Reliability of Portal Frames with Interacting Stress Resultants

By Luis Miguel da Cruz Simões[1]

Abstract: A general formulation based on mathematical programming and structural reliability theory is given for the analysis and synthesis of portal frames where yield with respect to plastic collapse is governed by several stress resultants. The mechanism compatibility equations are formulated using the generalized mesh description. The elasto-plastic material is assumed to satisfy Drucker's postulate of stability. The stochastically most important modes of the reliability assessment problem are found by minimizing a quadratic concave function over a linear domain. Mathematical programming techniques that include recent developments in concave quadratic minimization are presented. The paper also describes a first-order second-moment reliability-based approach to the optimum design of ductile frames. It involves an iterative process, which is repeated until the best reliability-based design is obtained. This design has a preassigned reliability level against plastic collapse and simultaneously minimizes the prescribed objective function. Examples are solved by employing the proposed computational techniques.

Introduction

Structural problems are nondeterministic and, consequently, engineering optimum design must cope with uncertainties. Clearly, the proper tool for the assessment and analysis of such uncertainties requires methods and concepts of reliability. Therefore, it is not an overstatement to affirm that the combination of reliability-based design procedures and optimization techniques are the only means of providing a powerful tool to obtain a practical optimum design solution. Most of the work on plastic reliability analysis is founded on the upper bound theorem of plasticity. According to this, an upper bound on the reliability can be evaluated on the basis of a set of plastic mechanisms. If the set of mechanisms is complete, the upper bound coincides with the exact reliability with respect to plastic collapse. For this reason an automatic method to find all modes can be employed, but for very large systems this procedure may become intractable. Most important in calculating the failure probability of structural systems is the search for the stochastically most relevant failure mechanisms. Variations of the parameters of the probability distributions, the respective types of distributions, and the correlation among the variables involved have quite a significant effect on the results. Approaches yielding a lower bound on the reliability with respect to plastic collapse were reported by Ditlevsen and Bjerager (1984). How[illegible], a close bound seems to be computationally expensive to obtain for more complex structures. The formulation of the plastic dissipation for a given mechanism in the case of yield surfaces of random shape can be rather involved. For this reason, most works considered structures without force interaction effects in the yield criteria. Bjerager (1989) considered force interaction effects by solving a nonlinear optimization problem. His method generates the failure modes in random order, and thus many of the important

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modes may be missed. There is no guarantee that the procedure will find the global optimum and generate the most important suboptimal solutions.

This paper describes a mathematical programming–based procedure for identifying the stochastically most relevant failure mechanisms. They are found by minimizing a qaudratic concave function over a linear domain. Mathematical programming techniques that include recent developments in concave quadratic minimization are presented. The generalized mesh description for frames of specified topology and geometry is used, leading to simpler, more efficient formulation of the mechanism compatibility equations (Munro 1979). The elasto-plastic material is assumed to satisfy Drucker's postulate of stability—a convex yield surface (Drucker 1951). An illustrative reliability assessment example is solved, showing the different probabilities of failure that are obtained when the structure is analyzed considering single and interacting stress resultants. Methods of optimization based on deterministic safety concepts, while minimizing the weight of an element, may change its level of safety. This is a major limitation of deterministic optimization formulations, in which the inherent random nature of both structural loading and strength is not included, and consequently, the safety criteria are not specified in terms of a risk value. This paper also describes a first-order second-moment reliability-based approach to the optimum design of ductile frames. The solution is obtained by solving alternately a reliability assessment problem and an optimal sizing program until the best reliability-based design is obtained. The proposed computational technique is illustrated on a portal frame with interacting stress resultants and some results are given concerning the sensitivity of the optimal solution to load and resistance correlations.

Structural Relations

Plasticity Relations

Following von Mises (1928), a plastic potential function $F(X)$ of the stress resultants X will be defined such that the strain resultant increments u are given by

$$u_j = \frac{\partial F}{\partial X_j \lambda} \qquad j = 1, 2, \ldots, 6 \qquad (1)$$

at critical section i where as many as six stress resultants may be interacting; and λ = a nonconstant parameter that is, however, the same for all stress resultants at the section i. Supposing that an element of material is subject to a set of stress resultants X_j^o, its stability may be tested by an external agency that can vary the applied stress resultants. Drucker's postulate for a stable material can be expressed by the following: (1) The plastic work done by the agency during application of these additional stress resultants is nonnegative; and (2) the net work done by the agency during the application of a cycle of additional stress resultants is nonnegative.

For the material loaded by a single stress resultant X, these ideas are obvious, as can be seen in Fig. 1. If this material is at state B, the addition of dX produces the triangle BCD of plastic work. The conditions

$$dXdu > 0; \qquad dXdu = 0; \qquad dXdu < 0 \qquad (2)$$

thus classify, respectively, a stable material, a perfectly plastic material, and an unstable material. Secondly, if the material is at state A, the cycle of loading $X^o \rightarrow X \rightarrow X + dX \rightarrow X \rightarrow X^o$ produces net positive work dW,

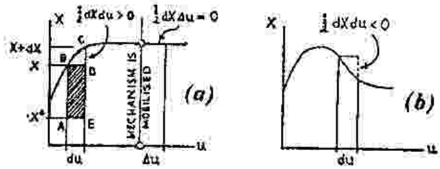


FIG. 1. Single Stress Resultant Material Characteristics: (*a*) Stable Material; and (*b*) Unstable Material

which is substantially *ABCDE* (neglecting second-order work).

$$dW = (X - X^o)du \geq 0 \quad \ldots\ldots (3)$$

A material loaded by several stress resultants (Fig. 2) may have some of these increasing, while others may be decreasing, and yet the material may still be stable under load. Drucker's discriminant of stability adds together the work contributions from all the stress resultants applied to the material element. Thus, for a perfectly plastic material

$$\mathbf{dX}^t du = 0 \quad \ldots\ldots (4a)$$

$$(\mathbf{X} - \mathbf{X}^o)^t \mathbf{du} \geq 0 \quad \ldots\ldots (4b)$$

If, corresponding to a resultant set $\mathbf{X}$, the strain resultant increment du is supposed known and a hyperplane H through $\mathbf{X}$ with $\mathbf{du}$ as normal is constructed, then Eq. 4*b* implies that the vector $(\mathbf{X} - \mathbf{X}^o)$ lies on the opposite side of H from $\mathbf{du}$ for any admissible H^o. Thus, H is a supporting hyperplane, so that the yield surface is convex and $\mathbf{du}$ follows the normality rule. Drucker's postulate guarantees the existence of a potential function and identifies one as the yield function $\phi(X)$. For this particular choice of potential function during the mechanism, increment u is normal to $\phi(X)$ and

$$u_j = \frac{\partial \phi}{\partial X_j} \lambda; \qquad \lambda \geq 0 \quad \ldots\ldots (5)$$

is said to be an associated flow rule. A generalization due to Koiter (Smith 1974) allows the safe region to be bounded by a finite set of yield functions $\phi_k(X) = 0$, whereupon Eq. 5 must be written

$$\Sigma_k \frac{\partial \phi_k}{\partial X_j} \lambda_k; \qquad \lambda_k \geq 0 \quad \ldots\ldots (6)$$

Then, at a singular point like A in Fig. 2, a vector $\mathbf{n}$ must lie in the cone of outward normals. At a critical section i several stress resultants may in-

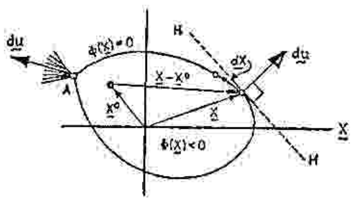


FIG. 2. Stress Resultant Space for Material Element

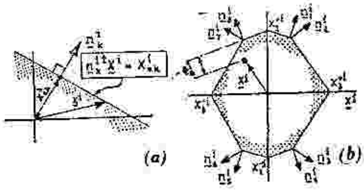


FIG. 3. Interaction Diagram for Single Critical Section: (*a*) Yield Hyperplane; and (*b*) Yield Polytope

teract, thus generating a safe polytope in a subspace X^i that would have as many as six dimensions. Usually, only two or three of these stress resultants produce significant interaction. Fig. 3 shows a subspace for the interaction of moment X_1^i and axial force X_3^i approximated by eight planes with unit normals $\mathbf{n}_k^i$.

Each such plane or, more generally, hyperplane, can be written in terms of these normals and corresponding perpendicular distances X_{*k}^i

$$\phi_k^i(\mathbf{X}^i) = (\mathbf{n}_k^i)^t\mathbf{X}^i - X_{*k}^i = 0 \quad (7)$$

This information can be assembled into the plasticity relations for all critical sections

$$\mathbf{N}^t\mathbf{X} - \mathbf{X}_* = \mathbf{0} \quad (8a)$$

$$\mathbf{u} = \mathbf{N}\mathbf{u}_* \quad (8b)$$

$$U = \mathbf{X}_*^t\mathbf{u} \quad (8c)$$

$$\boldsymbol{\phi}_* \leq \mathbf{0} \qquad \mathbf{u}_* \geq \mathbf{0} \qquad \boldsymbol{\phi}_*^t\mathbf{u}_* = 0 \quad (8d)$$

where $\mathbf{U}$ = the plastic energy dissipation. The number of hyperplanes being used to effect a linear approximation to the plasticity relations is h. Then $\mathbf{X}_*$ lists the h components X_{*k}^i, while the h rows of $\mathbf{N}^t$ contain the components of $\mathbf{n}_k^{i}$ ordered to correspond with the siting of X^i in the S-vector $\mathbf{X}$ of stress resultants. Finally the h-vector $\mathbf{u}_*$ of activation parameters represents the amplitudes of activation of the corresponding yield hyperplanes, $\boldsymbol{\phi}_* = \mathbf{0}$ being the parameters λ_k in the associated flow rule, while $\mathbf{N}$ corresponds to $\partial\phi_k/\partial X_j$.

Mesh Description of Statics and Kinematics

It is necessary to consider a structure for which a set of collapse stress resultants is $\mathbf{X}_c$ and an associated collapse mechanism is $\mathbf{u}_c$. At any critical section several of the collapse strain resultants $\mathbf{u}_c$ may be supported simultaneously. For this more general situation, Drucker's postulate for stable plastic materials is sufficient to ensure the convexity of this safe region and also the normality of the flow rule for $\mathbf{u}_c$. Fig. 4 shows an approximating yield polytope in stress resultant space for the whole structure and illustrates the fundamental inequality

$$(\mathbf{X} - \mathbf{X}_c)^t\mathbf{u} \geq 0 \quad (9)$$

where $\mathbf{u}$ = another kinematically admissible mechanism; and $\mathbf{X}$ = a set of stress resultants. This set does not satisfy equilibrium in general, but it is defined to be associated with $\mathbf{u}$, i.e., from Eq. 8*c*

$$\mathbf{X}^t\mathbf{u} = U = \mathbf{X}_*^t\mathbf{u}_* \quad (10)$$

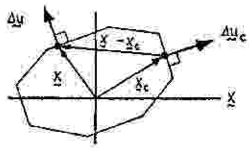


FIG. 4. Yield Polytope for Structure in Stress Resultant Space

If the flow rule is followed by $\mathbf{u}$, from Eq. 8*b*

$$\mathbf{u} = \mathbf{N}\mathbf{u}_* \quad (11)$$

It is assumed that several of the stress resultants incident at a critical section are sufficiently large that none of them can be considered to govern the criterion of plasticity or yield condition. The dual descriptions of statics and kinematics on a mesh basis are

$$\mathbf{X} = \mathbf{B}\mathbf{p} + \mathbf{B}_o\mathbf{L} \qquad \begin{matrix} \boldsymbol{\delta} = \mathbf{B}_o'\mathbf{u} \\ \mathbf{0} \quad \mathbf{B}' \end{matrix} \quad (12)$$

where $\mathbf{B}$ and $\mathbf{B}_o$ = matrices containing the significant stress resultants caused by unit magnitudes of indeterminate forces $\mathbf{p}$ and loading $\mathbf{L}$ acting upon the released or determinate structure. The most convenient basis for planar portal frames is that of regional meshes (Munro 1979). The $\mathbf{B}$ matrix can be assembled in terms of a small number of submatrices, and the elements of $\mathbf{B}_o$ can be obtained from any set of stress resultants that are in equilibrium with the unit loads. The kinematic relations for the mesh description to ensure compatibility, then, are

$$\mathbf{B}'\mathbf{N}\mathbf{u}_* = \mathbf{0} \quad (13)$$

The displacement rates δ that correspond to the loads $\mathbf{L}$ can be evaluated in terms of the strain resultant rates $\mathbf{u}_*$

$$\boldsymbol{\delta} = \mathbf{B}_o'\mathbf{N}\mathbf{u}_* \quad (14)$$

Reliability Assessment

Assumptions

The following assumptions are considered: (1) The general structural configuration including the lengths of all prismatic and straight members is specified in a fixed (deterministic) manner; (2) plastic collapse in which yield is governed by interacting stress resultants is the only possible failure mode; (3) the effects of shear and torsion are not considered; (4) the magnitudes of static loads that form the load vector $\mathbf{L}$ are random, but their locations are deterministic; (5) each yield surface has a deterministic shape with a random size and is described by a single random variable. Without any complexities, the random shape of each surface can be modeled by associating a random resistance variable to each face of the surface; and (6) the ultimate plastic capacities for both beam and column critical sections, which form the vector $\mathbf{X}$, are random, but their position is deterministic.

Collapse of Ductile Structures

The probability of failure via the *k*th individual collapse mode p_k can be obtained from the probability that a certain performance function Z_k

$$Z_k = U_k - E_k = \mathbf{X}'\boldsymbol{\Theta}_* - \mathbf{L}'\boldsymbol{\delta}_* \quad (15)$$

negative. In Eq. 15 U_k and E_k are the internal and external random works associated with the kth collapse mode. The strength of the structure with respect to the kth mode, U_k, is represented by a linear combination of random plastic capacities $\mathbf{X}$, and the load effect with respect to this mode, E_k, is represented by a linear combination of random loads $\mathbf{L}$. Consistent with first-order second-moment reliability analysis, the failure probability may be measured entirely with a function of the first and second moments of random parameters. It is assumed that safety with regard to plastic collapse via the failure mode k depends only on reliability index β_k, which is defined as the shortest distance from the origin to a failure surface in the reduced random variables coordinate system

$$\beta_k = \frac{\mu_{Z_k}}{\sigma_{Z_k}} \qquad (16)$$

Overall Probability of Failure

The exact evaluation of the overall probability of failure can be based on the following general result (Moses and Kinser 1967):

$$P_f = P(F_1) + \Sigma_{i=2,m} P(F_i)P(S_1 \cap S_2 \cap \ldots \cap S_{i-1}|F_i) \qquad (17)$$

in which $P(F_i)P(S_1 \cap S_2 \cap \ldots \cap S_{i-1}|F_i)$ = the conditional probability that the first $i - 1$ modes survive, given that mode i occurs. The numerical effort for the evaluation of the conditional probabilities is exorbitant, even for small systems. These difficulties can be overcome by limiting the scope of analysis to bounds on P_f. In general, the admissible failure probability for structural design is very low. Commonly used approximations of the overall probability of failure are based either on the assumption of perfect statistical dependence (Cornell's lower bound) or on that of their statistical independence (Cornell's upper bound). These upper and lower bounds may be widely different because the correlation between failure modes is not included in the formulation. Ditlevsen's method (1979), which incorporates the effects of the statistical dependence between any two failure modes, narrowed considerably the bounds on the system failure probability. The method introduced by Vanmarcke (1973) reduces the number of survival events in Eq. 17 to one, obtaining an upper bound to the overall probability of collapse. Using a first-order approach, Vanmarcke introduced a useful approximation of the conditional probability $P(S_j|F_i)$ in terms of the safety indices and the coefficient of correlation between the failure modes F_i and F_j. A different approximate method, which avoids calculating conditional probabilities resulting from conditions leading to failure via pairs of failure modes, is the PNET (Ang and Ma 1981). This method requires the determination of the coefficients of correlation between any two failure modes i and j and is bsaed on the notion of a demarcating correlation coefficient, assuming those failure modes with high correlation to be perfectly correlated and those with low correlation to be statistically independent.

Computation of Reliability Index

For Gaussian random variables, the identification of the stochastically most relevant mechanism consists of minimizing the reliability index β given by (Shinozuka 1983)

$$\beta = \frac{\mu_x'\theta_* - \mu_L'\delta_*}{\sqrt{\theta_*'\sigma_X'C_X\sigma_X\theta_* + \delta_*'\sigma_L'C_L\sigma_L\delta_*}} \qquad (18)$$

subject to the compatibility relations (Eq. 13), the linear incidence equations

$$\boldsymbol{\theta}_* = \mathbf{J}_\theta \mathbf{u}_*; \qquad \boldsymbol{\delta}_* = \mathbf{J}_\delta \boldsymbol{\delta} \quad (19)$$

where the incidence matrices $\mathbf{J}_\theta$ and $\mathbf{J}_\delta$ are obtained by associating the strain resultant-rates $\mathbf{u}_*$ represented by the same random variables $\boldsymbol{\theta}_*$ and the displacements of the point loads $\boldsymbol{\delta}$ linked by the same random variables $\boldsymbol{\delta}_*$. Sign constraints on the variables also need to be considered

$$\mathbf{u}_* \geq \mathbf{0}, \ \boldsymbol{\delta}_* \geq \mathbf{0} \quad (20)$$

If the probability distribution functions of the random variables are not Gaussian, the Rosenblatt transformation (1952) may be used. This mathematical program belongs to the class of fractional programming problems. The minimization of β shares its solutions with the quadratic concave minimization (Simões 1990)

$$\min \ \frac{-1}{\beta^2} = -\boldsymbol{\theta}'_* \boldsymbol{\sigma}'_X C_X \boldsymbol{\sigma}_X \boldsymbol{\theta}_* - \boldsymbol{\delta}'_* \boldsymbol{\sigma}'_L C_L \boldsymbol{\sigma}_L \boldsymbol{\delta}_* \quad (21a)$$

subject to

$$\boldsymbol{\mu}'_X \boldsymbol{\theta}_* - \boldsymbol{\mu}'_L \boldsymbol{\delta}_* = 1 \quad (21b)$$

$$\boldsymbol{\theta}_* = \mathbf{J}_\theta \mathbf{u}_*; \qquad \boldsymbol{\delta}_* = \mathbf{J}_\delta \boldsymbol{\delta} \quad (21c)$$

$$\mathbf{B}'\mathbf{N}\mathbf{u}_* = 0; \qquad \boldsymbol{\delta} = \mathbf{B}'_o \mathbf{N}\mathbf{u}_* \quad (21d)$$

$$\mathbf{u}_* \geq 0, \ \boldsymbol{\delta}_* \geq 0 \quad (21e)$$

This problem cannot be solved by convex programming techniques because of the possibility of nonglobal local minima. The global optimum of these programs gives the plastic deformations for the stochastically most important mechanism. The reduced random variables can be evaluated using

$$\mathbf{X}' = -\boldsymbol{\sigma}_X \boldsymbol{\theta}_* \beta^2; \qquad \mathbf{L}' = \boldsymbol{\sigma}_L \boldsymbol{\delta}_* \beta^2 \quad (22)$$

Solution of Concave Quadratic Minimization Problem

This constrained optimization problem is an NP-hard problem. From a computational complexity point of view, this means that in the worst case the computer time will grow exponentially with the number of nonlinear variables. The most general methods for global optimization that are appropriate to deal with nonconvexities can be divided in two classes: deterministic and stochastic. Among the former, the most important approaches for concave quadratic programming are enumerative techniques, cutting-plane methods, branch and bound, bilinear programming methods, or different combinations of these techniques. Computational results on most of these methods seem to be limited to cases where the number of nonlinear variables is less than 20. The few implementable approaches are for functions of a special structure such as quadratic or separable concave, and employ branch and bound techniques in conjunction with underestimating linear (or piecewise linear) problems (Falk and Soland 1969). The algorithm described here is more appropriate to solve concave quadratic minimization of large systems, solving reliability assessment problems very efficiently. It treats the linear variables in a different manner than those appearing linearly and provides narrow bounds for the objective function values.

Methods for Large-Scale Concave Quadratic Programming

The motivation for considering this type of problem is similar to that for problems zero-one integer linear programming. Large-scale zero-one mixed integer programming can be solved in a reasonable time (Marsten 1987), provided that most of the variables are continuous. The computational method presented by Rosen (1983) for finding the global minimum of a quadratic concave function over a polyhedral set takes advantage of the ellipsoid-like level surfaces of the objective function to find a good initial vertex and to eliminate a rectangular domain (enclosed in a level surface) from further consideration. The basic step is to determine initially a rectangular domain that contains the projection of the domain on the space of the nonlinear variables. This can be done by a multiple-cost-row LP with n objective functions. Then, a linear underestimating function is computed and a linear underestimating problem is solved to give lower and upper bounds for the global optimum. This solution also gives a bound on the relative error in the function value of this incumbent vertex. If the incumbent is not a satisfactory approximation to the global optimum, a guaranteed ϵ-approximate solution is obtained by solving a single zero-one mixed integer programming problem. This integer problem is formulated by a piecewise linear underestimation of the separable problem.

Reduction to Separable Form

This transformation is required only if the random variables are statistically dependent; Eq. 21 is already in the separable form if the nonlinear variables are uncorrelated. For simplicity of notation, the concave quadratic program can be written in the following form:

$$\min \psi(z, y) = -\frac{1}{2} \mathbf{z}'\mathbf{Q}\mathbf{z}; \quad \text{over } \Omega = [(\mathbf{z}, \mathbf{y}) : \mathbf{A}_1\mathbf{z} + \mathbf{A}_2\mathbf{y} = \mathbf{b}, \mathbf{z} \geq \mathbf{0}, \mathbf{y} \geq \mathbf{0}] \quad \ldots\ldots (23)$$

with $\mathbf{Q}$ = a positive definite symmetric matrix; and $\mathbf{A}_1$ and $\mathbf{A}_2$ having m rows. $\mathbf{z}$ and $\mathbf{y}$ are n and m vectors corresponding to the random variables and rotations of the critical sections, respectively. To carry out the reduction to separable form, it is necessary to compute the real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $\mathbf{Q}$ and the corresponding eigenvectors $u_1, u_2, \ldots, u_n$. Then $\mathbf{Q} = \mathbf{UDU}^t$, where $\mathbf{U} = [u_1, \ldots, u_n]$, $\mathbf{D} = \text{diag}[\lambda_1, \ldots \lambda_n]$. The multiple-cost-row linear program must be solved first

$$\max \mathbf{u}_i^t\mathbf{z}; \quad \text{subject to } (\mathbf{z}, \mathbf{y}) \in \Omega, \qquad i = 1, 2, \ldots, n \quad \ldots\ldots (24)$$

Denote by β_i the corresponding optimal values. The concave quadratic programming can be formulated as a separate program in terms of the new variables x_i

$$\min \phi(x) = \Sigma_{i=1,n} -\frac{1}{2} \lambda_i x_i^2 \quad \ldots\ldots (25a)$$

$$\text{subject to } \mathbf{A}_3\mathbf{x} + \mathbf{A}_2\mathbf{y} = \mathbf{b}, \qquad 0 \leq x_i \leq \beta_i, \qquad \mathbf{y} \geq 0 \quad \ldots\ldots (25b)$$

where $\mathbf{A}_3 = \mathbf{A}_1\mathbf{U}$.

Linear Underestimator and Error Bounds

The smallest rectangular domain R_x in the x-space can be constructed using β_i. A linear function $\Gamma(\mathbf{x})$, which interpolates $\phi(\mathbf{x})$ at every vertex of R_x and underestimates $\phi(\mathbf{x})$ on R_x, is given by

$$\Gamma(\mathbf{x}) = \Sigma_{i=1,n} -\frac{1}{2} \lambda_i \beta_i x_i; \quad \text{and} \quad R_x = (\mathbf{x}: 0 \leq x_i \leq \beta_i; i = 1, \ldots, n) \quad \ldots\ldots (26)$$

The following linear underestimating program, which differs from the multiple-cost row only in its objective function, must be solved:

$$\min \Gamma(\mathbf{x}) \text{ over } (\mathbf{x},\mathbf{y}) \in \Omega \quad (27)$$

The solution to this problem will give a vertex $\underline{\mathbf{v}} = (\underline{\mathbf{x}}, \underline{\mathbf{y}})$ that is a candidate for the global minimum ψ^* of the original problem and

$$\Gamma(\underline{\mathbf{x}}) \leq \psi^* \leq \phi(\underline{\mathbf{x}}) \quad (28)$$

Therefore, the error at $(\underline{\mathbf{x}}, \underline{\mathbf{y}})$ is given by $\phi(\underline{\mathbf{x}}) - \psi^*$, and this error is bounded by

$$E(\underline{\mathbf{x}}) = \phi(\underline{\mathbf{x}}) - \Gamma(\underline{\mathbf{x}}) \quad (29)$$

If $E(\underline{\mathbf{x}})$ is sufficiently small, $\psi(\underline{\mathbf{x}}, \underline{\mathbf{y}})$ is an acceptable approximation to the global optimum ψ^*. It is necessary to obtain bounds on $E(\underline{\mathbf{x}})$ relative to the range of $\phi(\mathbf{x})$ over R_x. The quantity

$$= -\min_{\mathbf{x}\in R_x} \phi(\mathbf{x}) \quad (30)$$

is used as a scaling factor to measure $E(\mathbf{x})$ on Ω.

Assuming, without loss of generality, that

$$\lambda_1\beta_1^2 \geq \lambda_i\beta_i^2; \qquad i = 2, \ldots, n \quad (31)$$

and defining the ratios

$$\rho_i = \lambda_i \frac{\beta_i^2}{\lambda_1} \beta_1^2; \qquad i = 2, \ldots, n \quad (32)$$

the lower bound on $\Delta\phi$ is given by

$$\Delta\phi \geq \frac{1}{2} \lambda_1\beta_1 \Sigma_{i=1,n} \rho_i \quad (33)$$

$E(x)$ attains its maximum at $x_i = \beta_i/2$, $i = 1, \ldots, n$, so that for any $x \in R_x$

$$E(\mathbf{x}) = \phi(\mathbf{x}) - \Gamma(\mathbf{x}) = \frac{1}{2} \Sigma_{i=1,n} \lambda_i(\beta_i - x_i)x_i \leq \frac{1}{8} \lambda_1\beta_1 \Sigma_{i=1,n} \rho_i$$

An a priori bound on the relative error is given by

$$\frac{\psi(\underline{\mathbf{x}}, \underline{\mathbf{y}}) - \psi^*}{\Delta\phi} \leq 0.25 \quad (34)$$

…ewise Linear Approximation and Zero-One Integer Formulation

Using piecewise linear approximation to each function $-1/2\ \lambda_i x_i^2$, a mixed integer zero-one LP can be formulated such that the finding of a solution $(\underline{\mathbf{x}}, \underline{\mathbf{y}})$ for which

$$\frac{\psi(\underline{\mathbf{x}}, \underline{\mathbf{y}}) - \psi^*}{\Delta\phi \leq \epsilon} \quad (35)$$

can be guaranteed for any specified tolerance ϵ. Each interval $(0, \beta_i)$ is partitioned into k_i equal subintervals of length $h_i = \beta_i/k_i$, and the new variables w_{ij} are introduced, such that the variable x_i is represented uniquely by w_{ij}

$$x_i = h_i \Sigma_{j=1,k_i} w_{ij} \quad (36)$$

where the variables w_{ij} are restricted to the range $(0, 1)$ and the vector $(w_{i1}, w_{i2}, \ldots, w_{ik_i})$ is restricted to have the form $(1, \ldots, 1, w_{il}, 0, \ldots, 0)$.

Assuming, without loss of generality, that

$$0 < p_n \le p_{n-1} \le \ldots \le p_1 = 1 \qquad (37)$$

k_i is the smallest integer chosen such that

$$k_i \ge \left[\frac{n\rho_i}{(4\epsilon \sum_{i=1,n} \rho_i)}\right]^{1/2} \quad \text{where} \quad 1 \le k_n \le k_{n-1} \le \ldots k_1 \qquad (38)$$

By defining

$$\Delta_{ij} = -\frac{1}{2}\lambda_i j^2 h_i^2 + \frac{1}{2}\lambda_i (j-1)^2 h_i^2; \qquad i = 1, \ldots, n; \qquad j = 1, \ldots, k_i \qquad (39)$$

It follows that the linear function

$$\Gamma_i(x_i) = \sum_{i=1,k_i} \Delta_{ij} w_{ij} \qquad (40)$$

interpolates $-1/2\ \lambda_i x_i^2$ at the points $x_i = jh_i j = 0, 1, \ldots, k_i$, and since $-1/2\ \lambda_i x_i^2$ is concave, it satisfies $\Gamma_i(x_i) \le -1/2\ \lambda_i x_i^2$ for $x_i \in (0, \beta_i)$. That is, $\Gamma_i(x_i)$ is a piecewise linear underestimating function for $-1/2\ \lambda_i x_i^2$.

Therefore, if the objective function $\psi(x, y)$ is approximated by the piecewise linear underestimating function $\sum_{i=1,n} \Gamma_i(x_i)$, the separable quadratic concave minimization can be approximated by the following zero-one mixed integer problem in the continuous variables w_{ij} and the binary variables z_{ij}:

$$\min \sum_{i=1,n} \sum_{j=1,k_i} \Delta_{ij} w_{ij} \qquad (41a)$$

$$\text{subject to } \sum_{i=1,n} h_i a_i \sum_{j=1,k_i} w_{ij} + A_2 y \qquad (41b)$$

$$0 \le w_{ij} \le 1,\ y \ge 0 \qquad (41c)$$

$$w_{ij+1} \le z_{ij} \le w_{ij}; \quad z_{ij} \in \{0, 1\}; \quad j = 1, \ldots, k_i - 1; \quad i = 1, \ldots, n \qquad (41d)$$

where a_i is the ith column of $\mathbf{A}_3$.

Parallel Branch and Bound Algorithm

An algorithm for concave quadratic minimization that is designed to be efficient for problems with many design variables and that can take full advantage of parallel processing was recently presented (Phillips and Rosen 1988). It considers linear underestimating functions and upper and lower bounds on the global minimum in the way described herein. Branch and bound techniques are then applied to reduce the feasible region under consideration and decrease the difference between the upper and lower bounds. The average computational performance for problems with 25 nonlinear and 400 linear variables and a maximum error bound of $\epsilon = 0.001$ using a four-processor Cray2 was 15 sec. Results for problems with as many as 50 nonlinear and 400 linear variables showed that an approximate solution with a minimum of computation but with a relatively large bound ($\epsilon = 0.1$) can be obtained in a computational time that depends linearly on the number of nonlinear variables.

Enumeration of Other Stochastically Important Mechanisms

An appraisal of the current procedures for generating the stochastically most representative failure modes indicates that they are variously dependent on simulation, trial and error, perturbation, human judgment, complex heuristic strategies, or approximations, either for choosing the appropriate starting points or for continuing the method at different stages. Some of the methods generate the modes in random order, and thus many of the important modes may be missed without one's ever knowing about them. The techniques described herein to find the stochastically most representative mechanism, either by mixed integer linear programming or by the parallel processing algorithm, are associated with branch and bound strategies that reduce the feasible region to decrease the difference between upper and lower bounds. Both methods can be employed to enumerate other stochastically important modes by assigning the incumbent solution at a desired level of significance, larger than the global solution. Since the domain is partitioned with respect to the nonlinear variables only (random loads and resisting moments), it is possible that within the same range of bounds other mechanisms exist (and are not identified). Moreover, mechanisms with plastic hinges at different locations but associated with the same values of the random variables might be overlooked. For this reason, after finding the stochastically dominant mode over each of the subregions, one of the two procedures described next must be employed to enumerate the remaining mechanisms.

Branch and Bound Tree

Once some of the critical sections participating in the most representative mechanisms over each subregion detected earlier are ruled out of the basis, other modes can be identified by a branch and bound-based strategy. A strong branching rule is employed: the number of nodes created at each stage from an intermediate node is equal to the number of critical sections participating in the mechanism associated with the intermediate node. The result obtained at any node is a lower bound on those obtained by branching from it, and if the reliability index associated with that node is larger that a prespecified value, or unfeasible, then the leaf of the combinatorial tree can be terminated. Since a large number of problems created by branching at intermediate nodes have no feasible solution or large lower bound, the procedure is reasonably efficient.

Vertex Enumeration and Ranking

Murty's method (Murty 1969) can be used for ordering the extreme points of a linear domain. It is based on a theorem that states that if $\mathbf{x}^1, \ldots, \mathbf{x}^r$ are r best points, $\mathbf{x}^{r+1}$ will be an adjacent point of one of the first extreme points. The new point is distinct from the first r and maximizes the objective function, giving $1/\beta^2$ among all the remaining extreme points. All the adjacent extreme points are found from the canonical tableau corresponding to the linear domain $\mathbf{Ax} = \mathbf{b}$ by bringing one by one all the nonbasic variables—only those that correspond to rotations in the critical sections and that do not participate in the mechanism—into the basis. Denoting the entries in any canonical tableau by the usual simplex notation $a_{i\ell}$ b_i in which ℓ is the column index for any nonbasic column, define

$$\Delta_\ell = \min \frac{b_i}{a_{i\ell}} \ \forall\, i, \ell \text{ such that } a_{i\ell} > 0; \quad \Delta_\ell = +\infty, \text{ if } a_{i\ell} \le 0 \ \forall\, i, \ell \quad \ldots\ldots\ (42)$$

Variables corresponding to index i yielding the minimum value of W_i indicate the basic variable that will leave the basis when the variable corresponding to ℓ enters the basis. For each ℓ with finite Δ_ℓ, an adjacent point of $\mathbf{x}$ is given by

$$x^\ell(\Delta_\ell) = [x_1^\ell(\Delta_\ell) \ldots x_n^\ell(\Delta_\ell)]^t \ \ldots\ldots\ (43)$$

whose basis vector is found from

$x^{\ell}_{s_i} = b_i - a_{i\ell}\Delta_\ell; \quad x^{\ell}_1 = \Delta_\ell$ .... (44)

where s denotes a basis. At least one of the $x^{\ell}_{s_i}$ will always be zero. The procedure is repeated until we find either a prespecified number of extreme points or obtain all the extreme points in the ranked sequence whose objective value gives a reliability index that is less than a given value of β_{max}. On the basis of plastic limit analysis, failure modes were generated by Murty's method (Nafday et al. 1987), although they ended up with a much larger number of degenerate mechanisms caused by the larger number of state variables that correspond to the nodal description. The smaller number of variables used in the mesh description reduces the likelihood of such solutions. Moreover, since this procedure is carried out over the subintervals, the number of adjacent feasible vertices is not very large.

,MERICAL EXAMPLE

This example consists of an unsymmetrical two-story two-bay frame represented in Fig. 5. $(\mathbf{X}^i_1, \mathbf{X}^i_3) = (M_j, N_j), (M_k, N_k), (M_l, N_l), (M_m, N_m), (M_n, N_n)$ are independent random variables. The loads are also independent random variables, except $\rho_{H_1,H_2} = 1.0$ (Table 1). For the single stress resultant (bending moment) predominant action, the mathematical programming problem

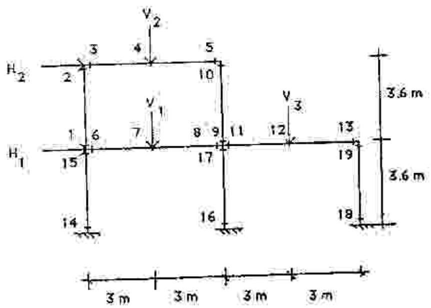


FIG. 5. Two-Bay Two-Story Frame

TABLE 1. Two-Bay Two-Story Example Data

Variable (1)	Mean (2)	Coefficient of variation (3)	Coefficient of correlation (4)
$(M_j, N_j)(j = 14, 15, 16, 17, 18, 19)$	95 kN · m	0.15	1.0
$(M_k, N_k)(k = 1, 2, 9, 10)$	95 kN · m	0.15	1.0
$(M_l, N_l)(l = 6, 7, 8)$	204 kN · m	0.15	1.0
$(M_m, N_m)(m = 11, 12, 13)$	122 kN · m	0.15	1.0
$(M_n, N_n)(n = 3, 4, 5)$	163 kN · m	0.15	1.0
V_1	169 kN	0.15	—
V_2	89 kN	0.25	—
V_3	116 kN	0.25	—
H_1	62 kN	0.25	—
H_2	31 kN	0.25	—

TABLE 2. Stochastically Most Important Mechanisms without Stress Resultant Interaction

Critical sections participating in mechanism (1)	Reliability index β (2)	Collapse load (PLA) (3)
14, 15, 16, 17, 18, 19	1.966	1.702
4, 7, 8, 9, 10, 14, 16, 17, 18, 19	1.994	1.324
2, 4, 10	2.054	1.625
11, 12, 19	2.058	1.678
4, 7, 8, 10, 12, 14, 16, 18, 19	2.060	1.308
1, 7, 8, 14, 16, 17, 18, 19	2.090	1.404
6, 7, 8	2.144	1.609
4, 5, 7, 8, 9, 14, 16, 17, 18, 19	2.200	1.368
3, 4, 10	2.215	1.727
2, 4, 5	2.215	1.727

tha. needs to be solved has nine nonlinear (random) and 19 linear (critical section rotations) variables. Since the lower bounds on the random variables is zero, it is necessary to compute narrow upper bounds. This is done by solving nine multiple-cost-row linear programs (Eq. 24). Many of the most important mechanisms will be available in this stage. Each interval of variation is then divided into equal subintervals computed according to Eq. 38 for $\epsilon = 0.01$. The zero-one mixed LP (Eq. 41) is then solved by an appropriate code. The stochastically most relevant mechanisms are listed in Table 2.

When the interaction between bending moment and axial force is considered, to reduce the size of the problem it is assumed that all members are in compression when the mechanisms are activated. A suitable approximation is obtained by four hyperplanes at each of the 19 critical sections, making 76 hyperplanes in all. This formulation leads to an increase in the number of linear variables: 76 strain-resultant rates. The linearized interaction diagram is represented in Fig. 6. The same type of interaction diagram applies to all the critical sections.

The yield conditions become

$$\mathbf{N}_i^{t}\mathbf{X}^{i} \le \mathbf{X}_*^{i}$$

that is

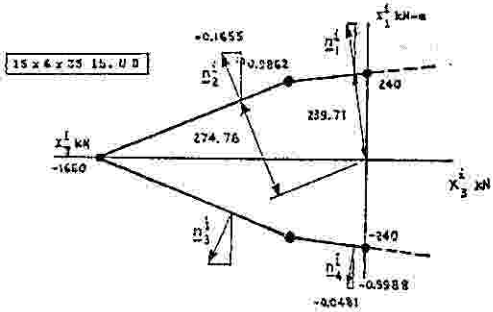


FIG. 6. Linearized Interaction Diagram for 15 × 6 × 35 lb (381 × 152.4 × 15.89 kg) UB Section

TABLE 3. Stochastically Most Important Mechanisms with Stress Resultant Interaction

Critical sections participating in mechanism (1)	Reliability index β (2)	Collapse load (PLA) (3)
12, 14, 15, 16, 17, 18, 19	1.435	1.390
14, 15, 16, 17, 18, 19	1.540	1.437
4, 7, 8, 9, 10, 14, 16, 17, 18, 19	1.550	1.243
1, 7, 8, 9, 10, 14, 16, 17, 18, 19	1.640	1.295
1, 5, 7, 8, 9, 14, 16, 17, 18, 19	1.644	1.296
1, 7, 8, 10, 12, 14, 16, 17, 18, 19	1.709	1.298
1, 5, 7, 8, 12, 14, 16, 17, 18, 19	1.713	1.299
4, 5, 7, 8, 9, 14, 16, 17, 18, 19	1.781	1.286
1, 2, 7, 8, 11, 14, 16, 17, 18, 19	1.788	1.322
4, 7, 8, 9, 10, 12, 14, 16, 19	1.792	1.264

$$\begin{vmatrix} 0.9988 & -0.0481 \\ 0.9862 & -0.1655 \\ -0.9862 & -0.1655 \\ -0.9988 & -0.0481 \end{vmatrix} \begin{vmatrix} X_1^i \\ X_3^i \end{vmatrix} \leq \begin{vmatrix} X_{1*}^i \\ X_{2*}^i \\ X_{2*}^i \\ X_{1*}^i \end{vmatrix} \qquad (45)$$

If the interaction of bending and axial forces is according to the yield diagram shown in Fig. 6, the stochastically most important modes become as shown in Table 3.

This type of stress resultant interaction leads to mechanisms with a larger number of plastic hinges activated. The increase in strength due to a larger number of plastic hinges is opposed by the reduction of strength due to the interaction of bending and compressive forces resulting in lower (higher) reliability indices (probabilities of failure). The probability of failure of the frame with and without stress resultant interaction given by Cornell's lower and upper bound (CLB, CUB), Ditlevsen's lower and upper bound (DLB, DUB), and Vanmarcke's upper bound (VUB) is represented in Fig. 7. The number of mechanisms considered in the cases without bending moment and with stress resultant interaction was 26 and 34, respectively. With the exception of Ditlevsen's lower bound, the probabilities of failure are increased by more than 80% when the stress resultant interaction is considered. The

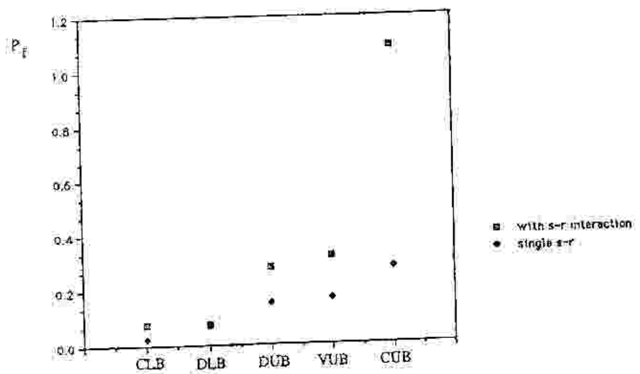


FIG. 7. Probabilities of Failure of Example Frame

rate of increase of the frame's probabilities of failure considering the linearized yield criterion of Fig. 7, with respect to the case where bending moment is the predominant action, gets smaller when the coefficients of variation of the plastic capacities and loading grow. The rate of increase becomes smaller with increasing correlation of the random variables.

Reliability-Based Design

Assumptions

Consistent with a first-order second-moment reliability approach, the minimum statistical information required for the evaluation of the optimum solution is: (1) The mean values of the loads that make up the vector $\boldsymbol{\mu}_L$, the coefficients of variation of the loads that make up the vector $\boldsymbol{\Omega}_L$, and the coefficients of correlation between pairs of loads that form a square symmetrical correlation matrix denoted $\mathbf{C}_L$; and (2) the coefficients of variation of the plastic capacities, which make up the vector $\boldsymbol{\Omega}_X$ and the coefficients of correlation between pairs of plastic capacities, which form a square symmetrical matrix denoted $\mathbf{C}_X$.

Proportionality of Plastic Capacities

At any particular critical section, the allowable values of the interacting stress resultants are specified by an interaction diagram. As in Fig. 3, the curved yield surfaces can be approximated by suitable hyperplanes with outward unit normal vectors $\mathbf{n}_k^i$. Now, synthesis is concerned with making comparisons between all members of a class of admissible designs. Clearly, the most general class must allow the full plastic capacities of all stress resultants at each critical section to be varied independently. Naturally, the normal vector $\mathbf{n}_k^i$ will consequently be altered in such a class of admissible variations, and since $\mathbf{n}_k^i$ then depends on the unknown plastic capacities, the problem increases its nonlinearity. However, suppose the class of admissible designs is restricted by the condition that all the fully plastic capacities at any particular section may only be varied in a preselected and fixed ratio to each other. In other words, at each critical section there is only a single parameter to be varied, and since for framed structures at least one of the interacting stress resultants will be a bending moment, its plastic capacity is conveniently chosen as the single design parameter at each critical section. The other plastic capacities are then written in terms of these design parameters. The imposition of proportionality of plastic capacities then produces the effect shown in Fig. 8.

All the possible yield surfaces from which that for the optimal design will

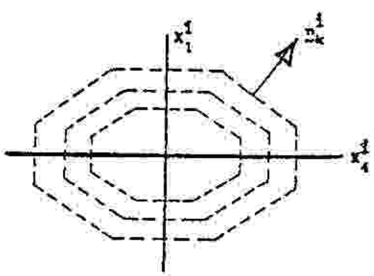


FIG. 8. Effect of Proportionality Condition on Interaction Diagram

be chosen bound geometrically similar convex polytopes in the stress resultant space of the ith critical section. From this follows the most important result: the unit normal $\mathbf{n}_k^i$ to the kth face of one of these polytopes is constant for all admissible variations of the single design parameter associated with the ith critical section. For the design of steelwork of commercially available rolled sections, this proportionality condition is realistic (Smith 1974).

Formulation

The process of selection of the optimum solution is highly complex, involving both qualitative and quantitative factors that must be considered simultaneously. There are diverging opinions on many basic issues from the very definition of reliability-based optimization, including the definition of the objective function and the constraints to its application in structural design practice. The reliability-based optimization problem generally adopted in the design process consists of member size selection for given detailing arrangements and specified probabilities of failure against collapse and unserviceability. Assuming the plastic capacities are proportional to the volume of material required, this problem can be stated as follows:

$$\min V(X) = \ell' \mu_{\mathbf{X}} \quad (46a)$$

$$\text{subject to } P_f = P_f(\mathbf{X}) \leq P_{f*} \quad (46b)$$

$$P_s = P_s(\mathbf{X}) \leq P_{s*} \quad (46c)$$

where ℓ = the vector of member lengths. The objective function is a linear function of the mean values of plastic capacitites, but the reliability constraints are nonlinear. An alternative approach minimizes the probability of collapse or unserviceability for a fixed volume of material. They are alternative formulations that give Pareto solutions to the general multi-objective reliability-based optimization.

Reliability against Unserviceablity

Assuming that the vector $\mathbf{X}_e$ represents the elastic envelope stress resultant coefficients obtained by the deterministic analysis and that $\mathbf{X}$ is the mean plastic capacities, the probability of unserviceability of individual sections is given by

$$P_{sj} = P(X_j - X_{ej} \leq 0); \qquad j = 1, \ldots, m \quad (47)$$

For completely correlated plastic capacities, the probability of failure is

$$P_s = \max_{j=1,m} P_{sj} \quad (48a)$$

and for uncorrelated plastic capacities

$$P_s = \max_{j=1,m} (\Sigma_{k=1,v_j} P_{sk}) \quad (48b)$$

where v_j represents the number of critical sections corresponding to all considered loading schemes. Serviceability constraints obtained from Eqs. 48*a*–*b* are nonlinear and pose a mild convex behavior.

Reliability against Collapse

By fixing the design variables, the reliability assessment problem (Eq. 21) gives the activation parameters $\boldsymbol{\theta}_*$ and displacement rates $\boldsymbol{\delta}_*$ associated with the stochastically most important mechanism and other relevant modes. Clearly, the reliability analysis for another set of design variables (but the same mechanism) would give proportional activation parameters and displacement rates. For a prespecified reliability index β_* and n mechanisms, single-mode probability constraints will be satisfied if

$$\mu_X'\theta_{*k} - \mu_L'\delta_{*k} \ge \beta_* \sqrt{\theta_*'\sigma_X'C_X\sigma_X\theta_* - \delta_*'\sigma_L'C_L\sigma_L\delta_*} \quad \text{(49)}$$

where $k = 1, \ldots, n$. It can be shown that these constraints are convex with respect to the design variables. Under mild requirements, the multimode constraints can be assumed to be convex.

Solution Method

The solution method is divided in the following two alternating subprocedures.

1. An optimization procedure for the nonconvex inner problem (Eq. 21) (reliability assessment), that finds the stochastically most important mechanism and enumerates other relevant collapse modes for a given value of the design variables (plastic capacities). An algorithm to solve the concave quadratic minimization, such as the zero-one mixed LP described here, must be used.
2. An optimization of the convex outer problem (Eqs. 46*a*–*c*) (optimal design), that is the vector of average plastic capacities giving a least-volume solution satisfying failure, serviceability, and technological requirements. Any convex programming technique can be employed (Vanderplaats 1984).

The procedure is repeated until the vector of design variables converges.

This form of reliability-based optimization is similar to the parametric optimization problem treated by Lee and Kwak (1987–1988) for elastic trusses, but it was assumed in their work that the solution of each inner problem is unique. This is not true for structures with plastic behavior, because each collapse mechanism is a local solutiton of the inner problem.

NUMERICAL EXAMPLE

It is intended to find the minimum volume of the rigid portal frame with fixed geometry represented in Fig. 9 that satisfies given reliability requirements. Five design variables corresponding to average bending moments of resistance of the columns and beams are considered. The mean values of the loads and the coefficients of variation of loads and plastic capacities are read as input data

$$\mu_{V1} = 169 \text{ kN};\quad \Omega_{V1} = 0.15;\quad \mu_{V2} = 89 \text{ kN};\quad \Omega_{V2} = 0.25;\quad \mu_{V3} = 116 \text{ kN};$$

$$\Omega_{V3} = 0.25;\quad \mu_{H1} = 62 \text{ kN};\quad \Omega_{H1} = 0.25;\quad \mu_{H2} = 31 \text{ kN};\quad \Omega_{H2} = 0.25;$$

$$\Omega_{x_1} = \Omega_{x_2} = \Omega_{x_3} = \Omega_{x_4} = \Omega_{x_5} = 0.15 \quad \text{(50)}$$

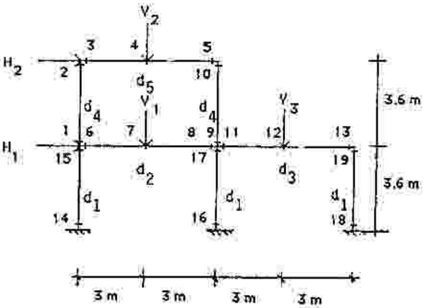


FIG. 9. Frame Example

TABLE 4. Probabilities of Failure of Deterministic Optimum Design

Type of probability measure (1)	Probabilities of failure (2)
CLB	1.16E-3
CUB	2.9 E-3
DLB	1.51E-3
DUB	2.06E-3
VUB	1.54E-3

TABLE 5. Probabilities of Failure of Reliability-Based Optimum Design

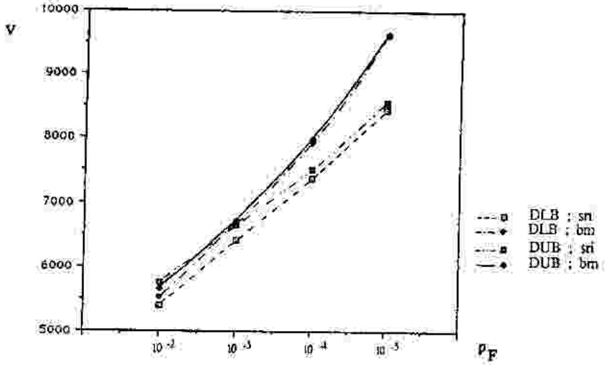
Type of probability measure (1)	Probabilities of failure (2)
CLB	6.6E-6
CUB	6.2E-5
DLB	2.7E-5
DUB	4.0E-5
VUB	4.0E-5

The linearized yield diagram of Fig. 6 for which the plastic capacities may vary proportionally is adopted whenever stress resultant interaction is assumed. For normally distributed load and plastic capacities, the following two types of correlation among the design variables are considered. Case I—perfect correlation within members and column-column correlation (statistically independent random loads, except the horizontal loads, which are perfectly correlated). Case II—perfect correlation among all plastic moments (statistically independent random loads, except the horizontal loads, which are perfectly correlated).

It will be illustrated next that the sizing given by deterministic plastic limit synthesis can be very inadequate when, for a fixed volume of material, the structural reliability is the major concern. By taking as average plastic capacities the design variables given by the plastic limit synthesis LP (with stress resultant interaction) multiplied by 2.5, the probabilities of failure given by Ditlevsen's lower and upper bounds (DLB, DUB), Vanmarcke's upper bound (VUB), and Cornell's upper bound (CUB) shown in Table 4 are obtained for case I. For the same material consumption, the reliability-based design leads to the solutions in Table 5. Therefore, the probability of failure against collapse can be reduced from 50 to 150 times just by redistributing the plastic capacities between the structural members according to the probability-based design.

The present investigation illustrates the influence of the statistical parameters of the random variables describing loads and strengths on the probabilities associated with collapse derived from the proposed reliability-based design solutions. Figs. 10–13 illustrate the results of the example frame with stress resultant interaction (sri) or without bending moment (bm) stress-resultant interaction. Figs. 10–11 are drawn for case I and show that the volume of material needed to satisfy the reliability requirements with increasing values of P_f is larger when the only stress resultant is the bending moment. This is caused by the larger number of plastic hinges required to form mechanisms when the bending moments interact with axial forces.

Fig. 12 shows the influence of the coefficient of variation of the plastic capacities on the probabilities associated with collapse evaluated according



.G. 10. Effect of Variation of Type of Probability Measure (DUB, DLB) on Optimum Solution

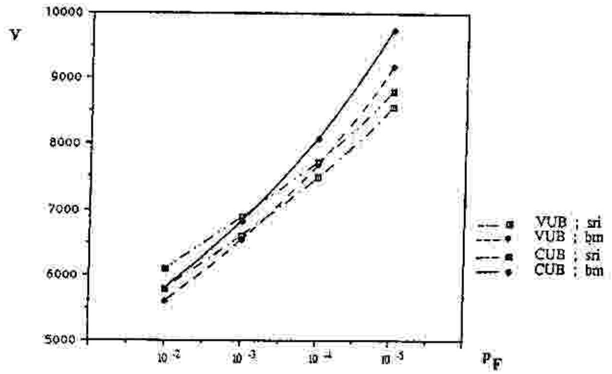


FIG. 11. Effect of Variation on Type of Probability Measure (DUB, VUB) on Optimum Solution

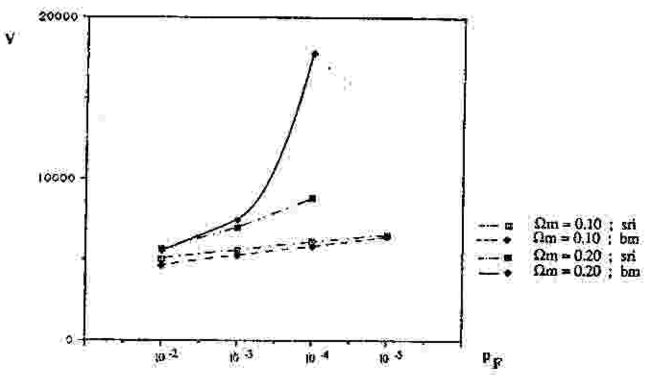


FIG. 12. Effect of Variation of Coefficient of Variation on Optimum Solution

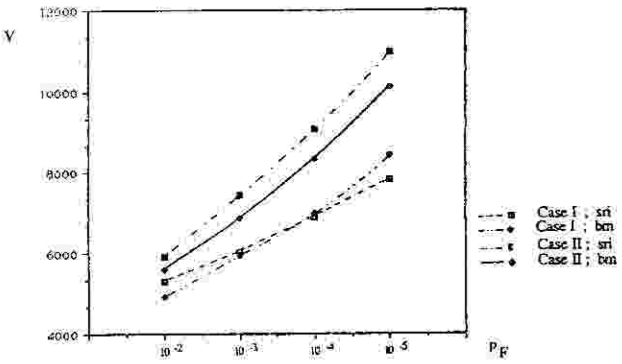


FIG. 13. Effect of Variation of Correlation on Optimum Solution

to Cornell's lower bound for case I. The influence of Ω_X is more significant with increasing values of P_f. For a given probability of failure and small coefficients of variation of the strengths, more material is required with interacting stress resultants than that needed when the predominant action is bending. When Ω_X increases, the graph indicates that the rate of increase in V is larger when the axial forces are not considered.

Fig. 13 enables the study of the sensitivity of V for various sets of specified probabilities of failure (obtained by Cornell's lower bound) and different types of correlation among the plastic capacities (cases I and II). Increasing the correlation among the critical sections makes the minimum volume of material required more sensitive to the effect of axial forces, so an assumption of independent beam-column capacities is on the unconservative side.

Conclusions

An upper bound theorem formulation of a discretized structure with interacting stress resultants is derived, enabling the reliability assessment by concave quadratic minimization within first-order second-moment reliability theory. Two algorithmic procedures to solve this nonconvex program are presented. The first employs a standard code for zero-one mixed linear programming. An alternative approach that uses parallel processing can be recommended if the software is developed for parallel processing computers, or transputers. The proposed technique illustrates that the stochastically dominant modes differ and the resulting probabilities of failure may change considerably, whenever the interaction of bending and axial forces is considered instead of a predominant bending action. The use of probabilistic concepts in structural design need not be restricted because of the ideal design problem's complexity. The solution method for the optimum design of portal frames with interacting stress resultants consists of two alternating subprocedures: (1) An optimization of the concave quadratic minimization giving the reliability indices of the stochastically most important modes: and (2) an optimization of the convex outer problem that includes the cost function. The present investigation has also discussed the influence of design parameters on the minimization of the total expected volume of material for a specified failure probability. The writer believes that an approximate design method based on probability safety concepts may be closer to reality than design methods based on deterministic safety concepts.

Acknowledgments

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Appendix II. Notation

The following symbols are used in this paper:

b_{ij} = element of the static matrix $\mathbf{B}$ giving the stress resultants at section i caused by the unit indeterminate force p_j;

b_{oij} = element of the static matrix $\mathbf{B}_o$ giving the stress resultants at section i caused by the applied load L_j;

E = external work, error function;

F = potential function;

$\mathbf{J}_\Theta, \mathbf{J}_\delta$ = binary incidence matrix obtained by associating either activation parameters or displacement rates sharing the same random variable X or L;

L = applied loading;

ℓ = vector of member lengths;

M_i, N_i = bending moment and axial force at section i;

$\mathbf{n}_k^i$ = element of the normal matrix $\mathbf{N}$ giving the kth normal to the linearized yield surface at section i;

$P(F_k)$ = probability of occurrence of mode F_k;

P_f = probability of failure against collapse;

P_s = probability of failure against unserviceability;

p_i = indeterminate mesh forces;

$\mathbf{U}$ = plastic energy dissipation, eigenvectors;

u = strain resultant increments;

u_* = mechanism activation parameters;

w_{ij} = real variable in zero-one mixed LP form;

X^i = stress resultants in critical section i;

X_{*k}^i = distance from the origin to the yield hyperplane k;

$\mathbf{X}, \mathbf{L}$ = vector of the random plastic capacities and applied loads, respectively;

$\mathbf{X}', \mathbf{L}'$ = vectors of reduced normal variables corresponding to $\mathbf{X}$ and $\mathbf{L}$, respectively;

z_{ij} = zero-one variable in zero-one mixed LP form;

β = reliability index, upper bound on random variable;

Γ = linear underestimating function;

δ = displacement rates;

$\boldsymbol{\delta}_*$ = vector of the sums of the displacement rates δ associated with the same random variable L;

θ_i = plastic rotation at critical section i;

$\boldsymbol{\theta}_*$ = vector of the sums of the activation parameters u_* associated with the same random variable X;

λ = parameter, eigenvalues;

$\boldsymbol{\mu}_X, \boldsymbol{\mu}_L$ = vector of the mean values of X and L, respectively;

$\rho(X_i, X_j), \rho(L_l, L_m)$ = coefficients of the correlation matrices $\mathbf{C}_X$ and $\mathbf{C}_L$, respectively;

ϕ = yield function; concave quadratic function in separable form;

$\boldsymbol{\sigma}_X, \boldsymbol{\sigma}_L$ = vector of the standard deviations of X and L, respectively;

ψ = concave quadratic function;

Ω = linear domain of the concave quadratic minimization; and

Ω_X, Ω_L = coefficient of variation of the random variables X and L, respectively.