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A STUDY ON DESIGN SENSITIVITY ANALYSIS

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ABSTRACT

The available strategies, methods and approaches of design sensitivity analysis are reviewed. They are classified into the following systematical categories: strategies, methods, and solutions. The number of combination is very large. For a general problem, none of the combination is guaranteed to preference to another. However, a "good practical" way can be obtained to an accepted level. The advantages and disadvantages of some combinations are discussed. Variational strategy for sizing problem and semi-analytic solution for general application are strongly recommended. Finally, Variational strategy, finite difference and semi-analytic solution are implemented with commercial finite element analysis (FEA) program ADINA.

INTRODUCTION

The development of commercial finite element analysis(FEA) techniques have provided the engineer with a useful tool. It is very important for the design engineer to know design sensitivity information, ie. the structural responses of a slight perturbing of the structure. In the field of structural optimization, most of the gradient-based optimization algorithms usually require the design sensitivity information in searching for the optimum design, and an efficient structural optimum design also uses the design sensitivity for avoiding many detailed finite element analysis by creating an approximate optimization. Design sensitivity analysis is therefore an important ingredient in engineering design. Even more and more commercial FEA programs have provided a design sensitivity analysis capability. However, for the designer using commercial FEA programs without the capability, there is a large catalog to chose a suitable way to calculate design sensitivity information. Therefore implementation of design sensitivity analysis(DSA) integrated with FEA becomes a quite complex problem. It is necessary to make a comprehensive study on the existing ways of calculation of design sensitivity information.

In the paper, first of all, the existed ways to calculate DSA are classified into the following systematical categories: two strategies (difference of a discrete system and variational strategy), two methods (direct eliminating and Lagrange Multiplier method), and three solutions (analytic, semi-analytic, and finite difference solutions). Then, the advantages and disadvantages of some combinations are discussed. *The Variational Strategy* is strongly recommended *in sizing optimization, as its FEA independence and easiness of implementation*. Also, the combination of the difference of a discrete system strategy, the direct eliminating method and the semi-analytic solution is recommended, based on the author's emphasis on general application and implementation of DSA outside the existing commercial FEA program. Finally, Variational strategy, finite difference and semi-analytic solutions for a very simple ten-bar problem were implemented with ADINA.

TWO STRATEGIES

The purpose of the design sensitivity analysis is to evaluate the derivatives of the structural response with respect to the size and shape design variables. There are two basic strategies for sensitivity derivatives calculation. The first is based on the differentiation of discrete (FE) system and the second is usually called variational approach which based on differentiating the continuum equations before discretized.

Strategy 1: differentiation of discretized system

The structure had been discreted into a finite element model, and structural responses were governed by algebraic equations. The sensitivity is then equivalent to the mathematical problem of obtaining the derivatives of the solutions of those equations with respect to their coefficients. Under the strategy there are several popular methods and solution widely used, which will be discussed later.

Strategy 2: Variational and continue-based sensitivity

Opposed to strategy of differentiation of discretized system, the strategy consider the differentiate the continuum equations before discretized. The resulting sensitivity equations can then be solved with the aid of a structural program.

For instance, we write the equation of equilibrium through the principle of virtual work as:

$$\int \sigma \delta \epsilon dV - \int f \delta u dV + \int T \delta u dS \tag{1}$$

where σ and ϵ are the stress and strain tensor, f is body force, T is a surface force on a part of surface S.

Equation (1) is complemented by a linear stress-strain law,

$$\sigma = D(\epsilon - \epsilon^{i}) + \sigma^{i} \tag{2}$$

where ϵ^{i} represents the initial strain and σ^{i} represents the initial stress field, and D is the matrix of the generalized Hook's law. In finite element method, the displacement field u is replaced by the nodal displacement vector U as

$$u = NU \tag{3}$$

The linear strain-displacement relation is

$$\epsilon = Lu = LNU = BU \tag{4}$$

The following equilibrium equation is obtained by substituting equation (2) -(4) into equation(1)

$$KU = F \tag{5}$$

where $$F = \int B^T D \epsilon^i dV - \int B^T \sigma^i dV + \int N^T f dV + \int N^T T dS \tag{6a}$$

$$K = \int (B^T D B) dV \tag{6b}$$

The design variable x_j are assumed here not to influence T, ϵ^i and σ^i. Taking the derivative of equations (1)-(3) with respect to the design variable x_j

$$\int \sigma_{x_j} \delta\epsilon dV = \int f_{x_j} \delta u dV \tag{7}$$

$$\sigma_{x_j} = D\epsilon_{x_j} + \sigma^i{}_{x_j}(\epsilon) \tag{8}$$

$$\epsilon_{x_j} = BU_{x_j} \tag{9}$$

Substituting equations (8)-(9) into equation (7) and considering δU as arbitrary,

$$KU_{x_j} = \int N^T f_{x_j} dV - \int B^T \sigma^i{}_{x_j}(\epsilon) dV \tag{10}$$

where the pseudo-load is

$$F_{ps} = \left(\int N^T f_{x_j} dV - \int B^T \sigma^i{}_{x_j}(\epsilon) dV\right) \tag{11}$$

The pseudo-load can therefore be computed by defining the fields of body force f_{xj} and initial stress σ^i_{xj}. The sensitivity information U_{xj} can be solved by equation (10) once the pseudo-load has been computed.

For *sizing optimization* as the geometry of mesh remains unchanged, the change of design variable x_j does not influence the body force f, therefore, the first part of equation (11) is nil. Concerning the second part of (11), the deviations of current stress field with fixed mesh could be anologied by a initial strain field load. As most of commercial FEA programs provided the initial strain field load, it is not difficult to implement. For example, the imposed pseudo-load for truss-type structures is formulated as follows,

$$F_{x_j} = \int B^T AE\left(-\frac{\sigma}{AE}\right) dl \tag{12}$$

where A is the cross sectional area of the bar, the initial strain is given as $-\sigma/AE$.

TWO METHODS

Under the strategy of differentiation of discretized systems, mathematically there are two ways of calculation of the derivatives of the solutions of those equations with respect to their coefficient, direct elimination and the adjoint (Lagrange Multipliers) method.

Mathematically, we consider the equations of constraints and its derivatives

$$g(u,x) \geq 0 \tag{13}$$

under the constraint of the equations of equilibrium

$$\mathbf{K}(\mathbf{x})\,\mathbf{U} = \mathbf{F} \tag{14}$$

Method 1: direct eliminating (peosudo load) method

Differentiating the equation above with respect to x_j, we obtain

$$KU_{x_j} = F_{x_j} - K_{x_j} \tag{15}$$

so that

$$\frac{dU}{dx} = K^{-1}F_{ps} = K^{-1}[F_{x_j} - K_{x_j}U] \tag{16}$$

is reached. We may obtain the value of pseudo load F_{ps} with analytic and semi-analytic iteration solutions.

Method 2: the adjoint (Lagrange Multipliers) method

We may state the sensitivity problem as follows: to find an expression for the differential change in a valued function

$$g(\mathbf{U},\mathbf{x}) \tag{17}$$

due to differential changes in x, in which u is determined implicitly by the design variables x through the constraint relations

$$\mathbf{f}(\mathbf{x},\mathbf{U}) = \mathbf{K}(\mathbf{x})\,\mathbf{U} - \mathbf{F} = \mathbf{0} \tag{18}$$

In the approach, g is adjoined to the constraint, to form the Lagrangian

$$\mathbf{L}(\mathbf{x},\mathbf{U},\lambda) = \mathrm{g}(\mathbf{U},\mathbf{x}) + \lambda^T(\mathbf{Ku}-\mathbf{F}) \tag{19}$$

in which $\lambda_1,\dots,\lambda_n$ are a set of under-determined multipliers or adjoint variables. Then a differential change in L due to differential changes in x and U is given by

$$\mathrm{dL} = \mathrm{L_x}\,\mathrm{d}\mathbf{x} + \mathrm{L_u}\,\mathrm{d}\mathbf{U} \tag{20}$$

Since we are interested in how L and, thus, g changes as **x** change, it is conventient to choose the λ vector so that

$$\mathrm{L_u} = \mathbf{g_u} + \lambda^T\mathbf{K} = 0 \tag{21}$$

$$\lambda^T = \mathbf{K}^{-1}(-\mathbf{g_u}) \tag{22}$$

Since U was found from $\mathbf{K}(\mathbf{x})\,\mathbf{U} - \mathbf{F} = \mathbf{0}$

$$dg \equiv dL = L_x dx = g_x + \lambda^T(F_x - K_x u) \tag{23}$$

The adjoint method is also known as the *dummy-load method* because g_u is often described as dummy load. When g in Eq.(22) is a upper limit on displacement, the dummy load also has a single nonzero component corresponding to the constrained displacement component. Similarly, when g_u is an upper limit on the stress in a truss member, the dummy load is composed of a pair of equal and opposite force acting on the two ends of the member. It is suitable that when we can write the constraint explicitly in the term of nodal displacement. The second part of equation (23) could be solved by different solutions discussed later.

Both methods require the solution for the additional loading cases. In pseudo-load method the number of additional loads is *n.c*, where *n* is the number of design variables and *c* is the number of applied loading case. With the adjoint method the number is *m*, where the *m* is the total number of constraints. When the problem has a large number of design and loading cases, the adjoint method has considerable advance.

THREE SOLUTIONS

Generally, the following three solutions are used under one of the methods discussed before in structural optimization:

(1) the *analytic solution*

The pseudo load F_{ps} in (16) is coded directly. The analytic method is popular due to its accuracy, but it requires the users' access to the detailed-level computation of each type of element in a finite element program. Most commercial FEA programs unfortunately do not provide user with that level of information.

(2) the overall finite difference method

The overall finite difference method uses the following equation

$$\frac{dU}{dx} \cong \frac{\Delta U}{\Delta x} = \frac{K^{-1}(x+\Delta x)F - K^{-1}(x)F}{\Delta x} \qquad (16)$$

Since it is very easy to implement, the overall finite difference method has its advantage. However, the solution has two serious shortcomings. First, its accuracy is very dependent on the choice of size if perturbation, the improper choice of perturbation may cause truncation or conditional error in computation. Second, it is computationally expensive especially for large scale problems which include a large number of design variables.

(3) the *semi-analytic solution*

The semi-analytic method is considered as compromise between the other two. In the solution, the equation (16) is calculated by finite deference, ie

$$\frac{dU}{dx} = K^{-1}[F_{ps}] \approx K^{-1}\left[\frac{\Delta F}{\Delta x} - \frac{\Delta K}{\Delta x}U\right] \qquad (17)$$

NUMERICAL TEST EXAMPLE

The well-known ten-bar truss problem is selected as an example, as shown in FIG.1. It is subjected to a loading condition P. The cross-sectional area of each member was selected as design variables. The values *L*, *P* and *E* are 250*cm*, 100*N*, and 200*GPa*, respectively. We considered the design sensitivity information of 8 nodal displacements with respect to design x_1 at ($\mathbf{x}$: $x_1 = ... = x_{10} = 5.0cm^2$) for illumination.

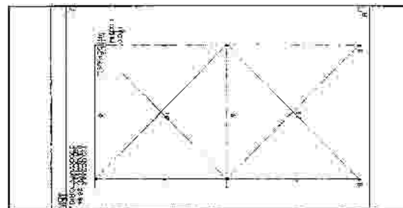

FIG.1 Ten-Bar Problem

The variational strategy, semi-analytic solution, and finite difference solution under the difference of discrete FE system strategy, with the direct eliminate method integrated with commercial finite element programs ADINA had been implemented, respectively. The flowcharts of different ways are shown in FIG.2.

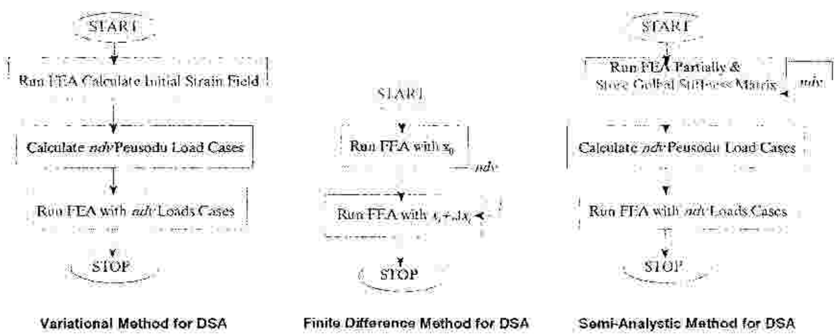

FIG.2 Flowcharts of difference Methods

Variational Strategy:

We can easily obtain the design sensitivity information by first running a structural analysis and then applying the initial strain by equation (12) to the original structure of the problem. The ADINA provide the initial strain loads case. The design sensitivity of nodal displacements with respect to design variable x_1 are shown in Table 1.

Finite Difference Method:

A set of design sensitivity information is gained by employing forward finite difference iterative method with step sizes from 10^{-1} to 10^{-8}. The design sensitivity of nodal displacements with respect to design variable x_1 are shown in the tables. The biggest disadvantage of this method is the truncated and conditional error problem. With the step sizes of 10^{-2} and 10^{-3}, the results are quite an acceptable comparison with the variational method (Table.1). The solution is not reliable when the step size is more than or/and equal 10^{-1} as well as less than 10^{-5}.

Semi-Analytic Method:

The same problem is implemented by using ADINA on IBM/RISC6000 as shown in the tables. First a perturbing stiffness matrix with defined step size was gained by using ADINA internal option (TAPESTORE IKMATRIX=1). Then pseudo load was calculated by equation (17) with user support load cases. After that, the design sensitivity information was gained by applying the pseudo load to the original structure. In the problem, the solution is reliable when the step size is from 10^{-1} to 10^{-7}.

Table 1.

	Variational Method (VM) $\times 10^{-8}$	Semi-Analytic Method (SAM)$\times 10^{-8}$		Finite Difference Method (FDM)$\times 10^{-8}$	
		$\Delta x_i/x_i$		$\Delta x_i/x_i$	
		0.001	0.0001	0.001	0.0001
du_1/dx_1	2.10143	2.09940	2.10121	2.09790	2.19978
du_2/dx_1	-1.25920	-1.25799	-1.25908	-1.25674	-1.25987
du_3/dx_1	2.11846	2.11441	2.11824	2.09790	1.99980
du_4/dx_1	.147425	.147282	.147410	.145854	.139986
du_5/dx_1	7.77026	7.76276	7.76948	7.79220	7.99920
du_6/dx_1	-.124217	-.124097	-.124205	-.124076	-.123988
du_7/dx_1	.629602	.628930	.629538	.639361	.799920
du_8/dx_1	.164454	.164296	.164438	.163836	.159984

Table 2. Ten-bar truss design sensitivity by FDM with difference step size

$\Delta x_i/x_i$	$du_1/dx_1 \times 10^{-7}$	$du_2/dx_1 \times 10^{-7}$	$du_3/d\ _1 \times 10^{-7}$	$du_4/dx_1 \times 10^{-8}$
10^{-1}	.175545	-.105185	.176945	.123145
10^{-2}	.206337	-.123584	.207723	.144554
10^{-3}	.209790	-.125674	.209790	.145854
10^{-4}	.219978	-.125987	.199980	.139986
10^{-5}	.199998	-.139999	.199989	0
10^{-6}	0	-.199998	0	0
10^{-7}	0	0	0	0
Relative error(%) with VM $[(y-y_0)/y_0]$				
10^{-1}	-16.46	16.47	-16.47	-16.47
10^{-2}	-1.18	1.86	-1.95	-1.95

10^{-3}	-0.17	0.20	-0.97	-1.07
10^{-4}	-4.68	-0.05	-5.60	-5.00
10^{-5}	-4.83	11.18	-5.60	-100.00
10^{-6}	-100.00	58.83	-100.00	-100.00
10^{-7}	-100.00	-100.00	-100.00	-100.00

Table 3. Ten-bar truss design sensitivity by SAM with difference step size

$\Delta x/x_i$	$du_i/dx_1 \times 10^{-7}$	$du_j/dx_1 \times 10^{-7}$	$du_j/dx_1 \times 10^{-7}$	$du_j/dx_1 \times 10^{-8}$
10^{-1}	.190913	-.114398	.192460	.133934
10^{-2}	.208048	-.124665	.209734	.145955
10^{-3}	.209940	-.125799	.211641	.147282
10^{-4}	.210121	-.125908	.211824	.147410
10^{-5}	.210141	-.125919	.211844	.147423
10^{-6}	.210143	-.125920	.211845	.147424
10^{-7}	.210143	-.125920	.211846	.147425
10^{-8}	.420284	-.251843	.423659	.294852
10^{-9}	0	0	0	0
Relative error(%) with VM $[(y-y_0)/y_0]$				
10^{-1}	-9.15	-9.15	-9.15	-9.15
10^{-2}	-1.00	-0.99	-1.00	-1.00
10^{-3}	-0.09	-0.10	-0.10	-0.10
10^{-4}	-0.01	-0.01	-0.01	-0.01
10^{-5}	0.0	0.0	0.0	0.0
10^{-6}	0.0	0.0	0.0	0.0
10^{-7}	0.0	0.0	0.0	0.0
10^{-8}	100.00	100.00	100.00	100.00
10^{-9}	-100.00	-100.00	-100.00	-100.00

CONCLUSION

The numerical results of test example have shown that finite difference method can be easily implemented into any commercial finite element programs. The finite difference is, however, very inefficient for large-scale problem which have large numbers of design variables. Furthermore, the accuracy of design sensitivity analysis is very dependent on the magnitude of step size.

The variational strategy is very attractive for its numerical efficiency and absence of the truncated error problem. It is easy to perform design sensitivity analysis in any commercial finite element program with a suitable pseudo-load. However, the initial strain of the new imposed pseudo load for different kinds of elements, with respects to different types of design variables, require a large research effort to find its exact form. As for size optimization, due to the geometry mesh unchanged, it is easy to achieve the simple form by equation (11). Also, because most commercial finite element programs provide initial strain field loads, it is strong recommended to use variational strategy with adequate formulation in size optimization.

The semi-analytic method is a practical compromise.

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