A MATHEMATICAL PROGRAMMING APPROACH FOR FINDING THE STOCHASTICALLY MOST RELEVANT FAILURE MECHANISM

L.M.C. SIMÕES
Departamento de Engenharia Civil
Faculdade de Ciências e Tecnologia
Universidade de Coimbra
Portugal

ABSTRACT. In calculating the failure probability of structural systems, the most important operation is the search for the stochasticly most relevant failure mechanism. The nodal and mesh description for the modelling of a flexural frame with fully plastic behaviour and slabs discretized into triangular finite elements whose behaviour conforms the yield line theory are considered. The mathematical programming method arising from these models can be formulated as the minimization of a quadratic concave function over a linear domain.

1. Introduction

Most of the research into the synthesis of structures is based on models with deterministic behaviour. Using these models, it is only possible to guarantee that the structure resists the most unfavorable static loading that is supposed to act during its life for a given behaviour of the constituent materials. Since neither the loading nor the materials are deterministic, research has been conducted to assess.the reliability of structures. To avoid the difficult numerical integration of the probability density functions involved, the reliability index is obtained from the limit state equation using the second moment approximation.

The characteristics of the algorithms used for mathematical programming require the discretization of trusses, frames, plates and shells. The finite element method was choosen owing to its versability in the automatic generation of the stochastically most relevant failure mechanism, that is the mechanism with the smallest reliability index β and the highest probability of failure p_F.

The efficiency in obtaining the mechanism with the highest probability of failure in 2-D structures has been impaired by difficulties in solving the corresponding mathematical programming problem. Even in simple portal frames the algorithms for convex programming used in the evaluation of the stochastically most important failure mechanism may end up with misleading results, because this problem is of nonconvex type. A mathematical programming method that leads to the stochasticly most important failure mechanism is deduced and strategies that give the global optimum (eliminating suboptima) and can be used to enumerate local solutions are presented. This formulation is limited to ductile structures such as steel frames and reinforced concrete slabs.

2. Fundamental Relations

There are two types of formulation available to describe the fundamental relations for the problems to be discussed. One formulation reflects the finite element connectivity at their

G. I. N. Rozvany and B. L. Karihaloo (eds.), Structural Optimization, 323–330.

common corner nodes and it is thereby called nodal description. Alternatively, the finite element connectivity may be reflected through their common sides leading to the formulation of meshes and it is thereby called mesh description.

The suggested finite element model can be regarded as a direct extension to slabs of the finite element modelling of flexural plane frames. Modal deformations are plastic rotations at pre-located critical sections in the case of frames and at pre-located element sides in the case of slabs. Nodes are positioned in between two or more critical sections in the case of frames and at the intersection of two or more element sides in the case of slabs. Futhermore, just as the bars of a frame form meshes, the finite elements modelling the slab can be grouped together forming also meshes as exemplified in Fig.1.

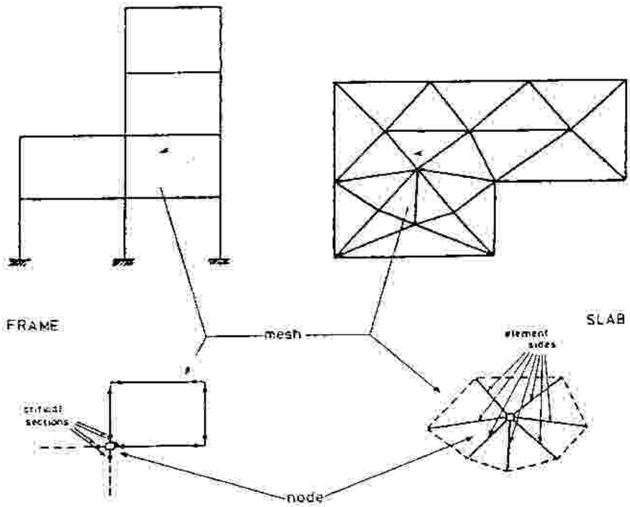

Figure 1

The frame shown in Fig.1 can be represented through a mesh model where an external mesh defines the frame in the two dimensional space. The slab can also be described through a mesh model where the meshes referring to the boundary nodes define the slab in the two-dimensioned space. Therefore, the structure (frame or slab) is replaced by a structural model which can be described in two-dimensional space either through the nodes (Nodal Descrition) defined at the intersection of the finite elements or through the meshes (Mesh Description) defined by chains of finite elements.

2.1. CONSTITUTIVE RELATIONS

The yield criterion for structures with perfectly plastic behaviour imposes bounds on the values of the moments in all critical sections. For example, if the negative and positive plastic resisting moments at the critical section i are m_{*i}^- and m_{*i}^+, respectively, then we have:

$$-m_{*i}^- \leq m_i \leq m_{*i}^+ \tag{1}$$

Similarly, the yield line method considers a very simple yield criterion involving solely the normal bending moment and the normal angular discontinuity at every element side.The yield conditions impose limit values on the magnitude of the resulting bending moments at the element side. Let the positive and negative bending moment capacities per unit length for the

i^{th} element side be, respectively $\underline{m}*_i^+$ and $\underline{m}*_i^-$ and the bending moment per unit length $\underline{m}_i$, with:

$$-\underline{m}^-*_i \le \underline{m}_i \le \underline{m}^+*_i \tag{2}$$

Now, if such an element side has a length l_i, the yield conditions in terms of total moments for the whole element side are:

$$-m*_i^- = -\underline{m}^-*_i\, l_i \le m_i \le \underline{m}^+*_i\, l_i = m^+*_i \tag{3}$$

The mechanism deformation can only take place at the element sides where the normal bending moment reaches one of its limiting values. That is to say the angular discontinuity $\Delta\theta_i$ at the element side can only take a positive value $\Delta\theta_i^+$ when m_i is equal to $m*_i^+$ and it can only take a negative value $\Delta\theta_i^-$ when m_i is equal to $m*_i^-$.

By neglecting the elastic deformations, the deformations of the mechanism are equal to the total deformations of the structure. Provided that at the incipient plastic collapse the displacements are small enough for the plastic analysis to be based on the underformed geometry of the structure, elastoplastic deformations need not to be considered. Clearly, whereas the plastification in frames can be considered to be restricted to pre-located critical sections, in the slabs the plastic behaviour is not necessarly restricted to the sides of the finite elements. Thus, the finite element modelling leads to a correct representation of the frame, but leads to a representation which is only approximate for the slab.

2.2. MESH AND NODAL DESCRIPTIONS

2.2.1. Mesh Description of Statics. The static indeterminacy number (α) of the frame represented in Fig.2 is 3 and the number of critical sections (c_r) that have to be considered is 7. This frame can be reduced to a statically determinate structure in many different ways, one of which is achived by introducing a cut adjacent to critical section 7.

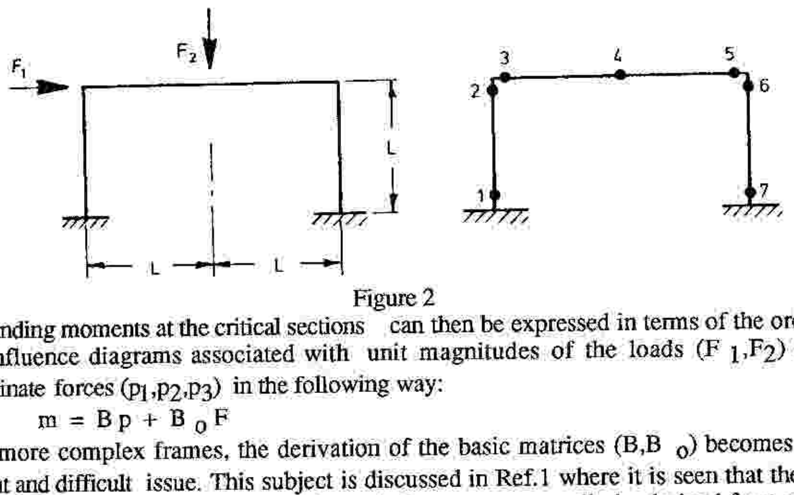

Figure 2

The bending moments at the critical sections can then be expressed in terms of the ordenates of the influence diagrams associated with unit magnitudes of the loads (F_1,F_2) and the indeterminate forces (p_1,p_2,p_3) in the following way:

$$m = B\,p + B_0\,F \tag{4}$$

With more complex frames, the derivation of the basic matrices (B,B_0) becomes a more important and difficult issue. This subject is discussed in Ref.1 where it is seen that the more convenient basis for generating the mesh matrix **B** cannot generally be derived from physical release systems.

2.2.2. Mesh Description of Kinematics. If a mechanism is built up due to the formation of hinges in the critical sections of the reduced structure, the angular discontinuities Δv can be written in terms of the rotation $\Delta\theta$ from the undeformed geometry of the frame, by means of the coefficients of the static matrix B:

$$\Delta v = B^t \Delta \theta \quad (5)$$

If the structure has a set of compatible displacements, then we have:

$$\Delta v = 0 \quad \Rightarrow \quad B^t \Delta \theta = 0 \quad (6)$$

These equations are valid for every mechanism. Similarly the displacements u corresponding to the loads F can be written in terms of the rotations $\Delta \theta$,

$$u = B_0^t \Delta \theta \quad (7)$$

Then the Kinematic relations for the mesh description in frames become:

$$\begin{bmatrix} 0 \\ u \end{bmatrix} = \begin{bmatrix} B^t \\ B_0^t \end{bmatrix} \Delta \theta \quad \text{or} \quad \begin{bmatrix} 0 \\ u \end{bmatrix} = \begin{bmatrix} B^t & -B^t \\ B_0^t & -B_0^t \end{bmatrix} \begin{bmatrix} \Delta \theta^+ \\ \Delta \theta^- \end{bmatrix} \quad (8)$$

where the rotation in the critical section i is decomposed in the pair of nonnegative variables $\Delta \theta^+$ and $\Delta \theta^-$, as in the mathematical programming algorithms:

$$\Delta \theta = \Delta \theta^+ - \Delta \theta^- \quad (9)$$

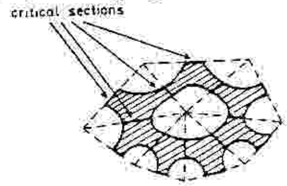

Figure 3

Acording to the yield line theory, the collapse is due to the formation of yield lines along which the slab folds when a mechanism is activated. In the mesh description compatibility conditions for slabs can be established directly, if the finite element meshes are enforced to remain as closed chains of elements during deformation. Such condition of compatibility may be stated for every mesh which can be defined in the discretized slab. A set of linearly independent meshes will be constituted by the meshes defined at each corner with the exception of one. It can be seen from Fig.3 that the static indeterminacy number associated to the finite element modelling of slabs is 2(NC-1), where NC is the number of corner nodes.

Taking one of these meshes, the corresponding compatibility conditions must ensure that when the collapse mechanism is activated, angular discontinuities develop along element sides at yield, maintaining continuity of vertical displacements. That is achieved if the two agebraic sums of the projections in two different directions of those angular discontinuities developing along a fixed sense around the mesh are set to zero. The two compatibility relations for the mesh represented in Fig.4 and for the projection in the two directions X and Y can be stated as follows.

$$\begin{bmatrix} \text{sen}\alpha_1 & \text{sen}\alpha_2 & \text{sen}\alpha_3 & \text{sen}\alpha_4 & \text{sen}\alpha_5 \\ \cos\alpha_1 & \cos\alpha_2 & \cos\alpha_3 & \cos\alpha_4 & \cos\alpha_5 \end{bmatrix} \begin{bmatrix} \Delta \theta_1 \\ \Delta \theta_2 \\ \Delta \theta_3 \\ \Delta \theta_4 \\ \Delta \theta_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (10)$$

where $\Delta \theta_i$ is the total rotation of the element side, that is the sum of the rotations of all element sides that share the same side.

Now, if these conditions are established for each mesh and they are all assembled, the resulting compatibility conditions may be written in the compact form:

$$\Delta V = B^t \Delta \theta = 0 \quad (11)$$

where the vector ΔV has 2 (NC -1) elements, the vector $\Delta \theta$ has NS elements - NS is the number of element sides - and the matrix B is called kinematic transformation matrix.

If all edges are clamped, the moments from which the matrix B_0 can be built are easily determined if the discretized slab is split into cantilever slabs formed by chains of finite elements. If all edges are either clamped or simply supported, the boundary can be considered to be provided by fixed finite elements. The same cannot be said with regard to free edges, but such a difficulty can be overcome if extra boundary finite elements are defined along them. New simply supported edges are thus obtained whilst the original free edges become internal lines along which the normal bending moment capacity is zero. In order to consider supporting columns, either provision is given to account for fixed external constraints or the contour defined by the column in the slab is taken as a clamped edge.

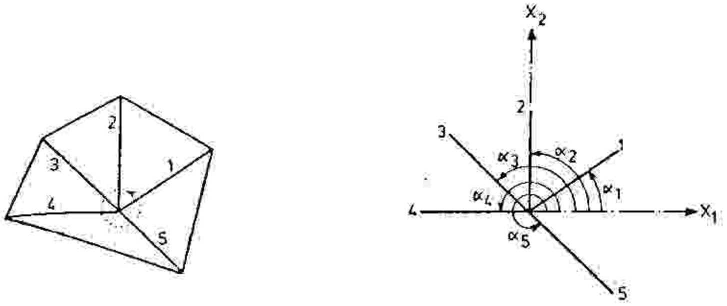

Figure 4

2.2.3. Nodal Description of Kinematics. The mesh description has its origins in the concept of static determinacy and in the expression of any static field in terms of the loading actions and of the mesh forces. The nodal description may for the present case be considered to have its bases on the concept of mechanisms. The elementary mechanisms for a frame are a collection of all the sway and joint mechanisms shown in Fig. 5 .

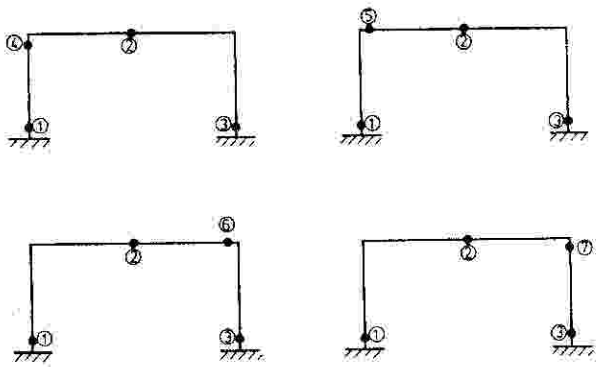

Figure 5

Any set of compatible deformation rates is associated with parameters that play the role of nodal displacements and can be written in the following form:

$$\Delta\theta = C \ \Delta q \tag{12}$$

Equations similar to these can be written for any number of critical sections c_r and any number of independent mechanisms in the basis $b = c_r - \alpha$.

Similarly the displacement corresponding to the applied loads can be expressed in terms of the nodal displacements:

$$u = C_0 \; \Delta q \tag{13}$$

Hence the nodal kinematic relations become:

$$\begin{bmatrix} \Delta \theta \\ u \end{bmatrix} = \begin{bmatrix} C \\ C_0 \end{bmatrix} \Delta q \qquad \text{or} \qquad \begin{bmatrix} \Delta \theta^+ - \Delta \theta^- \\ u \end{bmatrix} = \begin{bmatrix} C \\ C_0 \end{bmatrix} \Delta q \tag{14}$$

According to the hypothesis of the yield line theory, when the collapse mechanism is activated, the finite elements behave as rigid but angular discontinuities may be generated across the element sides whilst providing for continuity of vertical displacements (Ref.2).

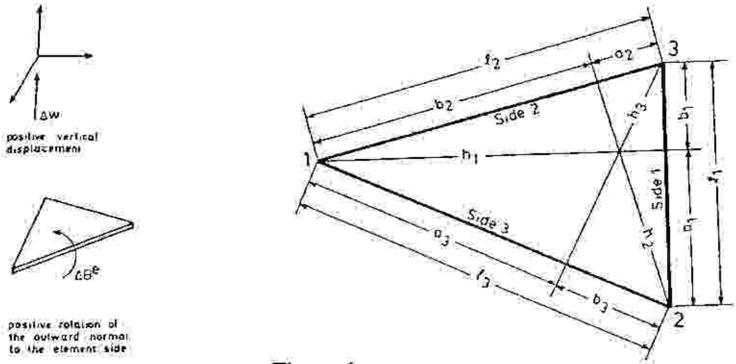

Figure 6

Taking one single finite element of a slab, as represented in Fig.6, the three relations $\Delta \theta_i^e$ of the outward normals to the three sides can then be expressed in terms of the three corner vertical displacements Δq in the following way:

$$\begin{bmatrix} \Delta \theta_1^e \\ \Delta \theta_2^e \\ \Delta \theta_3^e \end{bmatrix} = \begin{bmatrix} -1/h_1 & b_1/(l_1 h_1) & a_1/(l_1 h_1) \\ a_2/(l_2 h_2) & -1/h_2 & b_2/(l_2 h_2) \\ b_3/(l_3 h_3) & a_3/(l_3 h_3) & -1/h_3 \end{bmatrix} \begin{bmatrix} \Delta q_1 \\ \Delta q_2 \\ \Delta q_3 \end{bmatrix} \tag{15}$$

For an interelement side the total angular discontinuity $\Delta \theta_i$ is clearly the algebraic sum of the rotations $\Delta \theta_i^e$ of the two outward normals with respect to the two finite elements sharing such a side. The assemblage of relations (14) for all finite elements is thus readily performed and may be written in the following compact form:

$$\Delta \theta = C \; \Delta q \tag{16}$$

where, C is a (NS x NC) kinematic transformation matrix. Since these modal deformations are obtained as functions of the linearly independent modal displacements Δq it follows that such deformations are necessarily compatible and the rotation/displacement relations may be taken as the compatibility conditions.

3. Problem Statement

3.1. MATHEMATICAL PROGRAMMING FORMULATION

The identification of all the significant collapse modes of a ductile structural system is necessary in the analysis and evaluation of the system reliability, including the evaluation of the corresponding bounds. In this paper only the mathematical programming method corresponding to finding the most important mechanism is analysed. In the case of structures composed of ductile members such as components with elastic-perfectly plastic behaviour, the structural strength would be independent of the failure sequences of the components. It is usual to employ an approximate procedure called second order method (Ref.3) that only requires the mean and loading and structural resistance. For statistically independent random normal variables, the identification of the stochastically more important mechanism consists of finding the position of the limit-state equation closer to the origin of the reduced normal variables. This amounts to minimizing the distance - reliability index - β :

$$\min \ \beta = \sqrt{(m'_*)^2 + (\lambda'_F)^2} \tag{17}$$

By associating the rotations of the critical sections (or element sides) with the rotations of the members represented by the same random variable through the incidence matrix J_θ and the displacements (or deflections of the triangular elements centroids) corresponding to the loads linked by the same random variable through J_u, we have:

$$\Delta\theta_* = J_\theta \ \Delta\theta^+ + J_\theta \ \Delta\theta^- \quad ; \quad u_* = J_u \ u \tag{18}$$

The limit-state function represents the aptitude of the structure to support the loading equates the external and internal work produced by each mechanism:

$$g(m_*, \lambda) = m_*^T \Delta\theta_* - \lambda_F^T u_* = 0 \tag{19}$$

The relationships linking the reduced normal variables to the normal variables are:

$$m_* = \mu_M + \sigma_M \ m'_* \quad ; \quad \lambda_F = \mu_F + \sigma_F \ \lambda'_F \tag{20}$$

where μ_M, μ_F and σ_M, σ_F are the mean and standard deviation of the random variables m_* and λ_F, respectively. Another bilinear equation is yielded by substituting m_* and λ_F for (20) in the limit-state equation (19):

$$(\sigma_M \ m'_*)^T \Delta\theta_* - (\sigma_F \ \lambda'_F)^T u_* + \mu_M^T \Delta\theta_* - \mu_F^T u_* = 0 \tag{21}$$

In the limit-state equation (21) the plastic deformations of the mechanism are present as state variables. Since it is required to find the minimum distance from the origin of the reduced normal variates to the yield surface, the values of λ'_F and m'_* given by the minimum norm solution of (21) are:

$$m'_* = \frac{\sigma_M \ \Delta\theta_* \ [\mu_M^t \ \Delta\theta_* - \mu_F^t \ u_*]}{(\sigma_M)^{2t} (\Delta\theta_*)^2 + (\sigma_F)^{2t} \ (u_*)^2} \tag{22}$$

$$\lambda'_F = \frac{\sigma_F \ u_* \ [\mu_M^t \ \Delta\theta_* - \mu_F^t \ u_*]}{(\sigma_M)^{2t} (\Delta\theta_*)^2 + (\sigma_F)^{2t} \ (u_*)^2} \tag{23}$$

For positive $[\mu_M^t \ \Delta\theta_* - \mu_F^t \ u_*]$ the identification of the stocastic more relevant mechanism consists of finding:

$$\min \ \beta = \frac{\mu_M^t \ \Delta\theta_* - \mu_F^t \ u_*}{\sqrt{(\sigma_M)^{2t} (\Delta\theta_*)^2 + (\sigma_F)^{2t} \ (u_*)^2}} \tag{24}$$

subject to the linear incidence equations (18), the kinematic relations of the mesh description (9) and the sign constraints:

$$\Delta\theta^+ \geq 0, \ \Delta\theta^- \geq 0, \ u \geq 0, \ \Delta\theta_* \geq 0, \ u_* \geq 0 \tag{25}$$

In the nodal description, (14) is used instead of (9) to represent the kinematic relations and $\Delta\varphi \geq 0$ should be added to the sign constraint (25).

The solution of these quadratic fractional programming problems is obtained by minimizing the quadratic concave function:

$$\min - 1/\beta^2 = -(\sigma_M \ \Delta\theta_*)^2 - (\sigma_F \ u_*)^2 \tag{26}$$

subject to (9), (18), (25) in the mesh description - (14), (18), (25) in the nodal description - and

$$\mu_M \ \Delta\theta_* - \mu_F \ u_* = 1 \tag{27}$$

The global optimum of these programs gives the plastic deformations for the stocastic more important mechanism and the random variables can be evaluated using (22)-(23).

3.2. SOLUTION ALGORITHMS

Finding of the global optimum in mathematical programming problems with nonconvex objective function and/or constraints is normally considered a NP-hard problem: The computational effort required grows exponentially with the number of variables in the worst case. One of the few cases for which there are algorithms available that work reasonably well is the minimization of a concave quadratic objective function subject to linear constraints.

3.2.1. Branch and Bound Techniques - The general nonconvex domain is transformed in the branch and bound (B & B) strategy into a sequence of intersecting convex domains by the use of underestimating convex functions. The two main ingredients are a combinatorial tree with appropriately defined nodes and some upper and lower bounds to the final solution associated with each node of the tree.

Each node of the tree in the B & B strategy is associated with a linear programming problem. For the concave function $-x^2$ defined in the interval [a, b] the convex underestimate is:

$$-(a+b)\,x + a\,b \leq -x^2 \qquad (28)$$

More details on the implementation of this technique can be found in Ref.4.

3.2.2. Cutting Plane Methods - Since the nonconvexity is restricted to the objective function, it is possible to solve this problem through techniques that exploit its special structure: its local solutions are vertices of the domain. In Konnos's algorithm (Ref.5) a local maximum is found and a cut is generated by the Simplex algorithm. In the next iteration, this procedure either generates a point which is strictly better than the last local maximum found, or generates a cut which is deeper until the convergence criteria to a ε-solution is met. This algorithm claims to be more efficient than the cutting plane methods suggested by Tui and Ritter.

4. Discussion

Acording to the yield line theory, the collapse of the slab is due to the formation of yield lines along which the slab folds when a mechanism can be activated. An automatic procedure to derive the trial mechanisms can be devised if the yield line method is formulated as a form of triangular finite element representation in which the yield lines are restricted to element sides. Numerical experience suggests that although cutting plane methods are more efficient than branch and bound techniques when the number of constraints and variables is small, the latter is easier to implement and more reliable. Simplex based algorithms were employed in each iteration of the branch and bound strategy. Since its efficiency is directly related to the number of variables and the cube of the number of constraints, the mesh description should be used for frames and the nodal description for slabs because they lead to smaller problems.

References

1. Simões, L.M.C. 'On the Reliability of Ductile Structural Systems by Mathematical Programming', submitted to *J. Struct. Div.* , *ASCE* , (1988).
2. Munro,J. and Da Fonseca, A.M.A. 'Yield Line Method by Finite Elements and Linear Programming', *The Structural Engineer* , **2** (1978).
3. Ditlevsen, O. 'Generalized Second Moment Reliability Index', *J. Struct. Mech.* , **7** (1979).
4. Simões, L.M.C. 'A Branch and Bound strategy for finding the Reliability Index with Nonconvex Performance Functions', *Structural Safety* , in publication, (1988).
5. Konno, H. 'Maximization of a Convex Quadratic Function under Linear Constraints', *Math. Program.*, **11** (1976).