Then the equation of the tangent at this point is given by

$$x\frac{\partial f}{\partial x}(p,q,r)+y\frac{\partial f}{\partial y}(p,q,r)+z\frac{\partial f}{\partial z}(p,q,r)=0.$$

So $Xx+Yy+Zz=0$ is a tangent to the curve if

$$X=\lambda\frac{\partial f}{\partial x},\quad Y=\lambda\frac{\partial f}{\partial y},\quad Z=\lambda\frac{\partial f}{\partial z},\quad \lambda\in\mathbb{C}.$$

Eliminating $p,\ q,\ r$ and λ from these equations, along with $Xp+Yq+Zr=0$, gives the equation in $X,\ Y$ and Z of the dual curve.

We illustrate the above procedure for the conic C with equation $ax^2+by^2+cz^2=0$. Then the dual curve can be found by eliminating $p,\ q,\ r$, and λ from the equations

$$X=2\lambda ap,\quad Y=2\lambda bq,\quad Z=2\lambda cr,\qquad Xp+Yq+Zr=0,\quad \lambda\in\mathbb{C}.$$

The first three equations are easily solved for $p,\ q,\ r$ and substituting the solutions in the remaining equation yields

$$\frac{X^2}{2\lambda a}+\frac{Y^2}{2\lambda b}+\frac{Z^2}{2\lambda c}=0.$$

Removing 2λ from the denominators, the equation of the dual is

$$\frac{X^2}{a}+\frac{Y^2}{b}+\frac{Z^2}{c}=0.$$

In general, it is not easy to find the dual curve. However, it is quite easy to write down a parametric representation of a dual curve given a parametric representation of the original curve. Local parametric equations of a curve C have the form $x=x(t),\ y=y(t)$, where $t\in[t_0,t_1]\subset\mathbb{R}$ and $x(t),\ y(t)$ are real valued differentiable functions. So, assuming that $x'(t)\neq 0$, we have

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{y'(t)}{x'(t)}.$$

Hence, the equation of the tangent line to C at $(x(t),y(t))$ is

$$y-y(t)=\frac{y'(t)}{x'(t)}(x-x(t)),$$

Now, we can rewrite the above equation in the following form

$$y - y(t) = \frac{y'(t)x - y'(t)x(t)}{x'(t)}.$$

We have successively,

$$yx'(t) - y(t)x'(t) = y'(t)x - y'(t)x(t),$$

$$yx'(t) - y'(t)x = y(t)x'(t) - y'(t)x(t),$$

$$y\frac{x'(t)}{y(t)x'(t) - y'(t)x(t)} - x\frac{y'(t)}{y(t)x'(t) - y'(t)x(t)} - 1 = 0.$$

Thus, the parametric representation of C^* has the form

$$X(t) = \frac{-y'(t)}{y(t)x'(t) - y'(t)x(t)}, \tag{1.2}$$

$$Y(t) = \frac{x'(t)}{y(t)x'(t) - y'(t)x(t)}, \tag{1.3}$$

where $t \in [t_0, t_1]$.

Consider the ellipse C given by

$$f(x,y) = (\frac{x^2}{a^2}) + (\frac{y^2}{b^2}) = 1,$$

where $a \geqslant b \geqslant 0$. Firstly, we consider the parametric equations of the ellipse

$$x(t) = a\cos(t), \quad y(t) = b\sin(t), \quad t \in [0, 2\pi].$$

Having in mind equations (1.2) and (1.3), the dual parametric equations are

$$X(t) = \frac{\cos(t)}{a}, \quad Y(t) = \frac{\sin(t)}{b}.$$

Hence

$$a^2X^2 + b^2Y^2 = 1.$$

Now, substituting X, Y by $\frac{X}{Z}$, $\frac{Y}{Z}$, respectively, then homogenizing the equation, we conclude that C^* has the form

$$a^2X^2 + b^2Y^2 = Z^2.$$

1.2.3 Singular points

Let $f(x,y,z)=0$ be a curve of order d in $\mathbb{CP}^2$ and let l be a line. As l is determined by any two distinct points, say $A=(a_1,a_2,a_3)$ and $B=(b_1,b_2,b_3)$, then l can be parametrized as

$$x_1 = sa_1+tb_1, \quad x_2 = sa_2+tb_2, \quad x_3 = sa_3+tb_3,$$

where $s,\ t$ are real parameters. The intersection of f with l is given by $\varphi(s,t)=0$, where φ is defined by

$$\varphi(s,t) = f(sa_1+tb_1, sa_2+tb_2, sa_3+tb_3).$$

Let us look in more detail at the way which a line l intersects a projective curve f of order d. Assume l is not a component of f, that is, if l has equation $ax+by+cz=0$, then $ax+by+cz$ is not a factor of f. Let P be a point on l corresponding to the pair (s_0,t_0), in some parametrization of l. We define the **intersection number** $I(P,f,l)$ to be the multiplicity of (s_0,t_0) as a root of the equation $\varphi(s,t)=0$. This means that $\varphi(s,t)$ has a factor $(t_0s-s_0t)^m$ for some integer $m>0$ and that $I(P,f,l)=m$. Clearly, $I(P,f,l)$ is an integer greater than zero and vanishes if and only if P is not an intersection of f and l.

Definition 1.2.4. A point P on a projective curve f is **singular**, when it has multiplicity greater than or equal to two. The curve f itself is **singular** when it has at least one singular point, otherwise it is **nonsingular**.

It is well known that a point P on a curve f is a singular point if and only if

$$f(P)=0 \quad \textit{and} \quad f_x(P)=f_y(P)=f_z(P)=0.$$

For a proof see [8].

Points of multiplicity $1,2,3,\ldots$ are said to be **simple**, **double**, **triple**, ... points of f. (See figure 1.1.)

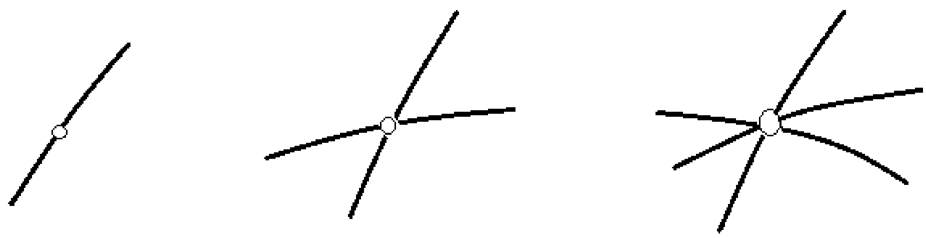

Fig. 1.1 Simple, double and triple points.

A **double tangent** to a curve is a line that is tangent to the curve at two distinct points. (See part(a) figure 1.3.)

An **ordinary double point** of a plane curve is a point where the curve intersects itself and the two branches of the curve have distinct tangent lines, and is also called a **node**. (Point R^* in part(b) figure 1.3.) A double point which has a double tangent is called a **cusp**. (Point R^* in part(b) figure 1.2.).

Moreover, an **acnode** is an isolated point, not on a curve, but whose coordinates satisfy the equation of the curve. A double point and an acnode are also called **conjugate points**.

Let P be a simple point of $f(x,y,z)=0$. Then the intersection number of the unique tangent line at P is greater than or equal to two. If the intersection number is greater than or equal to three we have a very special point on the curve called a **flex**. These points play a potentially important role in understanding the geometry of curves.

Consider the curve $y=x^3$, which has an inflection point at the origin. Next we obtain its dual curve. A local parametric equation of the curve has the form

$$x=t, \quad y=t^3, \quad t\in]-\infty,+\infty[.$$

Hence, the equation of the tangent line to the curve at the generic point (t,t^3) is

$$y-t^3=3t^2(x-t),$$

or

$$y-3t^2x+2t^3=0.$$

By substituting y and x with $\frac{y}{z}$ and $\frac{x}{z}$ respectively, and homogenizing the equation, we have

$$y-3t^2x+2t^3z=0.$$

Considering

$$Y(t)=1, \quad X(t)=-3t^2, \quad Z(t)=2t^3,$$

we conclude that the dual curve has the form

$$4X^3+27Z^2=0,$$

which is known as a **Neilian curve**. In this example, we have shown that the dual of an inflection point is a cusp. It can be shown that an inflection point of the curve C corresponds to a cusp R^* of the

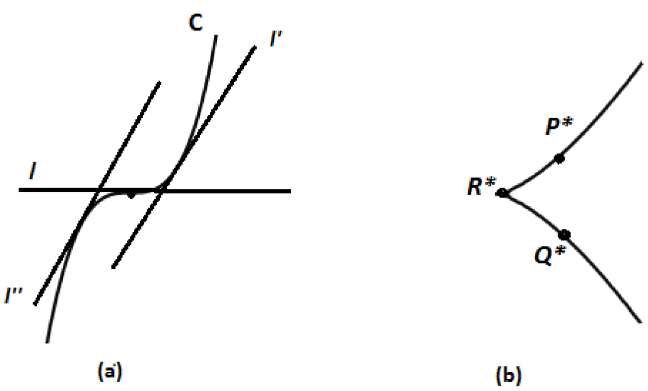

Fig. 1.2 The dual of the inflection point is a cusp. P^*, Q^*, R^* correspond to l', l'', l respectively.

dual curve. (See figure 1.2.)

When a plane curve has no singular points, the dual curve may however have singular points. For example, if the tangent line l to a curve C at a point P is also tangent to C at a point Q, this means that we have a double tangent, and the dual curve has a node. (See figure 1.3.)

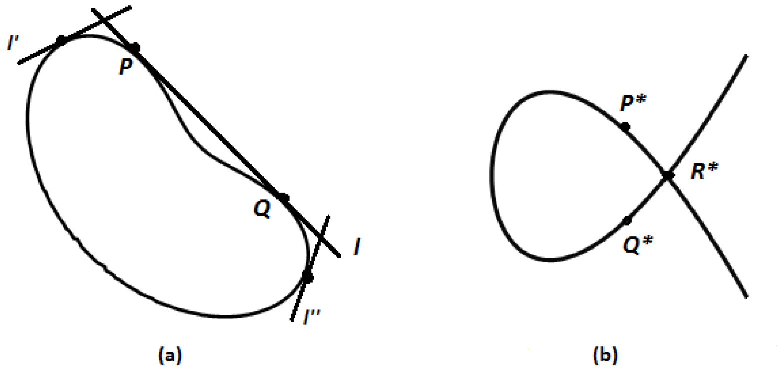

Fig. 1.3 The dual of the double tangent is a node: R^* is the dual of the double tangent l and P^* is the dual of the line l'. Also, Q^* is the dual of l''.

1.2.4 Characteristic curve of a matrix

Let A and B be a pair of n-by-n matrices over a field $\mathcal{F}$, $\mathbb{C}$ or $\mathbb{R}$. The **pencil** generated by A and B is the set of all linear combinations over $\mathcal{F}$ of A and B, and is denoted by $xA+yB$, where x and y are indeterminates over $\mathcal{F}$. Thus,

$$xA+yB=\{rA+sB : r,s\in\mathcal{F}\}.$$

The **characteristic polynomial** of the pencil $xA+yB$ such that A, B Hermitian, is the polynomial $P(x,y,z)=\det(zI-xA-yB)$.

The characteristic polynomial of the pencil $xReA+yImA$ associated with any matrix A can be used in two different ways to determine $W(A)$. One, is via Kippenhahn's result stating that the numerical range of A coincides with the convex hull of the real points of the dual curve of $\det(xReA+yImA+zI)=0$. As it will be shown in the sequel, in the cases $n=2$ and $n=3$, each possible type of boundary curve can be completely described. In this way, the classical algebraic curves theory is important in the study of the geometry of the numerical range. On the other hand, a parametric representation of the boundary of $W(A)$, denoted by $\partial W(A)$, can also be obtained by considering the largest eigenvalue of $Re\,A\cos\theta+Im\,A\sin\theta$ for θ ranging over $[0,2\pi]$. We give a brief account of both approaches related to the characteristic polynomial of the pencil $xRe\,A+yIm\,A$.

For A an n-by-n matrix, write $A=H+iK$, where H and K are Hermitian. The characteristic polynomial of $xH+yK$ is $P(x,y,z)=\det(zI-xH-yK)$, and $P(x,y,z)=0$ is the **characteristic curve of the matrix** A. In the special case $x=1,\ y=i$, the above polynomial reduces to $P(x,y,z)=\det(zI-(H+iK))=\det(zI-A)$.

Murnaghan [22] and Kippenhahn [17], independently, showed that the algebraic curve

$$P(x,y,z) = \det(zI + xH + yK) = 0,$$

determines the numerical range of $A = H + iK$ (cf. theorem 1.3.6). We shall discuss the connection between $P(x,y,z)$ and the numerical range of A. Next, we derive a basic result concerning $P(x,y,z)$.

Proposition 1.2.5. *Let H and K be n-by-n Hermitian matrices. Then the coefficients of the polynomial $P(x,y,z) = \det(zI - xH - yK)$ are real.*

Proof. Let $\overline{P(x,y,z)}$ denote the polynomial obtained by replacing the coefficients of $P(x,y,z)$ by their complex conjugates. Then

$$\overline{P(x,y,z)} = \det(zI - xH - yK)^* = \det(zI - xH - yK) = P(x,y,z),$$

so the polynomial coefficients are real. □

1.2.5 Classification of the cubic curves

The general cubic equation has the form

$$ax^3 + bx^2 + cx + d = 0 \tag{1.4}$$

with $a \neq 0$, and the coefficients a,b,c,d are generally assumed to be real numbers. Every cubic equation (1.4) with real coefficients has at least one solution x among the real numbers, a consequence of the **intermediate value theorem**. We can distinguish several possible cases using the **discriminant** of the cubic, which is defined as follows

$$\triangle = 18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2.$$

The following cases have to be considered:

1) If $\triangle > 0$, then the equation has three distinct real roots.

2) If $\triangle = 0$, then the equation has a multiple root and so all its roots are real.

3) If $\triangle < 0$, then the equation has one real root and two complex conjugate roots.

Definition 1.2.6. By **Weierstrass normal form** of a cubic, we mean a cubic in $\mathbb{CP}^2$ defined by an equation of the form $y^2z = f(x,z)$ with $f(x,z)$ a nonzero binary cubic form in x, z. Thus, in the affine view $z = 1$, such a curve has the form $y^2 = f(x)$ with $f(x)$ a cubic in x, that is,

$$y^2 = ax^3 + bx^2 + cx + d. \tag{1.5}$$

If we assume that the (complex) roots of the cubic polynomial in the right hand side are distinct, such a curve is called an **elliptic curve**.

It is well known that any general cubic curve f in $\mathbb{CP}^2$ is projectively equivalent to a Weierstrass normal form (for a proof see [8]).

Remark 1.2.7. By setting $x = \frac{X}{Z}$ and $y = \frac{Y}{Z}$ in the equation (1.5), the following homogenous form is obtained:

$$Y^2Z = aX^3 + bX^2Z + cXZ^2 + dZ^3.$$

Substituting $Z = 0$ into the equation, gives $X^3 = 0$, which has the triple root $X = 0$. This is the intersection of this cubic with the line at infinity $Z = 0$, meaning that the cubic meets the line at infinity in three points, being the three points all the same. So, the cubic has exactly one point at infinity. The point at infinity is an **inflection** point of the cubic and the tangent line at that point is the line at infinity, which meets it there at a point with multiplicity three. So, for a cubic in Weierstrass form there is one point at infinity and we denote that point by O.

If we change coordinates and move the point at infinity to a finite place, we take

$$t = \frac{x}{y} \text{ and } s = \frac{1}{y},$$

and $y^2 = ax^3 + bx^2 + cx + d$ becomes

$$s = at^3 + bt^2s + cts^2 + ds^3,$$

in the (t,s)-plane. We can always get back to the old coordinates, because $y = \frac{1}{s}$ and $x = \frac{t}{s}$. In the (t,s)-plane we have all the points of the old (x,y)-plane except the points where $y = 0$, and the zero element O on our curve is now at the origin $(0,0)$ in the (t,s)-plane. Hence, we have two views of the curve. The view in the (x,y)-plane shows us everything except O. The view in the (t,s)-plane shows us O and everything except the points of order two. Except for O and the points of order two, there is a one-to-one correspondence between points of the curve in the (x,y)-plane and points of the curve in the (t,s)-plane (see [26] and figure 1.4). For example, if we let $r = \frac{y}{x}$, then the equation $y^2 = x^2(x+1)$

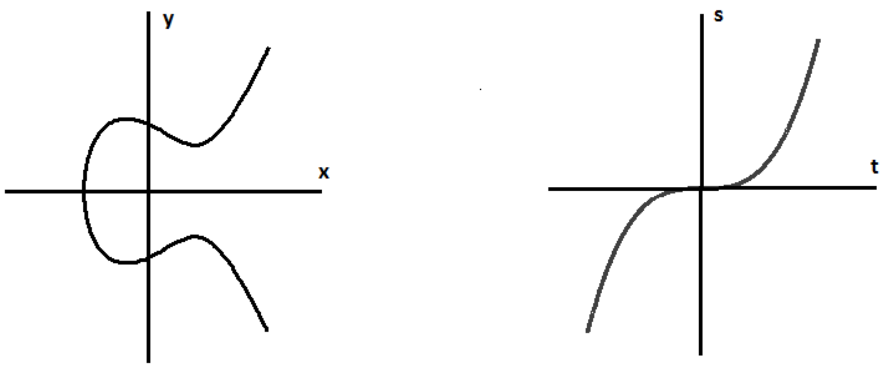

Fig. 1.4 Two views of a cubic curve.

becomes

$$r^2 = x + 1,$$

and so

$$x = r^2 - 1 \quad and \quad y = r^3 - r.$$

If we take a rational number r and define x and y in this way, then we obtain a rational point on the cubic, and if we start with a rational point (x, y) on the cubic, then we get the rational number r. These operations are inverse of each other, and are defined at all rational points except for the singular point $(0, 0)$ on the curve. So, in this way, we get all of the rational points on the curve. Also for the curve $y^2 = x^3$ we have

$$x = t^2 \quad and \quad y = t^3.$$

Studying more about singular points leads us to the concept of multiple points of curves. As already observed, a curve may, in a similar manner, have multiple tangents, or in other words, there may exist lines which touch the curve in two or more points, or which have with the curve a contact of the second or higher order. Commonly, the " singular points " are either multiple points, or points of contact of multiple tangents. It will be easier to our discussion of multiple tangents to consider that the axis $y = 0$ is a multiple tangent. We find the singular points where this line meets the curve by taking $y = 0$ in the cubic equation, i.e,

$$ax^3 + bx^2 + cx + d = 0,$$

which can be reduced to the form

$$a(x - \alpha)(x - \beta)(x - \gamma) = 0, \tag{1.6}$$

where α, β, γ, are the abscissas of the points where the real axis meets the curve

Newton´s classification of cubic curves appears in a book entitled **Lexicon Technicum** by John Harris, published in London in 1710. In his classification of cubics, Newton gives four classes of equations. The third class of equations is the one given below. Concerning this third case, Newton says:

" In the third case the equation defines a parabola whose legs diverge from one another and run out infinitely in contrary ways."

This case divides into five species and Newton gives a typical graph for each species. The five types depend on the roots of the cubic in x in the right hand side of the equation (1.5). These five types of curves are:

1. **An oval and an infinite branch** (**Bell** form) (cf. figure 1.5),

2. **A parabola of a Bell-like form** (cf. figure 1.6),

3. **A nodated**, also called **Tschirnhausen** cubic (cf. figure 1.7),

4. **A punctate** (by having the oval infinitely small) (cf. figure 1.8),

5. **A neilian** parabola, commonly called **semi cubical** (cf. figure 1.9).

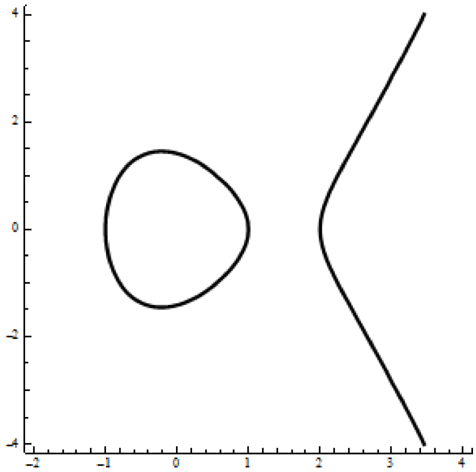

Fig. 1.5 The curve generated by the equation $y^2 = (x^2 - 1)(x - 2)$.

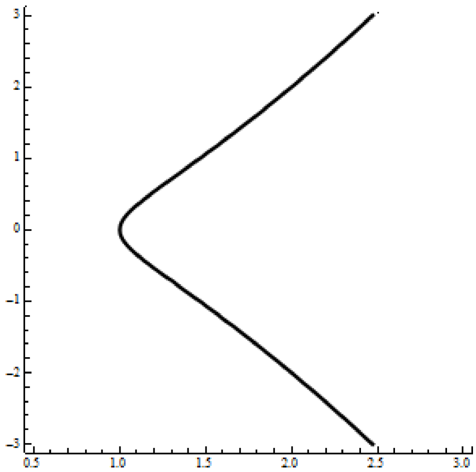

Fig. 1.6 The curve generated by the equation $y^2 = x(x^2 + 1)$.

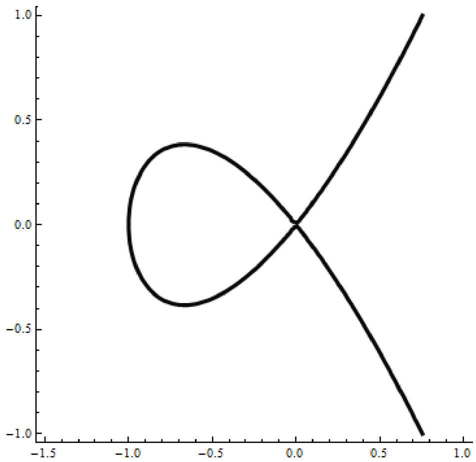

Fig. 1.7 The curve generated by the equation $y^2 = x^2(x + 1)$.

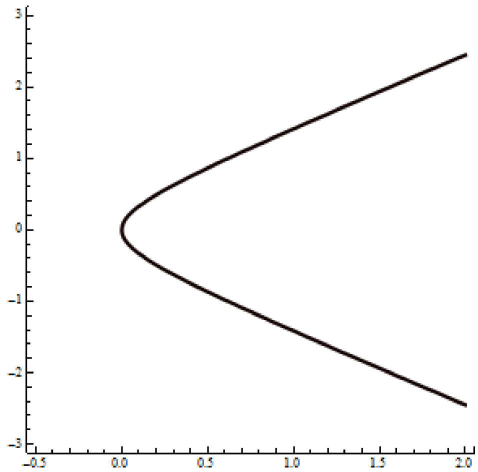

Fig. 1.8 The curve generated by the equation $y^2 = x^2(x-1)$.

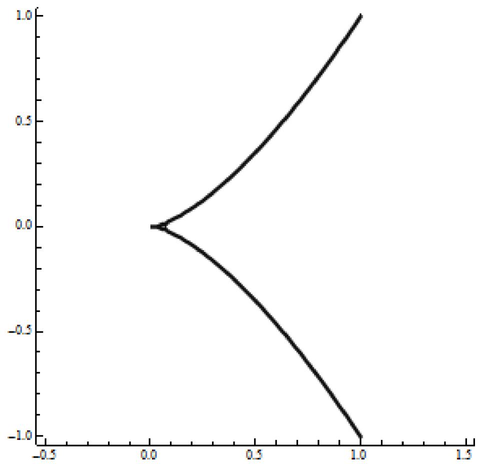

Fig. 1.9 The curve generated by the equation $y^2 = x^3$.

Among curves of second order there is no such distinction. In order to ascertain whether such distinctions exist among cubics, as mentioned before, it suffices to consider equations (1.5) and (1.6), to which every cubic may be reduced. Since we are now only concerned with varieties unaffected by projection, we may suppose the line z to be at infinity, and discuss the following form

$$y^2 = a(x-\alpha)(x-\beta)(x-\gamma), \tag{1.7}$$

where $a > 0$ and $\alpha,\ \beta,\ \gamma \in \mathbb{C}$. A classification of divergent parabolas is presented in the next theorem, taking into account the roots $\alpha,\ \beta,\ \gamma$ of the cubic.

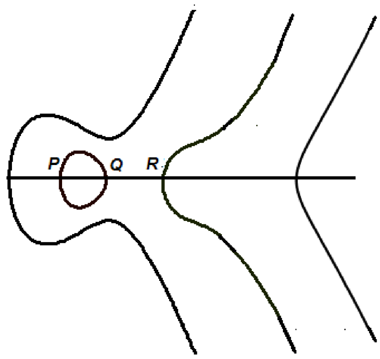

Fig. 1.10 Divergent parabolas.

Theorem 1.2.8. *Consider equation 1.7. The following possibilities may occur:*

i) If $\alpha < \beta < \gamma$ ***(all the roots are real and distinct)****, the right hand side of the equation (1.7) has three unequal factors, and the distinct factors are real. The curve consists of an oval and an infinite branch with three real inflection points lying on one line.*

ii) If ***two roots are complex conjugate*** *(only one root is real), one of the factors is real and two are complex, the oval then disappears and the infinite branch remains alone. The cubic has three real flexes. (This is a parabola of a Bell-like form.)*

iii) ***Two of the roots are equal****, the right hand side of the equation is factorized into two equal factors and one unequal factor:*

a) If $\alpha < \beta = \gamma$, the oval reduces to a conjugate point. The cubic consists of a unique piece with a node and a real inflection point . (This is the nodated or Tschirnhausen cubic.)

b) If $\alpha = \beta < \gamma$, the cubic consists of an oval and an infinite branch, being each sharpened out, so as to form a continuous self intersecting curve. The cubic has an isolated point and an infinite branch with three real flexes. (This is the punctate.)

iv) If $\alpha = \beta = \gamma$, ***three roots are equal****, the factors of the right hand side are all equal and the curve is cuspidal. The cubic consists of a unique piece with a real cusp and a real inflection point. (This is the Neilian parabola, or semi cubical.)*

The following discussion inspires a proof for the theorem.

i) Let us consider that α is less than β and γ greater than β. The curve 1.7 is symmetrical relatively to the real axis, since every value of x gives equal and symmetric values of y. The curve meets the axis of x at three points $x=\alpha$, $x=\beta$, $x=\gamma$. When x is less than α, y^2 is negative and therefore y is imaginary. Then y^2 becomes positive for values of x between α and β, negative for values between β and γ, and finally, positive for all values of x exceeding γ. The curve, therefore, consists of an oval lying between P and Q, and a branch starting at R, and extending indefinitely beyond it. To visualize the cubic represented by the equation (1.5), it will be convenient to take the origin at the middle point of the diameter of the oval and so the equation may be written

$$y^2 = (x^2-\alpha^2)(x-\beta), \tag{1.8}$$

where β is greater than α. By a standard differentiation, we find that the values of x which correspond to the maximum value of y, or to points where the tangent is parallel to the axis of x, are given by the equation

$$3x^2-2\beta x-\alpha^2=0,$$

where

$$x=\frac{1}{3}\{\beta\pm\sqrt{\beta^2+3\alpha^2}\}. \tag{1.9}$$

If we consider the negative sign in (1.9), we get the values of x corresponding to the highest point of the oval, and since this is negative, we see that the highest point of the oval is on the remote side of the infinite branch, and that the oval is therefore not symmetrical with regard to the two coordinates axes. This oval is symmetrical with regard to the axis of x, and not with regard to the axis of y, but rises more sharply on one side and slopes more gradually on the other.

The greater β is, for any given value of α, that is to say, the infinite part the more nearly the oval approaches to the elliptic form, while on the other hand, the difference is greatest when the oval closes up to the infinite part, that is, when the curve is Tschirnhausen.

If we consider the positive sign in (1.9) , the corresponding value of y is imaginary. The form of the equation shows that the point of contact with the curve of the line at infinity is on the line $x=0$, unlike the common parabola $y^2=px$, which is touched by the line at infinity at $y=0$. The infinite branches of the cubic, therefore, tend to become parallel to the axis of y and not to the axis of x, and there must be a finite point of inflection on each side of the diameter, where the curve changes from being concave to being convex towards the axis of x. Hence, the name **divergent parabola**. The form of the curve is then represented by the oval and the right hand infinite branch in the figure. By the remark 1.2.7, the curve has three inflection points.

ii) If we have in the equation (1.8), $+\alpha^2$ instead of $-\alpha^2$, then there will be no real oval, and the infinite branch will not be a real oval. The infinite branch will be either on the left hand side or on the right hand side, that is, there will, or will not, be points for which y is a maximum at which the tangent is parallel to the axis, according as $3\alpha^2$ is less or greater than β^2. There is also the intermediate case $3\alpha^2 = \beta^2$, where there is on each side of x a point of inflection, at which the tangent is parallel to this axis.

iii) Consider equation (1.7). It will be simpler to our discussion of multiple tangents to examine the case of the axis $y = 0$ being a multiple tangent. The axis will be a tangent when two of the points $\alpha,\ \beta,\ \gamma$ coincide. Assume that $\gamma = \alpha$, and so the equation is

$$(x-\alpha)^2(x-\beta) = 0.$$

The axis $y = 0$ touches the curve at the point $y = 0,\ x = \alpha$. If $d = 0,\ c = 0$ in equation (1.4) the axis $y = 0$ touches the curve at the origin.

iii-a) Let us suppose that $\beta = \gamma$ and so the equation (1.7) becomes

$$y^2 = (x-\alpha)(x-\beta)^2,$$

where β is greater than α. The point Q has now closed up to R, as Q approaches to R, the oval and the infinite branch sharpen out towards each other, and when ultimately two points are coincident the oval has joined the infinite branch, and the point Q has become a double point, with branches cutting at a certain angle.

iii-b) If $\alpha = \gamma$, then the equation (1.7) becomes

$$y^2 = (x-\alpha)^2(x-\beta),$$

where α is less than β , the oval has shrunk into the point P, so the curve consists of a point and an infinite branch.

iv) If we suppose $\alpha = \beta = \gamma$, the equation becomes

$$y^2 = (x-\alpha)^3,$$

the point P becomes a cusp, and the tangent at the cusp meets the curve in three coincident points.

The dual of the theorem of divergent parabolas, is presented in the next theorem.

Theorem 1.2.9. *(Dual of divergent parabolas) Consider an arbitrary cubic curve. The dual satisfies one of the following cases:*

i) Curve of order six, with three real cusps and an oval component.

ii) Curve of order four, with a real cusp and a real double tangent.

iii) Curve of order four, with three real cusps and a real double tangent.

iv) Cubic with a cusp and a real flex.

v) Curve of order six, with three real cusps, neither oval component nor real double tangent.

Proof. These five cases i), ii), iii), iv), v), are respectively the dual of the corresponding five cases of theorem 1.2.8. □

1.3 Kippenhahn theorem and the numerical range of three-by-three matrices

The numerical range, like the spectrum, is a subset of the complex plane, whose geometrical properties give information about the algebraic properties of the matrix. For example, in [16], a theorem asserting that, if A is a three-by-three matrix and $W(A)$ has a two dimensional shape with only one flat portion on its boundary, then A is an indecomposable matrix, which can be restored up to unitary similarity.

Although in Toeplitz paper [28] the geometry of the numerical range had been investigated, it was the work of R.Kippenhahn [17], that explicitly gave birth to this research avenue. In this section, we will concentrate on geometrical aspects of the theory.

Definition 1.3.1. A line of the complex plane is a **supporting line** of a convex set $S \subset \mathbb{C}$ if it has a boundary point of S and has no interior points of S.

The line on the plane at the distance d from the origin and whose normal has slope θ, has the following equation

$$x\cos\theta + y\sin\theta - d = 0, \tag{1.10}$$

which is called the **normal form** of the line.

Definition 1.3.2. An **extreme** point of a convex set $\mathbb{D}$, is a point $P \in \mathbb{D}$, with the property that if $P = uQ + (1-u)R$ with $Q, R \in \mathbb{D}$ and $u \in [0,1]$, then $Q = P$ and/or $R = P$.

Hence, the **extreme** points of a triangle are its vertices, and the extreme points of a disk are the points in boundary circle.

Since $W(A)$ is convex and compact, to characterize this set it suffices to determine the boundary of $W(A)$. A standard strategy is to calculate many well spaced points on $\partial W(A)$ and many supporting lines of $W(A)$ at these points. The convex hull of these boundary points is an outer of convex polygonal approximation to $W(A)$. The usefulness of the real part of the matrix A for the investigation of $W(A)$ will be shown in the consecutive lemmas.

Lemma 1.3.3. *(Projection) For all $A \in \mathbb{M}_n$*

$$Re(W(A)) = W(H(A)).$$

Proof. For $x \in \mathbb{C}^n$, we have

$$\begin{aligned} W(H(A)) &= \{(H(A)x,x) : x^*x = 1\} = \frac{1}{2}\{((A+A^*)x,x) : x^*x = 1\} \\ &= \frac{1}{2}\{(Ax,x) + (A^*x,x) : x^*x = 1\} = \frac{1}{2}\{(Ax,x) + \overline{(Ax,x)} : x^*x = 1\} \\ &= \{Re(Ax,x) : x^*x = 1\} = Re(W(A)). \end{aligned}$$

□

Lemma 1.3.4. *Let $A \in \mathbb{M}_n$, and let λ_{max} be the largest eigenvalue of $H(A)$. Then*

$$Max\ \ Re(W(A)) = Max\ \ W(H(A)) = \lambda_{max}(H(A)).$$

Proof. The first equality is easily proved by using the previous lemma and proposition 1.1.4.

For the second equality, we know that all eigenvalues of $H(A)$ are real. Since every Hermitian matrix is normal, by proposition 1.1.7, $W(H(A))$ is a compact convex set on the real line, so $W(H(A)) = [m, M]$, and we can easily conclude that M and m are the largest and the smallest eigenvalue of $H(A)$. □

Theorem 1.3.5. *Let A be an n-by-n complex matrix. If $ux + vy + w = 0$, with u, v, w nonzero complex numbers, is a supporting line of $W(A)$, then*

$$\det(uH + vK + wI) = 0.$$

Proof. Let λ_{max} be the largest eigenvalue of $H(A)$. According to the above lemma, $x = \lambda_{max}$ is a supporting line of $W(A)$. Now, if we rotate the numerical range by an angle $-\theta$, while we switch to the matrix $e^{-i\theta}A$, then for each value of θ, $x = \lambda_{max}(e^{-i\theta}A)$ is a supporting line of $W(e^{-i\theta}A)$. Now, $\lambda_{max}(e^{-i\theta}A)$ is the largest eigenvalue of the real part of

$$e^{-i\theta}A = (H\cos\theta + K\sin\theta) + i(K\cos\theta - H\sin\theta),$$

and therefore it is the largest eigenvalue of

$$H\cos\theta + K\sin\theta.$$

The eigenvalues of the latter matrix are obtained from the characteristic equation

$$\det(H\cos\theta + K\sin\theta - \lambda I) = 0.$$

If we rotate back the numerical range by the angle $+\theta$, then $W(e^{-i\theta}A)$ goes back to $W(A)$, while the line

$$x = \lambda_{max}(e^{-i\theta}A), \tag{1.11}$$

goes to the line

$$x\cos\theta + y\sin\theta - \lambda_{max} = 0. \tag{1.12}$$

This line is a supporting line of $W(A)$. If θ ranges over the interval $[0, 2\pi]$, then (1.12) yields every supporting line of $W(A)$. □

Let A be an n-by-n complex matrix. Consider the following set,

$$\Gamma = \{[u, v, w] \in \mathbb{CP}^2,\ \ P_A(u, v, w) = \det(uH + vK + wI) = 0\}.$$

The dual curve is given by

$$\Gamma^* = \{[x, y, z] \in \mathbb{CP}^2,\ \ ux + vy + wz = 0 \textit{ is a tangent of } \ \Gamma\},$$

and its real affine view is

$$C(A) = \{(x,y) \in \mathbb{R}^2, \;\; [x,y,1] \in \Gamma^*\}.$$

With the usual representation of complex numbers $x+iy$ as points (x,y) in the complex plane, one may regard $W(A)$ as a subset of the complex plane. The curve $C(A)$ has class n and is called the **associated curve**, or **boundary generating curve** of $W(A)$, or also **Kippenhahn curve** of A. In other words, we can say that $C(A)$ is the real part of the dual of

$$P_A(u,v,w) = \det(uH+vK+wI) = 0.$$

In 1951, Kippenhahn [17] showed that:

Theorem 1.3.6. *If A is an n-by-n matrix, then its numerical range $W(A)$ is the convex hull of the real points of the curve $C(A)$*

$$W(A) = Co(C(A)). \tag{1.13}$$

Before we prove Kippenhahn theorem, we need the following consequence of Hahn- Banach separation theorem.

Theorem 1.3.7. *Let C, D be two closed convex subsets of $\mathbb{R}^2$ such that $C \cap D = \emptyset$. Then there is a line l such that this line does not intersect these subsets and C, D are in two different sides of l.*

Proof. For a proof see cf. [25]. □

Theorem 1.3.8. *(Minkowski-Krein-Milman) A convex bounded planar set D is the convex closure of its extreme points.*

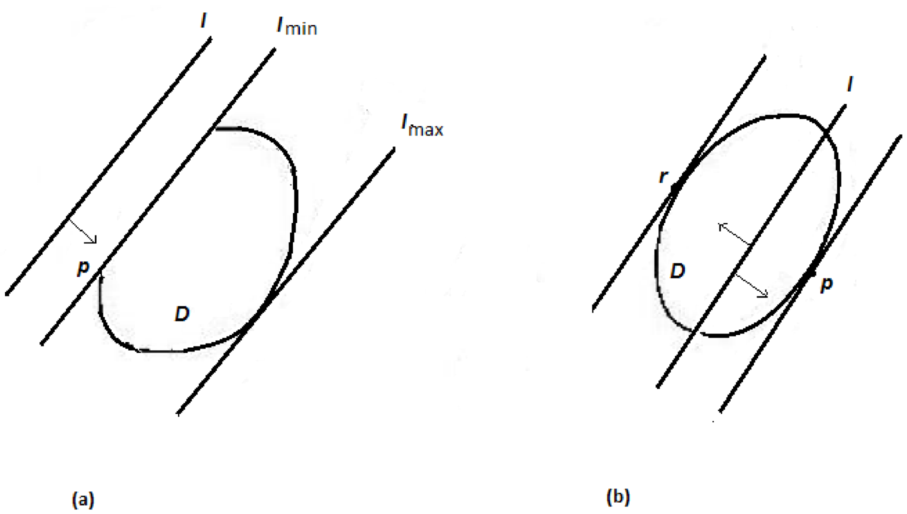

Fig. 1.11 a) P is an extreme point of the set D. b) P is an extreme point of the set D.

Proof. First of all we prove that the set of extreme points of the set D is nonempty. Let l be an arbitrary straight line. One possibility is that l has some points common with D and the set D lies entirely in one side of this line, in which case l is a supporting line of the set D. Otherwise, if l is not a supporting line, we will move it, keeping it parallel to itself, until it becomes a supporting line. (See figure 1.11.)

Consider the function on the plane, which is given by the distance from a point on the plane to l. This is a continuous function, hence by Weierstrass theorem, it attains its maximum and minimum in the set D. The parallel lines to l at the maximal and minimal distance from it are supporting lines. The intersection of a supporting line l_1 with the set D is an interval (which may degenerate into a point), since D is convex. By the definition of extreme point, we conclude that the endpoints of this interval will be extreme points of D. This interval is a convex and closed subset of D. Assume that there is a point P in the set D, but not in the set D_1. By theorem 1.3.7, the point P can be strictly separated from D_1 by a line l_1. Let us find in D the most distant point P_1 from l_1 in the same half-plane as P. By the above, the line l_1' parallel to l_1 and passing through P_1, contains an extreme point of D, which contradicts our construction. (See figure 1.12.) □

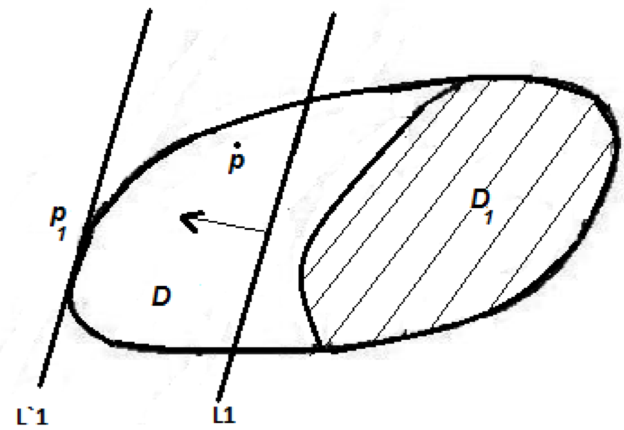

Fig. 1.12 The convex hull of a finite number of points.

Proof of theorem 1.3.6, The boundary generating curve is supported by the line with line coordinates $(\cos\theta, \sin\theta, -\lambda_{max})$ for arbitrary θ. Thus, the set of all supporting lines of $W(A)$ generates the curve and theorem 1.3.5 holds. According to theorem 1.3.8, the convex and bounded set $W(A)$ is the convex hull of its extreme points given by equation (1.11).

The following classification, based on factorability of P_A, was given by Kippenhahn [17], who showed that the numerical range of a three-by-three matrix A can assume four classes of shapes.

Theorem 1.3.9. *Let $A \in \mathbb{M}_3$, then the following types of curves can appear as boundary generating curves of $W(A)$:*

1) *Three points, and then $W(A)$ is the closed triangular region formed by these points.*

2) *A point and an ellipse, and so $W(A)$ is an elliptic disc if the point lies on or inside the ellipse, and a cone-like region otherwise.*

3) *A curve of order four with a double tangent and a cusp, and the boundary of $W(A)$ contains a line segment but no any corner.*

4) *A curve of order six, consisting of an oval and a curve with three cusps lying in its interior, and $W(A)$ is an oval region.*

Proof. If A is a three-by-three matrix, then one of the following cases holds:

1) If $P_A(u,v,w) = \det(uH + vK + wI)$ factorizes into three linear factors, since the dual of a line is a point, $C(A)$ consists of three (not necessarily distinct) points, which correspond to the eigenvalues of A. According to theorem 1.3.6, $W(A)$ is the closed triangular region formed by these points.

2) The polynomial $P_A(u,v,w)$ factorizes into $(w - \alpha u - \beta v)g(u,v,w)$, that is, a linear factor and a quadratic irreducible factor. Since the dual of an ellipse is an ellipse and the dual of a line is a point, then $C(A)$ consists of a point and an ellipse. As $W(A)$ is convex, it is an elliptic disk if the point lies on or inside the ellipse, otherwise it will be a cone-like region (the convex hull of a point and an ellipse).

3,4) If the matrix A is indecomposable, by theorem 1.2.8, there are different possibilities according to the nature of cubic polynomials. By theorem 1.2.9, we only have the following types of curves whose dual curves correspond to boundary generating curves:

a) **All the roots are real and unequal**: the curve $P_A(u,v,w) = 0$ consists of an oval and an infinite branch with three real points of inflection . Therefore, $C(A)$ is of order six and consists of two pieces, one inside the other, with the inner part having three cusps (dual of inflection points), and the outer part an oval (a strictly convex curve). So $W(A)$, being the convex hull of the outer part of $C(A)$, is an oval region. (Example 1.3.13, shows that such a curve can indeed appear as a boundary generating curve.)

b) **Two of the roots are equal**: In this case, there are two possible forms of curves:

   1) The curve $P_A(u,v,w) = 0$ has an infinite branch in length and an isolated point, so the dual curve contains a line and the dual of the infinite branch. Therefore, it is not finite and hence it can not be the boundary generating curve of a matrix.

   2) The curve $P_A(u,v,w) = 0$ has one real point of inflection and one node. Thus, $C(A)$ is of order four, has one real cusp (dual of the inflection point) and one double tangent (dual of the node).

c) **The three roots are equal**: The curve $P_A(u,v,w) = 0$, by theorem 1.2.8, is a Neilian curve. Then $C(A)$ can be found by theorem 1.2.9 and, as the numerical range should be bounded, the dual curve can not be a boundary generating curve.

   Finally, there remains the case:

d) **There is only one real root and two of the roots are complex conjugate**: In this case, by theorem 1.2.8, an infinite branch remains and therefore it can not be the boundary generating curve of a matrix A.

□

Before giving some illustrative examples of theorem 1.3.8, we recall two kinds of special points of the numerical range: a nondifferentiable boundary point of $W(A)$ called **sharp point** and the **foci**. We will show that they have an intimate relation with the spectral points of A.

For $A \in \mathbb{M}_n$, an extreme point P having supporting lines at least in two different directions is called a **sharp** point of $W(A)$.

In example 1.3.15, the point P is a sharp point.

Theorem 1.3.10. *Let $A \in \mathbb{M}_n$. If P is a sharp point of $W(A)$, then P is an eigenvalue of A.*

Proof. If there are supporting lines through $p = x+iy$ in two different directions θ_0 and θ_1, then there is an entire interval $[\theta_0, \theta_1]$ $(\theta_0 \neq \theta_1)$, so that for each value θ in that interval

$$x\cos\theta + y\sin\theta - \lambda_{max}(e^{-i\theta}(A)) = 0.$$

This may be equivalently written as

$$ux + vy + w = 0,$$

with

$$u = \cos\theta,\ v = \sin\theta,\ w = -\lambda_{max}(e^{-i\theta}(A)).$$

By theorem 1.3.5 we have

$$\det(uH + vK + wI) = 0,$$

so we can write

$$\det(uH + vK + wI) = (ux + vy + w)F(u, v, w),$$

in which $F(u, v, w)$ is an homogeneous polynomial of degree $n-1$. If we set $u = -1, v = -i$, then we find

$$\det(wI - A) = (w - (\alpha + i\beta))F(-1, -i, w),$$

therefore $x + iy$ is an eigenvalue of the matrix A. □

In projective geometry and in the complex projective plane $\mathbb{CP}^2$, there are two points at infinity, namely the complex conjugate points $I = (1, i, 0)$ and $J = (1, -i, 0)$. Let C be an algebraic curve, and let P be a point not equal to I or J. Let l_1, be the line through P and $(1, i, 0)$, and l_2 the line through P and $(1, -i, 0)$. If l_1 and l_2 are tangent to C, at points other than I and J, the point P is called a **focus** of C (P is a common point of l_1 and l_2). A curve of class n has n^2 foci, such that n are real points.

Consider now a standard ellipse

$$b^2x^2 + y^2a^2 - a^2b^2z^2 = 0,$$

where $0 < b < a$. The tangent to the ellipse at (X,Y,Z) is

$$xb^2X + ya^2Y - za^2b^2Z = 0. \tag{1.14}$$

If the tangent line passes through I, then $Y = \frac{ib^2X}{a^2}$. Substituting in (1.14), we have

$$Z = \pm X\frac{\sqrt{a^2-b^2}}{a^4}.$$

Thus, the points (X,Y,Z) of the conic where the tangents pass through I are

$$P = (a^2, ib^2, \frac{\sqrt{a^2-b^2}}{a^4}) \text{ and } Q = (a^2, ib^2, -\frac{\sqrt{a^2-b^2}}{a^4}),$$

and the tangents at these points are respectively

$$l_1 = x + iy - \frac{\sqrt{a^2-b^2}}{a^4}z = 0, \quad l_2 = x + iy + \frac{\sqrt{a^2-b^2}}{a^4}z = 0.$$

Similarly, for J we have two tangent lines l_1', l_2'. There are therefore two foci

$$F_1 = (\frac{\sqrt{a^2-b^2}}{a^4}, 0, 1), \quad F_2 = (-\frac{\sqrt{a^2-b^2}}{a^4}, 0, 1),$$

namely the points corresponding to the intersections of l_1, l_2 and l_1', l_2'.

Theorem 1.3.11. *The n real foci of the algebraic curve*

$$P_A(u,v,w) = \det(uH + vK + wI) = 0,$$

in homogenous linear coordinates, are the eigenvalues $\alpha_1, \dots, \alpha_n$ of the matrix $A = H + iK$.

Proof. If we consider $u = 1$, $v = i$, then the characteristic curve of the matrix A is

$$\det(H + iK + wI) = \det(A + wI) = 0,$$

and so w is equal to $-\alpha_i$ $(i = 1, \dots, n)$. Now consider $u = 1$, $v = -i$, so the characteristic curve of the matrix A is

$$\det(H - iK + wI) = \det(\bar{A} + wI) = 0.$$

Hence, w is equal to $-\bar{\alpha}_i$ $(i = 1, \dots, n)$. Let

$$l_p = x + iy + z = 0 \quad (p = 1, \dots, n),$$

be the line through the points $(1, i, 0)$ and $(1, i, -\alpha_i)$, and let

$$l_q = x - iy + z = 0 \quad (q = 1, \dots, n),$$

be the line through the points $(1,-i,0)$ and $(1,i,-\bar{\alpha}_i)$. The common points of the two lines l_p and l_q are $(Re(\alpha_i),Im(\alpha_i),1)$ $(i=1,...,n)$. $\square$

Next, we show that all the curves in theorem 1.3.9 can appear as a boundary generating curve of the numerical range.

Numerical examples

Example 1.3.12. This example shows that the curve in theorem 1.3.9 part $b.2$, appears as the boundary generating curve of the matrix:

$$A=\begin{bmatrix}0&0&1\\0&1&i\\1&i&0\end{bmatrix}\ \textit{with}\ H=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix},\ K=\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}.$$

Then, we easily obtain

$$P_A(u,v,w)=\det(uH+vK+wI)=-u^3-u^2w-v^2w+uw^2+w^3.$$

After solving $ux+vy+w=0$ (equation of a supporting line) with respect to u, replacing in $P_A(u,v,w)$, and substituting w by 1, we find

$$f(x,y,v)=P_A(-\frac{1+vy}{x},v,1)=1-v^2+\frac{1}{x^3}-\frac{1}{x^2}-\frac{1}{x}+\frac{3vy}{x^3}-\frac{2vy}{x^2}-\frac{vy}{x}+\frac{3v^2y^2}{x^3}-\frac{v^2y^2}{x^2}+\frac{v^3y^3}{x^3}.$$

Now, we compute the derivative of $f(x,y,z)$ with respect to v,

$$f_v(x,y,v)=-2v+\frac{3y}{x^3}-\frac{2y}{x^2}-\frac{y}{x}+\frac{6vy^2}{x^3}-\frac{2vy^2}{x^2}+\frac{v^3y^3}{x^3}.$$

The point equation of the curve may be obtained by eliminating v between $f(x,y,v)=0$ and $f_v(x,y,v)=0$. The boundary generating curve is given by

$$F(x,y)=0,$$

where

$$F(x,y)=4x^3+4x^4-27y^2-18xy^2+13x^2y^2+32y^4.$$

It should be noticed that $F(x,y)$ is a factor of the resultant of $f(x,y,v)$ and $f_v(x,y,v)$ with respect to v. The convex hull of $F(x,y)=0$ is the numerical range $W(A)$. The boundary of $W(A)$ contains a line segment, but no any corner.

Alternatively, we may solve $ux+vy+w=0$ (equation of a supporting line) with respect to w, replace in $P_A(u,v,w)$ and take $v=1$. Then we find

$$f(x,y,u)=P_A(u,1,-ux-y)=-u^3+ux+u^3x+u^3x^2-u^3x^3$$

$$+y+u^2y+2u^2xy-3u^2x^2y+uy^2-3uxy^2-y^3.$$

Now, we compute the derivative of $f(x,y,v)$ with respect to u,

$$f_u(x,y,u)=-3u^2+x+3u^2x+3u^2x^2-3u^2x^3+2uy+4uxy-6ux^2y+y^2-3xy^2.$$

As already mentioned, the point equation may be obtained by eliminating u between $f(x,y,v)=0$ and $f_u(x,y,u)=0$. Computing the resultant of $f(x,y,v)$ and $f_u(x,y,u)$ with respect to u, we find

$$F(x,y)=4x^3+4x^4-27y^2-18xy^2+13x^2y^2+32y^4.$$

The convex hull of $F(x,y)=0$ is $W(A)$, and is represented in figure 1.13. This example corresponds to the part 3,4(b.2) of theorem 1.3.9.

The following algorithm in Mathematica can help to draw the numerical range of the matrix A. This algorithm can be used for any matrix just changing the matrix A. (See figure 1.13.)

```
Id = {{1, 0, 0}, {0, 1, 0}, {0, 0, 1}}
A = {{0, 0, 1}, {0, 1, I}, {1, I, 0}}
MatrixForm[A2]
H = (A + Conjugate[Transpose[A2]])/2
K = (A - Conjugate[Transpose[A2]])/2/I
MatrixForm[H]
MatrixForm[K]
p = Det[H u + K v + Id w]
f = p /. w -> (-u x - v y) /. v -> 1
Df = D[f, u]
Factor[Resultant[f, Df, u]]
ContourPlot[ (-1 + x)^4 (4 x^3 +4 x^4 -27 y^2-
18 x y^2+ 13 x^2 y^2+ 32 y^4) == 0, {x, -1.5, 2.5}, {y, -1.5, 2.5}]
```

Example 1.3.13. Let

$$A=\begin{bmatrix}1 & i & i+1\\ \frac{1}{2} & 0 & 1\\ i+2 & 0 & 0\end{bmatrix}\ \textit{with}\ H=\begin{bmatrix}1 & \frac{1}{4}+\frac{i}{2} & \frac{3}{2}\\ \frac{1}{4}-\frac{i}{2} & 0 & \frac{1}{2}\\ \frac{3}{2} & \frac{1}{2} & 0\end{bmatrix},\ K=\begin{bmatrix}0 & \frac{1}{2}+\frac{i}{4} & 1+\frac{i}{2}\\ \frac{1}{2}-\frac{i}{4} & 0 & \frac{-i}{2}\\ 1\frac{-i}{2} & \frac{i}{2} & 0\end{bmatrix}.$$

Then

$$P_A(u,v,w)=\det(uH+vK+wI)=$$

$$\frac{u^3}{8}+2u^2v+\frac{9uv^2}{8}-\frac{45u^2w}{16}-\frac{7uvw}{2}-\frac{29v^2w}{16}+uw^2+w^3.$$

Now, we solve $ux+vy+w=0$ with respect to u, then we replace in $P_A(u,v,w)$ and we take w equal to one. We get $f(x,y,v)=P_A(-\frac{1+vy}{x},v,1)$, and we eliminate v between $f(x,y,v)$ and $f_v(x,y,v)$,

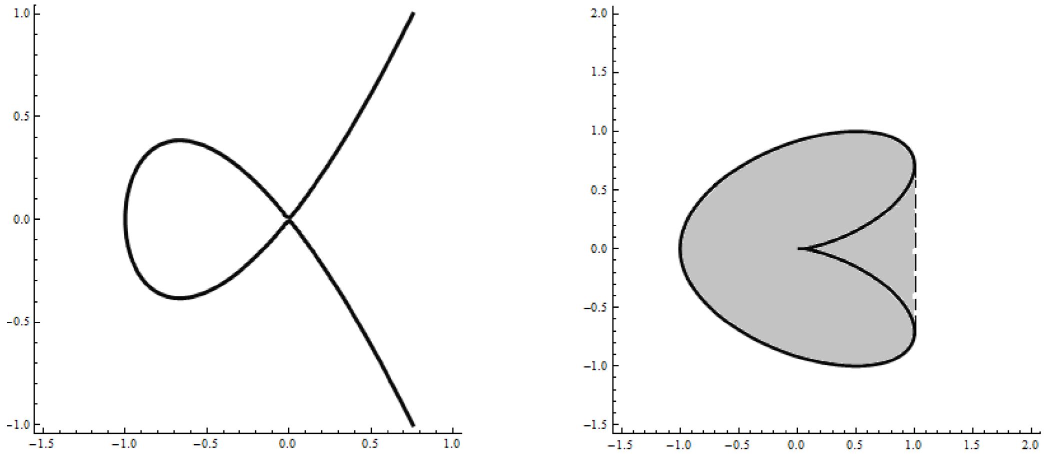

Fig. 1.13 Left: The characteristic curve. Right: $C(A)$ is represented by a thick line and the convex hull of this curve is the numerical range of the matrix in example 1.3.12.

computing the resultant of these polynomials with respect to v, $res_v = (f(x,y,v), f_v(x,y,v))$, gives us

$$\begin{aligned}F(x,y) = {} & 71280 + 238752x - 47864x^2 - 878072x^3 - 713777x^4 + 336400x^5 \\ & + 390224x^6 - 159424y + 62336xy + 2206672x^2y + 2725296x^3y - 1401280x^4y \\ & - 2260608x^5y - 1156520y^2 - 393336xy^2 - 1692482x^2y^2 + 1289504x^3y^2 + 5074480x^4y^2 \\ & - 797680y^3 - 1831760xy^3 + 1428096x^2y^3 - 5812480x^3y^3 + 2313775y^4 - 5513456xy^4 \\ & + 4928368x^2y^4 + 3679296y^5 - 3625600xy^5 + 1468816y^6.\end{aligned}$$

The convex hull of $F(x,y) = 0$ is the numerical range $W(A)$. This example corresponds to the part 3,4.(a) of theorem 1.3.9. (See figure 1.14.)

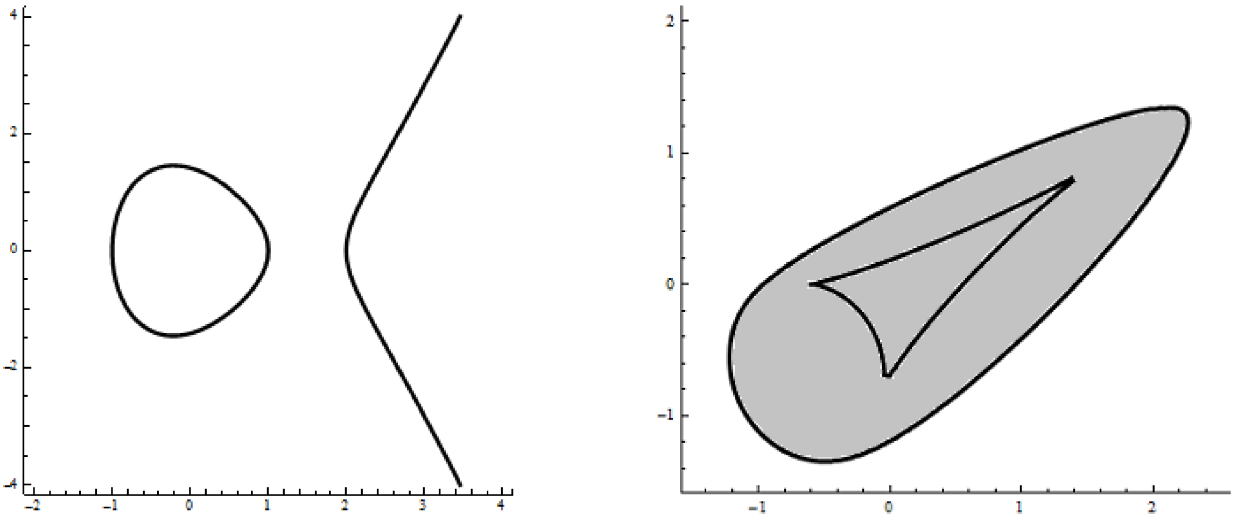

Fig. 1.14 Left: The characteristic curve P_A. Right: The curve $C(A)$ and the convex hull of this curve is the numerical range of the matrix in example 1.3.13.

Example 1.3.14. Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \quad with \;\; H = \begin{bmatrix} 1 & \frac{-1}{2} & 0 \\ \frac{-1}{2} & -1 & \frac{-1}{2} \\ 0 & \frac{-1}{2} & 1 \end{bmatrix}, \quad K = \begin{bmatrix} 0 & \frac{-i}{2} & 0 \\ \frac{i}{2} & 0 & \frac{-i}{2} \\ 0 & \frac{i}{2} & 0 \end{bmatrix}.$$

Then

$$P_A(u,v,w) = \det(uH + vK + wI) = \frac{-3u^3}{2} - \frac{uv^2}{2} - \frac{3u^2w}{2} - \frac{v^2w}{2} + uw^2 + w^3,$$

where P_A is the characteristic polynomial of the matrix A, and

$$F(x,y) = (1 - 2x + x^2 + y^2)^2(-3 + 2x^2 + 6y^2).$$

The convex hull of $F(x,y) = 0$ is the numerical range $W(A)$. This example corresponds to the part (2) of theorem 1.3.9. (See figure 1.15.)

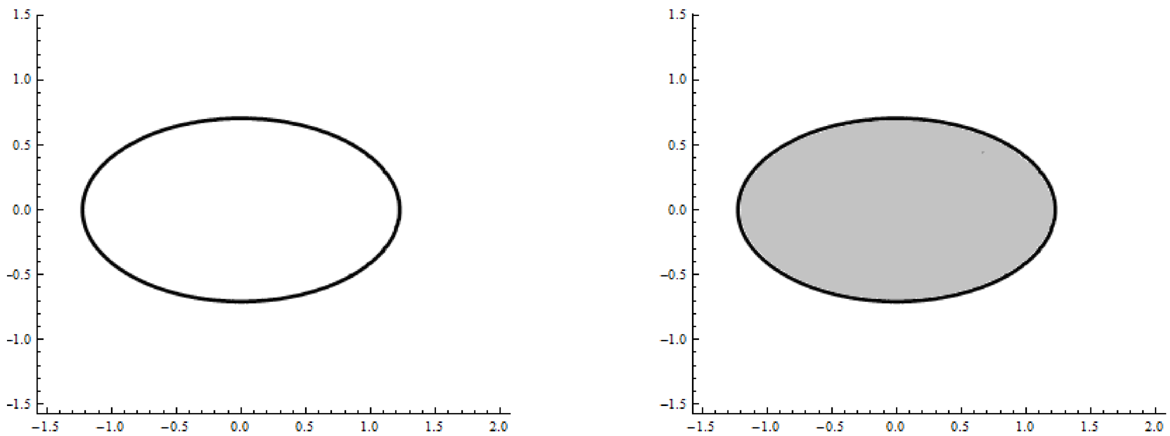

Fig. 1.15 The dual of an ellipse is an ellipse, the convex hull of this curve is the numerical range of the matrix is example 1.3.14.

Example 1.3.15. Let

$$A = \begin{bmatrix} 1+i & 0 & 0 \\ 0 & \frac{1}{2} & i \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \quad with \;\; H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{i}{2} \\ 0 & \frac{-i}{2} & \frac{-1}{2} \end{bmatrix}, \quad K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix}.$$

Then

$$P_A(u,v,w) = \det(uH + vK + wI) = (u + v + w)(\frac{-u^2}{2} - \frac{v^2}{4} + w^2),$$

where P_A is the characteristic polynomial of the matrix A and

$$F(x,y) = (x - y)(-x + x^2 + y^2).$$

The convex hull of $F(x,y) = 0$ is the numerical range $W(A)$. This example corresponds to the part (2) of theorem 1.3.9. (See figure 1.16.)

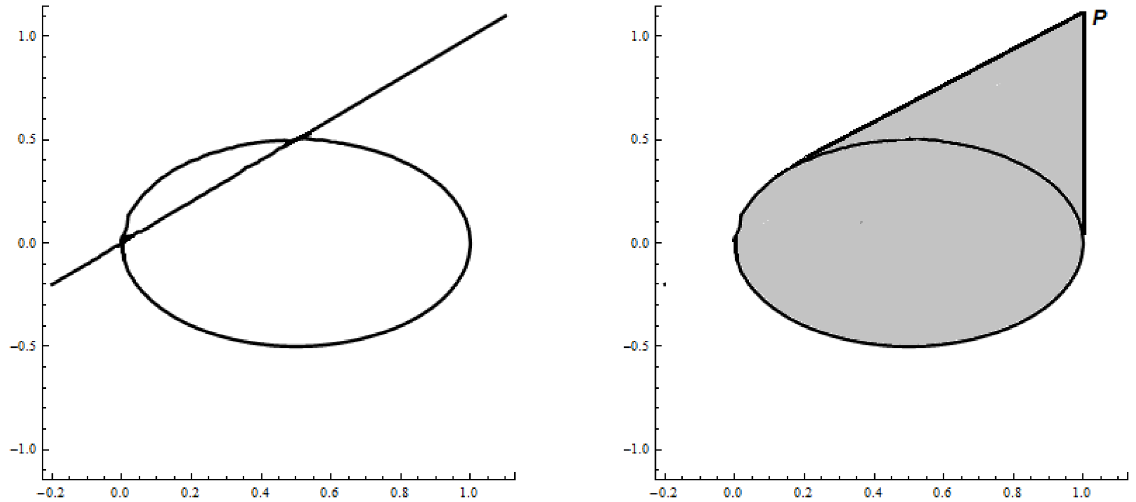

Fig. 1.16 Left: The characteristic curve P_A. Right: The dual of an ellipse is an ellipse and the dual of a line is a point. $C(A)$ consists of a point and an ellipse, so the convex hull of the point and the ellipse is the numerical range of the matrix in example 1.3.15. We notice that P is a sharp point.

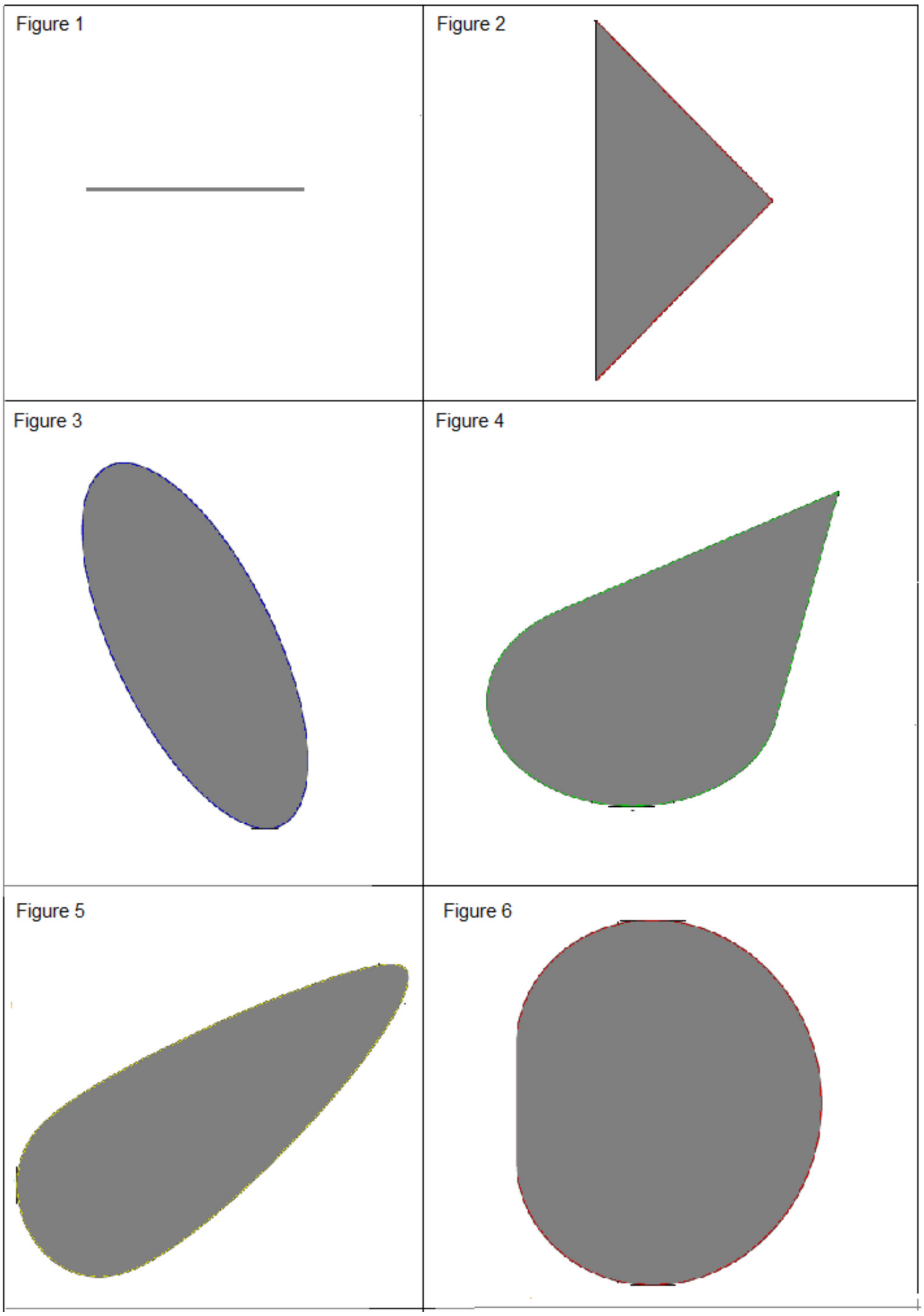

Fig. 1.17 Six possible shapes of the numerical range of three-by-three matrices.

Chapter 2

Numerical range of a self-adjoint pencil

2.1 Introduction

The spectral theory of pencils of linear operators has a long history, which has been developed by many authors, including Krein, Langer, Gohberg, Pontryagin and Shkalikov. The subject has several applications, for instance in control theory, mathematical physics and vibrating structures, which reveals that it has potentially considerable interest (cf. *e.g* [20] and [27]). In the investigation of pencils, the theory of Krein spaces plays an important role, [9]. There is a remarkable research activity on self-adjoint and quadratic pencils. (See *e.g* [23], [12] and references there in.)

Authors are mainly interested in the study of the spectral structure of a linear pencil, that is, a pencil of the form $L(\lambda) = A - \lambda B$, where A and B are complex matrices of the same size and λ is a complex number. Sometimes we may write the linear pencil as $A - \lambda B$, in alternative to $A + \lambda B$, as used by Gantmacher [7], and we also use the notation (A,B) for the linear pencil. The word pencil historically arises, due to the fact that the set of all matrices of the form $A - \lambda B$, for constant matrices A and B, and λ varying, forms a "line" of matrices in the linear space of matrices, and this resembles a "pencil" or "beam of light". The spectral theory of matrix pencils, like the spectral theory of matrices or linear operators, is built, for the most part, over $\mathbb{C}$, and in this case λ may take any complex value. However, the situation in which A and B are real matrices and λ is a real parameter is also important.

A matrix pencil (A,B) is said to be **regular** if A and B are square matrices of the same size and $det(A + \lambda B)$ is not identically zero. Otherwise, (A,B) is said to be **singular**.

A matrix pencil $A + \lambda B$ is a special case of a **polynomial matrix**, or **λ-matrix**, i.e., a matrix in which all elements are linear polynomials in λ. Thus, the tools used in the ordinary theory of λ-matrices, such as invariant polynomials or elementary divisors, can successfully be applied to the investigation of pencils.

The theory of matrix pencils (specially linear pencils) has an important role in modern linear algebra and its applications. If A is a square matrix and u is a real valued function of a real variable t, then the so called " time evolution equation "

$$\frac{du}{dt} = Au,$$

has solutions of the form

$$u(t) = e^{\lambda t} v, \tag{2.1}$$

where $u(t)$ is an analytic function of (time) t.

Suppose that the complex number λ is an eigenvalue of A and $v \neq 0$ is a corresponding eigenvector, so that

$$Av = \lambda v. \tag{2.2}$$

In some applications, this equation takes the more general form

$$B\frac{du}{dt} = Au,$$

where B is a square matrix of the same size of A. There will be solutions of the form (2.1), provided λ and v solve the **generalized eigenvalue problem**,

$$Av = \lambda Bv. \tag{2.3}$$

It is also common to write this equation as $(A - \lambda B)v = 0$, where the parameter-dependent matrix $A - \lambda B$ is a matrix pencil. In many applications B is Hermitian positive definite, but in other cases it may be indefinite Hermitian, non Hermitian, and even singular. These concepts arise in linear operator theory, but for simplicity we shall confine our attention to matrices. The problem of computing the characteristic values and vectors of (A, B) is also called **generalized characteristic value problem**, and the matrix equation

$$Ax = \lambda Bx, \tag{2.4}$$

is the generalized characteristic equation. If B is nonsingular, this problem can be reduced to the standard eigenvalue problem (2.2), considering

$$B^{-1}Av = \lambda v. \tag{2.5}$$

Normally, there would be two main reasons to work with the generalized eigenvalue problem (2.3) rather than its equivalent standard form (2.5). Firstly, one might be concerned with a case where B is singular. Secondly, the generalized form might be preferable for reasons of insight, scaling, or computation. For example, the matrix B may be mathematically invertible, but if it has a too large dimension to be inverted explicitly even with a computer, then it will be more adequate to consider (2.4).

Consider a linear pencil $L(\lambda) = A - \lambda B,\ \lambda \in \mathbb{C}$. Its spectrum is the set

$$\sigma(L) = \{\lambda \in \mathbb{C} :\ \det(A - \lambda B) = 0\}.$$

Thus, it coincides with the set of all roots of the polynomial

$$P(\lambda) = \det(A - \lambda B). \tag{2.6}$$

When B is the identity matrix, the spectrum of the pencil $A-\lambda B$ coincides with the spectrum of the matrix A, which is usually denoted by $\sigma(A)$. The first thing to observe about the generalized eigenvalue problem is that there exist n eigenvalues if and only if $rank(B)=n$. If B is **rank deficient** (i.e. if rank $B<n$), then $\sigma(L)$ may be finite, empty, or infinite. For example,

$$A=\begin{bmatrix}1&2\\0&3\end{bmatrix},\ B=\begin{bmatrix}1&0\\0&0\end{bmatrix}\Longrightarrow\quad \sigma(A,B)=\{1\},$$

$$A=\begin{bmatrix}1&2\\0&3\end{bmatrix},\ B=\begin{bmatrix}0&1\\0&0\end{bmatrix}\Longrightarrow\quad \sigma(A,B)=\emptyset,$$

$$A=\begin{bmatrix}1&2\\0&0\end{bmatrix},\ B=\begin{bmatrix}1&0\\0&0\end{bmatrix}\Longrightarrow\quad \sigma(A,B)=\mathbb{C}.$$

Note that if $0\neq\lambda\in\sigma(A,B)$, then $(\frac{1}{\lambda})\in\sigma(B,A)$.

We may also consider linear pencils of rectangular matrices, but in our study we shall not concentrate on this case.

Let $A,B\in\mathbb{M}_n$. As already mentioned, a linear matrix pencil is a family of matrices $A-\lambda B$, parametrized by a (complex) number λ. The associated generalized eigenvalue problem consists in finding the non trivial solutions of the equations

$$Ax=\lambda Bx \quad and/or \quad y^*A=\lambda y^*B,$$

where $x,\ y\in\mathbb{C}^n$, and $\lambda\in\mathbb{C}$.

Let $A,\ B\in\mathbb{M}_n$ with A a nonsingular matrix. The linear pencil $L(\lambda)=A-\lambda B$ is **equivalent** to the linear pencil $I-\lambda BA^{-1}$, because $A-\lambda B=(I-\lambda BA^{-1})A$, and the linear pencil $L(\lambda)$ is also equivalent to the linear pencil $I-\lambda A^{-1}B$, since $A-\lambda B=A(I-\lambda A^{-1}B)$. (Equivalence means that there exist nonsingular matrices S_1 and S_2 such that $L(A)=S_1(I-\lambda BA^{-1})S_2,\ \lambda\in\mathbb{C}$.) Equivalent pencils have the same spectrum.

The **field of values** or the **numerical range** of the linear pencil (A,B) is used as a rough estimate of a containment spectral region of the pencil.

The **numerical range of a pencil** (A,B), where $A,B\in\mathbb{M}_n$, is the set

$$W(A,B)=\{\lambda\in\mathbb{C}:\ x^*(A-\lambda B)x=0,\ \textit{for some}\ 0\neq x\in\mathbb{C}^n\}.$$

Moreover, if A and B have a common nullspace, that is, there is x such that $x^*Ax=x^*Bx=0$, then

$$W(A,B)=\mathbb{C}\cup\{\infty\},$$

otherwise, the numerical range of the linear pencil (A,B) can be equivalently defined by

$$W(A,B) = \{\frac{x^*Ax}{x^*Bx} : \ x \in \mathbb{C}^n, \ x \neq 0\}.$$

Throughout, we shall use indifferently the notation $W(A,B)$ or $W(A-\lambda B)$.

2.2 Basic properties of the numerical range of a linear pencil

We present some basic properties of the numerical range of linear pencils used throughout the text.

Proposition 2.2.1. *For all $A,\ B \in \mathbb{M}_n,\ \ \alpha \in \mathbb{C},\ \theta \in \mathbb{R}$, the following holds:*

a) If $A' = A + \alpha B$, then $W(A',B) = W(A,B) + \alpha$. (Translation)

b) If $A' = e^{i\theta}A$, then $W(A',B) = e^{i\theta}W(A,B)$. (Rotation)

c) If $k \neq 0$ is a complex number and $B' = \frac{B}{k}$, then $W(A,B') = kW(A,B)$, if $A' = kA$, then $W(A',B) = kW(A,B)$. (Scalar Multiplication)

Proof. a) We successively obtain,

$$\begin{aligned} W(A+\alpha B, B) &= \{\frac{x^*(A+\alpha B)x}{x^*Bx} : \ x \in \mathbb{C}^n, \ x \neq 0\} \\ &= \{\frac{x^*Ax + x^*\alpha Bx}{x^*Bx} : \ x \in \mathbb{C}^n, \ x \neq 0\} \\ &= \{\frac{x^*Ax}{x^*Bx} + \frac{x^*\alpha Bx}{x^*Bx} : \ x \in \mathbb{C}^n, \ x \neq 0\} \\ &= \{\frac{x^*Ax}{x^*Bx} + \frac{\alpha x^*Bx}{x^*Bx} : \ x \in \mathbb{C}^n, \ x \neq 0\} \\ &= \{\frac{x^*Ax}{x^*Bx} + \alpha : \ x \in \mathbb{C}^n, \ x \neq 0\} = W(A,B) + \alpha. \end{aligned}$$

b) Easy computations show that

$$\begin{aligned} W(e^{i\theta}A, B) &= \{\frac{x^*e^{i\theta}Ax}{x^*Bx} : \ x \in \mathbb{C}^n, \ x \neq 0\} \\ &= \{\frac{e^{i\theta}x^*Ax}{x^*Bx} : \ x \in \mathbb{C}^n, \ x \neq 0\} = e^{i\theta}W(A,B). \end{aligned}$$

c) We have

$$\begin{aligned} W(A, \frac{B}{k}) &= \{\frac{x^*Ax}{\frac{1}{k}x^*Bx} : \ x \in \mathbb{C}^n, \ x \neq 0\} \\ &= \{k\frac{x^*Ax}{x^*Bx} : \ x \in \mathbb{C}^n, \ x \neq 0\} = kW(A,B). \end{aligned}$$

$\square$

As it is well known, a Hermitian matrix $A \in \mathbb{M}_n$ is **positive definite** if $x^*Ax > 0$, for all nonzero $x \in \mathbb{C}^n$, and it is **positive semi-definite** if $x^*Ax \geqslant 0$ for all nonzero $x \in \mathbb{C}^n$. If A is positive definite, it is obviously positive semi-definite. A matrix is **indefinite** if it has positive and negative eigenvalues. It is well known that a complex Hermitian matrix is positive definite (respectively, semi-definite) if and only if all its eigenvalues are positive (respectively, nonnegative). For a proof see [15].

Theorem 2.2.2. *Let $A \in M_n$ be Hermitian.*

(a) A is positive semi-definite if and only if there is $B \in M_n$, such that $A = B^*B$.

(b) If $A = B^*B$ with $B \in M_n$, and if $x \in \mathbb{C}^n$, then $Ax = 0$ if and only if $Bx = 0$, so the nullspace $A =$ nullspace B and rank $A =$ rank B.

(c) If $A = B^*B$ with $B \in M_n$, then A is positive definite if and only if B has full rank.

Proof. a) If $A = B^*B$ for some $B \in M_{m,n}$, then $x^*Ax = x^*B^*Bx = \|Bx\|_2^2 \geqslant 0$ and equality occurs if and only if $Bx = 0$. The asserted factorization can be achieved, for example, with $B = A^{\frac{1}{2}}$.

b) If $Ax = 0$, then $x^*Ax = \|Bx\|_2^2 = 0$, if $Bx = 0$ then $Ax = B^*Bx = 0$, so A and B have the same null spaces and hence they have the same nullity and rank.

c) The nullity of A is zero if and only if the nullity of B is zero, and this occurs if and only if rank $B = n$.

□

Corollary 2.2.3. *Let $B \in M_n$, be a Hermitian positive definite matrix. There exists an n-by-n matrix $T \neq I_n$ such that*

$$T^*BT = B.$$

Proof. Let $B \in M_n$ be positive definite and let U be a unitary matrix. Then $U^*U = I_n$. Since

$$B^{\frac{-1}{2}}BB^{\frac{-1}{2}} = I_n,$$

we can write

$$B^{\frac{-1}{2}}BB^{\frac{-1}{2}} = U^*U.$$

Then

$$(U^*)^{-1}B^{\frac{-1}{2}}BB^{\frac{-1}{2}}U^{-1} = I_n.$$

Since U is a unitary matrix, we have

$$UB^{\frac{-1}{2}}BB^{\frac{-1}{2}}U^* = I_n.$$

Multiplying both sides of the above equation by $B^{\frac{1}{2}}$ we get

$$B^{\frac{1}{2}}UB^{\frac{-1}{2}}BB^{\frac{-1}{2}}U^*B^{\frac{1}{2}} = B.$$

So, we consider $T = B^{\frac{-1}{2}} U^* B^{\frac{1}{2}}$. $\square$

Proposition 2.2.4. *Let $A,\ B \in \mathbb{M}_n$ and let B be Hermitian positive definite. Then*

$$W(A,B) = W(B^{-\frac{1}{2}} A B^{-\frac{1}{2}}).$$

Proof. It is worth noting that as B is positive definite, then there exists an invertible matrix M such that $B = M^* M$ and so

$$\begin{aligned} W(A,B) &= \{\frac{x^*Ax}{x^*Bx} : \ x \in \mathbb{C}^n, \ \ x \neq 0\} \\ &= \{\frac{(x^*M^*)[(M^{-1})^*AM^{-1}](Mx)}{(x^*M^*)(Mx)} : \quad x \in \mathbb{C}^n, \ \ x \neq 0\} \\ &= W((M^{-1})^*AM^{-1}). \end{aligned}$$

We may consider $M = B^{\frac{1}{2}}$ and we have $W(A,B) = W(B^{-\frac{1}{2}} A B^{-\frac{1}{2}})$. $\square$

Contrarily to the classical numerical range of a matrix, which is a compact and convex set, $W(A,B)$ is not always convex, bounded or connected. For example, by definition of numerical range of a linear pencil, if $0 \in W(B)$ then $W(A,B)$ is unbounded [19].

If B is Hermitian positive definite, by proposition 2.2.4 and Toeplitz-Hausdorff theorem 1.1.3, $W(A,B)$ is a convex set. If B is nonsingular, the spectrum of $B^{-1}A$ coincides with that of the pencil (A,B). Henceforth,

$$W(B^{-1}A)$$

and

$$W(AB^{-1})$$

are inclusion regions for the eigenvalues of $Ax = \lambda Bx$. Interchanging the roles of A and B, and considering the generalized eigenvalue problem $Bx = \lambda^{-1}Ax$, the sets

$$\frac{1}{W(A^{-1}B)}$$

and

$$\frac{1}{W(BA^{-1})},$$

for nonsingular A, are also inclusion regions for the pencil eigenvalues.

The following properties of $W(A,B)$ can be easily verified [19].

Proposition 2.2.5. *Let $W(A,B)$ be an n-by-n linear pencil, where $B \neq 0$. The following holds:*

a) $W(A,B)$ *is closed in* $\mathbb{C}$.

b) *If the matrices A and B have a common nonzero isotropic vector* $0 \neq x \in \mathbb{C}^n$*, i.e.,* $x^*Ax = x^*Bx = 0$*, then* $W(A,B) = \mathbb{C}$.

c) *For any n-by-r matrix S of rank r, with* $r \leqslant n$*, we have* $W(S^*(A,B)S) \subseteq W(A,B)$*. Equality holds if* $r = n$.

d) $W(A,B) \backslash \{0\} = \{\lambda^{-1} \in \mathbb{C} : \ \lambda \in W(B,A), \ \ \lambda \neq 0\}$.

e) $W(A,B)$ *is bounded if and only if* $0 \notin W(B)$.

2.3 Compression to the two-by-two case

For A an n-square complex matrix, a two dimensional real orthogonal compression of A is a two-by-two square matrix

$$A_{\gamma\tau} = \begin{bmatrix} (A\gamma,\gamma) & (A\tau,\gamma) \\ (A\gamma,\tau) & (A\tau,\tau) \end{bmatrix}, \tag{2.7}$$

with γ, τ real orthonormal column n-tuples.

Theorem 2.3.1. *(Marcus-Pesce theorem [21]) Let A be an n-square complex matrix. Then*

$$W(A) = \bigcup_{\gamma,\tau} W(A_{\gamma\tau}),$$

where $A_{\gamma\tau}$ *is the matrix (2.7), and* γ, τ *run over all pairs of real orthonormal vectors.*

Proof. Let $w = u + iv \in \mathbb{C}^n$ be a complex unit vector in which u and v are real n-vectors. Then

$$1 = \|w\|^2 = \|u\|^2 + \|v\|^2. \tag{2.8}$$

Observe that if u and v are linearly dependent n-vectors, say $v = ru, \ r \in \mathbb{R}$, then

$$w = u + iru = (1 + ir)u,$$

and hence

$$(Aw, w) = (A|1 + ir|u, |1 + ir|u).$$

Also, $|1 + ir|u$ is a real unit vector. Thus, (Aw, w) is in the numerical range of A_{xv}, where $x = |1 + ir|u$ and v is chosen to be a real unit vector orthogonal to x.

Now, we assume that u and v are linearly independent, and use the Gram-Schmit process to obtain an orthonormal pair

$$\gamma = \frac{u}{\|u\|},$$
$$\tau = \frac{[v - (v,\gamma)\gamma]}{\|v - (v,\gamma)\gamma\|}.$$

Considering $R = \|v-(v,\gamma)\gamma\|$, we have

$$\begin{aligned} u &= \gamma\|u\|, \\ v &= (v,\gamma)\gamma + R\tau. \end{aligned}$$

Hence some computations yield

$$\begin{aligned} (Aw,w) &= (A(u+iv),u+iv) \\ &= (Au,u)+(Av,v)+i(Av,v)-i(Au,v) \\ &= \|u\|^2(A\gamma,\gamma)+(A[(v,\gamma)\gamma+R\tau],[(v,\gamma)\gamma+R\tau])+i(A[(v,\gamma)\gamma+R\tau],\|u\|\gamma) \\ &\quad - i(A\|u\|\gamma,[(v,\gamma)\gamma+R\tau])+\|u\|^2(A\gamma,\gamma)+R^2(A\tau,\tau)+R(v,\gamma)(A\tau,\gamma)+R(v,\gamma)(A\gamma,\tau) \\ &\quad + (v,\gamma)^2(A\gamma,\gamma)+iR\|u\|(A\tau,\gamma)+i\|u\|(v,\gamma)(A\gamma,\gamma)-iR\|u\|(A\gamma,\tau)-i\|u\|(v,\gamma)(A\gamma,\gamma) \\ &= [\|u\|^2+(v,\gamma)^2](A\gamma,\gamma)+R^2(A\tau,\tau)+[R(v,\gamma)-iR\|u\|](A\gamma,\tau)+[R(v,\gamma)+iR\|u\|](A\tau,\gamma). \end{aligned}$$

Let

$$A_{\gamma\tau} = \begin{bmatrix} (A\gamma,\gamma) & (A\tau,\gamma) \\ (A\gamma,\tau) & (A\tau,\tau) \end{bmatrix}.$$

Thus, $A_{\gamma\tau}$ is a two dimensional real orthogonal compression of A. Define $\xi = (v,\gamma)-i\|u\|$ so that

$$|\xi|^2 = (v,\gamma)^2+\|u\|^2, \tag{2.9}$$

and let y be the complex two-vector

$$y = [\xi,R]^T.$$

From (2.8) and (2.9) we obtain

$$\begin{aligned} \|y\|^2 = |\xi|^2+R^2 &= (v,\gamma)^2+\|u\|^2+(v-(v,\gamma)\gamma,v-(v,\gamma)\gamma) \\ &= (v,\gamma)^2+\|u\|^2+\|v\|^2-(v,\gamma)^2-(v,\gamma)^2+(v,\gamma)^2 = 1, \end{aligned}$$

and so y is a unit vector. Moreover,

$$(A_{\gamma\tau}y,y) = [\|u\|^2+(v,\gamma)^2](A\gamma,\gamma)+R^2(A\tau,\tau)+[R(v,\gamma)-iR\|u\|](A\gamma,\tau)+[R(v,\gamma)+iR\|u\|](A\tau,\gamma).$$

Thus,

$$(Aw,w) = (A_{\gamma\tau}y,y). \tag{2.10}$$

The equality 2.10 shows that any element (Aw,w) in $W(A)$ belongs to the numerical range of some two dimensional real orthogonal compression of A. Clearly, any two-by-two dimensional real orthogonal compression of A is a two dimensional square principal submatrix of a matrix orthogonaly similar to A. It is also obvious that if B is a principal submatrix of A, then $W(B) \subset W(A)$. $\square$

We denote by $S^{\perp}$ the orthogonal space to S.

Theorem 2.3.2. *(Chien and Nakazato [5]) For any $A, B \in \mathbb{M}_n$ and S a subspace of $\mathbb{C}^n$, we have*

$$W(A,B) = \bigcup_{u,v} W(A_{u,v} - \lambda B_{u,v}),$$

where $u,\ v$ vary over all pairs of orthonormal vectors in S and $S^{\perp}$, respectively, and

$$A_{u,v} = \begin{bmatrix} (Au,u) & (Av,u) \\ (Au,v) & (Av,v) \end{bmatrix}, \quad B_{u,v} = \begin{bmatrix} (Bu,u) & (Bv,u) \\ (Bu,v) & (Bv,v) \end{bmatrix}.$$

Proof. By theorem 2.3.1 we have that for every $t \in \mathbb{C}$,

$$W(A - tB) = \bigcup_{u,v} W(A_{uv} - tB_{uv}),$$

where u and v run over unit vectors in S and $S^{\perp}$, respectively. Then

$$t \in W(A,B) \Leftrightarrow 0 \in W(A - tB),$$

which is equivalent to

$$0 \in W(A_{uv} - tB_{uv}),$$

for some unit vectors $u \in S,\ v \in S^{\perp}$, which is also equivalent to

$$t \in W(A_{uv} - \lambda B_{uv}).$$

□

2.4 Linear pencils with Hermitian coefficients

An important class of generalized eigenvalue problems $Ax = \lambda Bx$ is that one in which A and B are Hermitian and some real linear combination of them is definite. Generally, eigenvalue problems $Ax = \lambda x$ with Hermitian A, have many desirable properties and are amenable to a variety of special algorithms. Here, we analize the closest analogues of this class of problems for the generalized eigenvalue problem.

Next, we recall some definitions and theorems useful to prove the main theorem 2.4.12.

If we assume that A and B are n-by-n Hermitian matrices, then we call $A - \lambda B$ a self-adjoint matrix pencil, or **self-adjoint pencil** for short.

Definition 2.4.1. Let $A \in M_n$ be Hermitian and let $\lambda_1 \geqslant ... \geqslant \lambda_n$ be its (nonincreasingly ordered) eigenvalues. Let $\Lambda = diag(\lambda_1, ..., \lambda_n)$ and let $U \in M_n$ be unitary such that $A = U\Lambda U^*$. Let $\lambda_i^+ = max\{\lambda_i, 0\}$ and $\lambda_i^- = min\{\lambda_i, 0\}$ for $i = 1, ..., n$. Let $\Lambda_+ = diag(\lambda_1^+, ..., \lambda_n^+)$ and $A_+ = U\Lambda_+ U^*$. The matrix A_+ is called the **positive semi-definite part** of A. Let $\Lambda_- = diag(\lambda_1^-, ..., \lambda_n^-)$ and $A_- = U\Lambda_- U^*$. The matrix A_- is the **negative semi-definite part** of A.

The **inertia** of a square matrix A with complex entries is defined to be the triple of integers $i(A) = (\pi(A), \nu(A), \delta(A))$, where $\pi(A)$ and $\nu(A)$ equal the number of eigenvalues of A in the open right and left half-plane, respectively, and $\delta(A)$ equals the number of eigenvalues in the imaginary axis.

Let $A \in \mathbb{M}_n$ be Hermitian. The inertia of A is the ordered triple

$$i(A) = (i_+(A), i_-(A), i_0(A)),$$

where $i_+(A)$ is the number of positive eigenvalues of A , $i_-(A)$ is the number of negative eigenvalues of A, and $i_0(A)$ is the number of vanishing eigenvalues of A.

Definition 2.4.2. Let $A, B \in \mathbb{M}_n$ be given. If there exists a nonsingular matrix S such that $B = SAS^*$, then B is said to be ***congruent** (" **star-congruent** ") to A.

Let $A \in \mathbb{M}_n$ be Hermitian. By the spectral theorem we may write $A = U\Lambda U^*$, in which $\Lambda = diag(\lambda_1, ..., \lambda_n)$ and U is unitary. It is convenient to assume that the positive eigenvalues occur first among the diagonal entries of Λ , then the negative eigenvalues appear, and finally the zero eigenvalues (if any). Thus, $\lambda_1 > 0, \lambda_2 > 0, ..., \lambda_{i_+(A)} > 0$, $\lambda_{i_+(A)+1} < 0, ..., \lambda_{i_+(A)+i_-(A)} < 0$, and $\lambda_{i_+(A)+i_-(A)+1} = ... = \lambda_n = 0$. Define the real diagonal nonsingular matrix

$$D = diag(\underbrace{\lambda_1^{\frac{1}{2}}, ..., \lambda_{i_+(A)}^{\frac{1}{2}}}_{i_+(A) entries}, \underbrace{(-\lambda_{i_+(A)+1})^{\frac{1}{2}}, ..., (-\lambda_{i_+(A)+i_-(A)})^{\frac{1}{2}}}_{i_-(A) entries}, \underbrace{1, ..., 1}_{i_0(A) entries}).$$

Then $\Lambda = DI(A)D$, in which the real matrix

$$I(A) = I_{i_+(A)} \oplus (-I_{i_-(A)}) \oplus 0_{i_0(A)},$$

is the inertia matrix of A. Finally,

$$A = U\Lambda U^* = UDI(A)DU^* = SI(A)S^*,$$

in which $S = UD$ is nonsingular. Thus, the following theorem holds.

Theorem 2.4.3. *Each Hermitian matrix is *congruent to its inertia matrix.*

Positive definite and positive semi-definite matrices have two special properties called **row and column inclusion**: Let $A \in \mathbb{M}_n$ be partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

in which $A_{11} \in \mathbb{M}_k$. We say that A has the column inclusion property if range $A_{12} \subset$ range A_{11} for every $k \in \{1, ..., n-1\}$. We say that A has the row inclusion property if A^* has the column inclusion property ([15], p.432).

Lemma 2.4.4. *Every positive semi-definite matrix has the row and column inclusion properties. In particular, if $A = (a_{ij})$ is positive semi-definite and $a_{kk} = 0$ for some $k \in \{1,...,n\}$, then $a_{ik} = a_{ki} = 0$ for each $i = 1,...,n$.*

Proof. For a proof see [15]. □

Theorem 2.4.5. *Let $A,\ B \in \mathbb{M}_n$ be Hermitian.*

(a) If A is positive definite, then there is a nonsingular $S \in \mathbb{M}_n$ such that $A = SIS^*$ and $B = S\Lambda S^*$, in which Λ is real diagonal. The inertias of B and Λ are the same, so Λ is nonnegative diagonal if B is positive semi-definite and Λ is positive diagonal if B is positive definite.

(b) If A and B are positive semi-definite and rank $A = r$, then there is a nonsingular $S \in \mathbb{M}_n$ such that $A = S(I_r \oplus 0_{n-r})S$ and $B = S\Lambda S^*$, in which Λ is nonnegative diagonal.

Proof. (a) Theorem 2.4.3 ensures that there is a nonsingular $T \in \mathbb{M}_n$ such that $T^{-1}A(T^{-1})^* = I$. The matrix $T^{-1}B(T^{-1})^*$ is Hermitian, so there is a unitary $U \in \mathbb{M}_n$ such that $U^*(T^{-1}B(T^{-1})^*)U = \Lambda$ is diagonal. Let $S = TU$. Then

$$S^{-1}A(S^{-1})^* = U^*T^{-1}A(T^{-1})^*U = U^*IU = I$$

and

$$S^{-1}B(S^{-1})^* = U^*T^{-1}B(T^{-1})^*U = \Lambda.$$

Silvester theorem ensures that B and Λ have the same inertia.

(b) Using theorem 2.4.3 again, choose a nonsingular $T \in \mathbb{M}_n$ such that $T^{-1}A(T^{-1})^* = I_r \oplus 0_{n-r}$ and consider

$$T^{-1}B(T^{-1})^* = \begin{bmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{bmatrix}$$

partitioned conformally to it. Since $T^{-1}B(T^{-1})^*$ is positive semi-definite, by lemma 2.4.4 there is X of size r-by-$(n-r)$ such that $B_{12} = B_{11}X$. Let

$$R = \begin{bmatrix} I_r & -X \\ 0 & I_{n-r} \end{bmatrix},$$

and compute

$$R^*(T^{-1}B(T^{-1})^*)R = B_{11} \oplus (B_{22} - X^*B_{11}X).$$

There are unitary matrices $U_1 \in \mathbb{M}_r$ and $U_2 \in \mathbb{M}_{n-r}$ such that $U_1^*B_{11}U_1 = \Lambda_1$ and $U_2^*(B_{22} - X^*B_{11}X)U_2 = \Lambda_2$ are real diagonal. Let $U = U_1 \oplus U_2$, $\Lambda = \Lambda_1 \oplus \Lambda_2$, and $S = TRU$. A computation shows that $S^{-1}A(S^{-1})^* = I_r \oplus 0_{n-r}$ and $B = S\Lambda S^*$.

□

Remark 2.4.6. In the proof of part (a) of the previous theorem, one possible choice for T is the matrix $A^{\frac{1}{2}}$, so $S = A^{\frac{1}{2}}U$, in which U is any unitary matrix such that $A^{\frac{1}{2}}BA^{\frac{1}{2}} = U\Lambda U^*$. If A and B are real matrices, this observation can be used to show that S may be chosen to be real.

Definition 2.4.7. Let B be an $n\times n$ Hermitian matrix. The B-**inner product** of two vectors x and y in $\mathbb{C}^n$ is defined as $(x,y)_B = y^*Bx$.

If B is indefinite Hermitian, then $(x,y)_B$ is a **pseudo-inner product** or **indefinite inner product**. This inner product violates the condition $(x,x)_B \geqslant 0$ for all $x\in\mathbb{C}^n$, valid in the definition of a standard inner product. The inner product $(x,y)_B$ can be used for normalizing purposes. Unlike in the positive definite case, there is a set of non vanishing vectors having **pseudolength** zero (as measured by $(x,x)_B$).

The **norm with respect to the** B **inner product**, is defined by $\|.\|_B^2 = (x^*Bx)$.

In fact, it is possible for a vector $x\neq 0$ to satisfy $x^*Ax = x^*Bx = 0$. This implies that the Rayleigh quotient $\frac{x^*Ax}{x^*Bx}$ is undefined.

As already said, a nonzero vector $x\neq 0$ satisfying $Ax = \lambda Bx$ is an eigenvector associated with the eigenvalue λ. The eigenvectors may be chosen to be B**-orthogonal**:

$$(Bx_j, x_i) = x_i^*Bx_j = 0 \ \ if \ \ i\neq j.$$

This orthogonality is defined with respect to the inner product induced by the Hermitian matrix B.

Theorem 2.4.8. *Let (A,B) be an n-by-n self-adjoint pencil such that the eigenvalues of the pencil are not all real, then $W(A,B)$ is the whole complex plane.*

Proof. Suppose that for $u\neq 0$, $Au = \lambda Bu$ and $\lambda\neq\bar{\lambda}$. Then, $u^*A = \bar{\lambda}u^*B$. Thus

$$u^*Au = \lambda u^*Bu = \bar{\lambda}u^*Bu.$$

Henceforth $u^*Au = u^*Bu = 0$, and by proposition 2.2.5.b $W(A,B) = \mathbb{C}$. ☐

Lemma 2.4.9. *Suppose that $A,\ B\in\mathbb{M}_n$ are Hermitian, $B = diag(\beta_1,...,\beta_r,0,...,0)$, and $W(A,B)\neq\mathbb{C}$. Let $\alpha_1,...,\alpha_r\in\mathbb{R}$ be the eigenvalues of the pencil. Then there exists a nonsingular T such that that*

$$T^*AT = diag(\alpha_1\beta_1,...,\alpha_r\beta_r,\alpha_{r+1},...,\alpha_n), \ \ T^*BT = B,$$

where $\alpha_{r+1},...,\alpha_n$ are real numbers.

Proof. Let $Av_j = \alpha_jBv_j, \ \ j = 1,...,r,$ that is, $\alpha_1,...,\alpha_r\in\mathbb{R}$, are the eigenvalues of the pencil and $v_1,...,v_r\in\mathbb{C}^n$ are the corresponding eigenvectors, which may be normalized such that $v_i^*Bv_j = \delta_{ij}\beta_j\ i,j = 1,...,r$. Let $v_{r+1},...,v_n\in\mathbb{C}^n$ be such that $v_i^*Bv_j = 0$. for $j = r+1,...,n,\ i = 1,...,r$. This vector system may be chosen so that it constitutes a basis of $\mathbb{C}^n$. Let $S\in\mathbb{M}_n$ be the nonsingular matrix whose columns are $v_1,...,v_n$. It may be shown that

$$S^*AS = diag(\alpha_1\beta_1,...,\alpha_r\beta_r)\oplus A_{22},$$

where A_{22} is Hermitian. Let $\alpha_{r+1},...,\alpha_n \in \mathbb{R}$ be the eigenvalues of the Hermitian matrix A_{22} and $V \in \mathbb{U}_{n-r}$ be such that

$$V^* A_{22} V = diag(\alpha_{r+1},...,\alpha_n).$$

Consider $T = S(I_r \oplus V)$. Then

$$T^* A T = diag(\alpha_1\beta_1,...,\alpha_r\beta_r,\alpha_{r+1},...,\alpha_n).$$

We show that $S^*BS = B$. It is clear that

$$S^*BS = \left(\begin{array}{ccc|c} \beta_1 & & O & \\ & \ddots & & O \\ O & & \beta_r & \\ \hline & O & & B'_{22} \end{array}\right),$$

where $B'_{22} \geqslant 0$. Since the inertias of S^*BS and B are the same, it follows that $B'_{22} = 0$. Thus, $S^*BS = B$. □

Throughout, according to proposition 2.2.5.b we assume that the matrices A and B have no common nonzero isotropic vectors x, i.e., $x^*Ax = 0$, $x^*Bx = 0$.

The next lemma will be used in the proof of theorem 2.4.12.

Lemma 2.4.10. *The product of two Hermitian matrices A and B is Hermitian if and only if $AB = BA$.*

Proof. We have $(AB)^* = B^*A^* = BA = AB$. The converse easily follows. □

For B Hermitian indefinite, consider $\mathbb{C}^n$ endowed with the B-inner product $(Bx,y) = y^*Bx$, and the corresponding B-norm $\|x\|_B^2 = (Bx,x)$.

Let (A,B) be an Hermitian pencil such that $W(A,B) \neq \mathbb{C}$. Hence, the norm of eigenvectors can not vanish.

In what follows, the eigenvalues corresponding to eigenvectors with positive (negative) B-norm constitute

$$\sigma_+(A,B) = \{\lambda \in \mathbb{C} : Au - \lambda Bu = 0, \textit{ for some } 0 \neq u \in \mathbb{C}^n,\ u^*Bu > 0\},$$

$$\sigma_-(A,B) = \{\lambda \in \mathbb{C} : Au - \lambda Bu = 0, \textit{ for some } 0 \neq u \in \mathbb{C}^n,\ u^*Bu < 0\}.$$

The characterization of $W(A,B)$ when A and B are Hermitian was obtained in theorem 4.1 of [19]. The last item of the theorem states that when both A and B are indefinite then $W(A,B) = \mathbb{R}$. We would like to point out that this result is not true as the following counterexample shows. Moreover, the result is also incorrectly reproduced in theorem 9 of [24].

Example 2.4.11. Let us consider the pencil (A,B) such that A,B are indefinite. Assume that $A = diag(-3,-5,1)$ and $B = diag(-1,1,1)$. The eigenvalues of (A,B) are

$$\lambda_1 = 3,\ \ \lambda_2 = -5,\ \ \lambda_3 = 1,$$

the B-norm of an eigenvector associated with λ_1 being negative, while the B- norms of eigenvectors associated with $\lambda_2,\ \lambda_3$, are positive. So, we have $\sigma_+(H,B) = \{-5,1\}$ and $\sigma_-(H,B) = \{3\}$. Thus $W(A,B) =]\infty,1] \cup [3,+\infty[$.

Next, we present the proper result and proof.

Theorem 2.4.12. *Let (A,B) be an n-by-n self-adjoint pencil with $W(A,B) \neq \mathbb{C}$.*

a) If B is positive (or negative) definite, then $W(A,B)$ is a closed interval in $\mathbb{R}$.

b) If B is positive (or negative) semi-definite, then $W(A,B)$ is an unbounded interval of the form $[a,+\infty[$ or $]-\infty,a]$.

c) If B is indefinite and A is positive (negative) definite, then $W(A,B)$ is the union of two disjoint unbounded intervals and $0 \notin W(A,B)$

d) If B is indefinite and A is positive (negative) semi-definite, then one of the following holds,

1) $W(A,B) =]-\infty,a] \cup [0,+\infty[$ with $a < 0$,

2) $W(A,B) =]-\infty,0] \cup [b,+\infty[$ with $0 < b$.

e) If both B and A are indefinite, then two possibilities may occur:

1) $W(A,B) =]-\infty,a] \cup [b,+\infty[$ with $0 \in W(A,B)$ and $a < b$.

2) $W(A,B) = \mathbb{R}$.

In all cases, the endpoints of the intervals are eigenvalues of the pencil.

Proof.

a) By proposition 2.2.4, $W(A,B) = W(B^{\frac{-1}{2}}AB^{\frac{-1}{2}})$, and by lemma 2.4.10 the matrix $B^{\frac{-1}{2}}AB^{\frac{-1}{2}}$ is Hermitian, so the result follows, because the classical field of values of a Hermitian matrix is a line segment whose endpoints are eigenvalues of the matrix.

b) Assume B is positive semi-definite with rank r. According to lemma 2.4.9 we can take $B = diag(\beta_1,...,\beta_r,0,...,0)$. Let $\sigma(A,B) = \{\alpha_1,...,\alpha_r\}$, $\alpha_1.....\alpha_r \in \mathbb{R}$. Further, there exists T such that

$$T^*BT = B \ \ and \ \ T^*AT = diag(\alpha_1\beta_1,...,\alpha_r\beta_r,\alpha_{r+1},...,\alpha_n),$$

and $\alpha_{r+1}.....\alpha_n \in \mathbb{R}$. Since $W(A,B) \neq \mathbb{C}$, then $\alpha_{r+1}.....\alpha_n \neq 0$ and the eigenvalues have all the same sign. In fact, suppose $\alpha_{r+1} > 0$ and $\alpha_{r+2} < 0$, with corresponding eigenvectors v_{r+1} and v_{r+2} respectively. Thus, it is possible to choose γ,δ conveniently so that $v = \gamma v_{r+1} + \delta v_{r+2}$ and

$$v^*T^*ATv = |\gamma|^2\alpha_{r+1} + |\delta|^2\alpha_{r+2} = 0.$$

This is impossible because by hypothesis there are no common isotropic vectors. Let$v = \sum_{i=1}^{n} \gamma_i e_i$, where $\gamma_i \in \mathbb{C}$ and e_i is the vector with one in place i and zeros every where else. Assume that $0 < \alpha_{r+1} \leqslant \ldots \leqslant \alpha_n$, thus,

$$v^*T^*ATv = \sum_{i=1}^{r} |\gamma_i|^2 \alpha_i \beta_i + \sum_{i=r+1}^{n} |\gamma_i|^2 \alpha_i \geqslant \sum_{i=1}^{n} |\gamma_i|^2 \alpha_i \beta_i,$$

and so we obtain

$$v^*Bv = \sum_{i=1}^{r} |\gamma_i|^2 \beta_i.$$

Hence assuming that $\alpha_1 \leqslant \alpha_2 \leqslant ... \leqslant \alpha_r$

$$\frac{v^*T^*ATv}{v^*Bv} \geqslant \frac{\sum_{i=1}^{n} |\gamma_i|^2 \alpha_i \beta_i}{\sum_{i=1}^{r} |\gamma_i|^2 \beta_i} \geqslant \alpha_1.$$

On the other hand, if $\alpha_{r+1} \leqslant \ldots \leqslant \alpha_n < 0$, we find

$$\frac{v^*T^*ATv}{v^*Bv} \leqslant \alpha_r.$$

c) Let B be indefinite with inertia $(r, n-r)$. Let $\frac{1}{\alpha} > 0 > \frac{1}{\beta}$ be the largest and the smallest eigenvalue of the pencil (B, A), so that $W(B,A) = [\frac{1}{\beta}, \frac{1}{\alpha}]$. Since $W(A,B) = \frac{1}{W(B,A)}$ and by proposition 2.2.5.d,

$$W(A,B) =]-\infty, \beta] \cup [\alpha, +\infty[,$$

and $0 \notin W(A,B)$.

d) Similar to c).

e) Let B and A be indefinite and B have inertia $(r, n-r)$.

$$B = diag(\beta_1, ..., \beta_n), \;\; \beta_1 > 0, ..., \beta_r > 0, \;\; 0 > \beta_{r+1}, ..., 0 > \beta_n.$$

According to hypothesis $W(A,B) \neq \mathbb{C}$, the eigenvalues of the pencil (A,B) are all and the associated eigenvectors are non-isotropic. Let $\sigma_+(A,B) = \{\alpha_1, ..., \alpha_r\}$, $\alpha_1 \geqslant \alpha_2 \geqslant ... \geqslant \alpha_r$ and $\sigma_-(A,B) = \{\alpha_{r+1}, ..., \alpha_n\}$, $\alpha_{r+1} \geqslant \alpha_{r+2} \geqslant ... \geqslant \alpha_n$. By similar arguments to those in lemma 2.4.9, it may be shown that there exists T such that

$$T^*AT = diag(\alpha_1\beta_1, ..., \alpha_n\beta_n), \quad T^*BT = B = diag(\beta_1, ..., \beta_n).$$

In fact, taking $v = \sum_{i=1}^{n} \gamma_i e_i$ with $\gamma_i \in \mathbb{C}$, we obtain

$$v^*T^*ATv = \sum_{i=1}^{n} |\gamma_i|^2 \alpha_i \beta_i$$

and

$$v^*T^*BTv = \sum_{i=1}^{n} |\gamma_i|^2 \beta_i.$$

and so

$$z = \frac{v^*T^*ATv}{v^*T^*BTv} = \frac{\sum_{i=1}^{n} |\gamma_i|^2 \alpha_i \beta_i}{\sum_{i=1}^{n} |\gamma_i|^2 \beta_i}.$$

Further, we may write $z = \dfrac{ap - bq}{p - q}$, where

$$a = \frac{\sum_{i=1}^{r} |\gamma_i|^2 \alpha_i \beta_i}{\sum_{i=1}^{r} |\gamma_i|^2 \beta_i}, \quad b = \frac{\sum_{i=r+1}^{n} |\gamma_i|^2 \alpha_i \beta_i}{\sum_{i=r+1}^{n} |\gamma_i|^2 \beta_i},$$

$$p = \sum_{i=1}^{r} |\gamma_i|^2 \beta_i > 0, \quad q = -\sum_{i=1}^{r+1} |\gamma_i|^2 \beta_i > 0,$$

Thus, $a \in [\alpha_r, \alpha_1]$ and $b \in [\alpha_{r+1}, \alpha_n]$. Moreover, $z \in]-\infty, b] \cup [a, +\infty[$ if $a > b$ and $z \in]-\infty, a] \cup [b, +\infty[$ if $a < b$, then these possibilities occur

1) If $\alpha_r > \alpha_{r+1}$, then $]-\infty, b[\cup[a, +\infty[\subseteq]-\infty, \alpha_{r+1}[\cup[\alpha_r, +\infty[$, so that α_r is the lowest value which z may assume if $p > q$, while α_{r+1} is the highest value which z may assume if $p < q$. Thus,

   $$W(A,B) =]-\infty, \alpha_{r+1}] \cup [\alpha_r, +\infty[.$$

   It may be seen that $\alpha_r \alpha_{r+1} > 0$ and so $0 \in W(A,B)$.

2) If $\alpha_n > \alpha_1$, then $]-\infty, a[\cup[b, +\infty[\subseteq]-\infty, \alpha_1[\cup[\alpha_n, +\infty[$ and $W(A,B) =]-\infty, \alpha_1] \cup [\alpha_n, +\infty[$, with $\alpha_n \alpha_1 > 0$ and so $0 \in W(A,B)$.

3) If $\alpha_r < \alpha_{r+1}$ and $\alpha_n < \alpha_1$, then $W(A,B) =]-\infty, +\infty[$.

Next, let A be indefinite and let B be singular, have inertia $(r, s-r, n-s)$. We may consider

$$B = diag(\beta_1, \beta_2, 0, ..., 0), \ \beta_1 \geqslant ... \geqslant \beta_r > 0 > \beta_{r+1} \geqslant ... \geqslant \beta_s.$$

According to hypothesis $W(A,B) \neq \mathbb{C}$, the eigenvalues of the pencil (A,B) are all and the associated eigenvectors are non-isotropic. Let $\sigma_+(A,B) = \{\alpha_1, ..., \alpha_r\}$, $\alpha_1 \geqslant \alpha_2 \geqslant ... \geqslant \alpha_r$ and $\sigma_-(A,B) = \{\alpha_{r+1}, ..., \alpha_s\}$, $\alpha_{r+1} \geqslant \alpha_{r+2} \geqslant ... \geqslant \alpha_s$. Also notice that $\alpha_{s+1}, ..., \alpha_n$ have the same sign otherwise there will exist common isotropic vectors for A and B. By similar arguments to those in lemma 2.4.9, it may be shown that there exists T such that

$$T^*AT = diag(\alpha_1\beta_1, ..., \alpha_s\beta_s, \alpha_{s+1}, ..., \alpha_n), \quad T^*BT = B,$$

where $\beta_1 > 0, \ldots, \beta_r > 0,\ \ 0 > \beta_{r+1}, \ldots, 0 > \beta_s$ and $\alpha_{s+1}, \ldots, \alpha_n \in \mathbb{R}$. In fact, taking $v = \sum_{i=1}^n \gamma_i e_i,\ \gamma_i \in \mathbb{C}$, we obtain

$$z = \frac{v^* T^* A T v}{v^* T^* B T v} = \frac{\sum_{i=1}^s |\gamma_i|^2 \alpha_i \beta_i + \sum_{i=s+1}^n |\gamma_i|^2 \alpha_i}{\sum_{i=1}^s |\gamma_i|^2 \beta_i}.$$

Further, we may write $z = \dfrac{ap - bq}{p - q} + t$, where

$$a = \frac{\sum_{i=1}^r |\gamma_i|^2 \alpha_i \beta_i}{\sum_{i=1}^r |\gamma_i|^2 \beta_i},\ \ b = \frac{\sum_{i=r+1}^s |\gamma_i|^2 \alpha_i \beta_i}{\sum_{i=r+1}^s |\gamma_i|^2 \beta_i},$$

$$p = \sum_{i=1}^r |\gamma_i|^2 \beta_i > 0,\ \ q = -\sum_{i=r+1}^s |\gamma_i|^2 \beta_i > 0,\ \ t = \sum_{i=s+1}^n \frac{|\gamma_i|^2}{p - q} \alpha_i.$$

Thus, $a \in [\alpha_r, \alpha_1]$ and $b \in [\alpha_{r+1}, \alpha_n]$. Moreover, $z \in]-\infty, a] \cup [b, +\infty[+ t$ if $a < b$, while $z \in]-\infty, b] \cup [a, +\infty[+ t$ if $a > b$.

1) If $\alpha_r > \alpha_{r+1}$ and $\alpha_i > 0,\ i = s+1, \ldots, n$. Then, $a > b$ and $t > 0$ if $p > q$, while $t < 0$ if $p < q$. Since $\dfrac{(ap - bq)}{(p - q)} \in]-\infty, b] \cup [a, +\infty[, a \geqslant \alpha_r$ and $b \leqslant \alpha_{r+1}$, it follows that

$$z \in]-\infty, b] \cup [a, +\infty[\subseteq]-\infty, \alpha_{r+1}] \cup [\alpha_r, +\infty[,$$

so that α_r is the lowest value which z may assume if $p > q$. Indeed, $z = \alpha_r$ if and only if all the γ_i vanish expect γ_r. On the other hand, α_{r+1} is the highest value which z may assume if $p < q$ and $z = \alpha_{r+1}$ if and only if all the γ_i vanish expect γ_{r+1}. Thus,

$$W(A, B) =]-\infty, \alpha_{r+1}] \cup [\alpha_r, +\infty[.$$

As $\alpha_r \alpha_{r+1} > 0,\ \alpha_r \neq \alpha_{r+1}$ we have $0 \in W(A, B)$.

2) If $\alpha_n > \alpha_1$ and $\alpha_i < 0,\ i = s+1, \ldots, n$, we get

$$z \in]-\infty, a] \cup [b, +\infty[\subseteq]-\infty, \alpha_1] \cup [\alpha_n, +\infty[,$$

then $W(A, B) =]-\infty, \alpha_1] \cup [\alpha_n, +\infty[$, with $0 \in W(A, B)$.

3) If neither item 1) nor item 2), then $W(A, B) = \mathbb{R}$, as follows,

- If $\alpha_n < \alpha_1$ and $\alpha_r < \alpha_{r+1}$, then $W(A, B) = \mathbb{R}$.
- If $\alpha_r > \alpha_{r+1}$ and $\alpha_i < 0,\ i = s+1, \ldots, n$, we may also conclude that $W(A, B) = \mathbb{R}$.

Remark 2.4.13. We observe that $W(A, B)$, for A and B Hermitian, is the convex hull of the eigenvalues of (A, B), if B is definite, and $W(A, B)$ is the pseudo-convex hull of the eigenvalues of (A, B), if B is nonsingular indefinite matrix.

□

Let us consider three three-by-three cases as an example. For a first example, let $A, B \in \mathbb{M}_3$ are Hermitian and A is Hermitian positive definite. Assume that $B = diag(b_1, b_2, -b_3),\ b_1,\ b_2,\ b_3 > 0$, and

$$\sigma_+(A,B) = \{\alpha, \beta\},\ \ \sigma_-(A,B) = \{\gamma\},\ \ \alpha \geqslant \beta > 0,\ \ \gamma < 0,$$

Without loss of generality, we may take $A = diag(\alpha b_1, \beta b_2, -\gamma b_3)$. Thus,

$$W(A,B) = \{\frac{\alpha b_1|x|^2 + \beta b_2|y|^2 - b_3\gamma|z|^2}{b_1|x|^2 + b_2|y|^2 - b_3|z|^2} :\ \ x,y,z \in \mathbb{C},\ \ b_1|x|^2 + b_2|y|^2 - b_3|z|^2 \neq 0\}.$$

So, $W(A,B) =]-\infty, \gamma] \cup [\beta, +\infty[$. Obviously $0 \notin W(A,B)$.

Now, let $A, B \in \mathbb{M}_3$ are Hermitian and $B = diag(b_1, b_2, b_3),\ \ b_1, b_2, b_3 > 0,\ \ \sigma(A,B) = \{\alpha, \beta, \gamma\}, \alpha \geqslant \beta \geqslant \gamma \in \mathbb{R}$. Without loss of generality, we may take $A = diag(\alpha b_1, \beta b_2, \gamma b_3) \in \mathbb{R}$. Then

$$W(A,B) = \{\frac{\alpha b_1|x|^2 + \beta b_2|y|^2 + b_3\gamma|z|^2}{b_1|x|^2 + b_2|y|^2 + b_3|z|^2} :\ \ \ x,y,z \in \mathbb{C},\ \ b_1|x|^2 + b_2|y|^2 + b|z|^2 \neq 0\}.$$

So, $W(A,B) = [\alpha, \gamma]$.

Finally, as a consequence of the previous result we can easily conclude the following. Let $A, B \in \mathbb{M}_3$, be Hermitian, $B = diag(b_1, b_2, 0), b_1, b_2 > 0, \sigma(A,B) = \{\alpha, \beta\}, \alpha, \beta \in \mathbb{R},\ \alpha \geqslant \beta$. Without loss of generality, we may take $A = diag(\alpha b_1, \beta b_2, a_3), a_3 \in \mathbb{R}$. Then

$$W(A,B) = \{\frac{\alpha b_1|x|^2 + \beta b_2|y|^2 + a_3\gamma|z|^2}{b_1|x|^2 + b_2|y|^2} :\ \ \ x,y,z \in \mathbb{C},\ \ b_1|x|^2 + b_2|y|^2 + b|z|^2 \neq 0\}.$$

So, if $a_3 > 0\ (a_3 < 0),\ \ W(A,B) = [\beta, +\infty[\ (W(A,B) =]-\infty, \alpha])$.

Moreover, let now $A, B \in \mathbb{M}_3$, be Hermitian, $B = diag(b_1, 0, 0),\ \ b_1 > 0,\ \ \sigma(A,B) = \{\alpha\},\ \ \alpha \in \mathbb{R}$. Without loss of generality, we may take $A = diag(\alpha b_1, a_2, a_3),\ \ a_2, a_3 \in \mathbb{R}$. Then

$$W(A,B) = \{\alpha + \frac{a_2}{b_1}|y|^2 + \frac{a_3}{b_1}|z|^2 :\ \ y,z \in \mathbb{C}\}.$$

So, if $a_2, a_3 > 0, W(A,B) = [\alpha, +\infty[$. If $a_2 < 0, a_3 > 0$,then $W(A,B) = \mathbb{R}$.

Chapter 3

Numerical range of two-by-two linear pencils with one hermitian coefficient

In this chapter we characterize $W(A,B)$ for A and B of size two, when one of these matrices is Hermitian. We recall some well known results useful for our approach. The proofs are here included for the sake of completeness.

3.1 Elliptical Range Theorem

Before presenting the main theorem of this section, we prove the following auxiliary result.

Lemma 3.1.1. *Let $A,\ B \in M_2$ with B positive definite in diagonal form $B = diag(f,g)$. Let $a \neq b$ be the eigenvalues of the pencil (A,B). Then there exists a two-by-two matrix T such that*

$$T^*BT = B, \quad T^*AT = A' = \begin{bmatrix} af & c\sqrt{fg} \\ 0 & bg \end{bmatrix}.$$

Proof. Let $T = [t_1,t_2]$ be the matrix whose columns are t_1 and t_2, where t_1 is an eigenvector of the pencil (A,B) associated with the eigenvalue a and such that

$$t_1^*Bt_1 = f,$$

t_2 is B-orthogonal to t_1,

$$t_2^*Bt_1 = 0, \quad t_1^*Bt_2 = 0,$$

and

$$t_2^*Bt_2 = g.$$

Then

$$T^*AT = \begin{bmatrix} t_1^* \\ t_2^* \end{bmatrix} A \begin{bmatrix} t_1 & t_2 \end{bmatrix} = \begin{bmatrix} t_1^*At_1 & t_1^*At_2 \\ t_2^*At_1 & t_2^*At_2 \end{bmatrix} = \begin{bmatrix} af & * \\ 0 & b' \end{bmatrix}.$$

Now

$$\det(T^*(A,B)T) = \det(T^*T)\det(A,B),$$

where $\det(A,B) = \det(A-\lambda B)$, moreover $\lambda = a$ and $\lambda = b$ are the eigenvalues of (A,B), so that $b' = \lambda b$. □

Theorem 3.1.2. *(Elliptical Range Theorem) Let $A,\ B \in \mathbb{M}_2$, with Hermitian positive definite B. Then $W(A,B)$ is a (possibly degenerate) closed elliptical disc, whose foci are the eigenvalues of $B^{-1}A$, a and b. The equation of the ellipse is*

$$\frac{X^2}{M^2} + \frac{Y^2}{N^2} = \frac{1}{4},$$

where

$$X = (x - Re\tilde{c})\cos\gamma - (y - Im\tilde{c})\sin\gamma, \quad Y = (x - Re\tilde{c})\sin\gamma + (y - \tilde{c})\cos\gamma,$$

$\tilde{c} = \frac{(\lambda_1+\lambda_2)}{2}$ is the center of the ellipse, and γ is the slope of the major axis. The length of the major axis is $M = \sqrt{tr(A^*B^{-1}AB^{-1}) - 2Re(\bar{a}b)}$, and $N = \sqrt{tr(A^*B^{-1}AB^{-1}) - |a|^2 - |b|^2}$ is the length of the minor axis.

Proof. Since B is Hermitian positive definite, without loss of generality we can assume that B is a diagonal matrix $B = diag(f,g)$ with $f > 0,\ \ g > 0$. According to lemma 3.1.1, we may assume A' as follows

$$A' = T^*AT = \begin{bmatrix} af & c\sqrt{fg} \\ 0 & bg \end{bmatrix}.$$

For $\xi, \eta \in \mathbb{C}^2$, we find

$$\frac{\xi^*A\xi}{\xi^*B\xi} = \frac{\eta^*A'\eta}{\eta^*B\eta} = \frac{\xi^*T^*AT\xi}{\xi^*T^*BT\xi},$$

provided $\xi = T\eta$, for T in lemma 3.1.1. Thus $W(A,B) = W(A',B)$. Furthermore, as B is Hermitian positive definite, we have $W(A',B) = W(B^{\frac{-1}{2}}A'B^{\frac{-1}{2}})$, and

$$A'' = B^{\frac{-1}{2}}A'B^{\frac{-1}{2}} = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix},$$

where $a,\ b$ are the eigenvalues of A''. Consider

$$A''' = \begin{bmatrix} \frac{a-b}{2} & c \\ 0 & \frac{b-a}{2} \end{bmatrix}.$$

Let $S = (p,q)$ be a unit vector in $\mathbb{C}^2$, $p = e^{i\alpha}\cos\theta,\ q = e^{i\beta}\sin\theta,\ \alpha \in [0,2\pi[,\ \beta \in [0,2\pi[,\ \theta \in [0,2\pi[$. Then we have

$$S^* A''' S = [\frac{a-b}{2}\cos 2\theta + \frac{c}{2}\ e^{i(\beta-\alpha)}\sin 2\theta] = x + iy,$$

where

$$x = \frac{a-b}{2}\cos 2\theta + \frac{c}{2}\ \sin 2\theta \cos(\beta - \alpha + \gamma),$$

$$y = \frac{c}{2}\sin 2\theta \sin(\beta - \alpha + \gamma), \quad \gamma = Arg(c).$$

Thus,

$$(x - \frac{a-b}{2}\cos 2\theta)^2 + y^2 = \frac{c^2}{4}\sin^2 2\theta.$$

Rewriting the last expression as

$$(x - \frac{a-b}{2}\cos\phi)^2 + y^2 = \frac{c^2}{4}\sin^2\phi, \quad 0 \leqslant \phi \leqslant \pi,$$

we conclude that this is a family of circles with radius $c|\sin\phi|$ and center $\frac{a-b}{2}\cos\phi$. We determine the envelope of this family of curves. Differentiating

$$F(x,y,\phi) = (x - \frac{a-b}{2}\cos\phi)^2 + y^2 - \frac{c^2}{4}\sin^2\phi,$$

with respect to ϕ, we get

$$F_\phi(x,y,\phi) = (a-b)x\sin\phi - \frac{1}{4}((a-b)^2 + c^2)\sin 2\phi.$$

Eliminating ϕ between $F(x,y,\phi) = 0$ and $F_\phi(x,y,\phi) = 0$, we obtain

$$\frac{x^2}{\frac{(a-b)^2}{4} + \frac{c^2}{4}} + \frac{y^2}{\frac{c^2}{4}} = 1,$$

or equivalently

$$\frac{x^2}{(a-b)^2 + c^2} + \frac{y^2}{c^2} = \frac{1}{4}.$$

So, $W(A''')$ is an ellipse with center at $(0,0)$, minor axis of length

$$N = 2c = \sqrt{tr((A''')^*A''') - |a|^2 - |b|^2},$$

and major axis of length

$$M = \sqrt{(a-b)^2 + 4c^2} = \sqrt{tr((A''')^*A''') - 2Re(\bar{a}b)}.$$

Thus, $W(A'')$ is an ellipse with center at

$$\tilde{c} = \frac{1}{2}tr(A'') = \frac{a+b}{2},$$

and the major axis has an inclination γ with the positive real axis. Since the larger denominator is under the term in x, the major axis of the ellipse is horizontal. The coordinates of a point on the ellipse, before translation and rotation, are $x,\ y$. After translation and rotation, we have

$$X = (x - Re\,\tilde{c})\cos\gamma - (y - Im\,\tilde{c})\sin\gamma, \quad Y = (x - Re\,\tilde{c})\sin\gamma + (y - Im\,\tilde{c})\cos\gamma,$$

and so

$$\frac{X^2}{M^2} + \frac{Y^2}{N^2} = 1.$$

□

In the case of degeneracy, $W(A,B)$ may reduce to a line segment whose endpoints are λ_1 and λ_2, or to a singleton if and only if $\lambda_1 = \lambda_2$.

3.2 Parabolical Range Theorem

Consider $W(A,B)$ for $A,\ B \in \mathbb{M}_2$, with B positive (negative) semidefinite, in the following form:

$$B = diag(f,0),$$

and

$$A = \begin{bmatrix} ae^{i\alpha} & ce^{i\gamma} \\ d & be^{i\beta} \end{bmatrix}, \quad b \neq 0.$$

According to proposition 2.2.1, we perform a translation and rotation so that may suppose, without loss of generality, in the matrix A, $\alpha = \gamma,\ \beta = 0,\ dc = ab$. For simplicity, we consider $f = 1$. Thus,

$$B = diag(1,0), \quad A = \begin{bmatrix} ae^{i\gamma} & ce^{i\gamma} \\ d & b \end{bmatrix}, \quad c,d \geqslant 0, \quad b > 0, \quad a = \frac{cd}{b}. \tag{3.1}$$

Therefore, $\det A = 0$. This means that we choose the focus of the conic to be the origin. Taking $\beta = 0$, means that the axis of the conic is along the x axis, and this is achieved by a rotation. Taking $f = 1$, means to perform a scaling transformation.

Theorem 3.2.1. *(Parabolical Range Theorem) Let $A, B \in \mathbb{M}_2$ be of the form (3.1). Then $W(A,B)$ is bounded by the (possibly degenerate) parabola with focus $\lambda_0 = 0$ and equation*

$$\frac{y^2}{4p^2} - \frac{x}{p} = 1,$$

where

$$p = \frac{a^2b^2 + c^4 - 2abc^2\cos\gamma}{4bc^2}. \tag{3.2}$$

Proof. Let $x\cos t + y\sin t + w = 0$ be a supporting line of $W(A,B)$. Writing $A = H(A) + iK(A)$, we have

$$H(A) = \begin{bmatrix} a\cos\gamma & \frac{ce^{i\gamma}+d}{2} \\ \frac{ce^{-i\gamma}+d}{2} & b \end{bmatrix}, \quad K(A) = \begin{bmatrix} a\sin\gamma & \frac{ce^{i\gamma}+d}{2} \\ \frac{d-ce^{-i\gamma}}{2i} & 0 \end{bmatrix}.$$

Thus,

$$P_{A,B}(t,w) = \det(H\cos t + K\sin t + wB) =$$

$$\det \begin{bmatrix} \cos t a\cos\gamma + \sin t a\sin\gamma + w & \cos t\frac{ce^{i\gamma}+d}{2} + \sin t\frac{ce^{i\gamma}+d}{2} \\ \cos t\frac{ce^{-i\gamma}+d}{2} + \sin t\frac{d-ce^{-i\gamma}}{2i} & \cos t bw \end{bmatrix} =$$

$$\cos^2 t(-\frac{c^2+d^2}{4} + \frac{1}{2}ab\cos\gamma) + \sin^2 t(-\frac{c^2+d^2}{4} + \frac{1}{2}ab\cos\gamma) + \cos t bw = 0.$$

Solving above equation, we find w. Then, replacing w by the obtained value in $\cos tx + \sin ty + w = 0$, we obtain

$$F(x,y,t) = x\cos t + y\sin t + \frac{1}{b\cos t}(\frac{c^2+d^2}{4} - \frac{1}{2}ab\cos\gamma).$$

Differentiating with respect to t, we get

$$F_t(x,y,t) = -x\sin t + y\cos t + \frac{\sin t}{b\cos^2 t}(\frac{c^2+d^2}{4} - \frac{1}{2}ab\cos\gamma).$$

Solving the system

$$F_t(x,y,t) = 0, \quad F(x,y,t) = 0,$$

with respect to x and y, we get

$$x = \frac{-2\tan t}{b}(\frac{c^4+a^2b^2}{4c^2} - \frac{1}{2}ab\cos\gamma) = \frac{2abc^2\cos\gamma - c^4 - a^2b^2}{2bc^2}\tan t,$$

and

$$y = \frac{-\cos 2t}{b\cos^2 t}\left(\frac{c^4+a^2b^2}{4c^2} - \frac{1}{2}ab\cos\gamma\right) = \frac{2abc^2\cos\gamma - c^4 - a^2b^2}{4bc^2}\cos 2t\sec^2 t.$$

By a simple computation we obtain the following equation of a parabola

$$\frac{y^2}{4p^2} - \frac{x}{p} = 1,$$

where

$$p = \frac{a^2b^2+c^4-2abc^2\cos\gamma}{4bc^2}.$$

□

In the case of degeneracy of the parabola, $W(A,B)$ may reduce to a half-line with $\lambda = 0$ as endpoint.

We remark that, in general, for $A = (a_{ij}) \in \mathbb{M}_2$, with $a_{22} \neq 0$ and $B = diag(1,0)$, the slope of the axis of the parabolic boundary, relatively to the positive semi real axis, is equal to $\theta_0 = Arg(a_{22})$, and the focus of the parabola is the (finite) eigenvalue of the pencil (A,B). The vertex of the parabola is the point

$$\frac{u_0^*Au_0}{u_0^*Bu_0},$$

where u_0 is an eigenvector of the Hermitian pencil

$$\left(\frac{1}{2}(e^{-i\theta_0}A + e^{i\theta_0}A), B\right),$$

corresponding to the finite eigenvalue.

The following algorithm in Mathematica can help to draw the shape of the numerical range of the pencil (A,B) with B positive (negative) semidefinite.

```
B = {{f, 0}, {0, 0}};
A = {{a E^(I al), c E^(I ga)}, {d, b E^(I be)}};
MatrixForm[A]
(*Computes the H (A)*)
HA = 1/2 (A + (Transpose[A] /. al -> -al /. be -> -be /. ga -> -ga));
(*Computes K (A)*)
KA = 1/(2 I) (A - (Transpose[A] /. al -> -al /. be -> -be /.
      ga -> -ga));
MatrixForm[HA]
MatrixForm[KA]
(*Computes det (uH (A)+vK (A)+wB)*)
P = Det[u HA + v KA + w B];
{{Sw}} = Solve[P == 0, w];
```

```
(*Determines F (x,y,t)*)
F = (x u + y v + (w /. Sw)) /. u -> Cos[t] /. v -> Sin[t];
(*Determines the derivative of F (x,y,t) with respect to t*)
DF = D[F, t];
(*Determines x (t), y (t)*)
{{Sx, Sy}} = Solve[{F == 0, DF == 0}, {x, y}];
X = FullSimplify[x /. Sx]
Y = FullSimplify[y /. Sy]
X /. be -> 0 /. al -> ga /. a -> c d /b
Y /. be -> 0 /. al -> ga /. a -> c d/b
 ParametricPlot[{X, Y} /. a -> 1 /. b -> 1 /. c -> 1 /. d -> 1 /.
      f -> 1 /. be -> 0 /. al -> ga /. ga -> Pi/4, {t, 0, Pi}]
```

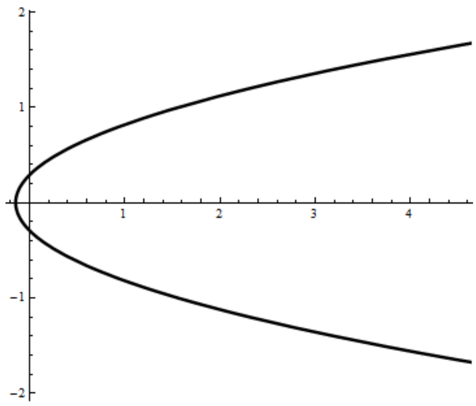

Fig. 3.1 The boundary of the numerical range of a two-by-two pencil (A,B), where A is arbitrary and B is semidefinite .

3.3 Hyperbolical Range Theorem

Theorem 3.3.1. *(Hyperbolical Range Theorem) Let A, $B \in \mathbb{M}_2$ with $B = diag(f,-g)$ Hermitian indefinite. Assume that the eigenvalues of (A,B) are $a > 0$ and $-a$. Then, $W(A,B)$ is bounded by the hyperbola with equation*

$$\frac{x^2}{4a^2 - c^2} - \frac{y^2}{c^2} = \frac{1}{4}.$$

Proof. By similar arguments to those in lemma 3.1.1, we may assume A' as follows

$$A' = \begin{bmatrix} af & c\sqrt{fg} \\ 0 & ag \end{bmatrix}.$$

So, we have

$$H(A') = \begin{bmatrix} af & \frac{c\sqrt{fg}}{2} \\ \frac{c\sqrt{fg}}{2} & ag \end{bmatrix}, \quad K(A') = \begin{bmatrix} 0 & \frac{-c\sqrt{fg}i}{2} \\ \frac{c\sqrt{fg}i}{2} & 0 \end{bmatrix}.$$

As a vector $\xi \in \mathbb{C}^2$ can be parametrized by $\xi = (\sqrt{g}\cos(s), \sqrt{f}\sin(s)e^{i\phi})^T$, where $0 \leqslant s \leqslant 2\pi,\ 0 \leqslant \phi \leqslant 2\pi$, we have

$$Re(\xi^* A'\xi) = \xi^* H(A')\xi = \begin{bmatrix} \sqrt{g}\cos(s) & \sqrt{f}\sin(s)e^{-i\phi} \end{bmatrix} \begin{bmatrix} af & \frac{c\sqrt{fg}}{2} \\ \frac{c\sqrt{fg}}{2} & ag \end{bmatrix} \begin{bmatrix} \sqrt{g}\cos(s) \\ \sqrt{f}\sin(s)e^{i\phi} \end{bmatrix}$$

$$= afg + \frac{cfg}{2}\sin(2s)\cos(\phi),$$

and

$$Im(\xi^* A'\xi) = \xi^* K(A')\xi = \begin{bmatrix} \sqrt{g}\cos(s) & \sqrt{f}\sin(s)e^{-i\phi} \end{bmatrix} \begin{bmatrix} 0 & \frac{-c\sqrt{fg}i}{2} \\ \frac{c\sqrt{fg}i}{2} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{g}\cos(s) \\ \sqrt{f}\sin(s)e^{i\phi} \end{bmatrix}$$

$$= \frac{cfg}{2}\sin(2s)\sin(\phi).$$

We also have

$$\xi^* B\xi = \begin{bmatrix} \sqrt{g}\cos(s) & \sqrt{f}\sin(s)e^{-i\phi} \end{bmatrix} \begin{bmatrix} f & 0 \\ 0 & -g \end{bmatrix} \begin{bmatrix} \sqrt{g}\cos(s) \\ \sqrt{f}\sin(s)e^{i\phi} \end{bmatrix} = fg\cos(2s).$$

Now, from the above expression we get

$$\frac{2afg + cfg\sin(2s)\cos(\phi)}{2fg\cos(2s)} + \frac{cfg\sin(2s)\sin(\phi)}{2fg\cos(2s)}i = x + iy.$$

Thus,

$$x = \frac{2afg + cfg\sin(2s)\cos(\phi)}{2fg\cos(2s)} = \frac{a}{\cos(2s)} + \frac{c\sin(2s)}{2\cos(2s)}\cos(\phi),$$

and

$$y = \frac{c\sin(2s)}{2\cos(2s)}\sin(\phi).$$

Hence

$$(x - \frac{a}{\cos(2s)})^2 + y^2 = \frac{c^2\sin^2(2s)}{2\cos^2(2s)}.$$

Rewriting the equation as

$$(x-\frac{a}{\cos(\theta)})^2+y^2=\frac{c^2\sin^2(\theta)}{4\cos^2(\theta)}, \tag{3.3}$$

we conclude that this is a family of circles with radius $\frac{1}{2}|c\tan\theta|$ and center $\frac{a}{\cos(\theta)}$. Next, we determine the envelope of this family of curves. Differentiating

$$F(x,y,\theta)=(x-\frac{a}{\cos(\theta)})^2+y^2-\frac{c^2\sin^2(\theta)}{4\cos^2(\theta)},$$

with respect to θ, we get

$$F_\theta(x,y,\theta)=-a(x-\frac{a}{\cos(\theta)})-\frac{c^2}{4\cos(\theta)}.$$

Eliminating θ between $F(x,y,\theta)=0$ and $F_\theta(x,y,\theta)=0$, we have

$$\cos\theta=\frac{4a^2-c^2}{4ax}.$$

By substituting the above expression of $\cos\theta$ in (3.3), we obtain,

$$(1-\frac{4a^2}{4a^2-c^2})^2x^2+y^2=\frac{4c^2a^2x^2}{(4a^2-c^2)^2}-\frac{c^2}{4}.$$

Thus,

$$\frac{-c^2}{4a^2-c^2}\,x^2+y^2=-\frac{c^2}{4},$$

and so

$$\frac{x^2}{4a^2-c^2}-\frac{y^2}{c^2}=\frac{1}{4}.$$

□

In the case of degeneracy of the hyperbola, $W(A,B)$ may reduce to two half-lines of the line defined by the eigenvalues of the pencil, and with these points as endpoints.

By a convenient translation, without loss of generality we may assume in theorem 3.3, the eigenvalues of (A,B) of the form $a>0$ and $-a$.

Chapter 4

Numerical range of three-by-three linear pencils with one Hermitian coefficient

The numerical range of a linear pencil of dimension n may be characterized in terms of a certain algebraic curve of class n, explicitly given by the characteristic polynomial of the pencil. For the case $n = 3$, each possible type of curve can be completely described using Newton's classification of cubic curves. Illustrative examples of all the different possibilities are given.

As we shall see in the sequel, the characteristic polynomial of (A,B) gives rise to the *boundary generating curve* of the numerical range $W(A,B)$. To investigate this relation and for the sake of completeness, we present some prerequisites concerning plane algebraic curves.

Theorem 4.0.2. *Let A,B be n-by-n complex matrices. If $ux+vy+w=0$ is the equation of a supporting line of $W(A,B)$, then*

$$P_{A,B}(u,v,w) = \det(uH + vK + wB) = 0, \tag{4.1}$$

where $H = \dfrac{A+A^*}{2}$ and $K = \dfrac{A-A^*}{2i}$.

It can be easily proved that the above result holds for B indefinite or semi-definite. Since $P_{A,B}(u,v,w)$ is a homogeneous polynomial of degree n, (4.1) may be viewed as the line equation of an algebraic curve in the complex projective plane $\mathbb{CP}^2$. The set of lines (u,v,w) (with equation $ux+vy+wz=0$) such that $P_{A,B}(u,v,w) = 0$, may be regarded as a set of lines in the plane whose envelope is a certain curve. Consider the following set

$$\Gamma = \{[u,v,w] \in \mathbb{CP}^2,\ \ P_{A,B}(u,v,w) = \det(uH + vK + wB) = 0\}.$$

The dual curve is

$$\Gamma^* = \{[x,y,z] \in \mathbb{CP}^2,\ \ ux+vy+wz = 0 \textit{ is a tangent of } \ \Gamma\},$$

and its real affine view is the boundary generating curve or associated curve

$$C(A,B) = \{(x,y) \in \mathbb{R}^2,\ \ [x,y,1] \in \Gamma^*\},$$

4.1 Characterization of $W(A,B)$ for $A,\ B \in M_3$

4.1.1 $C(A,B)$ for B positive definite

For B positive definite and arbitrary A , using the results of Kippenhahn for $W(A)$, we may easily characterize $W(A,B)$ which coincide with $W(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})$. Kippenhahn classified the associated curve $C(A,B)$, considering the factorizability of the characteristic polynomial of the pencil $P_{A,B}(u,v,w)$. Before analyzing the possibilities that may occur, we give a technical result.

Proposition 4.1.1. *Let* $A,\ B \in \mathbb{M}_3$, *with* $B = diag(b_1,b_2,b_3),\ \ b_1,\ b_2,\ b_3 > 0$ *and* $\sigma(A,B) = \{\alpha,\beta,\gamma\}$. *Then, there exists a nonsingular matrix* $T \in \mathbb{M}_3$ *such that*

$$T^*BT = B,\ \ T^*AT = A',$$

where

$$A' = \begin{bmatrix} \alpha b_1 & * & * \\ 0 & \beta b_2 & * \\ 0 & 0 & \gamma b_3 \end{bmatrix}. \tag{4.2}$$

Proof. Since, by hypothesis, $\alpha \in \sigma(A,B)$ there exists $u_1 \in \mathbb{C}^3$ such that $Au_1 = \alpha Bu_1$. We consider $u_2, u_3 \in \mathbb{C}^3$ such that $u_2^*Bu_1 = 0,\ u_3^*Bu_1 = 0,\ u_3^*Bu_2 = 0.$ Assume that the vectors $u_1,\ u_2,\ u_3$ are normalized according to

$$u_1^*Bu_1 = b_1,\ \ u_2^*Bu_2 = b_2,\ \ u_3^*Bu_3 = b_3,$$

and let us consider, the matrix $T_0 = [u_1,u_2,u_3] \in \mathbb{M}_3$ with columns u_1,u_2,u_3. It may be easily shown that

$$T_0^*BT_0 = B,\ \ T_0^*AT_0 = \begin{bmatrix} \alpha b_1 & * \\ 0_2 & A'' \end{bmatrix},$$

where $0_2 = [0,0]^T,\ \ A'' \in \mathbb{M}_2$. For $B'' = diag(b_2,b_3)$, it is clear that $\sigma(A'',B'') = \{\beta,\gamma\}$. The proof is completed by choosing

$$v_1 = \begin{bmatrix} 1 \\ 0_2 \end{bmatrix},\ \ v_2 = \begin{bmatrix} 0 \\ u_2' \end{bmatrix},\ \ v_3 = \begin{bmatrix} 0 \\ u_3' \end{bmatrix},$$

where $u_2', u_3' \in \mathbb{C}^2$ are B''-orthogonal such that

$$A''u_2' = \beta B''u_2',\ \ u_3'^*B''u_2' = 0.$$

Consider now $S = [v_1,v_2,v_3]$. The matrix $T = T_0S$ is nonsingular and yields the stated result. □

Definition 4.1.2. Let $A,\ B \in \mathbb{M}_n$. The matrix A is **B-decomposable** if there exists $T \in \mathbb{M}_n$ such that

$$T^*BT = B,\ \ T^*AT = A_1 \oplus A_2,$$

where A_1 and A_2 are matrices with sizes less than n.

We notice that by proposition 2.2.4, whenever B is positive definite $W(A,B)$ is convex, bounded and closed, since it reduces to $W(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})$, and inherits the properties of the classical numerical range. Following the arguments in [17] (theorem 10), we can prove the following

Theorem 4.1.3. *The convex hull of $C(A,B)$ is $W(A,B)$.*

Remark 4.1.4. As a consequence of a result, independently obtained by Murnaghan [22] and Kippenhahn [17], the real foci of the algebraic curve defined by $\det(uH+vK+wB)=0$, where B is positive definite, are the eigenvalues of the matrix $B^{-1}A$, with $A=H+iK$.

Kippenhahn classified the associated curve $C(A)$ of $W(A)$, considering the factorizability of the Kippenhahn polynomial $P_A(u,v,w)$. We shall use the same procedure for $W(A,B)$. If A and B are three-by-three matrices and B is positive definite, then one of the following cases holds:

1^{st} Case **The matrix A is B-decomposable and $P_{A,B}(u,v,w)$ is reducible,**

1) If $P_{A,B}(u,v,w)$ factorizes into three linear factors, then $C(A,B)$ consists of three points, which correspond to the eigenvalues of $B^{-1}A$, and $W(A,B)$ is the closed triangular region defined by these points. In this case, $B^{-1}A$ is normal.
2) Since B is Hermitian positive definite, without loss of generality, we can assume that B is a diagonal matrix with $b_1>0,\ b_2>0,\ b_3>0$. According to proposition 4.1.1, we may assume A' as follows

$$A' = \begin{bmatrix} cb_1 & 0 \\ 0 & A_1 \end{bmatrix},$$

where $c \in \mathbb{C}$ and $A_1 \in \mathbb{M}_2$. Thus, $P_{A,B}(u,v,w)$ factorizes into a linear factor and a quadratic irreducible one. The boundary generating curve of the pencil $(A_1, diag(b_2,b_3))$, according to (3.1), is an elliptical disc. Then $W(A',B)$ is the convex hull of c and $C(A_1, diag(b_2,b_3))$, and it is an elliptical disc if the point c lies on, or inside, the ellipse, and a cone-like region otherwise.

2^{nd} Case **The matrix A is B-indecomposable but $P_{A,B}(u,v,w)$ is reducible,**

The polynomial $P_{A,B}(u,v,w)$ factorizes into a linear and a quadratic factor. The quadratic factor corresponds to an ellipse. In fact, the conic can not be a parabola because one of its real foci is a point at infinity and this contradicts remark 4.1.4, and the conic can not be an hyperbola because this curve is unbounded. Therefore, $C(A,B)$ consists of an ellipse and a point. Moreover, the point must lie in the interior of the ellipse, because A is B-indecomposable, then $W(A,B)$ is an elliptical disc.

3^{rd} Case **The matrix A is B-indecomposable and $P_{A,B}(u,v,w)$ is irreducible,**

By proposition 4.1.1, we may consider $B=diag(b_1,b_2,b_3)$ and A of the form (4.2). The number of real cusps of an (irreducible) class three curve is one or three, and the order of the boundary generating curve is four or six.

By Newton's classification of cubic curves and dual considerations, there are the following possibilities for the associated curve:

1) $C(A,B)$ is of order six and consists of two pieces, one inside the other, namely a closed tricuspid curve lying inside an oval. Then, $W(A,B)$ is the convex hull of the outer part of $C(A,B)$, and so it is an oval region.
2) $C(A,B)$ is of order four, has one real cusp and one double tangent. Then the boundary of $W(A,B)$ contains a line segment, but no corners.

The examples in Chapter I show that all these types of curves may appear as $C(A,B)$ taking $B = I_3$.

4.1.2 $C(A,B)$ for B indefinite

For B Hermitian indefinite, consider $\mathbb{C}^n$ endowed with the B-inner product $(Bx,y) = y^*Bx$, and the corresponding B-norm $\|x\|_B^2 = (Bx,x)$.

Let $A \in \mathbb{M}_n$ be arbitrary and let $B \in \mathbb{M}_n$ be Hermitian indefinite nonsingular, then

$$W(A,B) = \{\frac{(Au,u)}{(Bu,u)} :\ u \in \mathbb{C}^n,\ (Bu,u) \neq 0\}.$$

We have

$$W(A,B) = W_+(A,B) \cup W_-(A,B),$$

where

$$W_+(A,B) = \{\frac{(Au,u)}{(Bu,u)} :\ u \in \mathbb{C}^n,\ (Bu,u) > 0\},$$

and

$$W_-(A,B) = \{\frac{(Au,u)}{(Bu,u)} :\ u \in \mathbb{C}^n,\ (Bu,u) < 0\}.$$

In our analysis, we have to consider the eigenvalues of **positive** and **negative** type, that is, the eigenvalues with associated eigenvectors with positive and negative B-norm, respectively. As in Chapter II, we denote by $\sigma_+(A,B)$ (*resp.* $\sigma_-(A,B)$) the set of eigenvalues of positive (negative) type.

The investigation of the projections of $W_+(A,B)$ ($W_-(A,B)$) on lines that pass through the origin and defining an angle θ with the real axis (projectively oriented) is crucial in our study. The projections are given by $W_+(H(e^{-i\theta}A),B)$ $(W_-(H(e^{-i\theta}A),B))$, and we shall use the characterization of the numerical range of self-adjoint pencils in theorem 2.4.12. If there exists a single direction $\theta \in \mathbb{R}$, such that $W_+(H(e^{-i\theta}A),B)$ is a half-lines, then $W_+(A,B)$ is a half-plane (possibly open), perpendicular to the direction θ. When the same happens for several directions, then $W_+(A,B)$ is the intersection of the corresponding half-planes, because the half-planes are defined by supporting lines of $W_+(A,B)$. These lines are tangent to the associated curve. In this case, the intersections of the referred half-plane coincide with the pseudo-convex hull of the associated curve. If no such a direction exists, $W_+(A,B)$ is the complex plane.

Let $X^+(X^-)$ be a set of points in $W_+(A,B)$ $(W_-(A,B))$. Let $\Omega^+(\Omega^-)$ be the convex hull of $X^+(X^-)$. Consider the lines defined by points z_+, z_-, with $z_+ \in \Omega^+, z_- \in \Omega^-$. The union of all half-lines with z_+ as endpoint not containing z_- and the half-lines with z_- as endpoint but not containing z_+, is the so called **pseudo-convex hull** of X^+ and X^-, throughout denoted by $PCo(X^+, X^-)$.

Theorem 4.1.5. *Let $A \in \mathbb{M}_3$ be arbitrary and let B be Hermitian indefinite. Then $W(A,B)$ is pseudo-convex.*

Proof. Let us consider $\lambda_1 \neq \lambda_2 \in \mathbb{C}$. Then, there exist $0 \neq v_1,\ 0 \neq v_2 \in \mathbb{C}^3$ such that $v_i^* A v_i = \lambda_i v_i^* B v_i,\ i = 1,2$. Let $\tilde{v}_1,\ \tilde{v}_2$ be orthonormal vectors belonging to the subspace $\mathscr{H}_2$ spanned by $v_1,\ v_2$. Let $A_{\tilde{v}_1,\tilde{v}_2}$ and $B_{\tilde{v}_1,\tilde{v}_2}$ be the compressions of A and B, respectively, to $\mathscr{H}_2$. Obviously, $W(A_{\tilde{v}_1,\tilde{v}_2}, B_{\tilde{v}_1,\tilde{v}_2})$ is either an elliptical, parabolical or hyperbolical domain, depending on $B_{\tilde{v}_1,\tilde{v}_2}$ being definite, semi-definite or indefinite. If $W(A_{\tilde{v}_1,\tilde{v}_2}, B_{\tilde{v}_1,\tilde{v}_2})$ is an elliptical or parabolical disc, we have that

$$\{\lambda_1 + x(\lambda_2 - \lambda_1) : 0 \leqslant x \leqslant 1\} \subseteq W(A_{\tilde{v}_1,\tilde{v}_2}, B_{\tilde{v}_1,\tilde{v}_2}) \subseteq W(A,B).$$

If $W(A_{\tilde{v}_1,\tilde{v}_2}, B_{\tilde{v}_1,\tilde{v}_2})$ is hyperbolical, either

$$\{\lambda_1 + x(\lambda_2 - \lambda_1) : 0 \leqslant x \leqslant 1\} \subseteq W(A_{\tilde{v}_1,\tilde{v}_2}, B_{\tilde{v}_1,\tilde{v}_2}) \subseteq W(A,B),$$

and $[\lambda_1, \lambda_2] \subseteq W_+(A,B)$ $([\lambda_1, \lambda_2] \subseteq W_-(A,B)$), or

$$\{\lambda_1 + x(\lambda_2 - \lambda_1) : x \leqslant 0 \ \text{ or } \ x \geqslant 1\} \subseteq W(A_{\tilde{v}_1,\tilde{v}_2}, B_{\tilde{v}_1,\tilde{v}_2}) \subseteq W(A,B),$$

and $\lambda_1 \in W_+(A,B),\ \lambda_2 \in W_-(A,B)$ ($\lambda_1 \in W_-(A,B), \lambda_2 \in W_+(A,B)$). In this case the line defined by λ_1, λ_2 excluding the open segment (λ_1, λ_2), belongs to $W(A,B)$. This completes the proof. $\square$

The curve $C(A,B)$ has branches of a well defined sign, either positive or negative, say $C_+(A,B)$ and $C_-(A,B)$. The sign is determined by considering, for the adequate root w of $P_{A,B}(u,v,w)$, an associated eigenvector x , such that

$$(uH + vK + wB)x = 0.$$

Each branch of $C(A,B)$ is characterized by the sign of the B-norm of x, which determines the sign of the branch. The pseudo-convex hull of $C_+(A,B)$ and $C_-(A,B)$ is $W(A,B)$. (See theorem 4.1.6)

We recall that an usual procedure to find the point equation of the associated curve $C(A,B)$ is to eliminate one of the indeterminate, say u, from

$$P_{A,B}(u,v,w) = \det(uH + vK + wB), \ \ A = H + iK,$$

and then the remaining parameter v from the equations $P_{A,B}(u,v,w) = 0$ and $P_v(u,v,w) = 0$. The curve $P_{A,B}(u,v,w) = 0$ has class n (because the defining polynomial has degree n), that is, through a general point in the plane there are n lines (may be complex) tangent to the curve.

We classify the associated curve $C(A,B)$, considering the factorizability of the polynomial $P_{A,B}(u,v,w)$. The proof is omitted since it is similar to the one of proposition 4.1.1.

We shall assume that A is arbitrary and B is indefinite. To avoid trivial cases of degeneracy of $W(A,B)$, we shall be specially concerned with the class of matrices in $\mathbb{M}_n$, for which there exists a real interval $[\theta_1,\theta_2]$, with $0<\theta_2-\theta_1<\pi$, such that for θ ranging over that interval, the Hermitian pencil $(H(e^{-i\theta}A),B)$, has real eigenvalues satisfying simultaneously the following conditions:

i) $\lambda_1(H(e^{-i\theta}A),B)\geqslant ...\geqslant \lambda_r(H(e^{-i\theta}A),B)\in\sigma_+(H(e^{-i\theta}A),B)$,

ii) $\lambda_{r+1}(H(e^{-i\theta}A),B)\geqslant ...\geqslant \lambda_n(H(e^{-i\theta}A),B)\in\sigma_-(H(e^{-i\theta}A),B)$,

iii) $\lambda_r(H(e^{-i\theta}A)),B)>\lambda_{r+1}(H(e^{-i\theta}A),B)$.

For the pencils of this class, $W(H(e^{-i\theta}A),B)$ is non-degenerate, that is, it is not a singleton, a whole line or the whole complex plane. This class of pencils will be called class $\mathscr{N}\mathscr{D}$, the acronym for non-degenerate.

Theorem 4.1.6. *If the pencil (A,B) belong to $\mathscr{N}\mathscr{D}$, then $W(A,B)$ is the pseudo-convex hull of $C_+(A,B)$ and $C_-(A,B)$.*

Proof. Let $ux+vy+w=0$ be the equation of a supporting line l of $W(A,B)$ with direction $e^{-i\theta}$ $0<\theta<\pi$ and let $A=H+iK$. Then $\det(uH+vK+wB)=0$. According to our conditions either l is tangent to $W_+(A,B)$ or to $W_-(A,B)$, the tangency points are the points $\frac{z^*Az}{z^*Bz}$ where with the minimum eigenvalue in $\sigma_+(H(e^{i\theta}A),B)$ or the maximum eigenvalue in $\sigma_+(H(e^{i\theta}A),B)$, respectively. In the first case, the tangency points belong to $C_+(A,B)$, in the second case they belong to $C_-(A,B)$. Since $W(A,B)$ is pseudo-convex, it coincides with the pseudo-convex hull of $C_+(A,B)$ and $C_-(A,B)$. $\square$

As a consequence of a result, independently obtained by Murnaghan [22] and Kippenhahn [17], the real foci of the algebraic curve $\det(uH+vK+wB)=0$, where B is positive definite, are the eigenvalues of the matrix $B^{-1}A$, with $A=H+iK$. The corresponding result for B indefinite is as follows [2].

Theorem 4.1.7. *Let $A,B\in M_n$, with B indefinite. The n real foci of the algebraic curve $P_{A,B}(u,v,w)=\det(uH+vK+wB)=0$ are the eigenvalues of the pencil (A,B), where $A=H+iK$ with H and K Hermitian.*

Proposition 4.1.8. *Let $A,B\in\mathbb{M}_3$, $B=diag(b_1,b_2,-b_3)$, $b_1,\ b_2,\ b_3>0$ and $\sigma(A,B)=\{\alpha,\beta,\gamma\}$. If there does not exist $u\neq 0$ such that $u^*Au=u^*Bu=0$ then, there exists $T\in\mathbb{M}_3$ nonsingular such that*

$$T^*BT=B,\ \ T^*AT=A',$$

where

$$A'=\begin{bmatrix}\alpha b_1 & * & * \\ 0 & \beta b_2 & * \\ 0 & 0 & -\gamma b_3\end{bmatrix}.$$

Proof. The proof is similar to the proof of proposition 4.1.1. $\square$

If A and B are three-by-three matrices and B is indefinite, then one of the following cases holds:

1^{st} Case **The matrix A is B-decomposable and $P_{A,B}(u,v,w)$ is reducible**,

1) If $P_{A,B}(u,v,w)$ factorizes into three linear factors, then $C(A,B)$ consists of three points which correspond to the eigenvalues of $B^{-1}A$, and $W(A,B)$ is the pseudo-convex hull of these points.
2) We assume now that A' is as follows

$$A' = \begin{bmatrix} cb_1 & 0 \\ 0 & A_1 \end{bmatrix} \tag{4.3}$$

or

$$A' = \begin{bmatrix} A_1 & 0 \\ 0 & -cb_3 \end{bmatrix}, \tag{4.4}$$

and $B = diag(b_1, b_2, -b_3)$.

Thus, $P_{A,B}(u,v,w)$ factorizes into a linear factor and a quadratic irreducible one. If A' is of the form (4.3), then the boundary generating curve $C(A',B)$ consists of a point c and of the boundary generating curve of the pencil $(A_1, diag(b_2, -b_3))$. According to theorem 3.3, this curve is an hyperbola with one branch in $W_+(A,B)$ and the other one in $W_-(A,B)$. We may write

$$C(A_1, diag(b_2,-b_3)) = C_+(A_1, diag(b_2,-b_3)) \cup C_-(A_1, diag(b_2,-b_3)),$$

where $C_\mp(A_1, diag(b_2,-b_3)) \subset W_\mp(A,B)$. Clearly, $c \in W_+(A,B)$.

Let $X_+ = Conv(c, C_+(A_1, diag(b_2,-b_3))$, the pseudo-convex hull of X_+ and $C_-(A_1, diag(b_2,-b_3))$ coincides with $W(A,B)$.

Suppose now that A' is of form (4.4). Notice that $c \in W_-(A,B)$ and $C(A_1, diag(b_1,b_2)) \subset W_+(A,B)$. Then $W(A,B)$ is the pseudo-convex hull of c and the ellipse (possibly degenerate) $C(A_1, diag(b_1,b_2))$.

2^{nd} Case **The matrix A is B-indecomposable but $P_{A,B}(u,v,w)$ is reducible**.

The polynomial $P_{A,B}(u,v,w)$ factorizes into a linear and a quadratic factor. The quadratic factor corresponds to a hyperbola or to an ellipse. In fact, the conic can not be a parabola because one of its real foci is a point at infinity and this contradicts theorem 4.1.7.

Therefore, $C(A,B)$ consists of: 1) one point, produced by vectors with a negative B-norm, and an ellipse produced by vectors with a positive B-norm, 2) one point, produced by vectors with a positive B-norm, and an hyperbola, with one branch produced by vectors with a negative B-norm and the other branch produced by vectors with a positive B-norm.

In case 1), $W(A,B) = \mathbb{C}$. In case 2), $W(A,B) = \mathbb{C}$, whenever the point lies inside the hyperbolic disc of negative type, otherwise $W(A,B)$ is a hyperbolical disc.

3^{rd} Case **The matrix A is B-indecomposable and $P_{A,B}(u,v,w)$ is irreducible.**

The number of real cusps of an (irreducible) class three curve is one or three, and the order of the boundary generating curve is four or six. By Newton's classification of cubic curves and dual considerations, there are the following possibilities for the associated curve:

C1) **All the roots of $P_{A,B}$ are real and distinct**: $C(A,B)$ is a sextic with three cusps and at least one oval component. (cf. example 4.1.10)

C2) **Two of the roots of $P_{A,B}(u,v,w)$ are equal**: in this case there are two possibilities:

a) $C(A,B)$ is sextic, with three cusps and not containing neither oval components nor ordinary double tangents. (cf. example 4.1.11)

b) $C(A,B)$ is a quartic with three cusps and one ordinary double tangent at two of its points. (cf. example 4.1.12)

C3) **The three roots of $P_{A,B}(u,v,w)$ are equal**: $C(A,B)$ is a cubic, with one real cusp and one real flex.

C4) **Only one root of $P_{A,B}(u,v,w)$ is real and two of the roots are complex conjugate:** $C(A,B)$ is of order four, has one real cusp and one ordinary double tangent (at two complex points). (cf. example 4.1.13)

In the examples here presented it is enough to determine the projection of $W(A,B)$ on the real axis, which is given by $W(H,B)$. This is a consequence of the matrices being real. In this case, the numerical range is symmetric relatively to the real axis. The figures have been produced with Mathematica 7. The associated curve is represented.

Numerical examples

Example 4.1.9. Let us consider the pencil (A,B) such that

$$A=\begin{bmatrix}0 & \frac{-1}{2} & 0\\ \frac{1}{2} & 0 & \frac{-1}{2}\\ 0 & \frac{-1}{2} & -\sqrt{2}\end{bmatrix},\ A=H+iK\ with,\ H=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & \frac{-1}{2}\\ 0 & \frac{-1}{2} & -\sqrt{2}\end{bmatrix},\ K=\begin{bmatrix}0 & \frac{i}{2} & 0\\ \frac{-i}{2} & 0 & 0\\ 0 & 0 & 0\end{bmatrix},$$

and $B=diag(1,1,-1)$. Then, the characteristic polynomial of the pencil (A,B) is

$$P_{A,B}(u,v,w)=\det(uH+vK+wB)=\frac{uv^2}{2\sqrt{2}}-\frac{u^2w}{4}+\frac{v^2w}{4}-\sqrt{2}uw^2-w^3.$$

Now, we solve $ux+vy+w=0$ with respect to u, then we replace in $P_{A,B}(u,v,w)$ and we take w equal to one. We get

$$f(x,y,v)=P_{A,B}(-\frac{1+vy}{x},v,1),$$

and we eliminate v between $f(x,y,v)$ and $f_v(x,y,v)$. Then, computing the resultant of these polynomials with respect to v, $res_v=(f(x,y,v),f_v(x,y,v))$, we obtain:

$$\begin{aligned}G(x,y)=&4\sqrt{2}x-76x^2+262\sqrt{2}x^3-834x^4+704\sqrt{2}x^5-656x^6+160\sqrt{2}x^7\\&-32x^8-y^2-18\sqrt{2}xy^2+400x^2y^2-1248\sqrt{2}x^3y^2+3048x^4y^2-1600\sqrt{2}x^5y^2\\&+608x^6y^2+6y^4-384x^2y^4+960\sqrt{2}x^3y^4-864x^4y^4-8y^6+32\sqrt{2}xy^6-32x^2y^6.\end{aligned}$$

The boundary generating curve is given by

$$F(x,y)=0,$$

where

$$\begin{aligned}F(x,y)=&\frac{1}{8}(-32\sqrt{2}x^5+8x^6+y^2-6y^4+8y^6+x^4(98-152y^2)\\&+2\sqrt{2}x^3(-35+124y^2)-2\sqrt{2}x(2-11y^2+12y^4)+4x^2(11-57y^2+54y^4).\end{aligned}$$

We easily find that

$$G(x,y)=(-8+32\sqrt{2}x-32x^2)F(x,y).$$

The pseudo-convex hull of $F(x,y)=0$ is the numerical range $W(A,B)$. So, it remains to specify which are the $C_+(A,B)$ branches and which are the $C_-(A,B)$ branches. Let us consider $u=1$ and $v=0$. We find that the eigenvalues of the pencil (H,B) are

$$\lambda_1=\frac{1}{2}(1+\sqrt{2}),\ \ \lambda_2=\frac{1}{2}(-1+\sqrt{2}),\ \ \lambda_3=0,$$

the B-norm of an eigenvector associated with λ_1 being positive, while the B- norms of eigenvectors associated with λ_2, λ_3, are negative. Thus, the $C_-(A,B)$ branches are on the left hand side of figure 4.1, and the $C_+(A,B)$ branches are on the right hand side of figure 4.1. The pseudo-convex hull of $C_+(A,B)$ and $C_-(A,B)$ is $W(A,B)$. This example corresponds to the 3^{rd} Case, first type.

```
A = {{0, -1/2, 0}, {1/2, 0, -1/2}, {0, -1/2, -Sqrt[2]}}
B = DiagonalMatrix[{1, 1, -1}]
MatrixForm[A]
H = (A + Transpose[A])/2
K = (A - Transpose[A])/2/I
p = Det[u H + v K + w B]
Solve[x u + y v + w == 0, w]
f = p /. w -> -x u - y v /. v -> 1
Df = D[f, u]
Res = Factor[Resultant[f, Df, u]]/x
Res0 = FullSimplify[Res 512/(-8 + 32 Sqrt[2] x - 32 x^2 )]
curve = ContourPlot[Res0 == 0, {x, -1, 2}, {y, -2, 2}]
Export["curve.eps", curve]
```

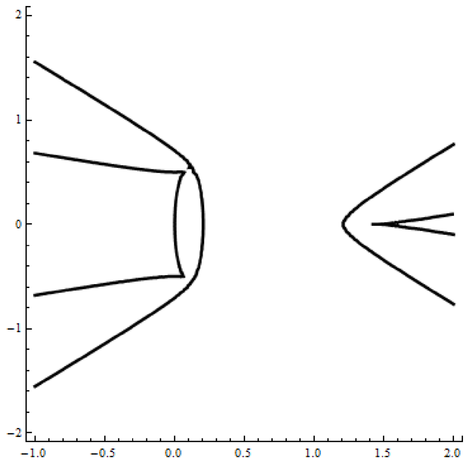

Fig. 4.1 Curve of type C1, with two oval components and two components with cusps.

Example 4.1.10. Let

$$A=\begin{bmatrix}3&2&1\\-2&-5&0\\-1&0&0\end{bmatrix}\ \textit{with}\ H=\begin{bmatrix}3&0&0\\0&-5&0\\0&0&0\end{bmatrix},\ K=\begin{bmatrix}0&-2i&-i\\2i&0&0\\i&0&0\end{bmatrix},$$

and $B=diag(1,1,-1)$. The characteristic polynomial of the pencil (A,B) is

$$P_{A,B}(u,v,w)=\det(uH+vK+wB)=5uv^2+15u^2w+3v^2w+2uw^2-w^3,$$

and we easily obtain

$$\begin{aligned}f(x,y,u)=&P_{A,B}(u,1,-ux-y)=5-3x-15u^2x+30u(-ux-y)\\&-4ux(-ux-y)+2(-ux-y)^2+3x(-ux-y)^2.\end{aligned}$$

The boundary generating curve is given by

$$F(x,y)=0,$$

where

$$\begin{aligned}F(x,y)=&\,7500x-14500x^2+9400x^3-1800x^4-324x^5+108x^6+5625y^2-61500xy^2\\&+11350x^2y^2+1860x^3y^2+441x^4y^2-36000y^4+33600xy^4+5600x^2y^4+14400y^6.\end{aligned}$$

It should be noticed that $F(x,y)$ is a factor of the resultant of of $f(x,y,u)=0$ and $f_u(x,y,u)=0$ with respect to u. It remains to specify the $C_+(A,B)$ branches and the $C_-(A,B)$ branches. Let us consider $u=1$ and $v=0$. We find that the eigenvalues of the pencil (H,B) are

$$\lambda_1=-5,\ \lambda_2=3,\ \lambda_3=0,$$

the B-norm of an eigenvector associated with λ_3 being negative, while the B- norms of eigenvectors associated with $\lambda_1,\ \lambda_2$, are positive. So, we have $\sigma_+(H,B) = \{0\}$ and $\sigma_-(H,B) = \{-5,3\}$ and by theorem 2.4.12 part(e) we conclude that

$$W(H,B) = W_+(H,B) = W_-(H,B) = \mathbb{R}.$$

Thus, $C_+(A,B)$ is the outer branch and $C_-(A,B)$ is the inner branch of the curve of figure 4.2. The pseudo-convex hull of $C_+(A,B)$ and $C_-(A,B)$ is $W(A,B)$. We easily conclude that $W(A,B)$ is the whole complex plane. This example corresponds to the 3^{rd} Case, first type.

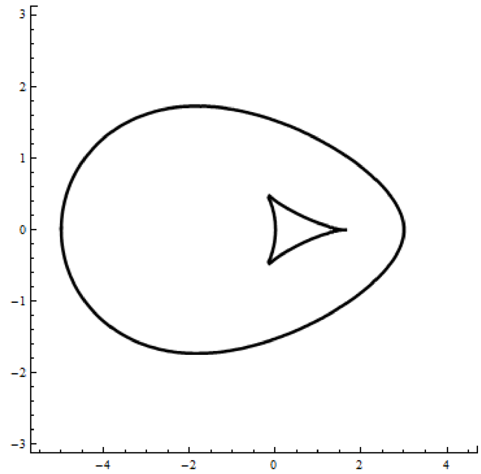

Fig. 4.2 Curve of type C1, with one closed oval and a deltoid.

Example 4.1.11. Let

$$A = \begin{bmatrix} 0 & 2\sqrt{2} & 0 \\ -2\sqrt{2} & 1 & \sqrt{5} \\ 0 & \sqrt{5} & 1 \end{bmatrix} \ \mathit{with}\ H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & \sqrt{5} \\ 0 & \sqrt{5} & 1 \end{bmatrix},\ K = \begin{bmatrix} 0 & -2i\sqrt{2} & 0 \\ 2i\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and $B = diag(1,1,-1)$. By easy calculations, the characteristic polynomial of the pencil (A,B) is obtained

$$P_{A,B}(u,v,w) = \det(uH + vK + wB) = -8uv^2 - 4u^2w + 8v^2w - w^3,$$

and

$$f(x,y,u) = P_{A,B}(u,1,-ux-y) = -8u + 8(-ux-y) - 4u^2(-ux-y) - (-ux-y)^3.$$

The boundary generating curve is given by

$$F(x,y) = 0,$$

where

$$F(x,y) = 128x + 384x^2 + 416x^3 + 224x^4 + 96x^5 + 32x^6 + 16y^2 + 320xy^2 \\ - 248x^2y^2 + 48x^3y^2 - 75x^4y^2 + 32y^4 - 48xy^4 + 24x^2y^4 - 4y^6.$$

The function $F(x,y)$ occurs as a factor of the resultant of $f(x,y,u) = 0$ and $f_u(x,y,u) = 0$, with respect to u. Finally, it remains to determine $W(A,B)$. For this purpose, let us consider $u = 1$ and $v = 0$. We find that the eigenvalues of the self-adjoint pencil (H,B) are

$$\lambda_1 = 2i,\ \lambda_2 = -2i,\ \lambda_3 = 0.$$

Since the pencil has complex eigenvalues, by theorem 2.4.8 $W(H,B) = \mathbb{C}$, so $W(A,B)$ is the whole complex plane. This example corresponds to the 3^{rd} Case, second type (a). (See figure 4.3.)

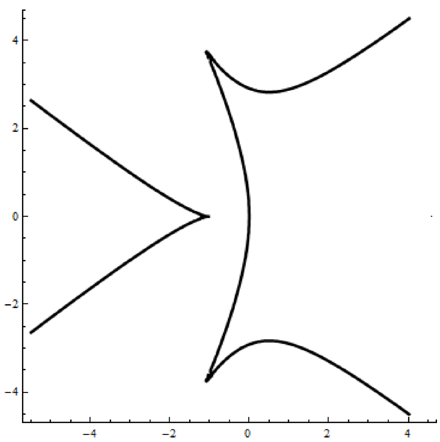

Fig. 4.3 Curve of type C2 (a), with two components with cusps.

Example 4.1.12. Let

$$A = \begin{bmatrix} \frac{3}{16} & \frac{-7}{4} & \frac{3}{4} \\ \frac{9}{4} & \frac{1}{3} & \frac{-3}{2} \\ \frac{-3}{4} & \frac{-7}{2} & -3 \end{bmatrix} \ \textit{with}\ H = \begin{bmatrix} \frac{3}{16} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{3} & \frac{-5}{2} \\ 0 & \frac{-5}{2} & -3 \end{bmatrix},\ K = \begin{bmatrix} 0 & 2i & \frac{-3i}{4} \\ -2i & 0 & -i \\ \frac{3i}{4} & i & 0 \end{bmatrix},$$

and $B = diag(1,1,-1)$. Then the characteristic polynomial of the pencil (A,B) is

$$P_{A,B}(u,v,w) = \det(uH + vK + wB) = \\ -\frac{75u^3}{64} + \frac{39uv^2}{2} - \frac{125u^2w}{16} + \frac{39v^2w}{16} - \frac{169uw^2}{48} - w^3,$$

and

$$
\begin{aligned}
F(x,y) = {} & 328010342400 - 2309739494400x + 1820890191744x^2 - 752604355152x^3 \\
& + 155429067456x^4 - 15045196608x^5 + 546683904 + x^6 + 113193314325y^2 \\
& + 389755033200xy^2 - 771845917752x^2y^2 + 316282531968x^3y^2 - 73023161328x^4y^2 \\
& - 45104599800 + y^4 + 151610877600xy^4 - 36657308896x^2y^4 - 7664793600y^6.
\end{aligned}
$$

Let us consider $u = 1$ and $v = 0$. We find that two eigenvalues of the pencil (H,B) are complex conjugate and one is real. Thus by theorem 2.4.8, $W(H,B)$ is the whole complex plane, and consequently $W(A,B) = \mathbb{C}$. This example corresponds to the 3^{rd} Case, second type (b). (See figure 4.4.)

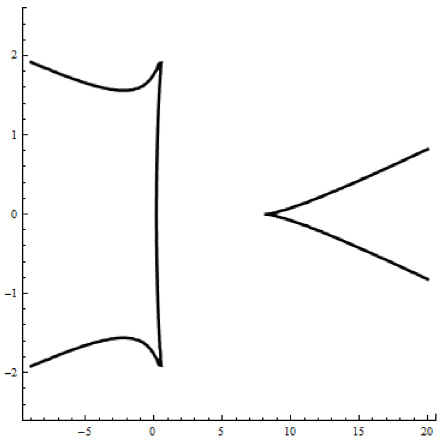

Fig. 4.4 Curve of type C2 (b), with two components.

Example 4.1.13. Let

$$
A = \begin{bmatrix} 0 & 4 & \sqrt{2} \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{bmatrix} \; with \;\; H = \begin{bmatrix} 0 & 2 & \frac{1}{\sqrt{2}} \\ 2 & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}, \;\; K = \begin{bmatrix} 0 & -2i & \frac{-i}{\sqrt{2}} \\ 2i & 0 & \frac{-i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{bmatrix},
$$

and $B = diag(1,1,-1)$. The characteristic polynomial of the pencil (A,B) is given by

$$
P_{A,B}(u,v,w) = \det(uH + vK + wB) = 2u^3 + 2uv^2 + 3u^2w + 3v^2w - w^3,
$$

and

$$
F(x,y) = -16 + 64x - 72x^2 + 27x^4 - 72y^2 + 54x^2y^2 + 27y^4.
$$

Let us consider $u = 1$ and $v = 0$. We find that the eigenvalues of the pencil (H,B) are

$$
\lambda_1 = -1,\ \lambda_2 = -1,\ \lambda_3 = 2,
$$

the B-norm of an eigenvector associated with λ_3 being positive, while the B- norms of eigenvectors associated with $\lambda_1,\ \lambda_2$, are zero. So $W(H,B)=\mathbb{C}$, therefore $W(A,B)$ is the whole complex plane. This example corresponds to the 3^{rd} Case, fourth type. (See figure 4.5.)

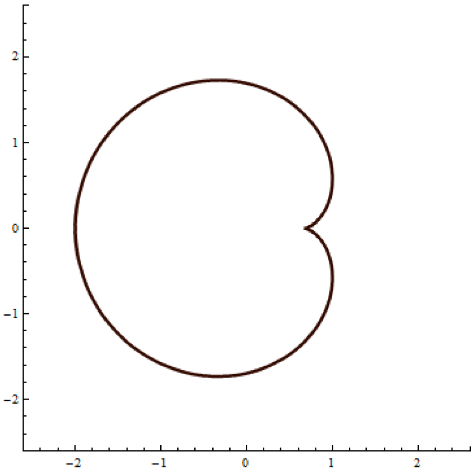

Fig. 4.5 Curve of type C4 , a cardioid with a cusp in the real axis and $x=1$ as a double tangent.

Example 4.1.14. Let

$$A=\begin{bmatrix}\sqrt{2} & 0 & 2\\ 0 & -\sqrt{2} & 4\\ 0 & 2 & 0\end{bmatrix}\ \mathit{with}\ H=\begin{bmatrix}\sqrt{2} & 0 & 1\\ 0 & -\sqrt{2} & 3\\ 1 & 3 & 0\end{bmatrix},\ K=\begin{bmatrix}0 & 0 & -i\\ 0 & 0 & -i\\ i & i & 0\end{bmatrix},$$

and $B=diag(1,1,-1)$. Then, the characteristic polynomial of the pencil (A,B) is

$$P_{A,B}(u,v,w)=\det(uH+vK+wB)=-8\sqrt{2}u^3-8u^2w-2v^2w-w^3,$$

and

$$\begin{aligned}G(x,y)=&\,128x^3-128\sqrt{2}x^4+64x^5-16\sqrt{2}x^6+16x^7+x^9-3456\sqrt{2}y^2+8064xy^2\\&-2816\sqrt{2}x^2y^2+1520x^3y^2-672\sqrt{2}x^4y^2+160x^5y^2-36\sqrt{2}x^6y^2+12x^7y^2-4480\sqrt{2}y^4\\&+6784xy^4-1536\sqrt{2}x^2y^4+944x^3y^4-144\sqrt{2}x^4y^4+48x^5y^4-1376\sqrt{2}y^6+1376xy^6+172x^3y^6.\end{aligned}$$

The boundary generating curve is given by

$$F(x,y)=0,$$

where

$$\begin{aligned}F(x,y)=&\,x^6-144\sqrt{2}xy^2(2+y^2)+4x^4(2+3y^2)+16x^2y^2(4+3y^2)\\&-4\sqrt{2}x^3(2+9y^2)+4y^2(2+y^2)(54+43y^2).\end{aligned}$$

We easily verify that

$$G(x,y) = (-1376\sqrt{2}y^6 + 1376xy^6 + 172x^3y^6)F(x,y).$$

Let us consider $u = 1$ and $v = 0$. We find that two eigenvalues of the pencil (H,B) are complex and one is real. By theorem 2.4.8 $W(H,B) = \mathbb{C}$ and consequently $W(A,B)$ is the whole complex plane. This example corresponds to the 3^{rd} Case, second type(b). (See figure 4.6.)

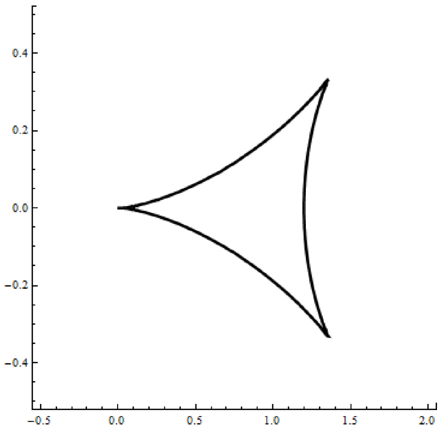

Fig. 4.6 Curve of type C2(a), a sextic reduced to one component with three cusps.

Example 4.1.15. Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \ with \ H = \begin{bmatrix} 1 & \frac{-1}{2} & 0 \\ \frac{-1}{2} & -1 & \frac{-1}{2} \\ 0 & \frac{-1}{2} & 1 \end{bmatrix}, \quad K = \begin{bmatrix} 0 & \frac{-i}{2} & 0 \\ \frac{i}{2} & 0 & \frac{-i}{2} \\ 0 & \frac{i}{2} & 0 \end{bmatrix},$$

and $B = diag(1,1,-1)$. The matrix A is B-indecomposable, and the characteristic polynomial of the pencil (A,B) is

$$P_{A,B}(u,v,w) = \det(uH + vK + wB) = -\frac{3u^3}{2} - \frac{uv^2}{2} + u^2w + uw^2 - w^3,$$

and

$$F(x,y) = 3 + 2x - 2x^2 - 2x^3 - 19y^2 - 62xy^2 - 22x^2y^2 + 36x^3y^2 + 27x^4y^2 + 94y^4 + 164xy^4 + 118x^2y^4 + 91y^6.$$

Let us consider $u = 1$ and $v = 0$. Since the eigenvalues of (H,B) are not all real, by theorem 2.4.8 $W(A,B)$ is the whole complex plane as well as $W(A,B)$. This example corresponds to the 3^{rd} Case, second part (a). (See figure 4.7.)

4.1.3 $C(A,B)$ for B singular-indefinite

Let A be arbitrary, $B = diag(b_1, -b_2, 0)$, with $b_1, b_2 > 0$. We say that $\theta \in [0, 2\pi[$ is an **admissible** direction if the Hermitian pencil $(H(e^{-i\theta}A), B)$ has real eigenvalues with associated non-isotropic

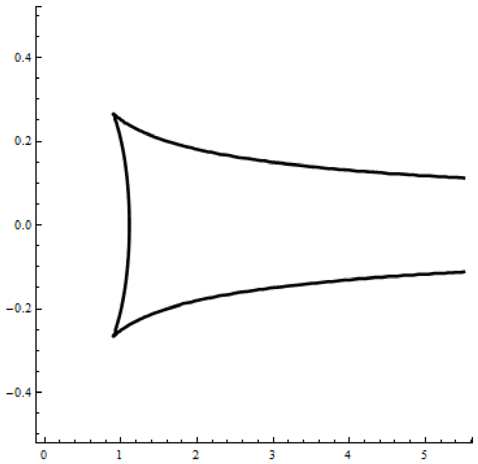

Fig. 4.7 Curve of type C2 (a), with three cusps and not containing neither oval components nor ordinary double tangents.

eigenvectors, and for $\sigma_+(H(e^{-i\theta}A),B)=\{\alpha_\theta\}$, $\sigma_-(H(e^{-i\theta}A),B)=\{\beta_\theta\}$, we have $(\alpha_\theta-\beta_\theta)u^*Au>0$, where $u=(0,0,1)^T$. The condition $(\alpha_\theta-\beta_\theta)u^*Au>0$ ensures that $W(H(e^{-i\theta}A),B)\neq\mathbb{R}$. If admissible directions do not exist, $W(A,B)=\mathbb{C}$.

Proposition 4.1.16. *Let (A,B) be a three-by-three linear pencil with $B=diag(b_1,-b_2,0)$ $b_1,b_2>0$ such that $W(A,B)\neq\mathbb{C}$. Let $u=(0,0,1)^T$, $\sigma_+(H(A),B)=\{\alpha\}$, $\sigma_-(H(A),B)=\{\beta\}$.*

i) If $(\alpha-\beta)u^*Au>0$, then $W(A,B)=]-\infty,\min(\alpha,\beta)]\cup[\max(\alpha,\beta),+\infty[$.

ii) If $(\alpha-\beta)u^*Au<0$, then $W(A,B)=\mathbb{R}$.

For $A\in\mathbb{M}_3$ and B singular-indefinite, the different possibilities that may occur for $C(A,B)$ can be identified according with the procedures in the previous sections.

Example 4.1.17. Let

$$A=\begin{bmatrix}2&2&1\\0&2&2\\0&0&1\end{bmatrix}\ \textit{with}\ H=\begin{bmatrix}2&1&\frac{1}{2}\\1&2&1\\\frac{1}{2}&1&1\end{bmatrix},\ K=\begin{bmatrix}0&-i&\frac{-i}{2}\\i&0&-i\\\frac{i}{2}&i&0\end{bmatrix},$$

and $B=diag(1,-1,0)$. Then, the characteristic polynomial of the pencil (A,B) is not factorizable and given by

$$P_{A,B}(u,v,w)=\frac{1}{4}(6u^3-10uv^2-3u^2w-3v^2w-4uw^2).$$

The boundary generating curve $C(A,B)$ is represented in figure 4.8, has Cartesian equation

$$\begin{aligned}F(x,y)=&\,6000-2400x-5080x^2+4248x^3-1161x^4+108x^5+2808y^2+1752xy^2\\&+1678x^2y^2-2184x^3y^2+36x^4y^2+2007y^4+2316xy^4-568x^2y^4+420y^6,\end{aligned}$$

Let us consider $u = 1$ and $v = 0$. We find that the eigenvalues of the pencil (H,B) are

$$\lambda_1 = \frac{-3-\sqrt{105}}{8}, \ \ \lambda_2 = \frac{-3+\sqrt{105}}{8}.$$

Thus, the $C_-(A,B)$ branch is on the left hand side of figure 4.8, and the $C_+(A,B)$ branches are on the right hand side of figure 4.8. This example corresponds to the 3^{rd} Case, first type of subsection 4.1.2. It is constituted of two branches, $C_-(A,B)$ for $x \leqslant \frac{(-3-\sqrt{105})}{8}$ and $C_+(A,B)$ for $x \geqslant \frac{(-3+\sqrt{105})}{8}$. The pseudo-convex hull of $C_+(A,B)$ and $C_-(A,B)$ is $W(A,B)$.

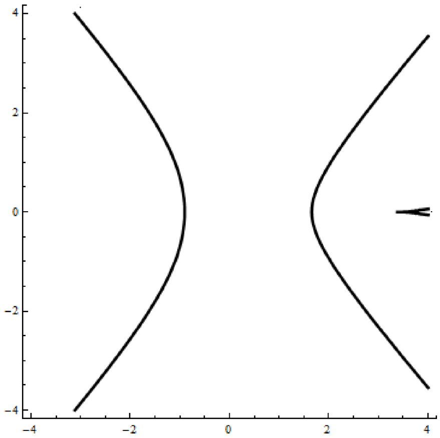

Fig. 4.8 Curve of type C1, with two oval components.

4.1.4 $C(A,B)$ for B positive semi-definite

Theorem 4.1.18. *Let $A \in \mathbb{M}_3$ be arbitrary and let $B \in \mathbb{M}_3$ be positive semi-definite. Then $W(A,B)$ is convex.*

Proof. Let us consider $\lambda_1 \neq \lambda_2 \in W(A,B)$. Then, there exist $0 \neq v_1,\ 0 \neq v_2 \in \mathbb{C}^3$ such that $v_i^* A v_i = \lambda_i v_i^* B v_i,\ i = 1,2$. Let $\tilde{v}_1,\ \tilde{v}_2$ be orthonormal vectors belonging to the subspace $\mathscr{H}_2$ spanned by $v_1,\ v_2$. Let $A_{\tilde{v}_1,\tilde{v}_2}$ and $B_{\tilde{v}_1,\tilde{v}_2}$ be the compressions of A and B, respectively, to $\mathscr{H}_2$. Obviously, $W(A_{\tilde{v}_1,\tilde{v}_2}, B_{\tilde{v}_1,\tilde{v}_2})$ is either a parabolic or elliptical disc, so it is convex. Thus, $[\lambda_1,\lambda_2] \in W(A_{\tilde{v}_1,\tilde{v}_2}, B_{\tilde{v}_1,\tilde{v}_2}) \subseteq W(A,B)$, which completes the proof. □

We next characterize $W(A,B)$, for B positive semi-definite and an arbitrary $A \in M_3$, using again Kippenhahn's approach. We classify the associated curve $C(A,B)$, considering the factorizability of the polynomial $P_{A,B}(u,v,w)$.

1^{st}*Case* **The matrix A is B-decomposable and $P_{A,B}(u,v,w)$ is reducible.**

1) Since B is Hermitian positive semi-definite, without loss of generality we can assume that $B = diag(b_1,b_2,0),\ \ b_1,b_2 > 0$. We assume A' as follows

$$A' = \begin{bmatrix} cb_1 & 0 \\ 0 & A_1 \end{bmatrix},$$

where $c \in \mathbb{C}$ and A_1 is a two-by-two matrix. The boundary generating curve of the pencil $(A_1, diag(b_2, 0))$, is a parabolical disc. Thus, $W(A', B)$ is the convex hull of c and $C(A_1, diag(b_2, 0))$, that is, a parabolical disc if the point lies on or inside the disc, and a cone-like region, otherwise.

2) Suppose that $B = diag(b_1, b_2, 0),\ \ b_1, b_2 > 0$, and A is a three-by-three B-decomposable matrix, i.e, there exists a matrix U, such that

$$U^*BU = B,$$

and

$$U^*AU = \begin{bmatrix} A_1 & 0 \\ 0 & c \end{bmatrix},$$

where $c \in \mathbb{C}$ and A_1 is a two-by-two matrix. Thus, $W(A, B)$ is the convex hull of a point at infinity and $C(A_1, diag(b_1, b_2))$ (cf. example 4.1.22).

2^{nd}*Case* **The matrix A is B-indecomposable and $P_{A,B}(u, v, w)$ is reducible.**

Suppose that $B = diag(b_1, b_2, 0),\ \ b_1, b_2 > 0$. The polynomial $P_{A,B}(u, v, w)$ factorizes into a linear and a quadratic factor. The linear factor corresponds to a point and the quadratic one corresponds to a parabola. Then, $C(A, B)$ consists of one real point and a parabola (cf. example 4.1.21).

3^{rd}*Case* **The matrix A is B-indecomposable and $P_{A,B}(u, v, w)$ is irreducible.**

Suppose that $B = diag(b_1, b_2, 0),\ \ b_1, b_2 > 0$. By Newton‘s classification of cubic curves and dual considerations, there are the following possibilities for the associated curve:

1) $C(A, B)$ is of order six (sextic), with three cusps and at least one oval component (cf. example 4.1.19).
2) $C(A, B)$ is of order four (quartic), with one cusp and an ordinary double tangent at two of its points (cf. example 4.1.20).

4^{th}*Case* The matrix A is B-indecomposable and $P_{A,B}(u, v, w)$ is irreducible,

Suppose that $B = diag(b_1, 0, 0),\ \ b_1 > 0$. By proposition 4.1.8, there exists a nonsingular matrix V, such that

$$V^*BV = B,$$

and

$$V^*AV = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}.$$

In order to avoid the existence of vectors $\xi \neq 0$ such that $\xi^* A\xi = \xi^* B\xi = 0$, we assume that $a_{22}a_{33} \neq 0$. We also assume that $\{a_{12}, a_{13}\} \neq \{0,0\}$, so that A is not B-decomposable. Take the compression of the pencil to the subspaces spanned by e_1, e_2 and by e_1, e_3. Notice that

$$W_1 = W(\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}, \begin{bmatrix} b_1 & 0 \\ 0 & 0 \end{bmatrix}),$$

is a nondegenerate parabola with focus $\frac{a_{11}}{b_1}$ and axis with slope equal to $Arg(a_{22})$, while

$$W_2 = W(\begin{bmatrix} a_{11} & a_{13} \\ 0 & a_{33} \end{bmatrix}, \begin{bmatrix} b_1 & 0 \\ 0 & 0 \end{bmatrix}),$$

is a nondegenerate parabola with focus $\frac{a_{11}}{b_1}$ and axis with slope equal to $Arg(a_{33})$. Considering $Co(W_1, W_2)$, we conclude the following.

- If $|Arg(\frac{a_{22}}{a_{33}})| \geqslant \frac{\pi}{2}$, then $W(A,B)$ is the whole complex plane (cf. example 4.1.24),
- If $|Arg(\frac{a_{22}}{a_{33}})| < \frac{\pi}{2}$, then $W(A,B)$ is a proper subset of the complex plane bounded by a certain algebraic curve, which is a quartic, whenever the characteristic polynomial is not factorizable (cf. example 4.1.23), and a conic if the characteristic polynomial is factorizable (cf. example 4.1.25).

Numerical examples

Example 4.1.19. Let

$$A = \begin{bmatrix} 1 & 1 & \frac{4}{5} \\ 0 & 1 & \frac{4}{5} \\ 0 & 0 & 1 \end{bmatrix} \ with\ H = \begin{bmatrix} 1 & \frac{1}{2} & \frac{2}{5} \\ \frac{1}{2} & 1 & \frac{2}{5} \\ \frac{2}{5} & \frac{2}{5} & 1 \end{bmatrix},\ K = \begin{bmatrix} 0 & \frac{-i}{2} & \frac{-2i}{5} \\ \frac{i}{2} & 0 & \frac{-2i}{5} \\ \frac{2i}{5} & \frac{2i}{5} & 0 \end{bmatrix},$$

and $B = diag(1,1,0)$. Then the characteristic polynomial of the pencil (A,B) is

$$P_{A,B}(u,v,w) = \det(uH + vK + wB) = \frac{1}{100}(71u^3 - 29uv^2 + 192u^2w - 8v^2w + 100uw^2),$$

and

$$\begin{aligned} F(x,y) = &-1731619 + 6115752x - 6709556x^2 + 3123808x^3 - 655104x^4 + 51200x^5 \\ &- 1891452y^2 + 7557408xy^2 - 17370208x^2y^2 + 9142400x^3y^2 - 160000x^4y^2 \\ &- 15865104y^4 + 51091200xy^4 - 21320000x^2y^4 - 21160000y^6. \end{aligned}$$

The convex hull of $F(x,y) = 0$ is the numerical range $W(A,B)$. (See figure 4.9.)

```
Id = {{1, 0, 0}, {0, 1, 0}, {0, 0, 1}}
A = {{1, 1, 4/5}, {0, 1, 4/5}, {0, 0, 1}}
B = {{1, 0, 0}, {0, 1, 0}, {0, 0, 0}}
MatrixForm[A2]
```

```
H = (A + Conjugate[Transpose[A2]])/2
K = (A - Conjugate[Transpose[A2]])/2/I
MatrixForm[H]
MatrixForm[K]
p = Det[H u + K v + B w]
f = p /. w -> (-u x - v y) /. v -> 1
Df = D[f, u]
Factor[Resultant[f, Df, u]]
ContourPlot[ (-4066339 + 21099912 x - 41434436 x^2 + 39230848 x^3 -
    18100224 x^4 + 3276800 x^5 - 16567212 y^2 + 60541248 x y^2 -
    72660448 x^2 y^2 + 31193600 x^3 y^2 - 2560000 x^4 y^2 -
    23710224 y^4 + 45916800 x y^4 - 14120000 x^2 y^4 -
    11560000 y^6) == 0, {x, 0, 2}, {y, -1, 1}]
```

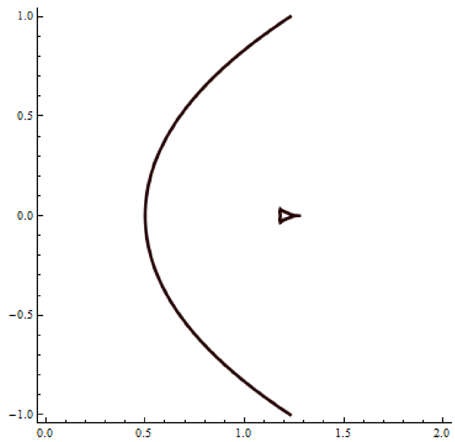

Fig. 4.9 The convex hull of this shape is the numerical range of the matrix in example 4.1.19.

Example 4.1.20. Let

$$A = \begin{bmatrix} 1 & \frac{1}{2} & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \text{ with } H = \begin{bmatrix} 1 & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}, \quad K = \begin{bmatrix} 0 & \frac{-i}{4} & \frac{-i}{2} \\ \frac{i}{4} & 0 & \frac{-i}{2} \\ \frac{i}{2} & \frac{i}{2} & 0 \end{bmatrix},$$

and $B = diag(1,1,0)$. Then the characteristic polynomial of the pencil (A,B) is

$$P_{A,B}(u,v,w) = \det(uH + vK + wB) = \frac{1}{16}(9u^3 - 7uv^2 + 24u^2w - 8v^2w + 16uw^2),$$

and

$$F(x,y) = -343 + 1176x - 1344x^2 + 512x^3 - 592y^2 + 1024xy^2 - 256x^2y^2 - 256y^4.$$

The convex hull of $F(x,y) = 0$ is the numerical range $W(A,B)$. (See figure 4.10.)

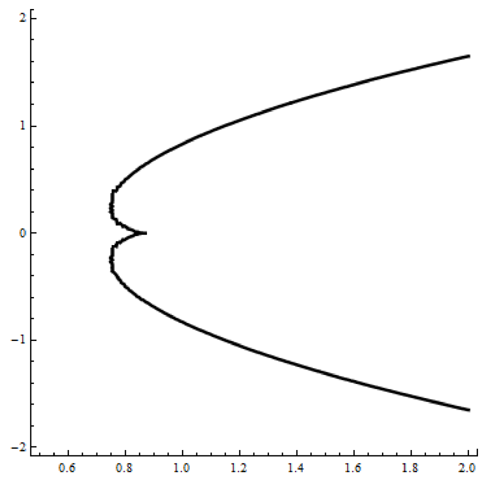

Fig. 4.10 The convex hull of this shape is the numerical range of the pencil in example 4.1.20.

Example 4.1.21. Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \; with \;\; H = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}, \;\; K = \begin{bmatrix} 0 & \frac{-i}{2} & \frac{-i}{2} \\ \frac{i}{2} & 0 & \frac{-i}{2} \\ \frac{i}{2} & \frac{i}{2} & 0 \end{bmatrix},$$

and $B = diag(1,1,0)$. Then the characteristic polynomial of the pencil (A,B) is

$$P_{A,B}(u,v,w) = \det(uH + vK + wB) = \frac{1}{2}(u^3 - uv^2 + 3u^2w - v^2w + uw^2),$$

and

$$F(x,y) = (-1+2x-y^2)((x-1)^2+y^2)^2.$$

The convex hull of $F(x,y) = 0$ is the numerical range $W(A,B)$. (See figure 4.11.)

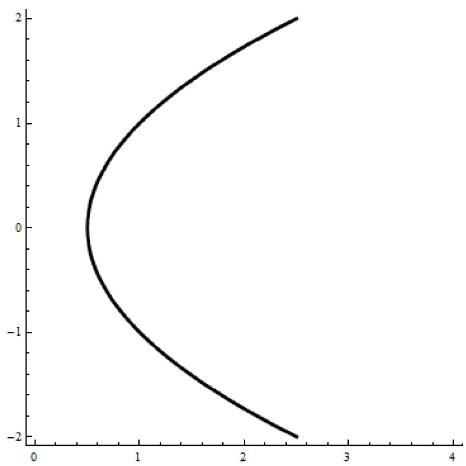

Fig. 4.11 The convex hull of this shape is the numerical range of the pencil in example 4.1.21.

Example 4.1.22. Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \; with \;\; H = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \;\; K = \begin{bmatrix} 0 & \frac{-i}{2} & 0 \\ \frac{i}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and $B = diag(1,1,0)$. Then the characteristic polynomial of the pencil (A,B) is

$$P_{A,B}(u,v,w) = \det(uH + vK + wB) = \frac{1}{4}u(3u^2 - v^2 + 8uw + w^2),$$

and

$$F(x,y) = (-1+2y)^2(1+2y)^2(4(x-1)^2 + 4y^2 - 1).$$

The Cartesian equation of the boundary generating curve is $(x-1)^2 + y^2 = \frac{1}{4}$, and $W(A,B)$ is the convex hull of a point at infinity and a circle. (See figure 4.12.)

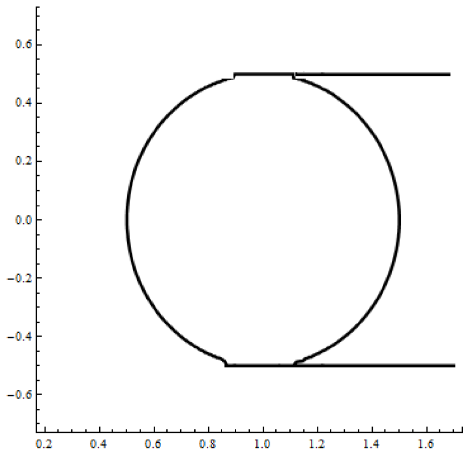

Fig. 4.12 The convex hull of the circle and $+\infty$ is the numerical range of the pencil in example 4.1.22.

Example 4.1.23. Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \; with \;\; H = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}, \;\; K = \begin{bmatrix} 0 & \frac{-i}{2} & \frac{-i}{2} \\ \frac{i}{2} & 0 & \frac{-i}{2} \\ \frac{i}{2} & \frac{i}{2} & 0 \end{bmatrix},$$

and $B = diag(1,0,0)$. Then the characteristic polynomial of the pencil (A,B) is

$$P_{A,B}(u,v,w) = \det(uH + vK + wB) = \frac{1}{4}(2u^3 - uv^2 + 3u^2w - v^2w),$$

and

$$F(x,y) = 16 - 48x + 48x^2 - 20x^3 + 3x^4 + 36y^2 - 36xy^2 - 18x^2y^2 + 27y^4.$$

We observe that $F(x,y)$ does not factorize and since $Arg(1)=0$, Case 4th in sub section 4.1.4 insures that $W(A,B)$ is a subset of the complex plane bounded by the bellow quartic. (See figure 4.13.)

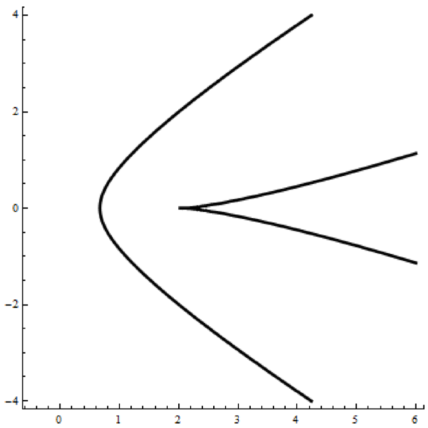

Fig. 4.13 The convex hull of this shape is the numerical range of the pencil in example 4.1.23.

Example 4.1.24. Let

$$A=\begin{bmatrix}1&1&1\\0&1&1\\0&0&-1\end{bmatrix}\ with\ H=\begin{bmatrix}1&\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&1&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}&-1\end{bmatrix},\ K=\begin{bmatrix}0&\frac{-i}{2}&\frac{-i}{2}\\\frac{i}{2}&0&\frac{-i}{2}\\\frac{i}{2}&\frac{i}{2}&0\end{bmatrix},$$

and $B=diag(1,0,0)$. Then the characteristic polynomial of the pencil (A,B) is

$$P_{A,B}(u,v,w)=\det(uH+vK+wB)=\frac{1}{4}(-4u^3-5u^2w-v^2w).$$

The Cartesian equation of the boundary generating curve of $W(A,B)$, represented in figure 4.14, is

$$F(x,y)=-4x^3+5x^4+108y^2-180xy^2+50x^2y^2+125y^4.$$

Since $Arg(-1)=\pi$, Case 4th in sub section 4.1.4 ensures that $W(A,B)$ is the whole complex plane.

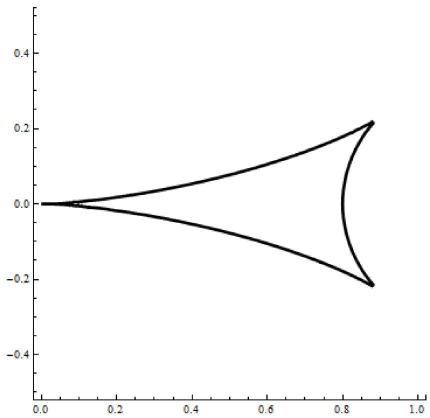

Fig. 4.14 $C(A,B)$ for example 4.1.24.

Example 4.1.25. Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \ with \ \ H = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix}, \ \ K = \begin{bmatrix} 0 & \frac{-i}{2} & \frac{-i}{2} \\ \frac{i}{2} & 0 & 0 \\ \frac{i}{2} & 0 & 0 \end{bmatrix},$$

and $B = diag(1,0,0)$. Then the characteristic polynomial of the pencil (A,B) is factorizable and given by

$$P_{A,B}(u,v,w) = \det(uH + vK + wB) = \frac{1}{2}u(u^2 - v^2 + 2uw).$$

We easily obtain

$$F(x,y) = -1 + 2x - y^2.$$

The convex hull of $F(x,y) = 0$ is the numerical range $W(A,B)$. The boundary of $W(A,B)$ is parabolic. (See figure 4.15.)

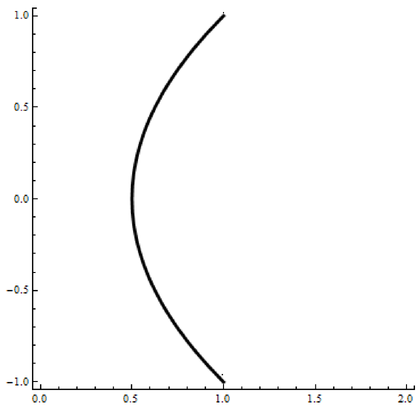

Fig. 4.15 The convex hull of this shape is the numerical range for example 4.1.25.

Chapter 5

Conclusion

The theory of numerical range was initiated by Toeplitz and Hausdorff, and dates back to the early decades of the twentieth century. Modern references are the book by Gustafson and Rao (Springer Verlag 1997) and Horn and Johnson (Cambridge University Press 1990). This theory has many applications in various branches of pure and applied mathematics, such as operator theory, functional analysis, C * - algebras, Banach algebras, matrix norms, inequalities, numerical analysis, perturbation theory, matrix polynomials, systems theory, quantum physics and quantum computing, etc. On the other hand, this is a rich interdisciplinary area which uses many different tools of algebra , analysis, geometry, combinatorial analysis and programming. There are good monographs on this area and many references can be found in platforms such as MathSciNet, Zentralblatt etc.

In this thesis we have discussed the classification of the boundary generating curves of the numerical range for three by three pencils with one hermitian coefficient. As a consequence of Kippenhahn's results [17], the cases C2, C3 of Newton cubics classification can not occur in the case of one of the matrices being positive (negative) definite. In the indefinite case, the examples of the last section show that all the five types of cubic Newton curves may occur. However, the associated curves of types C2, C3 have lead to degenerate cases, in which $W(A,B)$ coincides with $\mathbb{C}$. It is an open problem to prove (or disprove) the validity of this property, removing the restriction of one on the matrices being hermitian.

The curves of C1 and C4 types correspond to a closed oval with a deltoid in its interior and to a cardioid, respectively. In the last Section, we presented examples for unbounded associated curves $C(A,B)$ of type C1 (cf. Figures 4.1). We have obtained that $W(A,B)$ degenerates when those curves are bounded (cf. Figures 4.5 and 4.6). It is also an open problem to determine whether this is true in a more general setting.

Before finishing this thesis, some sentences which are at end of [11] by E. Gutkin are reproduced.

" Before stopping, I will give unsolicited advice to the reader. There is a pervasive custom of concentrating on the latest literature while doing research. I am no exception to this rule. However, my experience with the study of numerical range brought me to the conclusion: it is useful to read the work of "founding fathers"! ".

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