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# SPACES OF GENERALIZED SMOOTHNESS IN THE CRITICAL CASE: OPTIMAL EMBEDDINGS, CONTINUITY ENVELOPES AND APPROXIMATION NUMBERS

SUSANA D. MOURA, JÚLIO S. NEVES, AND CORNELIA SCHNEIDER

Abstract. We study necessary and sufficient conditions for embeddings of Besov spaces of generalized smoothness $B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}(\mathbb{R}^n)$ into generalized Hölder spaces $\Lambda^{\mu(\cdot)}_{\infty,r}(\mathbb{R}^n)$ when $\underline{s}(\boldsymbol{N}\boldsymbol{\tau}^{-1}) > 0$ and $\boldsymbol{\tau}^{-1} \in \ell_{q'}$, where $\boldsymbol{\tau} = \boldsymbol{\sigma}\boldsymbol{N}^{-n/p}$. A borderline situation, corresponding to the limiting situation in the classical case, is included and give new results. In particular, we characterize optimal embeddings for $B$-spaces.

As immediate applications of our results we obtain continuity envelopes and give upper and lower estimates for approximation numbers for the related embeddings.

We also consider the analogous results for the Triebel-Lizorkin spaces of generalized smoothness $F^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}(\mathbb{R}^n)$.

## 1. Introduction

Spaces of generalized smoothness have been studied by several authors, including different approaches. We follow the general Fourier-analytical approach as presented in [FL06]. There one can find more details and some history on these spaces.

The reason for the revived interest in the study of Besov and Triebel-Lizorkin spaces of generalized smoothness $B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}(\mathbb{R}^n)$ and $F^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}(\mathbb{R}^n)$ is its connection with applications for pseudo-differential operators (as generators of sub-Markovian semi-groups), cf. [FL06].

The aim of this paper is to complete the study of [HM08] by extending the results obtained in [MNS11] and [MNP09] to Besov and Triebel-Lizorkin spaces of generalized smoothness $B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}(\mathbb{R}^n)$ and $F^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}(\mathbb{R}^n)$, where a borderline situation, corresponding to the limiting situation in the classical case, is considered and gives new results. We also give examples which yield results that are not covered by the previous references. Moreover, we obtain new estimates for approximation numbers.

In the present paper (cf. Theorem 3.2 below), we give necessary and sufficient conditions for embeddings of $B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}(\mathbb{R}^n)$ into generalized Hölder spaces $\Lambda^{\mu(\cdot)}_{\infty,r}(\mathbb{R}^n)$ when $\underline{s}(\boldsymbol{N}\boldsymbol{\tau}^{-1}) > 0$ and $\boldsymbol{\tau}^{-1} \in \ell_{q'}$, where $\boldsymbol{\tau} = \boldsymbol{\sigma}\boldsymbol{N}^{-n/p}$, *i.e.*, we show that

$$B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}(\mathbb{R}^n) \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}(\mathbb{R}^n), \tag{1.1}$$

if, and only if,

$$\sup_{M\geq 0} \left( \sum_{j=0}^{M} \int_{N_{j+1}^{-1}}^{N_j^{-1}} (\mu(t))^{-r} \frac{\mathrm{d}t}{t} \right)^{\frac{1}{r}} \left( \sum_{k=M}^{\infty} \tau_k^{-q'} \right)^{\frac{1}{q'}} < \infty$$

(with the usual modification if $r = \infty$ and/or $q' = \infty$), provided that $0 < p \leq \infty$, $0 < q \leq r \leq \infty$, $\boldsymbol{\sigma} = \{\sigma_j\}_{j\in\mathbb{N}_0}$ and $\boldsymbol{N} = \{N_j\}_{j\in\mathbb{N}_0}$ are admissible sequences, the latter satisfying $\underline{N}_1 > 1$, and $\mu \in \mathcal{L}_r$ (see Section 3 for precise definitions).

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Furthermore, (cf. Corollary 3.4), when $q > 1$ and $r \in [q, \infty]$, the embedding (1.1) with $\mu = \lambda_{qr}$,

$$\lambda_{qr}(t) = \left(\Lambda(t^{-1})t^{n/p}\right)^{\frac{q'}{r}} \left(\int_0^t \left(\Lambda(s^{-1})s^{n/p}\right)^{-q'} \frac{\mathrm{d}s}{s}\right)^{\frac{1}{q'}+\frac{1}{r}}, \qquad t \in (0, N_0^{-1}],$$

where $\Lambda$ is an admissible function such that $\Lambda(z) \sim \sigma_j,\ \ z \in [N_j, N_{j+1}],\ \ j \in \mathbb{N}_0$, with equivalence constants independent of $j$, is sharp with respect to the parameter $\mu$, that is, the target space $\Lambda^{\mu(\cdot)}_{\infty,r}(\mathbb{R}^n)$ in (1.1) and the space $\Lambda^{\lambda_{qr}(\cdot)}_{\infty,r}(\mathbb{R}^n)$ (*i.e.*, the target space in (1.1) with $\mu = \lambda_{qr}$) satisfy $\Lambda^{\lambda_{qr}(\cdot)}_{\infty,r}(\mathbb{R}^n) \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}(\mathbb{R}^n)$. The embedding with $r = q$ and $\mu = \lambda_{qr} = \lambda_{qq}$ is optimal (*i.e.*, it is the best possible embedding among all the embeddings considered in (1.1)). The case $0 < q \leq 1$ is considered in Remark 3.5.

We also consider the analogous results for the Triebel-Lizorkin spaces of generalized smoothness $F^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}(\mathbb{R}^n)$ (cf. Theorem 3.6, Corollary 3.8 and Remark 3.9 below).

Concerning applications, we compute continuity envelopes $\mathfrak{E}_{\mathsf{C}}(X) = \left(\mathcal{E}^X_{\mathsf{C}}(t), u^X_{\mathsf{C}}\right)$ (cf. Definition 4.1 below), which are closely related with sharp embeddings, where $\mathcal{E}^X_{\mathsf{C}}(t) := \sup_{\|f|X\|\leq 1} \frac{\omega(f,t)}{t}$, $t > 0$, together with some fine index $u^X_{\mathsf{C}}$; here $\omega(f,t)$ stands for the modulus of continuity.

In Proposition 4.4, under the same assumptions of Theorem 3.2, we show, in particular, that

$$\mathcal{E}^{B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}}_{\mathsf{C}}(t) \sim \frac{\lambda_{q\infty}(t)}{t}, \quad t \in (0, \varepsilon].$$

where

$$\lambda_{q\infty}(t) := \left(\int_0^t \left(\Lambda(y^{-1})\, y^{n/p}\right)^{-q'} \frac{\mathrm{d}y}{y}\right)^{\frac{1}{q'}}, \qquad t \in (0, N_0^{-1}], \qquad \text{if} \quad 1 < q \leq \infty,$$

and

$$\lambda_{q\infty}(t) := \sup_{y \in (0,t)} (\Lambda(y^{-1}))^{-1}\, y^{-n/p}, \qquad t \in (0, N_0^{-1}], \qquad \text{if} \quad 0 < q \leq 1.$$

These results generalize those previously obtained in [MNP09, Proposition 3.2]. Additionally, if $\underline{s}(\boldsymbol{\tau}) > 0$, we recover results from [HM08] regarding continuity envelopes. In [HM08] it was proved that in such a case the fine index of the continuity envelope is $q$ for $B$-spaces, and in this case the continuity envelope yields the optimal embedding. The new results in this paper, regarding continuity envelopes, correspond to the situation when, additionally, $\underline{s}(\boldsymbol{\tau}) \leq 0$.

Under some additional assumptions, we obtain in Theorem 4.5, that $\mathfrak{E}_{\mathsf{C}}\left(B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}\right) = \left(\frac{\lambda_{q\infty}(t)}{t}, \infty\right)$, which extends [MNP09, Theorems 3.4 (i) and 3.5 (i)].

In terms of $F$-spaces the results are similar, with the usual replacement of $q$ by $p$.

As for approximation numbers, Theorem 4.7 provides upper and lower bounds for approximation numbers of the embedding of the spaces $B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}(U)$ and $F^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}(U)$, where $N = (2^j)_{j\in\mathbb{N}_0}$, into $C(U)$.

The paper is organized as follows. Section 2 contains notation, definitions, preliminary assertions and auxiliary results. In Section 3 we state our main results, providing necessary and sufficient conditions for the embeddings to hold, and derive optimal weights and sharp embedding assertions. Finally, Section 4 contains some interesting applications concerning continuity envelopes and estimates for approximation numbers.

## 2. Preliminaries

For a real number $a$, let $a_+ := \max(a, 0)$ and let $[a]$ denote its integer part. For $p \in (0, \infty]$, the number $p'$ is defined by $1/p' := (1 - 1/p)_+$ with the convention that $1/\infty = 0$. By $c$, $c_1$, $c_2$, etc. we denote positive constants independent of appropriate quantities. For two non-negative expressions (i.e. functions or functionals) $\mathcal{A}$, $\mathcal{B}$, the symbol $\mathcal{A} \lesssim \mathcal{B}$ (or $\mathcal{A} \gtrsim \mathcal{B}$) means that $\mathcal{A} \leq c\,\mathcal{B}$ (or $c\,\mathcal{A} \geq \mathcal{B}$). If $\mathcal{A} \lesssim \mathcal{B}$ and $\mathcal{A} \gtrsim \mathcal{B}$, we write $\mathcal{A} \sim \mathcal{B}$ and say that $\mathcal{A}$ and $\mathcal{B}$ are equivalent. If not otherwise indicated, log is always taken with respect to base 2. Given two quasi-Banach spaces $X$ and $Y$, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding is bounded. Furthermore, $L_p(\mathbb{R}^n)$, with $0 < p \le \infty$, is the usual Lebesgue space, with respect to the Lebesgue measure, endowed with the usual quasi-norm $\|\cdot \mid L_p(\mathbb{R}^n)\|$. The space of all scalar-valued (real or complex), bounded and continuous functions on $\mathbb{R}^n$ is denoted by $C_B(\mathbb{R}^n)$, which is equipped with the $L_\infty(\mathbb{R}^n)$-norm.

With the exception of the last section, we consider here only function spaces defined on $\mathbb{R}^n$; so for convenience we shall usually omit the "$\mathbb{R}^n$" from their notation.

2.1. **Admissible sequences and admissible functions.** In this subsection we explain the class of sequences we shall be interested in and some related basic results.

A sequence $\boldsymbol{\gamma} = (\gamma_j)_{j\in\mathbb{N}_0}$ of positive real numbers is said to be *admissible* if there exist two positive constants $d_0$ and $d_1$ such that

$$d_0\, \gamma_j \le \gamma_{j+1} \le d_1\, \gamma_j, \qquad j \in \mathbb{N}_0. \tag{2.1}$$

Clearly, for admissible sequences $\boldsymbol{\gamma}$ and $\boldsymbol{\tau}$, $\boldsymbol{\gamma\tau} := (\gamma_j\tau_j)_{j\in\mathbb{N}_0}$ and $\boldsymbol{\gamma}^r := (\gamma_j^r)_{j\in\mathbb{N}_0}$, $r \in \mathbb{R}$, are admissible, too.

For an admissible sequence $\boldsymbol{\gamma} = (\gamma_j)_{j\in\mathbb{N}_0}$, let

$$\underline{\gamma}_j := \inf_{k\ge 0} \frac{\gamma_{j+k}}{\gamma_k} \qquad \text{and} \qquad \overline{\gamma}_j := \sup_{k\ge 0} \frac{\gamma_{j+k}}{\gamma_k}, \qquad j \in \mathbb{N}_0. \tag{2.2}$$

Then clearly $\underline{\gamma}_j\gamma_k \le \gamma_{j+k} \le \gamma_k\overline{\gamma}_j$, for any $j,k \in \mathbb{N}_0$. In particular, $\underline{\gamma}_1$ and $\overline{\gamma}_1$ are the best possible constants $d_0$ and $d_1$ in (2.1), respectively. The lower and upper Boyd indices of the sequence $\boldsymbol{\gamma}$ are defined, respectively, by

$$\underline{s}(\boldsymbol{\gamma}) := \lim_{j\to\infty} \frac{\log \underline{\gamma}_j}{j} \qquad \text{and} \qquad \overline{s}(\boldsymbol{\gamma}) := \lim_{j\to\infty} \frac{\log \overline{\gamma}_j}{j}. \tag{2.3}$$

The above definition is well posed: the sequence $(\log \overline{\gamma}_j)_{j\in\mathbb{N}}$ is sub-additive and hence the right-hand side limit in (2.3) exists, it is finite (since $\boldsymbol{\gamma}$ is an admissible sequence) and it coincides with $\inf_{j>0} \log \overline{\gamma}_j/j$. The corresponding assertions for the lower counterpart $\underline{s}(\boldsymbol{\gamma})$ can be read off observing that $\log \underline{\gamma}_j = -\log(\overline{\gamma^{-1}})_j$. Moreover, $\underline{s}(\boldsymbol{\gamma}) = -\overline{s}(\boldsymbol{\gamma}^{-1})$ and $\overline{s}(\boldsymbol{\gamma}) = -\underline{s}(\boldsymbol{\gamma}^{-1})$.

**Remark 2.1.** The Boyd index $\overline{s}(\boldsymbol{\gamma})$ of an admissible sequence $\boldsymbol{\gamma}$ describes the asymptotic behaviour of the $\overline{\gamma}_j$'s and provides more information than simply $\overline{\gamma}_1$ and, what is more, is stable under the equivalence of sequences: if $\boldsymbol{\gamma} \sim \boldsymbol{\tau}$, then $\overline{s}(\boldsymbol{\gamma}) = \overline{s}(\boldsymbol{\tau})$ as one readily verifies. In general, we have for admissible sequences $\boldsymbol{\gamma}, \boldsymbol{\tau}$ that

$$\overline{s}(\boldsymbol{\gamma}^r) = r\ \overline{s}(\boldsymbol{\gamma}), \quad r \ge 0, \qquad \overline{s}(\boldsymbol{\gamma\tau}) \le \overline{s}(\boldsymbol{\gamma}) + \overline{s}(\boldsymbol{\tau}), \tag{2.4}$$

and

$$\underline{s}(\boldsymbol{\gamma}^r) = r\underline{s}(\boldsymbol{\gamma}), \quad r \ge 0, \qquad \underline{s}(\boldsymbol{\gamma\tau}) \ge \underline{s}(\boldsymbol{\gamma}) + \underline{s}(\boldsymbol{\tau}). \tag{2.5}$$

Observe that, given $\varepsilon > 0$, there are two positive constants $c_1 = c_1(\varepsilon)$ and $c_2 = c_2(\varepsilon)$ such that

$$c_1\ 2^{(\underline{s}(\boldsymbol{\gamma})-\varepsilon)j} \le \underline{\gamma}_j \le \overline{\gamma}_j \le c_2\ 2^{(\overline{s}(\boldsymbol{\gamma})+\varepsilon)j}, \qquad j \in \mathbb{N}_0. \tag{2.6}$$

From (2.6) it follows that if $\underline{s}(\boldsymbol{\gamma}) > 0$, then $\boldsymbol{\gamma}^{-1} \in \ell_u$ for arbitrary $u \in (0,\infty]$. Conversely, $\overline{s}(\boldsymbol{\gamma}) < 0$ implies $\boldsymbol{\gamma}^{-1} \notin \ell_\infty$, that is, $\boldsymbol{\gamma}^{-1}$ does not belong to any $\ell_u$, $0 < u \le \infty$.

**Examples 2.2.** We consider some examples of admissible sequences.

(i) The sequence $\boldsymbol{\gamma} = (\gamma_j)_{j\in\mathbb{N}_0}$,

$$\gamma_j = 2^{sj}\,(1+j)^b\,(1+\log(1+j))^c$$

with arbitrary fixed real numbers $s$, $b$ and $c$ is a standard example of an admissible sequence with $\underline{s}(\boldsymbol{\gamma}) = \overline{s}(\boldsymbol{\gamma}) = s$.

(ii) Let $\Phi : (0,1] \to \mathbb{R}$ be a slowly varying function (or equivalent to a slowly varying one) in the sense of [BGT89]. Then, for $s \in \mathbb{R}$ the sequence $\boldsymbol{\gamma} = (2^{sj}\,\Phi(2^{-j}))_{j\in\mathbb{N}_0}$ is an admissible sequence. Also here we have $\underline{s}(\boldsymbol{\gamma}) = \overline{s}(\boldsymbol{\gamma}) = s$.

(iii) In view of Proposition 1.9.7 of [Bri02], the case $\boldsymbol{\gamma} = (2^{sj}\,\Psi(2^{-j}))_{j\in\mathbb{N}_0}$, where now $\Psi$ is an admissible function in the sense of [ET98] (i.e., a positive monotone function defined on $(0,1]$ such that $\Psi(2^{-2j}) \sim \Psi(2^{-j})$, $j \in \mathbb{N}_0$), can be regarded as a special case of (ii).

(iv) Take $s_1 > 0$. Consider the following recursively defined sequence:

$$j_0 = 0, \quad j_1 = 1, \quad j_{2\ell} = 2j_{2\ell-1} - j_{2\ell-2} \quad \text{and} \quad j_{2\ell+1} = 2^{j_{2\ell}}, \quad \ell \in \mathbb{N}.$$

The sequence $\boldsymbol{\gamma} = (\gamma_j)_{j\in\mathbb{N}_0}$, defined by

$$\gamma_j = \begin{cases} 2^{\frac{s_1}{2} j_{2\ell}} & \text{if } j_{2\ell} \le j < j_{2\ell+1}, \\ 2^{s_1(j-j_{2\ell+1})} 2^{\frac{s_1}{2} j_{2\ell}} & \text{if } j_{2\ell+1} \le j < j_{2\ell+2}, \end{cases} \qquad \ell \in \mathbb{N}_0,$$

is an example of an admissible sequence with $\underline{s}(\boldsymbol{\gamma}) = 0$, $\overline{s}(\boldsymbol{\gamma}) = \frac{s_1}{2}$. Moreover, sequence oscillates between $(j^{\frac{s_1}{2}})_{j\in\mathbb{N}_0}$ and $(2^{j\frac{s_1}{2}})_{j\in\mathbb{N}_0}$, *i.e.*,

$$j^{\frac{s_1}{2}} \lesssim \gamma_j \lesssim 2^{j\frac{s_1}{2}}, \quad j \in \mathbb{N},$$

and there exist infinitely many $j'$ and $j''$ such that $\gamma_{j'} = {j'}^{\frac{s_1}{2}}$ and $\gamma_{j''} = 2^{j''\frac{s_1}{2}}$. The case $s_1 = 2$ has been treated in [FL06, Leo98].

(v) Take $s_0 \ge 0$ and $s_1 > 0$. Consider the sequence $(j_k)_{k\in\mathbb{N}_0}$ from the previous example. The sequence $\boldsymbol{\gamma} = (\gamma_j)_{j\in\mathbb{N}_0}$,

$$\gamma_j := \exp\left(\int_1^{2^j} \xi(u)\,\frac{\mathrm{d}u}{u}\right), \quad j \ge 0,$$

where

$$\xi(u) := \begin{cases} s_0 & \text{if } 2^{j_{2\ell}} \le u < 2^{j_{2\ell+1}}, \\ s_0 + s_1 & \text{if } 2^{j_{2\ell+1}} \le u < 2^{j_{2\ell+2}}, \end{cases} \qquad \ell \in \mathbb{N}_0,$$

is an example of an admissible sequence with $\underline{s}(\boldsymbol{\gamma}) = s_0$ and $\overline{s}(\boldsymbol{\gamma}) = s_0 + \frac{1}{2}s_1$ (we refer to [KLSS06, Example 4.13]). The case $s_0 = 0$ and $s_1 = 2$ has been treated in [FL06, Leo98].

**Remark 2.3.** The Examples 2.2 (i)-(iii), above, have in common the fact that their upper and lower Boyd indices coincide. However, this is not in general the case. Example 2.2 (iv), due to [KLSS06], shows that an admissible sequence has not necessarily a fixed main order and their upper and lower Boyd indices do not coincide. Moreover, one can easily see that there is an admissible sequence $\boldsymbol{\gamma}$ with $\underline{s}(\boldsymbol{\gamma}) = a$ and $\overline{s}(\boldsymbol{\gamma}) = b$, for any $-\infty < a \le b < \infty$, that is, with prescribed upper and lower Boyd indices.

The following Lemma provides a substitute of [MNS11, Lemma 2.8].

**Lemma 2.4.** *Let $\boldsymbol{\gamma} = (\gamma_j)_{j\in\mathbb{N}_0}$ be an admissible sequence and let $0 < u \le \infty$.*

(i) *If $\underline{s}(\boldsymbol{\gamma}) > 0$, then*

$$\left(\sum_{j=0}^{k} \gamma_j^u\right)^{\frac{1}{u}} \sim \gamma_k, \quad k \in \mathbb{N}_0 \tag{2.7}$$

(*with the usual modification if $u = \infty$*).

(ii) *If $\overline{s}(\boldsymbol{\gamma}) < 0$, then*

$$\left(\sum_{j=k}^{\infty} \gamma_j^u\right)^{\frac{1}{u}} \sim \gamma_k, \quad k \in \mathbb{N}_0 \tag{2.8}$$

(*with the usual modification if $u = \infty$*).

*Proof.* We present the proof for $u \in (0, \infty)$, since the case $u = \infty$ follows by obvious modifications.

(i) Let $k \in \mathbb{N}_0$. For $0 \le j \le k$, we have

$$\underline{\gamma}_{k-j} \le \frac{\gamma_k}{\gamma_j}.$$

This and (2.6), together with the fact that $\underline{s}(\boldsymbol{\gamma}) > 0$, yield

$$\left(\sum_{j=0}^{k} \gamma_j^u\right)^{\frac{1}{u}} \lesssim \gamma_k \left(\sum_{j=0}^{k} 2^{-(\underline{s}(\boldsymbol{\gamma})-\varepsilon)(k-j)u}\right)^{\frac{1}{u}} \lesssim \gamma_k, \quad k \in \mathbb{N}_0,$$

by choosing $\varepsilon \in (0, \underline{s}(\boldsymbol{\gamma}))$. The rest is clear.

(ii) Let $k \in \mathbb{N}_0$. For $j \ge k$,

$$\overline{\gamma}_{j-k} \ge \frac{\gamma_j}{\gamma_k}.$$

This and (2.6), together with the fact that $\overline{s}(\boldsymbol{\gamma}) < 0$, yield

$$\left(\sum_{j=k}^{\infty} \gamma_j^u\right)^{\frac{1}{u}} \lesssim \gamma_k \left(\sum_{j=k}^{\infty} 2^{(\overline{s}(\boldsymbol{\gamma})+\varepsilon)(j-k)u}\right)^{\frac{1}{u}} \lesssim \gamma_k, \quad k \in \mathbb{N}_0,$$

by choosing $\varepsilon \in (0, -\overline{s}(\boldsymbol{\gamma}))$. The reverse estimate is clear. □

The functions we are going to introduce now will be central for the estimates which will be presented later. In this context we also refer to [CF06, Section 2.2].

**Definition 2.5.** *A function $\Lambda : (0,\infty) \to (0,\infty)$ will be called admissible if it is continuous and if for any $b > 0$ it satisfies*

$$\Lambda(bz) \sim \Lambda(z) \quad \textit{for any } z > 0.$$

**Example 2.6.** Let $(N_j)_{j\in\mathbb{N}_0}$ be a sequence of positive numbers such that $\underline{N}_1 > 1$. Let $(\sigma_j)_{j\in\mathbb{N}_0}$ be an admissible sequence. Then the function $\Lambda : (0,\infty) \to (0,\infty)$ defined by

$$\Lambda(z) = \begin{cases} \frac{\sigma_{j+1}-\sigma_j}{N_{j+1}-N_j} z + \sigma_j - \frac{(\sigma_{j+1}-\sigma_j)N_j}{N_{j+1}-N_j}, & \text{if } z \in [N_j, N_{j+1}),\ j \in \mathbb{N}_0 \\ \sigma_0, & \text{if } z \in (0, N_0) \end{cases}$$

is admissible and satisfies $\Lambda(N_j) = \sigma_j$ for any $j \in \mathbb{N}_0$. Moreover, $\Lambda(z) \sim \sigma_j$ for $z \in [N_j, N_{j+1}]$, $j \in \mathbb{N}_0$, with equivalence constants independent of $j$.

The next proposition provides a very useful discretization method, which coincides partially with [MNS11, Proposition 2.7] and generalizes [MNP09, Proposition 2.5].

**Proposition 2.7.** *Let $0 < u \le \infty$, $\boldsymbol{\sigma}$ and $\boldsymbol{N}$ be admissible sequences, the latter satisfying $\underline{N}_1 > 1$. Furthermore, let $\Lambda$ be an admissible function such that $\Lambda(z) \sim \Lambda(N_j) = \sigma_j$ for $z \in [N_j, N_{j+1}]$, $j \in \mathbb{N}_0$. Put $\boldsymbol{\tau} = \boldsymbol{\sigma N}^{-n/p}$ and assume $\boldsymbol{\tau^{-1}} \in \ell_u$. Then for $k \in \mathbb{N}_0$, we have*

$$\left(\int_0^t \left(\Lambda(y^{-1})y^{n/p}\right)^{-u} \frac{\mathrm{d}y}{y}\right)^{\frac{1}{u}} \sim \left(\sum_{j=k}^{\infty} \tau_j^{-u}\right)^{\frac{1}{u}}, \qquad \text{if} \quad t \in [N_{k+1}^{-1}, N_k^{-1}],$$

*with equivalence constants independent of $k \in \mathbb{N}_0$. In the case $u = \infty$ the usual modification with supremum is required.*

*Proof.* Let $u \in (0, \infty)$. For $k \in \mathbb{N}_0$ and $t \in [N_{k+1}^{-1}, N_k^{-1}]$, taking advantage of the hypotheses on $\boldsymbol{\sigma}$, $\boldsymbol{N}$, and $\Lambda$, we obtain

$$\begin{aligned}
\sum_{j=k}^{\infty} \tau_j^{-u} &= \sum_{j=k}^{\infty} \left(\sigma_j N_j^{-n/p}\right)^{-u} \lesssim \sum_{j=k+1}^{\infty} \left(\sigma_j N_j^{-n/p}\right)^{-u} \\
&\sim \sum_{j=k+1}^{\infty} \int_{N_{j+1}^{-1}}^{N_j^{-1}} \left(\Lambda(y^{-1})y^{n/p}\right)^{-u} \frac{\mathrm{d}y}{y} = \int_0^{N_{k+1}^{-1}} \left(\Lambda(y^{-1})y^{n/p}\right)^{-u} \frac{\mathrm{d}y}{y} \\
&\lesssim \int_0^t \left(\Lambda(y^{-1})y^{n/p}\right)^{-u} \frac{\mathrm{d}y}{y} \\
&\lesssim \int_0^{N_k^{-1}} \left(\Lambda(y^{-1})y^{n/p}\right)^{-u} \frac{\mathrm{d}y}{y} = \sum_{j=k}^{\infty} \int_{N_{j+1}^{-1}}^{N_j^{-1}} \left(\Lambda(y^{-1})y^{n/p}\right)^{-u} \frac{\mathrm{d}y}{y} \\
&\sim \sum_{j=k}^{\infty} \left(\sigma_j N_j^{-n/p}\right)^{-u} = \sum_{j=k}^{\infty} \tau_j^{-u}.
\end{aligned}$$

When $u = \infty$, the proof is analogous to the previous case. □

2.2. **Function spaces of generalized smoothness.** Let $\boldsymbol{N} = (N_j)_{j\in\mathbb{N}_0}$ be an admissible sequence with $\underline{N}_1 > 1$ (recall (2.2)). In particular $\boldsymbol{N}$ is a so-called strongly increasing sequence (cf. [FL06, Def. 2.2.1]) which guarantees the existence of a number $l_0 \in \mathbb{N}_0$ such that

$$N_k \ge 2N_j \quad \text{for any} \ \ k, j \quad \text{such that} \quad k \ge j + l_0. \tag{2.9}$$

It should be noted that the sequence $\boldsymbol{N} = (N_j)_{j\in\mathbb{N}_0}$ plays the same role as the sequence $(2^j)_{j\in\mathbb{N}_0}$ in the classical construction of the spaces $B^s_{pq}$ and $F^s_{pq}$. This will be clear from the following considerations.

For a fixed sequence $\boldsymbol{N}$ as above we define the associated covering $\Omega^{\boldsymbol{N}} = (\Omega_j^{\boldsymbol{N}})_{j\in\mathbb{N}_0}$ of $\mathbb{R}^n$ by

$$\Omega_j^{\boldsymbol{N}} = \{\xi \in \mathbb{R}^n : |\xi| \le N_{j+l_0}\}, \quad j = 0, \cdots, l_0 - 1,$$

and

$$\Omega_j^{\boldsymbol{N}} = \{\xi \in \mathbb{R}^n : N_{j-l_0} \le |\xi| \le N_{j+l_0}\}, \quad j \ge l_0,$$

with $l_0$ according to (2.9).

**Definition 2.8.** *For a fixed admissible sequence $\boldsymbol{N}$ with $\underline{N}_1 > 1$, and for the associated covering $\Omega^{\boldsymbol{N}} = (\Omega_j^{\boldsymbol{N}})_{j\in\mathbb{N}_0}$ of $\mathbb{R}^n$, a system $\varphi^{\boldsymbol{N}} = (\varphi_j^{\boldsymbol{N}})_{j\in\mathbb{N}_0}$ will be called a (generalized) partition of unity subordinate to $\Omega^{\boldsymbol{N}}$ if:*

(i) $\varphi_j^{\boldsymbol{N}} \in C_0^\infty$ *and* $\varphi_j^{\boldsymbol{N}}(\xi) \ge 0$ *if* $\xi \in \mathbb{R}^n$ *for any* $j \in \mathbb{N}_0$;
(ii) $\operatorname{supp} \varphi_j^{\boldsymbol{N}} \subset \Omega_j^{\boldsymbol{N}}$ *for any* $j \in \mathbb{N}_0$;

(iii) *for any* $\alpha \in \mathbb{N}_0^n$ *there exists a constant* $c_\alpha > 0$ *such that for any* $j \in \mathbb{N}_0$

$$|\mathrm{D}^\alpha \varphi_j^{\boldsymbol{N}}(\xi)| \le c_\alpha \, (1+|\xi|^2)^{-|\alpha|/2} \quad \text{for any} \quad \xi \in \mathbb{R}^n;$$

(iv) *there exists a constant* $c_\varphi > 0$ *such that*

$$0 < \sum_{j=0}^{\infty} \varphi_j^{\boldsymbol{N}}(\xi) = c_\varphi < \infty \quad \text{for any} \quad \xi \in \mathbb{R}^n.$$

Before turning to the definition of the spaces of generalized smoothness let us recall that $\mathcal{S}$ denotes the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$ equipped with the usual topology and by $\mathcal{S}'$ we denote its topological dual, the space of all tempered distributions on $\mathbb{R}^n$. For $\varphi \in \mathcal{S}$ and $f \in \mathcal{S}'$ we will use the notation $\varphi(\mathrm{D})f = \mathcal{F}^{-1}(\varphi \mathcal{F} f)$, where $\mathcal{F}$ and $\mathcal{F}^{-1}$ stand, respectively, for the Fourier and inverse Fourier transform. Furthermore, if $0 < p \le \infty$ and $0 < q \le \infty$, then $L_p$ and $\ell_q$ have the standard meaning and, if $(f_j)_{j\in\mathbb{N}_0}$ is a sequence of complex-valued Lebesgue measurable functions on $\mathbb{R}^n$, then

$$\|(f_j)_{j\in\mathbb{N}_0} \mid \ell_q(L_p)\| := \Big(\sum_{j=0}^{\infty} \|f_j \mid L_p\|^q\Big)^{1/q}$$

and

$$\|(f_j)_{j\in\mathbb{N}_0} \mid L_p(\ell_q)\| := \Big\|\Big(\sum_{j=0}^{\infty} |f_j(\cdot)|^q\Big)^{1/q} \mid L_p\Big\|$$

with the appropriate modification if $q = \infty$.

**Definition 2.9.** *Let* $\boldsymbol{N} = (N_j)_{j\in\mathbb{N}_0}$ *be an admissible sequence with* $\underline{N}_1 > 1$ *and* $\varphi^{\boldsymbol{N}}$ *be a system of functions as in Definition* 2.8*. Let* $0 < q \le \infty$ *and* $\boldsymbol{\sigma} = (\sigma_j)_{j\in\mathbb{N}_0}$ *be an admissible sequence.*

(i) *Let* $0 < p \le \infty$*. The Besov space of generalized smoothness* $B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}$ *is the set of all tempered distributions* $f$ *such that the quasi-norm*

$$\|f \mid B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}\| := \|(\sigma_j \, \varphi_j^{\boldsymbol{N}}(\mathrm{D})f)_{j\in\mathbb{N}_0} \mid \ell_q(L_p)\|$$

*is finite.*

(ii) *Let* $0 < p < \infty$*. The Triebel-Lizorkin space of generalized smoothness* $F^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}$ *is the set of all tempered distributions* $f$ *such that the quasi-norm*

$$\|f \mid F^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}\| := \|(\sigma_j \, \varphi_j^{\boldsymbol{N}}(\mathrm{D})f(\cdot))_{j\in\mathbb{N}_0} \mid L_p(\ell_q)\|$$

*is finite.*

**Remark 2.10.** We refer to [FL06] for some historical references on the subject and a systematic study of these spaces, including a characterization by local means and atomic decomposition. If $\boldsymbol{\sigma} = (2^{sj})_{j\in\mathbb{N}_0}$, with $s$ a real number, and $\boldsymbol{N} = (2^j)_{j\in\mathbb{N}_0}$, then the spaces $B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}$ and $F^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}$ coincide with the usual Besov or Triebel-Lizorkin spaces $B^s_{p,q}$ and $F^s_{p,q}$, respectively. If we let $\boldsymbol{\sigma} = (2^{sj}\Psi(2^{-j}))_{j\in\mathbb{N}_0}$, where $\Psi$ is an admissible function in the sense of [ET98, ET99] (see Example 2.2(iii)), the corresponding Besov space coincides with the space $B^{(s,\Psi)}_{p,q}$ introduced by Edmunds and Triebel in [ET98, ET99] and also considered by Moura in [Mou01a, Mou01b]. Similarly for the $F$-counterpart.

In what follows we present some embedding results for spaces of generalized smoothness. The following proposition may be found in [CF06, Theorem 3.7]:

**Proposition 2.11.** *Let* $0 < p_1 \le p_2 \le \infty$*,* $0 < q_1, q_2 \le \infty$*, and* $\boldsymbol{\sigma}, \boldsymbol{\tau}$ *be admissible sequences,* $\boldsymbol{N} = (N_j)_{j\in\mathbb{N}_0}$ *admissible with* $\underline{N}_1 > 1$*. Let* $q^*$ *be given by*

$$\frac{1}{q^*} := \left(\frac{1}{q_2} - \frac{1}{q_1}\right)_+ . \tag{2.10}$$

*If*

$$\boldsymbol{\sigma}^{-1}\boldsymbol{\tau}\boldsymbol{N}^{n(\frac{1}{p_1}-\frac{1}{p_2})} \in \ell_{q^*} \ , \tag{2.11}$$

*then*

$$B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p_1,q_1} \ \hookrightarrow \ B^{\boldsymbol{\tau},\boldsymbol{N}}_{p_2,q_2}. \tag{2.12}$$

We refer to [CL06, Lemma 1] for the following generalization of the Franke-Jawerth-embedding (see [Fra86, Jaw77] for the classical situation):

**Proposition 2.12.** *Let $0 < p_1 < p < p_2 \le \infty$, $0 < q \le \infty$, $\boldsymbol{\sigma}$ and $\boldsymbol{N}$ be admissible sequences, the latter satisfying $\underline{N}_1 > 1$. Let $\boldsymbol{\sigma'}$ and $\boldsymbol{\sigma''}$ be the (clearly admissible) sequences defined, respectively, by*

$$\boldsymbol{\sigma'} := \boldsymbol{N}^{n(\frac{1}{p_1}-\frac{1}{p})}\boldsymbol{\sigma} \quad \textit{and} \quad \boldsymbol{\sigma''} := \boldsymbol{N}^{n(\frac{1}{p_2}-\frac{1}{p})}\boldsymbol{\sigma}.$$

*Then*

$$B^{\boldsymbol{\sigma'},\boldsymbol{N}}_{p_1,u} \hookrightarrow F^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q} \hookrightarrow B^{\boldsymbol{\sigma''},\boldsymbol{N}}_{p_2,v}$$

*if, and only if, $0 < u \le p \le v \le \infty$.*

For each $f \in C_B$, $\omega(f,\cdot)$ stands for the modulus of continuity of $f$ and it is defined by

$$\omega(f,t) := \sup_{|h|\le t}\ \sup_{x\in\mathbb{R}^n} |\Delta_h f(x)| = \sup_{|h|\le t} \|\Delta_h f|L_\infty(\mathbb{R}^n)\|, \quad t>0,$$

with $\Delta_h f(x) := f(x+h) - f(x),\ x,h \in \mathbb{R}^n$.

Regarding the embeddings into $C_B$, we have the following result.

**Proposition 2.13.** *Let $0 < p,q \le \infty$, $\boldsymbol{\sigma}$ and $\boldsymbol{N}$ be admissible sequences, the latter satisfying $\underline{N}_1 > 1$.*

(i) *Then*

$$B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q} \hookrightarrow C_B \quad \textit{if, and only if,} \quad \boldsymbol{\sigma}^{-1}\boldsymbol{N}^{n/p} \in \ell_{q'}.$$

(ii) *If $p < \infty$, then*

$$F^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q} \hookrightarrow C_B \quad \textit{if, and only if,} \quad \boldsymbol{\sigma}^{-1}\boldsymbol{N}^{n/p} \in \ell_{p'}.$$

We refer to [CF06, Corollary 3.10 & Remark 3.11] and to [CL06, Proposition 4.4] concerning part (i) and part (ii), respectively, of the proposition above.

Let $r \in (0,\infty]$ and let $\mathcal{L}_r$ be the class of all continuous functions $\lambda : (0,1] \to (0,\infty)$ such that

$$\left(\int_0^1 \frac{1}{(\lambda(t))^r}\frac{\mathrm{d}t}{t}\right)^{\frac{1}{r}} = \infty \tag{2.13}$$

and

$$\left(\int_0^1 \frac{t^r}{(\lambda(t))^r}\frac{\mathrm{d}t}{t}\right)^{\frac{1}{r}} < \infty \tag{2.14}$$

(with the usual modification if $r = \infty$).

**Definition 2.14.** *Let $0 < r \le \infty$, $\mu \in \mathcal{L}_r$. The generalized Hölder space $\Lambda^{\mu(\cdot)}_{\infty,r}$ consists of all functions $f \in C_B$ for which the quasi-norm*

$$||f|\Lambda^{\mu(\cdot)}_{\infty,r}|| := \|f|L_\infty\| + \left(\int_0^1 \left(\frac{\omega(f,t)}{\mu(t)}\right)^r \frac{\mathrm{d}t}{t}\right)^{\frac{1}{r}} \tag{2.15}$$

*is finite (with the usual modification if $r = \infty$).*

One can replace the integral $\int_0^1$ in (2.15) by $\int_0^\varepsilon$ for $\varepsilon > 0$ and obtain an equivalent quasi-norm. Standard arguments show that the space $\Lambda^{\mu(\cdot)}_{\infty,r}$ is complete, cf. [Nev01b, Theorem 3.1.4]. Conditions (2.13) and (2.14) are natural. In fact, if (2.13) does not hold, then $\Lambda^{\mu(\cdot)}_{\infty,r}$ coincides with $C_B$. If (2.14) does not hold, then the space $\Lambda^{\mu(\cdot)}_{\infty,r}$ contains only constant functions. If $r = \infty$, we can assume without loss of generality in the definition of $\Lambda^{\mu(\cdot)}_{\infty,r}$ that all the elements $\mu$ of $\mathcal{L}_r$ are continuous increasing functions on the interval $(0, 1]$ such that $\lim_{t\to 0^+} \mu(t) = 0$ (cf. [GNO10]).

The space $\Lambda^{\mu(\cdot)}_{\infty,\infty}$, cf. [Nev01a, Proposition 3.5], coincides with the space $C^{0,\mu(\cdot)}$ defined by

$$\|f|C^{0,\mu(\cdot)}\| := \sup_{x\in\mathbb{R}^n} |f(x)| + \sup_{x,y\in\mathbb{R}^n,\, 0<|x-y|\le 1} \frac{|f(x) - f(y)|}{\mu(|x - y|)} < \infty.$$

If $\mu(t) = t$, $t \in (0, 1]$, then $\Lambda^{\mu(\cdot)}_{\infty,\infty}$ coincides with the space of the Lipschitz functions, which is denoted by $\mathrm{Lip}^1$. If $\mu(t) = t^\alpha$, $\alpha \in (0, 1]$, then the space $\Lambda^{\mu(\cdot)}_{\infty,r}$ coincides with the space $C^{0,\alpha,r}$ introduced in [AF03]. Furthermore, if $\mu(t) = t|\log t|^\beta$, $\beta > \frac{1}{r}$ (with $\beta \ge 0$ if $r = \infty$), the space $\Lambda^{\mu(\cdot)}_{\infty,r}$ coincides with the space $\mathrm{Lip}^{(1,-\beta)}_{\infty,r}$ of generalized Lipschitz functions presented and studied in [EH99], [EH00], and [Har00].

2.3. **Hardy inequalities.** In the sequel, discrete weighted Hardy inequalities will be indispensable for our proofs. There is a vast amount of literature concerning this topic. We merely rely on results as can be found in [Gol98, pp. 17-20], adapted to our situation. In this context we refer as well to [Ben91, Theorem 1.5] and [OK90].

Let $0 < q, r \le \infty$ and $(b_n)_{n\in\mathbb{N}_0}$, $(d_n)_{n\in\mathbb{N}_0}$ be non-negative sequences. Consider the inequalities

$$\left(\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j} a_k d_k\right)^r b_j^r\right)^{\frac{1}{r}} \lesssim \left(\sum_{n=0}^{\infty} a_n^q\right)^{\frac{1}{q}} \quad \text{for all non-negative sequences} \quad (a_n)_{n\in\mathbb{N}_0} \tag{2.16}$$

and

$$\left(\sum_{j=0}^{\infty}\left(\sum_{k=j}^{\infty} a_k d_k\right)^r b_j^r\right)^{\frac{1}{r}} \lesssim \left(\sum_{n=0}^{\infty} a_n^q\right)^{\frac{1}{q}} \quad \text{for all non-negative sequences} \quad (a_n)_{n\in\mathbb{N}_0} \tag{2.17}$$

(with the usual modification if $r = \infty$ or $q = \infty$).

**Theorem 2.15.** (i) *Let $0 < q \le r \le \infty$. Then,* (2.16) *is satisfied if, and only if,*

$$\sup_{N\ge 0}\left(\sum_{j=N}^{\infty} b_j^r\right)^{\frac{1}{r}}\left(\sum_{k=0}^{N} d_k^{q'}\right)^{\frac{1}{q'}} < \infty \tag{2.18}$$

*and, furthermore,* (2.17) *is satisfied if, and only if,*

$$\sup_{N\ge 0}\left(\sum_{j=0}^{N} b_j^r\right)^{\frac{1}{r}}\left(\sum_{k=N}^{\infty} d_k^{q'}\right)^{\frac{1}{q'}} < \infty \tag{2.19}$$

*(with the usual modification if $r = \infty$ or $q = \infty$ or $q' = \infty$).*

(ii) *Let $0 < r < q \le \infty$. Then,* (2.16) *is satisfied if, and only if,*

$$\left\{\sum_{N=0}^{\infty}\left(\sum_{j=N}^{\infty} b_j^r\right)^{\frac{u}{q}} b_N^r \left(\sum_{k=0}^{N} d_k^{q'}\right)^{\frac{u}{q'}}\right\}^{\frac{1}{u}} < \infty \tag{2.20}$$

*and, furthermore,* (2.17) *is satisfied if, and only if,*

$$\left\{ \sum_{N=0}^{\infty} \left( \sum_{j=0}^{N} b_j^r \right)^{\frac{u}{q}} b_N^r \left( \sum_{k=N}^{\infty} d_k^{q'} \right)^{\frac{u}{q'}} \right\}^{\frac{1}{u}} < \infty \tag{2.21}$$

(*with the usual modification if* $q' = \infty$), *where* $\frac{1}{u} := \frac{1}{r} - \frac{1}{q}$.

## 3. Main results

The following Proposition generalizes [MNS11, Proposition 3.1] and will play a key role for proving necessity in the main theorem below. For related assertions, but different, see [Tri01, pp. 220-221], and [CH05, Proposition 2.4].

**Proposition 3.1.** *Let* $0 < p, q \leq \infty$ *and let* $\boldsymbol{\sigma}$ *and* $\boldsymbol{N}$ *be admissible sequences, the latter satisfying* $\underline{N}_1 > 1$. *Consider*

$$h(y) := e^{-1/(1-y^2)} \quad \text{if} \quad |y| < 1 \quad \text{and} \quad h(y) := 0 \quad \text{if} \quad |y| \geq 1.$$

*Let* $L + 1 \in \mathbb{N}_0$. *Given* $L \in \mathbb{N}_0$ *and* $\delta \in (0, 1]$, *define*

$$h_{\delta,L}(y) := h(y) - \sum_{l=0}^{L} \rho_{\delta,l}\, h^{(l)}\big(\delta^{-1}(y - 1 - \delta)\big),$$

*where the coefficients* $\rho_{\delta,l}$ *are uniquely determined by imposing that* $h_{\delta,L}$ *shall obey the following set of conditions:*

$$\int_{\mathbb{R}} y^k\, h_{\delta,L}(y)\, \mathrm{d}y = 0, \quad k = 0, \cdots, L. \tag{3.1}$$

*Given* $L = -1$ *complement* (3.1) *by* $h_{\delta,-1} = h$ (*then* (3.1) *is empty*). *Let now*

$$\phi_{\delta,L}(x) := h_{\delta,L}(x_1) \prod_{m=2}^{n} h(x_m), \quad x = (x_j)_{j=1}^n \in \mathbb{R}^n. \tag{3.2}$$

*For a fixed* $L + 1 \in \mathbb{N}_0$ *with*

$$L > -1 + n \left( \frac{\log \overline{N}_1}{\log \underline{N}_1} \frac{1}{\min(1, p)} - 1 \right) - \frac{\log \underline{\sigma}_1}{\log \underline{N}_1},$$

*and* $\boldsymbol{b} = (b_j)_{j \in \mathbb{N}_0}$ *a sequence of non-negative numbers in* $\ell_q$, *let* $f_{\boldsymbol{b}}$ *be given by*

$$f^{\boldsymbol{b}}(x) := \sum_{j=0}^{\infty} b_j\, \sigma_j^{-1}\, N_j^{n/p}\, \phi_{\delta,L}(N_j x), \quad x \in \mathbb{R}^n. \tag{3.3}$$

(i) *Then* $f^{\boldsymbol{b}} \in B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}$ *and*

$$\| f^{\boldsymbol{b}} \,| B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q} \| \leq c_1\, \| \boldsymbol{b} \,| \ell_q \| \tag{3.4}$$

*for some* $c_1 > 0$ *independent of* $\boldsymbol{b}$.

(ii) *Moreover, it holds*

$$\frac{\omega(f^{\boldsymbol{b}}, N_k^{-1})}{N_k^{-1}} \geq c_2\, N_k \sum_{j=k}^{\infty} b_j \sigma_j^{-1} N_j^{n/p}, \quad k \in \mathbb{N}_0, \tag{3.5}$$

*and*

$$\frac{\omega(f^{\boldsymbol{b}}, N_k^{-1})}{N_k^{-1}} \geq c_3 \sum_{j=0}^{k} b_j \sigma_j^{-1} N_j^{n/p+1}, \quad k \in \mathbb{N}_0, \tag{3.6}$$

*for some $c_2, c_3 > 0$ depending only on the function $\phi_{\delta,L}$.*

*Proof.* We use the atomic decomposition theorem for $B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}$ as stated in [FL06, Theorem 4.4.3]. Since the functions

$$a_j(x) := \sigma_j^{-1}\, N_j^{n/p}\, \phi_{\delta,L}(N_j x), \quad x \in \mathbb{R}^n, \quad j \in \mathbb{N}_0,$$

are (up to constants, independently of $j$) $(\boldsymbol{\sigma}, p)_{M,L} - \boldsymbol{N}$-atoms, for some fixed $M \in \mathbb{N}$ with $M > \frac{\log \overline{\sigma}_1}{\log \underline{N}_1}$, the atomic decomposition theorem, cf. [FL06, Theorem 4.4.3]), yields that $f^{\boldsymbol{b}} \in B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}$ and

$$\|f^{\boldsymbol{b}} \mid B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}\| \le c\, \|\boldsymbol{b} \mid \ell_q\|, \tag{3.7}$$

for some positive constant $c$ (independent of $\boldsymbol{b}$).

Let us now prove (ii). Let $k \in \mathbb{N}_0$ and let $\eta \in (0,1)$ be fixed. Then, putting temporarily $c_1 = \prod_{i=2}^n h(0)$, we obtain

$$\begin{aligned}
f^{\boldsymbol{b}}(0,0,\cdots,0) - &f^{\boldsymbol{b}}(-\eta\, N_k^{-1}, 0, \cdots, 0) = \\
&= c_1 \sum_{j=0}^{\infty} b_j\, \sigma_j^{-1}\, N_j^{n/p}\, \big(h_{\delta,L}(0) - h_{\delta,L}(-\eta N_j N_k^{-1})\big) \\
&= c_1 \sum_{j=0}^{\infty} b_j\, \sigma_j^{-1}\, N_j^{n/p}\, \big(h(0) - h(-\eta N_j N_k^{-1})\big) \\
&\ge c_1 \sum_{j=k}^{\infty} b_j\, \sigma_j^{-1}\, N_j^{n/p}\, \big(h(0) - h(-\eta N_j N_k^{-1})\big) \\
&\ge c_1 \sum_{j=k}^{\infty} b_j\, \sigma_j^{-1}\, N_j^{n/p}\, (h(0) - h(-\eta)) \\
&\ge c_2 \sum_{j=k}^{\infty} b_j\, \sigma_j^{-1}\, N_j^{n/p},
\end{aligned}$$

where the third step above holds true, since $h(0) - h(-\eta N_j N_k^{-1}) \ge 0$ for all $j, k \in \mathbb{N}_0$. The two last inequalities above follow from the fact that $h(0) - h(-\eta N_j N_k^{-1}) \ge h(0) - h(-\eta) \ge c > 0$ for all $j \ge k$. Therefore,

$$\frac{\omega(f^{\boldsymbol{b}}, N_k^{-1})}{N_k^{-1}} \gtrsim N_k \sum_{j=k}^{\infty} b_j\, \sigma_j^{-1}\, N_j^{n/p}, \quad k \in \mathbb{N}_0.$$

This proves (3.5). The proof of (3.6) is similar. We estimate

$$\begin{aligned}
f^{\boldsymbol{b}}(0,0,\cdots,0) - &f^{\boldsymbol{b}}(-\eta\, N_k^{-1}, 0, \cdots, 0) = \\
&= c_1 \sum_{j=0}^{\infty} b_j\, \sigma_j^{-1}\, N_j^{n/p}\, \big(h(0) - h(-\eta N_j N_k^{-1})\big) \\
&= c_2 \sum_{j=0}^{\infty} b_j\, \sigma_j^{-1}\, N_j^{n/p+1}\, N_k^{-1}\, h'(z_{j,k}) \\
&\ge c_2 \sum_{j=0}^{k} b_j\, \sigma_j^{-1}\, N_j^{n/p+1}\, N_k^{-1}\, h'(z_{j,k}),
\end{aligned}$$

for some $z_{j,k} \in (-\eta N_j N_k^{-1}, 0)$. For $j \le k$, $z_{j,k} \in [-\eta, 0)$, and hence $h'(z_{j,k}) \ge c > 0$ for some $c$ which is independent of $j$ and $k$, leading to

$$f^{\boldsymbol{b}}(0,0,\cdots,0) - f^{\boldsymbol{b}}(-\eta\, N_k^{-1}, 0, \cdots, 0) \ge c \sum_{j=0}^{k} b_j\, \sigma_j^{-1}\, N_j^{n/p+1}\, N_k^{-1}.$$

Therefore,

$$\frac{\omega(f^{\boldsymbol{b}}, N_k^{-1})}{N_k^{-1}} \ge c \sum_{j=0}^{k} b_j\, \sigma_j^{-1}\, N_j^{n/p+1}, \quad k \in \mathbb{N}_0.$$

□

The following theorem characterizes optimal embeddings of Besov spaces with generalized smoothness $B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}$ into generalized Hölder spaces when $\underline{s}(\boldsymbol{N}\boldsymbol{\tau}^{-1}) > 0$ and $\boldsymbol{\tau}^{-1} \in \ell_{q'}$, where $\boldsymbol{\tau} = \boldsymbol{\sigma}\boldsymbol{N}^{-n/p}$. In particular, our results generalize those previously obtained in [MNS11, Theorem 3.2] and, if additionally $\underline{s}(\boldsymbol{\tau}) > 0$, we recover results from [HM08] regarding continuity envelopes, as we shall see in the next section.

In this context we also refer to [GNO07, Theorem 4] and [GNO10, Theorem 1.6, Corollary 1.7], where the authors obtained similar embedding results for Bessel-potential-type spaces in the limiting case. There, the technics were completely different from the ones considered here.

**Theorem 3.2.** *Let $0 < p \le \infty$, $0 < q, r \le \infty$, $\mu \in \mathcal{L}_r$. Furthermore, let $\boldsymbol{\sigma}$ and $\boldsymbol{N}$ be admissible sequences, the latter satisfying $\underline{N}_1 > 1$. Put $\boldsymbol{\tau} = \boldsymbol{\sigma}\boldsymbol{N}^{-n/p}$. Assume that*

$$\underline{s}(\boldsymbol{N}\boldsymbol{\tau}^{-1}) > 0 \tag{3.8}$$

*and*

$$\boldsymbol{\tau}^{-1} \in \ell_{q'}. \tag{3.9}$$

(i) *If $0 < q \le r \le \infty$, then*

$$B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q} \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}, \tag{3.10}$$

*if, and only if,*

$$\sup_{M \ge 0} \left( \sum_{j=0}^{M} \int_{N_{j+1}^{-1}}^{N_j^{-1}} (\mu(t))^{-r} \frac{\mathrm{d}t}{t} \right)^{\frac{1}{r}} \left( \sum_{k=M}^{\infty} \tau_k^{-q'} \right)^{\frac{1}{q'}} < \infty \tag{3.11}$$

(*with the usual modification if $r = \infty$ and/or $q' = \infty$*).

(ii) *If $0 < r < q \le \infty$, then*

$$B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q} \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}, \tag{3.12}$$

*if, and only if,*

$$\left\{ \sum_{M=0}^{\infty} \left( \sum_{j=0}^{M} \int_{N_{j+1}^{-1}}^{N_j^{-1}} (\mu(t))^{-r} \frac{\mathrm{d}t}{t} \right)^{\frac{u}{q}} \cdot \left( \int_{N_{M+1}^{-1}}^{N_M^{-1}} (\mu(t))^{-r} \frac{\mathrm{d}t}{t} \right) \cdot \left( \sum_{k=M}^{\infty} \tau_k^{-q'} \right)^{\frac{u}{q'}} \right\}^{\frac{1}{u}} < \infty \tag{3.13}$$

*and*

$$\left\{\sum_{M=0}^{\infty}\left(\sum_{j=M}^{\infty} N_j^{-r}\int_{N_{j+1}^{-1}}^{N_j^{-1}}(\mu(t))^{-r}\,\frac{\mathrm{d}t}{t}\right)^{\frac{u}{q}}\cdot N_M^{-r}\left(\int_{N_{M+1}^{-1}}^{N_M^{-1}}(\mu(t))^{-r}\,\frac{\mathrm{d}t}{t}\right)\cdot\left(\sum_{k=0}^{M}\left(N_k^{-1}\tau_k\right)^{-q'}\right)^{\frac{u}{q'}}\right\}^{\frac{1}{u}}<\infty \quad (3.14)$$

(*with the usual modification if* $q'=\infty$), *where* $\frac{1}{u}:=\frac{1}{r}-\frac{1}{q}$.

*Proof.* In the sequel we shall always assume that $q$ and $r$ are finite, since the limiting situations ($q=\infty$ and/or $r=\infty$) are proven in the same way with the obvious modifications.

Step 1: In order to prove sufficiency in (i), assume that (3.11) holds. Let $f\in B^{\boldsymbol{\tau},\boldsymbol{N}}_{\infty,q}$. Then, by Proposition 2.13(i), $B^{\boldsymbol{\tau},\boldsymbol{N}}_{\infty,q}\subset C_B$ and we can thus make use of the following estimate which can be found in [Mou07, formulas (4.4), (4.5), pp. 1196, 1197], stating that for some fixed $a>0$ and for $|h|\le N_j^{-1}$,

$$\|\Delta_h f|L_\infty\|\le c\sum_{k=0}^{j}N_kN_j^{-1}\|(\varphi_k^{\boldsymbol{N}*}f)_a|L_\infty\|+\sum_{k=j+1}^{\infty}\|(\varphi_k^{\boldsymbol{N}*}f)_a|L_\infty\| \quad (3.15)$$

(the constant involved is independent of $f$). Using the fact that $\omega(f,\cdot)$ is monotonically increasing, together with (3.15), leads to

$$\begin{aligned}\left(\int_0^{N_0^{-1}}\left(\frac{\omega(f,t)}{\mu(t)}\right)^r\frac{\mathrm{d}t}{t}\right)^{\frac{1}{r}}&=\left(\sum_{j=0}^{\infty}\int_{N_{j+1}^{-1}}^{N_j^{-1}}(\omega(f,t))^r(\mu(t))^{-r}\,\frac{\mathrm{d}t}{t}\right)^{\frac{1}{r}}\\&\lesssim\left(\sum_{j=0}^{\infty}(\omega(f,N_j^{-1}))^r\underbrace{\int_{N_{j+1}^{-1}}^{N_j^{-1}}(\mu(t))^{-r}\,\frac{\mathrm{d}t}{t}}_{=:m_j}\right)^{\frac{1}{r}}\\&\lesssim\left(\sum_{j=0}^{\infty}m_j\left(\sum_{k=0}^{j}N_kN_j^{-1}\|(\varphi_k^{\boldsymbol{N}*}f)_a|L_\infty\|+\sum_{k=j+1}^{\infty}\|(\varphi_k^{\boldsymbol{N}*}f)_a|L_\infty\|\right)^r\right)^{\frac{1}{r}}\\&\lesssim\underbrace{\left(\sum_{j=0}^{\infty}N_j^{-r}m_j\left(\sum_{k=0}^{j}N_k\|(\varphi_k^{\boldsymbol{N}*}f)_a|L_\infty\|\right)^r\right)^{\frac{1}{r}}}_{=(I)}+\underbrace{\left(\sum_{j=0}^{\infty}m_j\left(\sum_{k=j}^{\infty}\|(\varphi_k^{\boldsymbol{N}*}f)_a|L_\infty\|\right)^r\right)^{\frac{1}{r}}}_{=(II)}.\end{aligned} \quad (3.16)$$

Setting

$$b_j:=N_j^{-1}m_j^{\frac{1}{r}},\qquad a_k:=\tau_k\|(\varphi_k^{\boldsymbol{N}*}f)_a|L_\infty\|,\qquad\text{and}\qquad d_k:=N_k\tau_k^{-1}, \quad (3.17)$$

an application of Theorem 2.15(i), to the first term of (3.16), and the equivalent characterization of our function spaces via maximal function from [FL06, Theorem 4.3.4] yield

$$(I)\lesssim\left(\sum_{l=0}^{\infty}\tau_l^q\|(\varphi_l^{\boldsymbol{N}*}f)_a|L_\infty\|^q\right)^{\frac{1}{q}}\sim\|f|B^{\boldsymbol{\tau},\boldsymbol{N}}_{\infty,q}\|\quad\text{for all}\quad f\in B^{\boldsymbol{\tau},\boldsymbol{N}}_{\infty,q}. \quad (3.18)$$

The first inequality above can be justified as follows. Condition (3.11) gives

$$\int_{N_{M+1}^{-1}}^{N_M^{-1}} (\mu(t))^{-r} \frac{\mathrm{d}t}{t} \lesssim \tau_M^r \qquad \text{for all } M. \tag{3.19}$$

Since $\underline{s}(\boldsymbol{N}\boldsymbol{\tau}^{-1}) > 0$, Lemma 2.4 (i) and Lemma 2.4 (ii), yield

$$\begin{aligned} &\sup_{M\geq 0} \left(\sum_{j=M}^{\infty} N_j^{-r} m_j\right)^{\frac{1}{r}} \left(\sum_{k=0}^{M} N_k^{q'} \tau_k^{-q'}\right)^{\frac{1}{q'}} \\ &\sim \sup_{M\geq 0} \left(\sum_{j=M}^{\infty} N_j^{-r} m_j\right)^{\frac{1}{r}} N_M \tau_M^{-1} \\ &\lesssim \sup_{M\geq 0} \left(\sum_{j=M}^{\infty} N_j^{-r} \tau_j^{r}\right)^{\frac{1}{r}} N_M \tau_M^{-1} \\ &\lesssim N_M^{-1} \tau_M N_M \tau_M^{-1} \lesssim 1 < \infty \end{aligned} \tag{3.20}$$

and thus (2.18) is satisfied. For the second term of (3.16), we put

$$b_j := m_j^{\frac{1}{r}}, \qquad a_k := \tau_k \|(\varphi_k^{\boldsymbol{N}*} f)_a | L_\infty\|, \qquad \text{and} \qquad d_k := \tau_k^{-1}. \tag{3.21}$$

An application of Theorem 2.15(i) and the equivalent characterization of our function spaces via maximal functions from [FL06, Theorem 4.3.4] give

$$(II) \lesssim \left(\sum_{l=0}^{\infty} \tau_l^q \|(\varphi_l^{\boldsymbol{N}*} f)_a | L_\infty\|^q\right)^{\frac{1}{q}} \sim \|f | B^{\boldsymbol{\tau},\boldsymbol{N}}_{\infty,q}\| \quad \text{for all} \quad f \in B^{\boldsymbol{\tau},\boldsymbol{N}}_{\infty,q}, \tag{3.22}$$

since, by (3.11),

$$\begin{aligned} &\sup_{M\geq 0} \left(\sum_{j=0}^{M} m_j\right)^{\frac{1}{r}} \left(\sum_{k=M}^{\infty} \tau_k^{-q'}\right)^{\frac{1}{q'}} \\ &= \sup_{M\geq 0} \left(\sum_{j=0}^{M} \int_{N_{j+1}^{-1}}^{N_j^{-1}} (\mu(t))^{-r} \frac{\mathrm{d}t}{t}\right)^{\frac{1}{r}} \left(\sum_{k=M}^{\infty} \tau_k^{-q'}\right)^{\frac{1}{q'}} < \infty \end{aligned} \tag{3.23}$$

and (2.19) is satisfied. Now, (3.16), together with (3.18), (3.22) and Proposition 2.13(i), yields

$$B^{\boldsymbol{\tau},\boldsymbol{N}}_{\infty,q} \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}.$$

Since, by Proposition 2.11,

$$B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q} \hookrightarrow B^{\boldsymbol{\tau},\boldsymbol{N}}_{\infty,q},$$

we have the desired embedding

$$B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q} \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}.$$

Step 2: Concerning sufficiency in (ii) again we have (3.16). Let $\frac{1}{u} := \frac{1}{r} - \frac{1}{q}$. Applying (2.20), using (3.17) we obtain for the first integral (I) the estimate (3.18), since

$$\left\{ \sum_{M=0}^{\infty} \left( \sum_{j=M}^{\infty} N_j^{-r} \int_{N_{j+1}^{-1}}^{N_j^{-1}} (\mu(t))^{-r} \frac{\mathrm{d}t}{t} \right)^{\frac{u}{q}} N_M^{-r} \left( \int_{N_{M+1}^{-1}}^{N_M^{-1}} (\mu(t))^{-r} \frac{\mathrm{d}t}{t} \right) \left( \sum_{k=0}^{M} N_k^{q'} \tau_k^{-q'} \right)^{\frac{u}{q'}} \right\}^{\frac{1}{u}}$$

is bounded by (3.14). For the second integral (II) in (3.16), an application of (2.21) yields (3.22), since inserting (3.21) we obtain

$$\left\{ \sum_{M=0}^{\infty} \left( \sum_{j=0}^{M} \int_{N_{j+1}^{-1}}^{N_j^{-1}} (\mu(t))^{-r} \frac{\mathrm{d}t}{t} \right)^{\frac{u}{q}} \left( \int_{N_M^{-1}}^{N_M} (\mu(t))^{-r} \frac{\mathrm{d}t}{t} \right) \left( \sum_{k=M}^{\infty} \tau_k^{-q'} \right)^{\frac{u}{q'}} \right\}^{\frac{1}{u}},$$

which is bounded by (3.13).

Step 3: Concerning necessity in (i) and (ii), assume we have the embedding

$$B_{p,q}^{\boldsymbol{\sigma},\boldsymbol{N}} \hookrightarrow \Lambda_{\infty,r}^{\mu(\cdot)}, \qquad 0 < q, r \le \infty,$$

which means that

$$\left( \int_0^{N_0^{-1}} \left( \frac{\omega(f,t)}{\mu(t)} \right)^r \frac{\mathrm{d}t}{t} \right)^{\frac{1}{r}} \lesssim \| f | B_{p,q}^{\boldsymbol{\sigma},\boldsymbol{N}} \| \qquad \text{for all } f \in B_{p,q}^{\boldsymbol{\sigma},\boldsymbol{N}}.$$

In particular, for each non-negative sequence $(a_n)_{n\in\mathbb{N}_0}$, using the function $f^a$ constructed in Proposition 3.1, we have

$$\begin{aligned}
\|a|\ell_q\| &\gtrsim \left( \int_0^{N_0^{-1}} \left( \frac{\omega(f^a,t)}{\mu(t)} \right)^r \frac{\mathrm{d}t}{t} \right)^{\frac{1}{r}} \\
&= \left( \sum_{k=0}^{\infty} \int_{N_{k+1}^{-1}}^{N_k^{-1}} \left( \frac{\omega(f^a,t)}{t} \right)^r \frac{t^{r-1}}{(\mu(t))^r} \,\mathrm{d}t \right)^{\frac{1}{r}} \\
&\gtrsim \left( \sum_{k=0}^{\infty} \left( \frac{\omega(f^a,N_k^{-1})}{N_k^{-1}} \right)^r \int_{N_{k+1}^{-1}}^{N_k^{-1}} (\mu(t))^{-r} t^{r-1} \,\mathrm{d}t \right)^{\frac{1}{r}} \qquad (3.24) \\
&\gtrsim \left( \sum_{k=0}^{\infty} \left( N_k \sum_{j=k}^{\infty} a_j \tau_j^{-1} \right)^r \int_{N_{k+1}^{-1}}^{N_k^{-1}} (\mu(t))^{-r} t^{r-1} \,\mathrm{d}t \right)^{\frac{1}{r}} \\
&\sim \left( \sum_{k=0}^{\infty} \left( \sum_{j=k}^{\infty} a_j \tau_j^{-1} \right)^r \int_{N_{k+1}^{-1}}^{N_k^{-1}} (\mu(t))^{-r} \frac{\mathrm{d}t}{t} \right)^{\frac{1}{r}}, \qquad (3.25)
\end{aligned}$$

where we used the fact that $\frac{\omega(f,t)}{t}$ is equivalent to a monotonically decreasing function and the estimate (3.5). Putting

$$d_j = \tau_j^{-1} \qquad \text{and} \qquad b_k = \left( \int_{N_{k+1}^{-1}}^{N_k^{-1}} (\mu(t))^{-r} \frac{\mathrm{d}t}{t} \right)^{\frac{1}{r}}, \qquad (3.26)$$

from (3.25) we obtain

$$\|a|\ell_q\| \gtrsim \left(\sum_{k=0}^{\infty}\left(\sum_{j=k}^{\infty} a_j d_j\right)^r b_k^r\right)^{\frac{1}{r}} \quad \text{for all non-negative sequences} \quad (a_n)_{n\in\mathbb{N}_0}, \tag{3.27}$$

which is the Hardy-type inequality (2.17). Now the necessary conditions (3.11) and (3.13) follow from Theorem 2.15. If we apply the estimate (3.6) instead of (3.5) in (3.24), we obtain

$$\|a|\ell_q\| \gtrsim \left(\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} a_j \tau_j^{-1} N_j\right)^r N_k^{-r}\int_{N_{k+1}^{-1}}^{N_k^{-1}} (\mu(t))^{-r}\,\frac{\mathrm{d}t}{t}\right)^{\frac{1}{r}} \tag{3.28}$$

for all non-negative sequences $(a_n)_{n\in\mathbb{N}_0}$. Now setting

$$d_j = \tau_j^{-1} N_j \qquad \text{and} \qquad b_k = N_k^{-1}\left(\int_{N_{k+1}^{-1}}^{N_k^{-1}} (\mu(t))^{-r}\,\frac{\mathrm{d}t}{t}\right)^{\frac{1}{r}}, \tag{3.29}$$

we obtain

$$\|a|\ell_q\| \gtrsim \left(\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} a_j d_j\right)^r b_k^r\right)^{\frac{1}{r}} \quad \text{for all non-negative sequences} \quad (a_n)_{n\in\mathbb{N}_0}, \tag{3.30}$$

which is the Hardy-type inequality (2.16). Theorem 2.15 now yields (3.14). This finally completes the proof. □

**Remark 3.3.** (i) Note that if $\overline{s}(\boldsymbol{\tau}) < 0$, then $\boldsymbol{\tau}^{-1} \notin \ell_u$ for any $u \in (0,\infty]$ (cf. Remark 2.1). Therefore, only the case $\overline{s}(\boldsymbol{\tau}) \geq 0$ is allowed in the previous theorem. Moreover, by (2.5), if $\overline{s}(\boldsymbol{\tau}) < \underline{s}(\boldsymbol{N})$, then (3.8) is satisfied.

(ii) By (3.8) and (3.9), we always have

$$B_{p,q}^{\boldsymbol{\sigma},\boldsymbol{N}} \not\hookrightarrow \mathrm{Lip}^1. \tag{3.31}$$

In fact, by the previous theorem,

$$B_{p,q}^{\boldsymbol{\sigma},\boldsymbol{N}} \hookrightarrow \mathrm{Lip}^1$$

if, and only if,

$$\sup_{M\geq 0} N_M \left(\sum_{k=M}^{\infty} \tau_k^{-q'}\right)^{\frac{1}{q'}} < \infty. \tag{3.32}$$

But (3.8) implies that $\boldsymbol{N}^{-1}\boldsymbol{\tau} \in \ell_u$ for any $u \in (0,\infty]$ (cf. Remark 2.1). Hence, $N_M \tau_M^{-1} \to \infty$ as $M \to \infty$. Therefore, (3.32) is not satisfied.

(iii) Regarding Examples 2.2 (ii), (iv), it is possible to consider examples that verify the conditions of the previous theorem.

(iv) Let $\Lambda$ be any admissible function such that $\Lambda(z) \sim \sigma_j,\ z \in [N_j, N_{j+1}],\ j \in \mathbb{N}_0$, with equivalence constants independent of $j$. Note that, by Proposition 2.7, condition (3.11) is equivalent to the following integral version,

$$\sup_{\varkappa\in(0,N_0^{-1})} \left(\int_{\varkappa}^{N_0^{-1}} (\mu(t))^{-r}\,\frac{\mathrm{d}t}{t}\right)^{\frac{1}{r}} \left(\int_0^{\varkappa} \left(\Lambda(s^{-1})s^{n/p}\right)^{-q'}\frac{\mathrm{d}s}{s}\right)^{\frac{1}{q'}} < \infty \tag{3.33}$$

(with the usual modification if $r = \infty$ and/or $q' = \infty$).

(v) When $0 < q \le 1$, on exchanging suprema, we can rewrite (3.11) as

$$\sup_{k\ge 0}\left(\sum_{j=0}^{k}\int_{N_{j+1}^{-1}}^{N_j^{-1}}(\mu(t))^{-r}\,\frac{\mathrm{d}t}{t}\right)^{\frac{1}{r}}\tau_k^{-1}<\infty \tag{3.34}$$

and we can rewrite (3.33) as

$$\sup_{s\in(0,N_0^{-1})}\left(\int_s^{N_0^{-1}}(\mu(t))^{-r}\,\frac{\mathrm{d}t}{t}\right)^{\frac{1}{r}}\left(\Lambda(s^{-1})s^{n/p}\right)^{-1}<\infty. \tag{3.35}$$

In terms of optimal weights we have the following result when $1 < q \le \infty$. For the case $0 < q \le 1$, see Remark 3.5 below.

**Corollary 3.4.** *Let $1 < q \le \infty$, $0 < p, r \le \infty$, $\mu \in \mathcal{L}_r$. Let $\boldsymbol{\sigma}$ and $\boldsymbol{N}$ be admissible sequences, the latter satisfying $\underline{N}_1 > 1$. Put $\boldsymbol{\tau} = \boldsymbol{\sigma}\boldsymbol{N}^{-n/p}$ and assume that*

$$\underline{s}(\boldsymbol{N}\boldsymbol{\tau}^{-1}) > 0 \tag{3.36}$$

*and*

$$\boldsymbol{\tau}^{-1} \in \ell_{q'}.$$

*Furthermore, let $\lambda_{qr} \in \mathcal{L}_r$ be defined by*

$$\lambda_{qr}(t) := \left(\Lambda(t^{-1})t^{n/p}\right)^{\frac{q'}{r}}\left(\int_0^t\left(\Lambda(s^{-1})s^{n/p}\right)^{-q'}\frac{\mathrm{d}s}{s}\right)^{\frac{1}{q'}+\frac{1}{r}}, \qquad t\in(0,N_0^{-1}], \tag{3.37}$$

*where $\Lambda$ is an admissible function such that $\Lambda(z) \sim \Lambda(N_j) = \sigma_j$ for $z \in [N_j, N_{j+1}]$, $j \in \mathbb{N}_0$. We consider the embedding*

$$B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q} \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}. \tag{3.38}$$

(i) *If $1 < q \le r \le \infty$, then (3.38) holds if, and only if,*

$$\sup_{M\ge 0}\frac{\left(\sum_{j=0}^{M}\int_{N_{j+1}^{-1}}^{N_j^{-1}}(\mu(t))^{-r}\,\frac{\mathrm{d}t}{t}\right)^{\frac{1}{r}}}{\left(\sum_{j=0}^{M}\int_{N_{j+1}^{-1}}^{N_j^{-1}}(\lambda_{qr}(t))^{-r}\,\frac{\mathrm{d}t}{t}\right)^{\frac{1}{r}}}<\infty \tag{3.39}$$

*(with the usual modification if $r = \infty$).*

(ii) *If $0 < r < q \le \infty$ and $q > 1$, then (3.38) holds if, and only if,*

$$\left\{\sum_{M=0}^{\infty}\frac{\left(\sum_{j=0}^{M}\int_{N_{j+1}^{-1}}^{N_j^{-1}}(\mu(t))^{-r}\,\frac{\mathrm{d}t}{t}\right)^{u/q}}{\left(\sum_{j=0}^{M}\int_{N_{j+1}^{-1}}^{N_j^{-1}}(\lambda_{qr}(t))^{-r}\,\frac{\mathrm{d}t}{t}\right)^{u/r}}\int_{N_{M+1}^{-1}}^{N_M^{-1}}(\mu(s))^{-r}\,\frac{\mathrm{d}s}{s}\right\}^{\frac{1}{u}}<\infty \tag{3.40}$$

*(with the usual modification if $q = \infty$), where $\frac{1}{u} := \frac{1}{r} - \frac{1}{q}$.*

(iii) *Let $r \in [q, \infty]$. Among the embeddings in (3.38), that one with $\mu = \lambda_{qr}$ is sharp with respect to the parameter $\mu$.*

(iv) *Among the embeddings in (3.38), that one with $\mu = \lambda_{qq}$ and $r = q$, i.e.,*

$$B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q} \hookrightarrow \Lambda^{\lambda_{qq}(\cdot)}_{\infty,q}, \tag{3.41}$$

*is optimal.*

*Proof.* Concerning (i), Theorem 3.2 shows that (3.38) holds if, and only if,

$$\sup_{M\geq 0}\left(\sum_{j=0}^{M}\int_{N_{j+1}^{-1}}^{N_j^{-1}}(\mu(t))^{-r}\,\frac{\mathrm{d}t}{t}\right)^{\frac{1}{r}}\left(\sum_{k=M}^{\infty}\tau_k^{-q'}\right)^{\frac{1}{q'}}<\infty,$$

which is equivalent to

$$\sup_{\varkappa\in(0,N_0^{-1})}\left(\int_{\varkappa}^{N_0^{-1}}(\mu(t))^{-r}\,\frac{\mathrm{d}t}{t}\right)^{\frac{1}{r}}\left(\int_0^{\varkappa}\left(\Lambda(s^{-1})s^{n/p}\right)^{-q'}\frac{\mathrm{d}s}{s}\right)^{\frac{1}{q'}}<\infty. \tag{3.42}$$

Since

$$\left(\int_{\varkappa}^{N_0^{-1}}(\lambda_{qr}(t))^{-r}\,\frac{\mathrm{d}t}{t}\right)^{-\frac{1}{r}}=\left(\int_{\varkappa}^{N_0^{-1}}\left(\Lambda(t^{-1})t^{n/p}\right)^{-q'}\left(\int_0^t\left(\Lambda(s^{-1})s^{n/p}\right)^{-q'}\frac{\mathrm{d}s}{s}\right)^{-\frac{r}{q'}-1}\frac{\mathrm{d}t}{t}\right)^{-\frac{1}{r}}$$

$$\sim\left(\int_0^{\varkappa}\left(\Lambda(t^{-1})t^{n/p}\right)^{-q'}\frac{\mathrm{d}t}{t}\right)^{\frac{1}{q'}}\qquad\text{for all }\ \varkappa\in\left(0,N_1^{-1}\right], \tag{3.43}$$

and as singularities of functions in question are only at 0, this means that (3.42) is equivalent to (3.39).

Turning towards (ii) the same argument used above shows that (3.13) is equivalent to (3.40). Now, necessity follows from Theorem 3.2(ii). Let $\frac{1}{u}:=\frac{1}{r}-\frac{1}{q}$. As for sufficiency, we observe that

$$A_1:=\left\{\int_0^{N_1^{-1}}\left(\int_{\varkappa}^{N_0^{-1}}\frac{(\mu(t))^{-r}}{t}\,\mathrm{d}t\right)^{\frac{u}{q}}\cdot\frac{(\mu(\varkappa))^{-r}}{\varkappa}\cdot\left(\int_{\varkappa}^{N_0^{-1}}\frac{(\lambda_{qq}(t))^{-q}}{t}\,\mathrm{d}t\right)^{-\frac{u}{q}}\mathrm{d}\varkappa\right\}^{\frac{1}{u}}$$

$$\lesssim\left\{\int_0^{N_0^{-1}}\left(\int_{\varkappa}^{N_0^{-1}}\frac{(\mu(t))^{-r}}{t}\,\mathrm{d}t\right)^{\frac{u}{q}}\cdot\frac{(\mu(\varkappa))^{-r}}{\varkappa}\cdot\left(\int_0^{\varkappa}\left(\Lambda(t^{-1})t^{n/p}\right)^{-q'}\frac{\mathrm{d}t}{t}\right)^{\frac{u}{q'}}\mathrm{d}\varkappa\right\}^{\frac{1}{u}}$$

$$\lesssim\left\{\sum_{M=0}^{\infty}\left(\int_{N_{M+1}^{-1}}^{N_0^{-1}}(\mu(t))^{-r}\,\frac{\mathrm{d}t}{t}\right)^{\frac{u}{q}}\left(\int_0^{N_M^{-1}}\left(\Lambda(t^{-1})t^{n/p}\right)^{-q'}\frac{\mathrm{d}t}{t}\right)^{\frac{u}{q'}}\int_{N_{M+1}^{-1}}^{N_M^{-1}}(\mu(\varkappa))^{-r}\,\frac{\mathrm{d}\varkappa}{\varkappa}\right\}^{\frac{1}{u}} \tag{3.44}$$

is bounded by (3.13). But now, since $\omega(f,\cdot)$ is increasing, [HS93, Proposition 2.1(ii)] implies

$$\Lambda_{\infty,q}^{\lambda_{qq}(\cdot)}\hookrightarrow\Lambda_{\infty,r}^{\mu(\cdot)}.$$

This and (3.38) (with $\mu=\lambda_{qq}$ and $r=q$, which follows from part (i)), yield

$$B_{p,q}^{\boldsymbol{\sigma},\boldsymbol{N}}\hookrightarrow\Lambda_{\infty,q}^{\lambda_{qq}(\cdot)}\hookrightarrow\Lambda_{\infty,r}^{\mu(\cdot)},\qquad 0<r<q\leq\infty,\ \ q>1. \tag{3.45}$$

This completes the proof of (ii).

Let us now prove (iii). We need to show that the target space $\Lambda_{\infty,r}^{\mu(\cdot)}$ in (3.38) and the space $\Lambda_{\infty,r}^{\lambda_{qr}(\cdot)}$ (that is, the target space in (3.38) with $\mu=\lambda_{qr}$) satisfy

$$\Lambda_{\infty,r}^{\lambda_{qr}(\cdot)}\hookrightarrow\Lambda_{\infty,r}^{\mu(\cdot)}. \tag{3.46}$$

Indeed, since $\omega(f,\cdot)$ is increasing, this last embedding holds if

$$\sup_{\varkappa\in(0,N_1^{-1})}\frac{\left(\int_{\varkappa}^{N_0^{-1}}(\mu(t))^{-r}\,\frac{\mathrm{d}t}{t}\right)^{\frac{1}{r}}}{\left(\int_{\varkappa}^{N_0^{-1}}(\lambda_{qr}(t))^{-r}\,\frac{\mathrm{d}t}{t}\right)^{\frac{1}{r}}}<\infty \tag{3.47}$$

(cf. [HS93, Proposition 2.1(i)], see also [GNO10, Theorem 3.6(i)]), which is equivalent to (3.39) and completes the proof of (iii).

Turning our attention towards (iv), we need to show that the target space $\Lambda^{\mu(\cdot)}_{\infty,r}$ in (3.38) and the space $\Lambda^{\lambda_{qq}(\cdot)}_{\infty,q}$ (that is, the target space in (3.38) with $\mu = \lambda_{qq}$ and $r = q$) satisfy

$$\Lambda^{\lambda_{qq}(\cdot)}_{\infty,q} \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}. \tag{3.48}$$

Since $\omega(f,\cdot)$ is increasing, this last embedding holds, for $q \le r$, if

$$\sup_{\varkappa\in(0,N_1^{-1})} \frac{\left(\int_\varkappa^{N_0^{-1}} (\mu(t))^{-r}\, \frac{\mathrm{d}t}{t}\right)^{\frac{1}{r}}}{\left(\int_\varkappa^{N_0^{-1}} (\lambda_{qq}(t))^{-q}\, \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}} \sim \sup_{\varkappa\in(0,N_1^{-1})} \frac{\left(\int_\varkappa^{N_0^{-1}} (\mu(t))^{-r}\, \frac{\mathrm{d}t}{t}\right)^{\frac{1}{r}}}{\left(\int_\varkappa^{N_0^{-1}} (\lambda_{qr}(t))^{-r}\, \frac{\mathrm{d}t}{t}\right)^{\frac{1}{r}}} < \infty \tag{3.49}$$

(cf. [HS93, Proposition 2.1(i)], see also [GNO10, Theorem 3.6(i)]), which is equivalent to (3.39). In the case $r < q$ we obtained (3.45) when proving (ii), which gives the desired embedding. □

**Remark 3.5.** Let $0 < q \le 1$, $0 < p, r \le \infty$, $\mu \in \mathcal{L}_r$. Let $\boldsymbol{\sigma}$ and $\boldsymbol{N}$ be admissible sequences, the latter satisfying $\underline{N}_1 > 1$. Put $\boldsymbol{\tau} = \boldsymbol{\sigma}\boldsymbol{N}^{-n/p}$ and assume that $\underline{s}(\boldsymbol{N}\boldsymbol{\tau}^{-1}) > 0$ and $\tau_j^{-1} \to 0$ as $j \to \infty$.

Let $\Lambda$ be an admissible function such that $\Lambda(z) \sim \Lambda(N_j) = \sigma_j$ for $z \in [N_j, N_{j+1}]$, $j \in \mathbb{N}_0$. Let $H(t) := \inf_{s\in(0,t)} \Lambda(s^{-1})s^{n/p}$, $t \in (0, N_0^{-1})$, and suppose that $H$ is differentiable with $H'(t) \ne 0$, $t \in (0, N_0^{-1})$. Furthermore, let $\lambda_{qr} \in \mathcal{L}_r$ be defined by

$$\lambda_{qr}(t) := \lambda_r(t) := t^{-\frac{1}{r}} \left(\inf_{s\in(0,t)} \Lambda(s^{-1})s^{n/p}\right)^{\frac{1}{r}-1} \left(-\frac{d}{dt} \inf_{s\in(0,t)} \Lambda(s^{-1})s^{n/p}\right)^{-\frac{1}{r}}, \qquad t \in (0, N_0^{-1}]. \tag{3.50}$$

Then, (3.43) holds, with the usual modification since $q' = \infty$. Now, proceeding as in the proof of Corollary 3.4, conditions (3.39), (3.40) remain the same and the sharp embeddings with respect to parameter $\mu$, when $0 < q \le 1$, are obtained by taking $r \ge q$ and $\mu := \lambda_{qr}(t) = \lambda_r$. Moreover, we obtain the optimal embedding by putting $r = q$ and $\mu := \lambda_{qq} = \lambda_q$.

In terms of the Triebel-Lizorkin spaces our results read as follows.

**Theorem 3.6.** *Let $0 < p < \infty$, $0 < q, r \le \infty$, $\mu \in \mathcal{L}_r$. Furthermore, let $\boldsymbol{\sigma}$ and $\boldsymbol{N}$ be admissible sequences, the latter satisfying $\underline{N}_1 > 1$. Put $\boldsymbol{\tau} = \boldsymbol{\sigma}\boldsymbol{N}^{-n/p}$ and assume that*

$$\underline{s}(\boldsymbol{N}\boldsymbol{\tau}^{-1}) > 0 \tag{3.51}$$

*and*

$$\boldsymbol{\tau}^{-1} \in \ell_{p'}. \tag{3.52}$$

(i) *If $0 < p \le r < \infty$ and $p < r$ if $r = \infty$, then*

$$F^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q} \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}, \tag{3.53}$$

*if, and only if,*

$$\sup_{M\ge 0} \left(\sum_{j=0}^{M} \int_{N_{j+1}^{-1}}^{N_j^{-1}} (\mu(t))^{-r}\, \frac{\mathrm{d}t}{t}\right)^{\frac{1}{r}} \left(\sum_{k=M}^{\infty} \tau_k^{-p'}\right)^{\frac{1}{p'}} < \infty. \tag{3.54}$$

*(with the usual modification if $r = \infty$ and/or $p' = \infty$).*

(ii) *If $0 < r < p \le \infty$, then*

$$F^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q} \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}, \tag{3.55}$$

*if, and only if,*

$$\left\{ \sum_{M=0}^{\infty} \left( \sum_{j=0}^{M} \int_{N_{j+1}^{-1}}^{N_j^{-1}} (\mu(t))^{-r} \frac{\mathrm{d}t}{t} \right)^{\frac{u}{p}} \cdot \left( \int_{N_{M+1}^{-1}}^{N_M^{-1}} (\mu(t))^{-r} \frac{\mathrm{d}t}{t} \right) \cdot \left( \sum_{k=M}^{\infty} \tau_k^{-p'} \right)^{\frac{u}{p'}} \right\}^{\frac{1}{u}} < \infty \tag{3.56}$$

*and*

$$\left\{ \sum_{M=0}^{\infty} \left( \sum_{j=M}^{\infty} N_j^{-r} \int_{N_{j+1}^{-1}}^{N_j^{-1}} (\mu(t))^{-r} \frac{\mathrm{d}t}{t} \right)^{\frac{u}{p}} \cdot N_M^{-r} \left( \int_{N_{M+1}^{-1}}^{N_M^{-1}} (\mu(t))^{-r} \frac{\mathrm{d}t}{t} \right) \cdot \left( \sum_{k=0}^{M} \left( N_k^{-1} \tau_k \right)^{-p'} \right)^{\frac{u}{p'}} \right\}^{\frac{1}{u}} < \infty \tag{3.57}$$

(*with the usual modification if* $p' = \infty$), *where* $\frac{1}{u} := \frac{1}{r} - \frac{1}{p}$.

*Proof.* The proof is a consequence of Proposition 2.12 and Theorem 3.2. □

**Remark 3.7.** (i) In particular, it turns out that for the $F$-spaces our results are independent of the parameter $q$.

(ii) By (3.51) and (3.52), we always have

$$F_{p,q}^{\boldsymbol{\sigma},\boldsymbol{N}} \not\hookrightarrow \mathrm{Lip}^1. \tag{3.58}$$

(iii) Regarding condition (3.54), similar observations as the ones in Remark 3.3 (iv)-(v) can be made with $q$ replaced by $p$.

The counterpart of Corollary 3.4 in terms of the Triebel-Lizorkin spaces is as follows.

**Corollary 3.8.** *Let* $1 < p \le \infty$, $0 < q, r \le \infty$, $\mu \in \mathcal{L}_r$. *Let* $\boldsymbol{\sigma}$ *and* $\boldsymbol{N}$ *be admissible sequences, the latter satisfying* $\underline{N}_1 > 1$. *Put* $\boldsymbol{\tau} = \boldsymbol{\sigma}\boldsymbol{N}^{-n/p}$ *and assume that*

$$\underline{s}(\boldsymbol{N}\boldsymbol{\tau}^{-1}) > 0 \tag{3.59}$$

*and*

$$\boldsymbol{\tau}^{-1} \in \ell_{p'}.$$

*Furthermore, let* $\lambda_{pr} \in \mathcal{L}_r$ *be defined by*

$$\lambda_{pr}(t) := \left( \Lambda(t^{-1}) t^{n/p} \right)^{\frac{p'}{r}} \left( \int_0^t \left( \Lambda(s^{-1}) s^{n/p} \right)^{-p'} \frac{\mathrm{d}s}{s} \right)^{\frac{1}{p'} + \frac{1}{r}}, \qquad t \in (0, N_0^{-1}], \tag{3.60}$$

*where* $\Lambda$ *is an admissible function such that* $\Lambda(z) \sim \Lambda(N_j) = \sigma_j$ *for* $z \in [N_j, N_{j+1}]$, $j \in \mathbb{N}_0$. *We consider the embedding*

$$F_{p,q}^{\boldsymbol{\sigma},\boldsymbol{N}} \hookrightarrow \Lambda_{\infty,r}^{\mu(\cdot)}. \tag{3.61}$$

(i) *If* $1 < p \le r < \infty$ *and* $1 < p < r$ *if* $r = \infty$, *then* (3.61) *holds if, and only if,*

$$\sup_{M \ge 0} \frac{\left( \sum_{j=0}^{M} \int_{N_{j+1}^{-1}}^{N_j^{-1}} (\mu(t))^{-r} \frac{\mathrm{d}t}{t} \right)^{\frac{1}{r}}}{\left( \sum_{j=0}^{M} \int_{N_{j+1}^{-1}}^{N_j^{-1}} (\lambda_{pr}(t))^{-r} \frac{\mathrm{d}t}{t} \right)^{\frac{1}{r}}} < \infty \tag{3.62}$$

(*with the usual modification if* $r = \infty$).

(ii) *If* $0 < r < p < \infty$ *and* $p > 1$, *then* (3.61) *holds if, and only if,*

$$\left\{ \sum_{M=0}^{\infty} \frac{\left( \sum_{j=0}^{M} \int_{N_{j+1}^{-1}}^{N_j^{-1}} (\mu(t))^{-r} \frac{\mathrm{d}t}{t} \right)^{u/p}}{\left( \sum_{j=0}^{M} \int_{N_{j+1}^{-1}}^{N_j^{-1}} (\lambda_{pr}(t))^{-r} \frac{\mathrm{d}t}{t} \right)^{u/r}} \int_{N_{M+1}^{-1}}^{N_M^{-1}} (\mu(s))^{-r} \frac{\mathrm{d}s}{s} \right\}^{\frac{1}{u}} < \infty, \qquad (3.63)$$

*where* $\frac{1}{u} := \frac{1}{r} - \frac{1}{p}$.

(iii) *Let* $r \in [p, \infty]$. *Among the embeddings in* (3.61), *that one with* $\mu = \lambda_{pr}$, *is sharp with respect to the parameter* $\mu$.

(iv) *Among the embeddings in* (3.61), *that one with* $\mu = \lambda_{pp}$ *and* $r = p$, *i.e.,*

$$F_{p,q}^{\boldsymbol{\sigma},\boldsymbol{N}} \hookrightarrow \Lambda_{\infty,p}^{\lambda_{pp}(\cdot)}, \qquad (3.64)$$

*is optimal.*

**Remark 3.9.** Let $0 < p \le 1$, $0 < q, r \le \infty$, $\mu \in \mathcal{L}_r$. Let $\boldsymbol{\sigma}$ and $\boldsymbol{N}$ be admissible sequences, the latter satisfying $\underline{N}_1 > 1$. Put $\boldsymbol{\tau} = \boldsymbol{\sigma}\boldsymbol{N}^{-n/p}$ and assume that $\underline{s}(\boldsymbol{N}\boldsymbol{\tau}^{-1}) > 0$ and $\tau_j^{-1} \to 0$ as $j \to \infty$.

Let $\Lambda$ be an admissible function such that $\Lambda(z) \sim \Lambda(N_j) = \sigma_j$ for $z \in [N_j, N_{j+1}]$, $j \in \mathbb{N}_0$. Let $H(t) := \inf_{s\in(0,t)} \Lambda(s^{-1})s^{n/p}$, $t \in (0, N_0^{-1})$, and suppose that $H$ is differentiable with $H'(t) \neq 0$, $t \in (0, N_0^{-1})$. Furthermore, let $\lambda_{pr} \in \mathcal{L}_r$ be defined by

$$\lambda_{pr}(t) := t^{-\frac{1}{r}} \left( \inf_{s\in(0,t)} \Lambda(s^{-1})s^{n/p} \right)^{\frac{1}{r}-1} \left( -\frac{d}{dt} \inf_{s\in(0,t)} \Lambda(s^{-1})s^{n/p} \right)^{-\frac{1}{r}}, \qquad t \in (0, N_0^{-1}]. \qquad (3.65)$$

Then, conditions (3.62), (3.63) remain the same and the sharp embeddings with respect to parameter $\mu$, when $0 < p \le 1$, are obtained by taking $r \ge p$ and $\mu := \lambda_{pr}(t)$. Moreover, we obtain the optimal embedding by putting $r = p$ and $\mu := \lambda_{pp}$.

## 4. Applications

4.1. **Continuity envelopes for $A_{p,q}^{\boldsymbol{\sigma},\boldsymbol{N}}$.** The concept of continuity envelopes has been introduced by Haroske in [Har02] and Triebel in [Tri01]. Here we quote the basic definitions and results concerning continuity envelopes. We refer to [Har07, Tri01] for heuristics, motivations and details on this subject.

**Definition 4.1.** *Let* $X \hookrightarrow C_B$ *be some function space on* $\mathbb{R}^n$.

(i) *Let* $\mathcal{E}_{\mathsf{C}}^X : (0,\infty) \to [0,\infty)$ *be defined by*

$$\mathcal{E}_{\mathsf{C}}^X(t) := \sup_{\|f|X\|\le 1} \frac{\omega(f,t)}{t}, \qquad t > 0. \qquad (4.1)$$

*The* continuity envelope function *of* $X$ *is the class* $[\mathcal{E}_{\mathsf{C}}^X]$ *of functions* $f : (0,\varepsilon] \to [0,\infty)$, *for some* $\varepsilon > 0$, *such that* $f(\cdot) \sim \mathcal{E}_{\mathsf{C}}^X(\cdot)$ *in* $(0,\varepsilon]$. *For convenience, we do not distinguish between representative and equivalence class. Therefore, any representative function of the class will be called as well continuity envelope function and sometimes we also denote a particular representative by* $\mathcal{E}_{\mathsf{C}}^X(\cdot)$.

(ii) *Assume* $X \not\hookrightarrow \mathrm{Lip}^1$. *Let* $\varepsilon \in (0,1)$, $H(t) := -\log \mathcal{E}_{\mathsf{C}}^X(t)$, $t \in (0,\varepsilon]$, *and let* $\mu_H$ *be the associated Borel measure. The number* $u_X$, $0 < u_X \le \infty$, *is defined as the infimum of all numbers* $v$, $0 < v \le \infty$, *such that*

$$\left( \int_0^{\varepsilon} \left( \frac{\omega(f,t)}{t\, \mathcal{E}_{\mathsf{C}}^X(t)} \right)^v \mu_H(\mathrm{d}t) \right)^{1/v} \le c\, \|f|X\| \qquad (4.2)$$

(*with the usual modification if* $v = \infty$) *holds for some* $c > 0$ *and all* $f \in X$. *The couple*

$$\mathfrak{E}_{\mathsf{C}}(X) = \left(\mathcal{E}_{\mathsf{C}}^X(\cdot), u_X\right)$$

*is called* continuity envelope *for the function space* $X$.

**Remark 4.2.** (i) Note that the function $\mathcal{E}_{\mathsf{C}}^X$, defined by (4.1), is equivalent to some monotonically decreasing function; for a proof and further properties we refer to [Har07]. Concerning Definition 4.1(ii) we shall assume that we can choose a continuous representative in the equivalence class $[\mathcal{E}_{\mathsf{C}}^X]$, for convenience denoted again by $\mathcal{E}_{\mathsf{C}}^X$.

(ii) Note that $H(t) = -\log \mathcal{E}_{\mathsf{C}}^X(t)$ is a (finite) real increasing function on $(0, \varepsilon]$, which tends to $-\infty$ when $t$ goes to 0. There is only a Borel measure (i.e., a measure defined on the Borel sets) $\mu_H$ in $(0, \varepsilon]$ such that $\mu_H([a, b]) = H(b) - H(a)$, for all $[a, b] \subset (0, \varepsilon]$. Its restriction to each such $[a, b]$ is the Stieltjes-Borel measure associated with $H\big|_{[a,b]}$.

In the important case when $H$ happens to be continuously differentiable in $(0, \varepsilon]$, we have $\mu_H(\mathrm{d}t) = H'\mathrm{d}t$, and for the functions we want to integrate we can calculate the integrals as improper Riemann integrals.

(iii) Furthermore, (4.2) holds with $v = \infty$ in any case, but – depending upon the underlying function space $X$ – there might be some smaller $v_0$ such that (4.2) is still satisfied (and therefore also for all $v \in [v_0, \infty]$), cf. [Har07, Remark 6.2].

As it will be useful in the sequel, we recall some properties of the continuity envelopes.

**Proposition 4.3.** (i) *Let* $X_i \hookrightarrow C_B$, $i = 1, 2$, *be some function spaces on* $\mathbb{R}^n$. *Then* $X_1 \hookrightarrow X_2$ *implies that there is some positive constant* $c$ *such that for all* $t > 0$,

$$\mathcal{E}_{\mathsf{C}}^{X_1}(t) \;\le\; c\, \mathcal{E}_{\mathsf{C}}^{X_2}(t).$$

(ii) *We have* $X \hookrightarrow \mathrm{Lip}^1$ *if, and only if,* $\mathcal{E}_{\mathsf{C}}^X$ *is bounded.*

(iii) *Let* $X_i \hookrightarrow C_B$, $i = 1, 2$, *be some function spaces on* $\mathbb{R}^n$ *with* $X_1 \hookrightarrow X_2$. *Assume for their continuity envelope functions*

$$\mathcal{E}_{\mathsf{C}}^{X_1}(t) \;\sim\; \mathcal{E}_{\mathsf{C}}^{X_2}(t), \qquad t \in (0, \varepsilon),$$

*for some* $\varepsilon > 0$. *Then we get for the corresponding indices* $u_{X_i}$, $i = 1, 2$, *that*

$$u_{X_1} \;\le\; u_{X_2}.$$

Let $A \in \{B, F\}$. Regarding the study of continuity envelopes in the context of the spaces of generalized smoothness $A_{p,q}^{\boldsymbol{\sigma},\boldsymbol{N}}$, of interest are those spaces with

$$A_{p,q}^{\boldsymbol{\sigma},\boldsymbol{N}} \hookrightarrow C_B \qquad \text{but} \qquad A_{p,q}^{\boldsymbol{\sigma},\boldsymbol{N}} \not\hookrightarrow \mathrm{Lip}^1.$$

Taking into consideration Proposition 2.13, and Remarks 3.3 (ii) and 3.7 (ii), we shall be concerned with the investigation of the continuity envelopes of the spaces $A_{p,q}^{\boldsymbol{\sigma},\boldsymbol{N}}$ with

$$\underline{s}(\boldsymbol{N}\boldsymbol{\tau}^{-1}) > 0 \qquad \text{and} \qquad \boldsymbol{\tau}^{-1} \in \ell_{u'}, \tag{4.3}$$

where $\boldsymbol{\tau} = \boldsymbol{\sigma}\boldsymbol{N}^{-n/p}$, setting $u = q$ if $A = B$ and $u = p$ if $A = F$, respectively.

In particular, our results generalize those previously obtained in [MNP09, Proposition 3.2] and, if additionally $\underline{s}(\boldsymbol{\tau}) > 0$, we recover results from [HM08] regarding continuity envelopes. The new results in this paper, regarding continuity envelopes, correspond to consider the situation when (4.3) is satisfied and $\underline{s}(\boldsymbol{\tau}) \le 0$.

**Proposition 4.4.** *Let* $0 < p \le \infty$ ($p < \infty$ *in the* $F$*-case*), $0 < q \le \infty$, $\boldsymbol{\sigma}$ *and* $\boldsymbol{N}$ *be admissible sequences, the latter satisfying* $\underline{N}_1 > 1$. *Assume further that* (4.3) *holds. Let* $\Lambda$ *be any admissible function such that*

*$\Lambda(z) \sim \sigma_j$, $z \in [N_j, N_{j+1}]$, $j \in \mathbb{N}_0$, with equivalence constants independent of $j$, and let $\lambda_{u\infty}$ defined in $(0, N_0^{-1}]$ be the optimal weights from Corollary 3.4 with $r = \infty$, i.e.,*

$$\lambda_{u\infty}(t) := \left( \int_0^t \Big( \Lambda(y^{-1})\, y^{n/p} \Big)^{-u'} \frac{\mathrm{d}y}{y} \right)^{\frac{1}{u'}} \qquad \text{if} \quad 1 < u \le \infty, \tag{4.4}$$

*and*

$$\lambda_{u\infty}(t) := \sup_{y \in (0,t)} (\Lambda(y^{-1}))^{-1}\, y^{-n/p} \qquad \text{if} \quad 0 < u \le 1. \tag{4.5}$$

*Then, there exists $\varepsilon > 0$ such that*

$$\mathcal{E}_{\mathsf{C}}^{A^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}}(t) \sim \frac{\lambda_{u\infty}(t)}{t}, \qquad t \in (0, \varepsilon], \tag{4.6}$$

*where $u = q$ if $A = B$ and $u = p$ if $A = F$, respectively.*

*Proof.* Using Proposition 2.7 we observe that

$$\lambda_{u\infty}(t) \sim \left( \sum_{j=k}^{\infty} \tau_j^{-u'} \right)^{\frac{1}{u'}}, \qquad t \in [N_{k+1}^{-1}, N_k^{-1}], \quad k \in \mathbb{N}_0. \tag{4.7}$$

By Theorems 3.2(i) and 3.6(i), we have $A^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q} \hookrightarrow \Lambda^{\lambda_{u\infty}(\cdot)}_{\infty,\infty}$, which implies for $\varepsilon < \min(N_0^{-1}, 1)$,

$$\sup_{0<t\le\varepsilon} \frac{\omega(f,t)}{\lambda_{u\infty}(t)} \lesssim \|f|A^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}\|,$$

leading to

$$\frac{\omega(f,t)}{t} \lesssim \frac{\lambda_{u\infty}(t)}{t} \|f|A^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}\| \qquad \text{for all} \quad t \in (0,\varepsilon].$$

Considering only functions $f \in A^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}$ with $\|f|A^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}\| \le 1$, taking the supremum yields

$$\mathcal{E}_{\mathsf{C}}^{A^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}}(t) \lesssim \frac{\lambda_{u\infty}(t)}{t} \qquad \text{for all} \quad t \in (0,\varepsilon],$$

giving the desired upper bound.

On the other hand, it is obvious that (4.2) holds for $v = \infty$ and any $X$. This together with that fact that we only consider spaces $A^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q} \hookrightarrow C_B$ yields for $\Phi(t) = t\mathcal{E}_{\mathsf{C}}^{A^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}}(t)$,

$$A^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q} \hookrightarrow \Lambda^{\Phi(\cdot)}_{\infty,\infty},$$

which according to Theorem 3.2(i) and Theorem 3.6(i) holds only if

$$\sup_{M\ge 0} \sup_{t\in[N_{M+1}^{-1}, N_0^{-1}]} \left( t\mathcal{E}_{\mathsf{C}}^{A^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}}(t) \right)^{-1} \left( \sum_{k=M}^{\infty} \tau_k^{-u'} \right)^{\frac{1}{u'}} < \infty.$$

Since $\Phi(t)$ is monotonically increasing in $t$, using (4.7), we obtain

$$\left( N_{M+1}^{-1} \mathcal{E}_{\mathsf{C}}^{A^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}}(N_{M+1}^{-1}) \right)^{-1} \le c\, (\lambda_{u\infty}(N_{M+1}^{-1}))^{-1} \qquad \text{for all } M \ge 0.$$

A monotonicity argument finally gives the lower bound

$$\mathcal{E}_{\mathsf{C}}^{A^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}}(t) \gtrsim \frac{\lambda_{q\infty}(t)}{t}, \qquad t \in (0,\varepsilon].$$

This completes the proof. □

Under the conditions of the previous theorem, if we additionally require $\underline{s}(\boldsymbol{\tau}) > 0$, then one can easily verify that (4.6) is equivalent to [HM08, eqs. (26), (27)]. Hence, in the previous theorem we recover partially the results from [HM08, Proposition 4.1] regarding continuity envelopes. In [HM08] it was proved that in such a case the fine index of the continuity envelope is $q$ for $B$-spaces and $p$ for $F$-spaces, where the continuity envelope yields the optimal embedding.

Provided condition (4.8) (below) is satisfied, we have an instance of the phenomenon where the continuity envelope does not yield the optimal embedding given by Theorem 3.2 and Corollary 3.4. Similar observations were already made for Besov spaces of type $B^{(s,\Psi)}_{p,q}$ (which are covered by our studies), we refer to [MNS11, Remark 3.4(iii)] for more details. A similar situation occurs as well for Bessel-potential-type spaces in the limiting case, cf. [GNO07, Theorem 4, Remark 5] and [GNO10, Theorem 1.6, Corollary 1.7].

**Theorem 4.5.** *Let $0 < p \le \infty$ ( $p < \infty$ in the F-case), $0 < q \le \infty$, $\boldsymbol{\sigma}$ and $\boldsymbol{N}$ be admissible sequences, the latter satisfying $\underline{N}_1 > 1$. Let $u = q$ for $B$-spaces and $u = p$ for $F$-spaces, respectively. Assume that* (4.3) *holds. Additionally, suppose that either*

$$\lim_{M\to\infty} \frac{\sum_{j=M}^{\infty} \tau_j^{-u'}}{\tau_M^{-u'}} = \infty, \qquad \text{if} \quad u > 1, \tag{4.8}$$

*or $\boldsymbol{\tau} = \{\tau_j\}_{j\in\mathbb{N}_0}$ is monotonically non-decreasing and satisfies*

$$\lim_{M\to\infty} \tau_M^{-1} \left( \sum_{j=0}^{M} \tau_j^{v} \right)^{\frac{1}{v}} = \infty, \quad \text{for} \quad v \in [u,\infty), \qquad \text{if} \quad 0 < u \le 1. \tag{4.9}$$

*Then*

$$\mathfrak{E}_{\mathsf{C}}\left(A^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}\right) = \left(\frac{\lambda_{u\infty}(t)}{t}, \infty\right),$$

*where $\lambda_{u\infty}$ is defined by* (4.4) *or by* (4.5).

*Proof.* By virtue of Proposition 4.4, it only remains to prove that $u_{B^{\boldsymbol{\sigma},\boldsymbol{N}}_{pq}} = \infty$ and $u_{F^{\boldsymbol{\sigma},\boldsymbol{N}}_{pq}} = \infty$. Furthermore, taking into account Propositions 2.12, 4.4 and Proposition 4.3, it is sufficient to prove that assertion for the Besov space case.

Step 1: Let $1 < q < \infty$. By Proposition 4.4, take $\mathcal{E}_{\mathsf{C}}^{B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}}(t) := \frac{\lambda_{q\infty}(t)}{t}$, $t \in (0,\varepsilon]$, for some small $\varepsilon > 0$. Assume that for some $v \in [q,\infty)$ there is a positive constant $c(v)$ such that

$$\left( \int_0^{\varepsilon} \left( \frac{\omega(f,t)}{t\, \mathcal{E}_{\mathsf{C}}^{B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}}(t)} \right)^{v} \mu_{q'}(\mathrm{d}t) \right)^{\frac{1}{v}} \le c(v)\, \|f \mid B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}\| \tag{4.10}$$

holds for all $f \in B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}$, where $\mu_{q'}$ is the Borel measure associated with $-\log \mathcal{E}_{\mathsf{C}}^{B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}}$ in $(0,\varepsilon]$. Since $\mathcal{E}_{\mathsf{C}}^{B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}}(\cdot)$ is continuously differentiable in $(0,\varepsilon]$ (see [CF06, Lemma 2.5] for a similar assertion), the integral on the left hand side of (4.10) can be calculated as the improper Riemann integral

$$\int_0^{\varepsilon} \left( \frac{\omega(f,t)}{t\, \mathcal{E}_{\mathsf{C}}^{B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}}(t)} \right)^{v} \frac{\left(-\mathcal{E}_{\mathsf{C}}^{B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}}\right)'(t)}{\mathcal{E}_{\mathsf{C}}^{B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}}(t)}\, \mathrm{d}t. \tag{4.11}$$

By the definition of $\lambda_{q\infty}(t)$, we have

$$-\frac{\left(\mathcal{E}_{\mathsf{C}}^{B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}}\right)'(t)}{\mathcal{E}_{\mathsf{C}}^{B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}}(t)} = \frac{1}{t}\left(1-\frac{1}{q'}\frac{\left(\Lambda(t^{-1})t^{n/p}\right)^{-q'}}{\int_0^t\left(\Lambda(s^{-1})s^{n/p}\right)^{-q'}\frac{\mathrm{d}s}{s}}\right), \qquad \text{for} \quad t\in(0,\varepsilon)$$

and from (4.8) it follows that there is $\delta>0$ such that

$$-\frac{\left(\mathcal{E}_{\mathsf{C}}^{B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}}\right)'(t)}{\mathcal{E}_{\mathsf{C}}^{B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}}(t)} \sim \frac{1}{t}, \qquad 0<t<\delta. \tag{4.12}$$

Since $\int_0^1$ in (2.15) can be replaced by $\int_0^\varepsilon$ and $B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}\hookrightarrow C_B$, we see that (4.10) is equivalent to

$$B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}\hookrightarrow \Lambda_{\infty,v}^{\lambda_{q\infty}(\cdot)}, \tag{4.13}$$

for $v\in[q,\infty)$. Theorem 3.2 provides necessary conditions for this embedding. But (4.8) implies that

$$\sup_{M\geq 0}\frac{\sum_{k=0}^{M}\left(\sum_{j=k}^{\infty}\tau_j^{-q'}\right)^{-v/q'}}{\left(\sum_{j=M}^{\infty}\tau_j^{-q'}\right)^{-v/q'}}=\infty \tag{4.14}$$

because, using l'Hôpital's rule (since the numerator and the denominator tend to $\infty$),

$$\lim_{x\to 0+}\frac{\int_x^{N_0^{-1}}\left(\int_0^t\left(\Lambda(s^{-1})s^{n/p}\right)^{-q'}\frac{\mathrm{d}s}{s}\right)^{-v/q'}\frac{\mathrm{d}t}{t}}{\left(\int_0^x\left(\Lambda(s^{-1})s^{n/p}\right)^{-q'}\frac{\mathrm{d}s}{s}\right)^{-v/q'}}\sim\lim_{x\to 0+}\frac{\int_0^x\left(\Lambda(s^{-1})s^{n/p}\right)^{-q'}\frac{\mathrm{d}s}{s}}{\left(\Lambda(x^{-1})x^{n/p}\right)^{-q'}}=\infty,$$

which contradicts (3.11). Therefore, we have no embedding (4.13) and there is no $v\in[q,\infty)$ such that (4.10) holds. Hence, $u_{B^{\boldsymbol{\sigma},\boldsymbol{N}}_{pq}}=\infty$.

If $q=\infty$, we assume that there is $v\in(0,\infty)$ such that (4.10) holds for all $f\in B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}$. Proceeding as before, (4.14) would contradict (3.13). Thus, we also have in this situation that $u_{B^{\boldsymbol{\sigma},\boldsymbol{N}}_{pq}}=\infty$.

Step 2: Let $0<q\leq 1$. Recall that, by Propositions 4.4 and 2.4 and the monotonicity of $\boldsymbol{\tau}$, we have

$$\mathcal{E}_{\mathsf{C}}^{B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}}(t)=\frac{\lambda_{q\infty}(t)}{t}=\frac{1}{t}\sup_{s\in(0,t)}(\Lambda(s^{-1}))^{-1}s^{-n/p}\sim N_k\sup_{j\geq k}\tau_j^{-1}=N_k\tau_k^{-1}, \tag{4.15}$$

for $t\in[N_{k+1}^{-1},N_k^{-1}]$, $k\in\mathbb{N}_0$. We remark that, due to the hypothesis (4.3),

$$\underline{s}(\boldsymbol{\tau}^{-1}\boldsymbol{N})>0,$$

implies the existence of a natural number $k_0$ and a positive constant $c_1$ such that

$$\log\left(\frac{\tau_{j+k}^{-1}N_{j+k}}{\tau_j^{-1}N_j}\right)\geq c_1k, \quad j\in\mathbb{N}_0, \quad k\geq k_0. \tag{4.16}$$

As in Step 1, let us assume that for some $v\in[q,\infty)$ there is a positive constant $c(v)$ such that

$$\left(\int_0^\varepsilon\left(\frac{\omega(f,t)}{t\,\mathcal{E}_{\mathsf{C}}^{B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}}(t)}\right)^v\mu_\infty(\mathrm{d}t)\right)^{\frac{1}{v}}\leq c(v)\,\|f\mid B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}\| \tag{4.17}$$

holds for all $f \in B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}(\mathbb{R}^n)$. Let $d \in \mathbb{N}$ such that $N^{-1}_{dk_0} \le \varepsilon$ be fixed. Let $\boldsymbol{b} = (b_j)_{j\in\mathbb{N}} \in \ell_q$ be a sequence of non-negative numbers and let $f^{\boldsymbol{b}}$ be the corresponding function according to (3.3). Using (4.17), (3.4), (3.5), (4.15), and (4.16), as well as the admissibility of $\boldsymbol{N}$ and $\boldsymbol{\tau}$, we obtain

$$\begin{aligned}
\|b\,|\ell_q\| &\gtrsim \left(\int_0^{N^{-1}_{dk_0}} \left(\frac{\omega(f^b,t)}{t\,\mathcal{E}_{\mathsf{C}}^{B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}}(t)}\right)^v \mu_\infty(\mathrm{d}t)\right)^{\frac{1}{v}} \\
&= \left(\sum_{j=d}^{\infty} \int_{N^{-1}_{(j+1)k_0}}^{N^{-1}_{jk_0}} \left(\frac{\omega(f^b,t)}{t\,\mathcal{E}_{\mathsf{C}}^{B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}}(t)}\right)^v \mu_\infty(\mathrm{d}t)\right)^{\frac{1}{v}} \\
&\gtrsim \left(\sum_{j=d}^{\infty} \left(\frac{\omega(f^b,N^{-1}_{jk_0})}{N^{-1}_{jk_0}}\,\frac{1}{\mathcal{E}_{\mathsf{C}}^{B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}}(N^{-1}_{(j+1)k_0})}\right)^v \mu_\infty[N^{-1}_{(j+1)k_0},N^{-1}_{jk_0}]\right)^{\frac{1}{v}} \\
&\gtrsim \left(\sum_{j=d}^{\infty} \left(N_{jk_0}\sum_{\ell=jk_0}^{\infty} b_\ell\,\tau_\ell^{-1}\right)^v N^{-v}_{(j+1)k_0}\tau^v_{(j+1)k_0}\log\left(\frac{N_{(j+1)k_0}\tau^{-1}_{(j+1)k_0}}{N_{jk_0}\tau^{-1}_{jk_0}}\right)\right)^{\frac{1}{v}} \\
&\gtrsim \left(\sum_{j=d}^{\infty}\tau^v_{jk_0}\left(\sum_{\ell=jk_0}^{\infty} b_\ell\,\tau_\ell^{-1}\right)^v\right)^{\frac{1}{v}}.
\end{aligned} \tag{4.18}$$

For fixed $m \in \mathbb{N}$ with $m > d$, we get from (4.18),

$$\begin{aligned}
\|b\,|\ell_q\| &\gtrsim \left(\sum_{j=d}^{m}\tau^v_{jk_0}\left(\sum_{\ell=jk_0}^{\infty} b_\ell\,\tau_\ell^{-1}\right)^v\right)^{\frac{1}{v}} \\
&\gtrsim \left(\sum_{j=d}^{m}\tau^v_{jk_0}\left(\sum_{\ell=mk_0}^{\infty} b_\ell\,\tau_\ell^{-1}\right)^v\right)^{\frac{1}{v}} \\
&\sim \left(\sum_{\ell=mk_0}^{\infty} b_\ell\,\tau_\ell^{-1}\right)\left(\sum_{j=d}^{m}\tau^v_{jk_0}\right)^{\frac{1}{v}}.
\end{aligned}$$

Then, choosing

$$b_\ell := \begin{cases} 1, & \text{for} \quad \ell = mk_0, \\ 0, & \text{for} \quad \ell \ne mk_0 \end{cases}$$

we arrive at

$$1 \gtrsim \tau^{-1}_{mk_0}\left(\sum_{j=d}^{m}\tau^v_{jk_0}\right)^{\frac{1}{v}}, \qquad \text{for any} \quad m > d. \tag{4.19}$$

Note that, since $\boldsymbol{\tau}$ is an admissible sequence, it holds

$$\tau_\ell \lesssim \tau_{dk_0} \qquad \text{for all} \quad \ell \in \{0,\cdots,dk_0\}, \tag{4.20}$$

and

$$\tau_{sk_0+\ell} \lesssim \tau_{sk_0} \qquad \text{for all} \quad s\in\mathbb{N}_0, \quad \ell \in \{0,\cdots,dk_0\}, \tag{4.21}$$

with equivalence constants being independent of $s$ and $\ell$ (depend only on $\boldsymbol{\tau}$, $d$ and $k_0$). By (4.20) and (4.21), we have

$$\sum_{j=0}^{k_0 m} \tau_j^v = \sum_{j=0}^{dk_0-1} \tau_j^v + \sum_{\ell=d}^{m-1} \sum_{t=k_0\ell}^{k_0(\ell+1)} \tau_t^v \lesssim \sum_{\ell=d}^{m} \tau_{k_0\ell}^v, \qquad \text{for all} \quad m \in \mathbb{N} \quad \text{with} \quad m > d. \tag{4.22}$$

The hypothesis (4.9) implies, in particular, that

$$\lim_{m\to\infty} \tau_{k_0 m}^{-1} \left( \sum_{j=0}^{k_0 m} \tau_j^v \right)^{\frac{1}{v}} = \infty,$$

which, due to (4.22), yields

$$\lim_{m\to\infty} \tau_{k_0 m}^{-1} \left( \sum_{\ell=d}^{m} \tau_{k_0\ell}^v \right)^{\frac{1}{v}} = \infty,$$

but this contradicts (4.19). Therefore, there is no $v \in [q, \infty)$ such that (4.17) holds and hence $u_{B_{pq}^{\boldsymbol{\sigma},\boldsymbol{N}}(\mathbb{R}^n)} = \infty$. □

4.2. **Approximation numbers for $A_{p,q}^{\boldsymbol{\sigma},\boldsymbol{N}}$.** As an immediate consequence of our results for continuity envelopes, we obtain an upper estimate for approximation numbers.

The following result can be found in [CH03].

**Proposition 4.6.** *Let $X$ be some Banach space of functions defined on the unit ball $U$ in $\mathbb{R}^n$ with $X(U) \hookrightarrow C(U)$, where $C(U)$ is the space of bounded continuous functions defined on $U$. Then there is some $c > 0$ such that for all $k \in \mathbb{N}$*

$$a_k\big(\mathrm{id} : X(U) \longrightarrow C(U)\big) \;\le\; c\, k^{-1/n}\, \mathcal{E}_{\mathsf{C}}^X(k^{-1/n}), \tag{4.23}$$

*where the $k$-th approximation number $a_k$ of $\mathrm{id} : X(U) \longrightarrow C(U)$ is defined by*

$$a_k(\mathrm{id} : X(U) \longrightarrow C(U)) := \inf \{\|\mathrm{id} - L\| : L \in L(X(U), C(U)),\ \mathrm{rank} L < k\},$$

*with* rank$L$ *as the dimension of the range of $L$.*

We return to the function spaces studied above. Note that there cannot be a compact embedding between spaces on $\mathbb{R}^n$; the counterpart for spaces $A_{p,q}^{\boldsymbol{\sigma},\boldsymbol{N}}(\mathbb{R}^n)$ follows immediately from the well-known fact for spaces $A_{p,q}^s(\mathbb{R}^n)$ and $A \in \{B, F\}$. Let $U$ be the unit ball in $\mathbb{R}^n$; we deal with spaces $A_{p,q}^{\boldsymbol{\sigma},\boldsymbol{N}}(U)$ now defined by restriction from their $\mathbb{R}^n$-counterparts. One immediately verifies that Theorem 4.5 can be transferred to spaces on domains without any difficulty, i.e., we have for the local continuity envelopes $\mathfrak{E}_{\mathsf{C}}\left(A_{p,q}^{\boldsymbol{\sigma},\boldsymbol{N}}(U)\right) = \mathfrak{E}_{\mathsf{C}}\left(A_{p,q}^{\boldsymbol{\sigma},\boldsymbol{N}}(\mathbb{R}^n)\right)$.

In the following theorem we give upper and lower bounds for approximation numbers of the embedding of the spaces $A_{p,q}^{\boldsymbol{\sigma},\boldsymbol{N}}(U)$, where $N = (2^j)_{j\in\mathbb{N}_0}$, into $C(U)$.

When $N = (2^j)_{j\in\mathbb{N}_0}$, we denote the spaces $A_{p,q}^{\boldsymbol{\sigma},\boldsymbol{N}}$ by $A_{p,q}^{\boldsymbol{\sigma}}$.

**Theorem 4.7.** *Let $2 < p \le \infty$, $0 < q \le \infty$, $\boldsymbol{N} = (2^j)_{j\in\mathbb{N}_0}$, and $\boldsymbol{\sigma}$ be an admissible sequence. Put $\boldsymbol{\tau} = \boldsymbol{\sigma}\boldsymbol{N}^{-n/p}$ and let $u = q$ for $B$-spaces and $u = p$ for $F$-spaces, respectively. Assume that*

$$\boldsymbol{\tau}^{-1} \in \ell_{u'} \qquad \text{and} \qquad \overline{s}(\boldsymbol{\tau}) < \min\{1, \underline{s}(\boldsymbol{\tau}) + 1\}. \tag{4.24}$$

*Then there are positive numbers $c_1$, $c_2$ such that, for all $k \in \mathbb{N}$,*

$$c_1\, \tau_l^{-1} \;\le\; a_k\big(\mathrm{id} : A_{p,q}^{\boldsymbol{\sigma}}(U) \longrightarrow C(U)\big) \le\; c_2 \left( \sum_{j=l}^{\infty} \tau_j^{-u'} \right)^{\frac{1}{u'}} \tag{4.25}$$

(*with the usual modification if* $u' = \infty$)*, where* $l \in \mathbb{N}_0$ *is such that*

$$k^{-1/n} \in [2^{-(l+1)}, 2^{-l}]. \tag{4.26}$$

*Proof.* Step 1: We start establishing the results for $B$-spaces. Note that for the upper estimate we only need (4.3), which is verified if (4.24) is satisfied. The restrictions $p \geq 2$ and $\overline{s}(\boldsymbol{\tau}) < \min\{1, \underline{s}(\boldsymbol{\tau}) + 1\}$ are due to the lower estimate.

By our initial assumption (4.26) we have $k^{-1/n} \sim 2^{-l}$. Since $\boldsymbol{\tau}^{-1} \in \ell_{q'}$, by Proposition 2.13, $B^{\boldsymbol{\sigma}}_{p,q}(U) \hookrightarrow C(U)$. Moreover, as $\underline{s}(\boldsymbol{N}\boldsymbol{\tau}^{-1}) > 0$, combining (4.23) with Propositions 4.4 and 2.7 immediately leads to the upper estimates. The difficulty with $0 < p, q < 1$, when the spaces $B^{\boldsymbol{\sigma}}_{p,q}(U)$ are not Banach spaces and hence Proposition 4.6 cannot be applied directly, can easily be surmounted by a continuous embedding argument, $B^{\boldsymbol{\sigma}}_{p,q} \hookrightarrow B^{\boldsymbol{\tau}}_{r,\hat{q}}$, where $p < 1 < r$, $\sigma_j = \tau_j 2^{jn\left(\frac{1}{p} - \frac{1}{r}\right)}$, $j \in \mathbb{N}_0$, and $\hat{q} = \max(q, 1)$, cf. Proposition 2.11. The rest follows in view of the multiplicativity of approximation numbers.

It remains to verify the lower bound. We proceed similar as in [HM04, Proposition 4.4] and make use of the special lift property, cf. [FL06, Theorem 3.1.9], together with related results for the classical spaces when $\boldsymbol{\sigma} = (\sigma_j)_{j\in\mathbb{N}_0} = (2^{sj})_{j\in\mathbb{N}_0}$. Note that $\boldsymbol{\tau}^{-1} \in \ell_{q'}$ implies $\overline{s}(\boldsymbol{\tau}) \geq 0$ (cf. Remark 3.3 (i)). Let $\mu \in (\overline{s}(\boldsymbol{\tau}), \min\{1, \underline{s}(\boldsymbol{\tau}) + 1\})$, define

$$\boldsymbol{\mu} = (\mu_j)_{j\in\mathbb{N}_0} = (\tau_j^{-1} 2^{j\mu})_{j\in\mathbb{N}_0},$$

and let $\boldsymbol{\sigma}^0$ denote the sequence with all terms equal to 1, i.e., $B^{\boldsymbol{\sigma}^0}_{\infty,\infty} = B^0_{\infty,\infty}$. Then we have

$$B^{\boldsymbol{\sigma}}_{p,q}(U) \hookrightarrow C(U) \hookrightarrow B^{\boldsymbol{\sigma}^0}_{\infty,\infty}(U) \hookrightarrow B^{\boldsymbol{\mu}^{-1}}_{\infty,\infty}(U). \tag{4.27}$$

In order for the last embedding in (4.27) to hold we have to assume that $\boldsymbol{\mu}^{-1} \in \ell_\infty$, cf. Proposition 2.11. But this is true since

$$\underline{s}(\boldsymbol{\mu}) \geq \underline{s}(\boldsymbol{\tau}^{-1}) + \mu = -\overline{s}(\boldsymbol{\tau}) + \mu > 0, \tag{4.28}$$

implying $\boldsymbol{\mu}^{-1} \in \ell_u$ for any $u \in (0, \infty]$. By the multiplicativity of approximation numbers,

$$a_{2k}\left(\mathrm{id} : B^{\boldsymbol{\sigma}}_{p,q}(U) \longrightarrow B^{\boldsymbol{\mu}^{-1}}_{\infty,\infty}(U)\right) \leq a_k\left(\mathrm{id} : B^{\boldsymbol{\sigma}}_{p,q}(U) \longrightarrow C(U)\right) a_k\left(\mathrm{id} : C(U) \longrightarrow B^{\boldsymbol{\mu}^{-1}}_{\infty,\infty}(U)\right).$$

It is thus sufficient to show that

$$a_{2k}\left(\mathrm{id} : B^{\boldsymbol{\sigma}}_{p,q}(U) \longrightarrow B^{\boldsymbol{\mu}^{-1}}_{\infty,\infty}(U)\right) \; \geq \; c_1 k^{-\frac{\mu}{n}}, \tag{4.29}$$

for $2 \leq p \leq \infty$, and

$$a_k\left(\mathrm{id} : C(U) \longrightarrow B^{\boldsymbol{\mu}^{-1}}_{\infty,\infty}(U)\right) \; \leq \; c_2 k^{-\frac{\mu}{n}} \tau_l, \tag{4.30}$$

in order to verify the estimate from below. In view of [FL06, Theorem 3.1.9] we can simplify (4.29) by

$$\begin{aligned} a_{2k}\left(\mathrm{id} : B^{\boldsymbol{\sigma}}_{p,q}(U) \longrightarrow B^{\boldsymbol{\mu}^{-1}}_{\infty,\infty}(U)\right) &\sim a_{2k}\left(\mathrm{id} : B^{(2^{j\frac{n}{p}})}_{p,q}(U) \longrightarrow B^{(2^{-j\mu})}_{\infty,\infty}(U)\right) \\ &= a_{2k}\left(\mathrm{id} : B^{n/p}_{p,q}(U) \longrightarrow B^{-\mu}_{\infty,\infty}(U)\right) \geq c_1 k^{-\frac{\mu}{n}}, \end{aligned}$$

the rest being a consequence of the well-known result [ET96, Theorem 3.3.4, p.119], see also [Har07, p.202]. Note that [FL06, Theorem 3.1.9] works on $\mathbb{R}^n$ originally, but due to [Lop09, Theorem 5.3.15], there is a linear extension operator, such that by usual extension-restriction procedures we can apply it to our situation, too. Concerning (4.30), $C(U) \hookrightarrow B^{\boldsymbol{\sigma}^0}_{\infty,\infty}(U)$ leads to

$$a_k\left(\mathrm{id} : C(U) \longrightarrow B^{\boldsymbol{\mu}^{-1}}_{\infty,\infty}(U)\right) \leq c\, a_k\left(\mathrm{id} : B^{\boldsymbol{\sigma}^0}_{\infty,\infty}(U) \longrightarrow B^{\boldsymbol{\mu}^{-1}}_{\infty,\infty}(U)\right).$$

Another application of [FL06, Theorem 3.1.9], $C(U) \hookrightarrow B^{\boldsymbol{\sigma}^0}_{\infty,\infty}(U)$, (4.23), Proposition 4.4 with $u=\infty$ (note that $\underline{s}(\boldsymbol{\mu})>0$ by (4.28) and $\underline{s}(\boldsymbol{N}\boldsymbol{\mu}^{-1})>0$) provide

$$\begin{aligned} a_k\left(\mathrm{id}: B^{\boldsymbol{\sigma}^0}_{\infty,\infty}(U) \longrightarrow B^{\boldsymbol{\mu}^{-1}}_{\infty,\infty}(U)\right) &\le c_1\, a_k\left(\mathrm{id}: B^{\boldsymbol{\mu}}_{\infty,\infty}(U) \longrightarrow B^{\boldsymbol{\sigma}^0}_{\infty,\infty}(U)\right) \\ &\le c_2\, a_k\left(\mathrm{id}: B^{\boldsymbol{\mu}}_{\infty,\infty}(U) \longrightarrow C(U)\right) \\ &\le c_3\, k^{-\frac{1}{n}} \mathcal{E}_{\mathsf{C}}^{B^{\boldsymbol{\mu}}_{\infty,\infty}}(k^{-\frac{1}{n}}) \\ &\sim \sum_{j=l}^{\infty} \tau_j 2^{-j\mu} \;\sim\; 2^{-l\mu}\tau_l \\ &\sim\; k^{-\frac{\mu}{n}}\tau_l. \end{aligned} \tag{4.31}$$

This yields (4.30) and finishes the proof for $B$-spaces.

Step 2: Concerning the results for $F$-spaces, we assume $\boldsymbol{\tau}^{-1}\in \ell_{p'}$. Let $0<p_1<p$ and define $\boldsymbol{\sigma'} = \boldsymbol{N}^{n\left(\frac{1}{p_1}-\frac{1}{p}\right)}\boldsymbol{\sigma}$. Then

$$B^{\boldsymbol{\sigma}'}_{p_1,p}(U) \hookrightarrow F^{\boldsymbol{\sigma}}_{p,q}(U) \hookrightarrow C(U)$$

and hence

$$a_k\big(\mathrm{id}:\ B^{\boldsymbol{\sigma'}}_{p_1,p}(U) \hookrightarrow C(U)\big) \lesssim a_k\big(\mathrm{id}:\ F^{\boldsymbol{\sigma}}_{p,q}(U) \hookrightarrow C(U)\big).$$

Since $\boldsymbol{\tau'} = \boldsymbol{\sigma'}\boldsymbol{N}^{-\frac{n}{p_1}} = \boldsymbol{N}^{n\left(\frac{1}{p_1}-\frac{1}{p}\right)}\boldsymbol{\sigma}\boldsymbol{N}^{-\frac{n}{p_1}} = \boldsymbol{\tau}$, using the results obtained in Step 1, we get

$$a_k\big(\mathrm{id}:\ F^{\boldsymbol{\sigma}}_{p,q}(U) \hookrightarrow C(U)\big) \gtrsim \tau_l^{-1}. \tag{4.32}$$

On the other hand, let $p<p_2<\infty$ and define $\boldsymbol{\sigma''} = \boldsymbol{N}^{n\left(\frac{1}{p_2}-\frac{1}{p}\right)}\boldsymbol{\sigma}$. Then $F^{\boldsymbol{\sigma}}_{p,q}(U) \hookrightarrow B^{\boldsymbol{\sigma}''}_{p_2,p}(U)$. Since $\boldsymbol{\sigma}^{-1}\boldsymbol{N}^{\frac{n}{p}} \in \ell_{p'}$ and $(\boldsymbol{\sigma''})^{-1}\boldsymbol{N}^{\frac{n}{p_2}} = \boldsymbol{N}^{-n\left(\frac{1}{p_2}-\frac{1}{p}\right)}\boldsymbol{\sigma}^{-1}\boldsymbol{N}^{\frac{n}{p_2}} = \boldsymbol{\sigma}^{-1}\boldsymbol{N}^{\frac{n}{p}}$, it holds $B^{\boldsymbol{\sigma}''}_{p_2,q}(U) \hookrightarrow C(U)$. And for the approximation numbers we get

$$a_k\big(\mathrm{id}:\ F^{\boldsymbol{\sigma}}_{p,q}(U) \hookrightarrow C(U)\big) \lesssim a_k\big(\mathrm{id}:\ B^{\boldsymbol{\sigma}''}_{p_2,p}(U) \hookrightarrow C(U)\big) \lesssim \left(\sum_{j=l}^{\infty} (\tau''_j)^{-p'}\right)^{\frac{1}{p'}} = \left(\sum_{j=l}^{\infty} \tau_j^{-p'}\right)^{\frac{1}{p'}}, \tag{4.33}$$

since $\boldsymbol{\tau}'' = \boldsymbol{\sigma}''\boldsymbol{N}^{-\frac{n}{p_2}} = \boldsymbol{\sigma}\boldsymbol{N}^{-\frac{n}{p}} = \boldsymbol{\tau}$ in the last step. Now (4.32) and (4.33) yield the desired lower and upper estimates, respectively. This finally completes the proof. □

**Remark 4.8.** (i) The loss of sharpness of our applied tools is not very surprising, see [Har01, 6.4]. However, if $0<u\le 1$, thus $u'=\infty$, then assuming that $\boldsymbol{\tau}=\{\tau_j\}_{j\in\mathbb{N}_0}$ is monotonically non-decreasing leads to

$$\sup_{j=l,\dots,\infty} \tau_j^{-1} = \tau_l^{-1},$$

showing the sharpness of the estimates in (4.25).

(ii) If $\boldsymbol{\sigma} = (2^{j\frac{n}{p}}\Psi(2^{-j}))_{j\in\mathbb{N}_0}$, $\Psi$ a slowly varying function, and $\boldsymbol{N}=(2^j)_{j\in\mathbb{N}_0}$, we recover the spaces $A^{(n/p,\Psi)}_{p,q}$, where $A\in\{B,F\}$, studied in [MNS11]. In terms of these spaces our results now read as follows. Let $2<p\le\infty$ (with $p<\infty$ in the $F$-case), $0<q\le\infty$. Assume $\big((\Psi(2^{-j}))^{-1}\big)_{j\in\mathbb{N}}\in\ell_{u'}$, where $u=q$ if $A=B$ and $u=p$ if $A=F$, respectively. Then there are positive numbers $c_1$, $c_2$ such that, for all $k\in\mathbb{N}$,

$$c_1\,\left(\Psi(k^{-1/n})\right)^{-1} \;\le\; a_k\big(\mathrm{id}: A^{(n/p,\Psi)}_{pq}(U) \longrightarrow C(U)\big) \;\le\; c_2 \left(\sum_{j=[\frac{\log k}{n}]}^{\infty} (\Psi(2^{-j}))^{-u'}\right)^{\frac{1}{u'}} \tag{4.34}$$

(usual modification if $u'=\infty$). In particular, this result improves [MNP09, Proposition 3.8].

(iii) Note that the restriction $p \geq 2$ is due to the lower estimates, similarly to [HM04, Proposition 4.4], where it was proved that

$$a_{2k}\left(\mathrm{id}: B^{(s,\Psi)}_{p,q}(U) \longrightarrow C(U)\right) \sim k^{-\frac{s}{n}+\frac{1}{p}}(\Psi(k^{-\frac{1}{n}}))^{-1}, \qquad k \in \mathbb{N}, \tag{4.35}$$

assuming that $2 < p \leq \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$ with $\frac{n}{p} < s < \frac{n}{p} + 1$, and $\Psi$ a slowly varying function.

Concerning estimates of approximation numbers for spaces $B^{(s,\Psi)}_{p,q}(U)$ in the super-limiting case when $s = \frac{n}{p} + 1$ we refer to [CH05, Proposition 4.9].

(iv) Regarding estimates for approximation numbers of embeddings of the spaces $A^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}(U)$ into $C(U)$, with general admissible sequences $\boldsymbol{N}$, satisfying $\underline{N}_1 > 1$, one can make use of the previous theorem and the standardization procedure [CL06, Theorem 1], that is,

$$B^{\boldsymbol{\sigma},\boldsymbol{N}}_{p,q}(U) = B^{\boldsymbol{\beta}}_{p,q}(U)$$

where for all $j \in \mathbb{N}_0$,

$$\beta_j = \sigma_{k(j)}, \qquad \text{with} \qquad k(j) = \min\{k \in \mathbb{N}_0 :\ 2^{j-1} \leq N_{k+l_0}\},$$

with $l_0$ satisfying (2.9).

(v) In view of what has been done in [HM04, Corollary 4.9] one can even expect that our results on approximation numbers yield rough estimates for entropy numbers. But this will be done elsewhere.

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