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Discrete Optimum Design of Cable-Stayed Bridges

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1. Abstract
This work presents a procedure for finding the discrete optimum design of cable-stayed bridges. The minimization problem is stated as the minimization of stresses and bridge cost. A finite-element approach is used for structural analysis. It includes a direct analytic sensitivity analysis module, which provides the structural behaviour responses to changes in the design variables. To solve the continuous design problem an equivalent multi-criteria approach was used turning the original problem into the sequential minimization of unconstrained convex scalar functions, from which a Pareto optimum can be obtained. The segmental method which uses linear programming is adopted for solve the discrete design problem, using the Revised Simplex Method. The design sought to ensure that stresses are within acceptable limits, using discrete sections for member sizing. A steel cable-stayed bridge example is presented to illustrate the features of this design method.
2. Keywords: Structural optimization, cable-stayed bridges, discrete optimization.

3. Introduction
The optimization of cable-stayed bridges can be stated as the minimization of structural cost or volume and the maximum stresses and displacements throughout the structure. Additional objectives are aimed at deflections or displacements and to guarantee that the design variables are at least specified minimum values. The work started [1] with the shape and sizing optimization by using a two-dimensional finite-element model for the analysis. An equivalent multi-criteria approach was used to solve the continuous design problem turning the original problem into the sequential minimization of unconstrained convex scalar functions from which a Pareto optimum is obtained. It included a direct analytic sensitivity analysis module, which provides the structural behaviour responses to changes in the design variables. The problem was extended to three-dimensional analysis and the consideration of erection stages under static loading [2]. Seismic effects were considered in the optimization both by a modal-spectral approach and a time-history based procedure [3]. In most of the previous studies, a grid solution was adopted for modelling the deck, with stiffening girders supporting transverse beams, although box-girder sections were employed [4]. Pre-stressing design variables were also considered for the problems of optimal correction of cable forces during erection. Cable-stayed bridge optimization was formulated [5] within a level II probabilistic framework to achieve a reliability-based optimum design.
The rigorous discrete optimum design is a NP-hard problem significantly more difficult than the continuous problem. A starting continuous solution was obtained by an entropy-based procedure. This design forms a lower bound to the discrete optimum and it is usually assumed that the continuous sizes should be somehow rounded up or down to discrete sizes. The rounding process turns out to be a combinatorial method. The segmental method [6], [7] introduces the artificial concepts of segmental members and segmental optimum design and provides a close bound to the discrete optimum problem. The discrete optimum design assumes that each member is of known length and has unknown, but uniform, cross sectional properties. It is assumed instead that each member of the structure is composed of several segments each with a cross sectional area equal to one of the discrete sizes, such that all sizes are represented among the segments. The areas of all segments are known but the segment lengths are unknown. The segmental solution is obtained via linear programming. A cable-stayed bridge optimization problem illustrates the features of this procedure.

4. Structural Analysis
The structural analysis was done by means of a finite element computer program developed specifically for that purpose, because code availability was a fundamental requirement in order to the necessary further developments, namely, sensitivity analysis and structural optimization.
The bridge was modelled as a two-dimensional structure using bar and beam (Euler-Bernoulli formulation) elements.

5. Continuous Optimization

5.1. Decision variables

The shapes of the cross-sections of the main girder and pylon were assumed to be the box types represented in Figure 1 and Figure 2. The span lengths, number of cables, height and width of the elements of the main girder and pylon, material types to be used for each structural element, are pre-assigned constant design parameters. To perform the sizing optimization, the decision variables selected were:

- The equivalent plate thickness of the upper and bottom flanges of each main girder element (t_g in Figure 1);
- The equivalent plate thickness of each pylon element (t_t in Figure 2);
- Cross-sectional area of each cable (A_c in Figure 3).

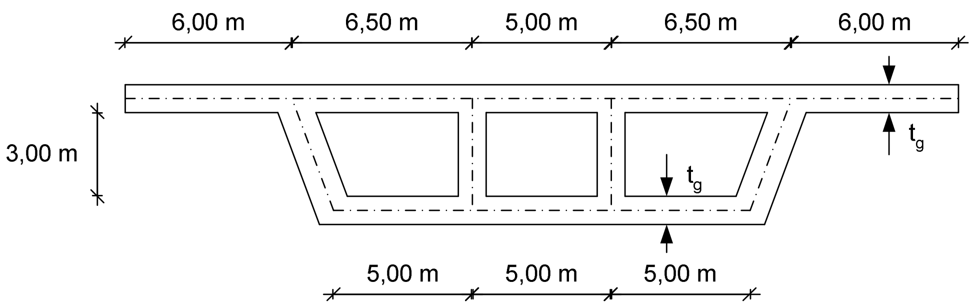

Figure 1: Deck cross-section

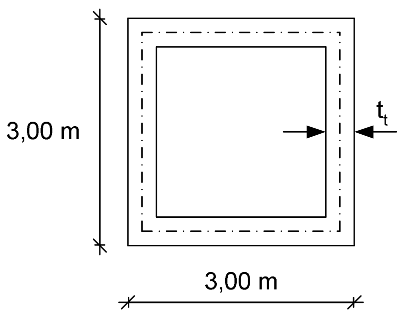

Figure 2: Pylon cross-section

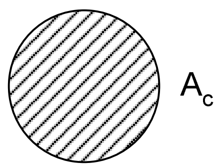

Figure 3: Cable cross-section

The thickness of the flange plates are dealt with as the converted thickness which includes the contributions of the longitudinal stiffeners. To simplify notation the vector of these sizing design variables will be referred to as $\boldsymbol{x}$ in the following sections.

In order to achieve computing time savings, it was assumed a symmetrical structural model. The Figure 4 identifies the considered decision variables, distributed by four zones in the deck, three zones in the pylons and one by each cable.

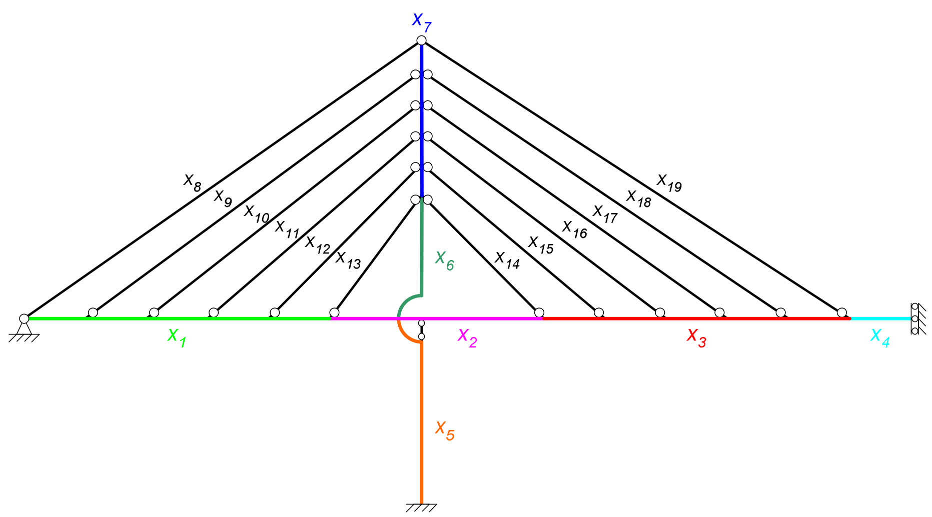

Figure 4: Decision variables

5.2. Multi-objective formulation

As an initial design a continuous optimum was sought by an entropy-based procedure. In minimization problems, a solution vector is said to be Pareto optimal if no other feasible vector exists that could decrease one objective function without increasing at least another one. The optimum vector usually exists in practical problems and is not unique. Cross-sectional design variables are considered, represented by x_i, respectively, and the global design variable vector is

$$\boldsymbol{x} = \{x_1, x_2, x_3, ..., x_n\}^T \ . \tag{1}$$

Bounds must be set for these variables in order to achieve executable solutions. The overall objective of cable-stayed bridge design is to achieve an economic, and yet safe, solution. In this study it is not intended to include all factors influencing the economics of a design, but only the cost of material, one of the conventionally adopted factors, and the achieving of the smallest stresses as possible. The optimization method described in the next section requires that all these goals should be cast in a normalized form. If some reference cost V_0 is specified, this goal can be written in the form

$$g_1(\boldsymbol{x}) = V(\boldsymbol{x}) / V_0 - 1 \le 0 \ . \tag{2a}$$

A second set of goals arises from the imposition of lower and upper limits on the sizing variables, namely minimum cable cross-sections to prevent topology changes and feasible dimensions for the deck and pylons cross-sections

$$g_2(\boldsymbol{x}) = -x_i / x_L + 1 \le 0 \tag{2b}$$

$$g_3(\boldsymbol{x}) = x_i / x_U - 1 \le 0 \ , \tag{2c}$$

where x_i is the i^{th} sizing variable and x_L and x_U its lower and upper bounds.

Further goals arise from the requirement that the stresses at each cable and elements of the main girder and pylon should be as small as possible

$$g_4(\boldsymbol{x}) = \frac{\sigma}{\sigma_t} - 1 \le 0 \tag{2d}$$

$$g_5(\boldsymbol{x}) = \frac{\sigma}{\sigma_c} - 1 \le 0 \ , \tag{2e}$$

where σ , σ_t and σ_c are the acting stress and the allowable stresses in tension and compression, respectively.

Another set of goals arise from the imposition of limit values, δ_0 , for the deflections or displacements in certain points of the structure

$$g_6(\boldsymbol{x}) = \frac{|\delta|}{\delta_0} - 1 \le 0 \ . \tag{2f}$$

The objective is to minimize all of these objectives over sizing variables $\boldsymbol{x}$. Different weights can be attributed to different goals just by changing the reference cost or stress limits. The objective of this Pareto optimization is to obtain an unbiased improvement of the current design. Templeman [8] has shown, by using an entropy-based approach, that the problem solution is equivalent to that of an unconstrained convex scalar function, which may be solved by conventional quasi-Newton methods

$$F(\boldsymbol{x}) = \frac{1}{\rho} \ln \left[\sum_{j=1}^{M} e^{\rho (g(\boldsymbol{x}))} \right] . \tag{3}$$

This function depends only on one control parameter, ρ, which must be steadily increased through the optimisation process.

5.3. Scalar function optimization

The goal functions $g_j(\boldsymbol{x})$ do not have an explicit algebraic form in most cases and the strategy adopted was to solve Eq.(3) by means of an iterative sequence of explicit approximation models. An explicit approximation can be formulated by taking Taylor series expansions of all the goal functions $g_j(\boldsymbol{x})$ truncated after the linear term. This gives:

$$\min F(\boldsymbol{x}) = \frac{1}{\rho}\ln\left[\sum_{j=1}^{M} e^{\rho\left(g_0(\boldsymbol{x}) + \sum_{i=1}^{N}\frac{dg_{0j}(\boldsymbol{x})}{dx_i}\Delta x_i\right)}\right], \tag{4}$$

where N and M are, respectively, the number of sizing design variables and the number of goal functions. g_{0j} and dg_{0j}/dx_i are the goals and their derivatives evaluated for the current design variable vector (x_0), at which the Taylor series expansion is made.

Solving Eq.(4) for particular numerical values of g_{0j} forms only one iteration of the complete solution of problem, Eq.(3). The solution vector (x_1) of such iteration represents a new design that must be analysed and gives new values for g_{1j}, dg_{1j}/dx_i and (x_1), to replace those corresponding to (x_0) in Eq.(4). Iterations continue until changes in the design variables become small. Move limits are imposed to ensure the accuracy of the explicit approximation. During these iterations the control parameter ρ must not be decreased to ensure that a multi-objective solution is found.

6. Sensitivity Analysis

Iterative optimization algorithms need to know the way a change in each design variable will affect the requirements expressed as goals. This is the task of the sensitivity analysis and represents most of the computational effort required for structural optimization. The evolution of the problem depends on a critical way on the accuracy with which these values are computed. Given the availability of the source code, the discrete nature of cable-stayed bridge structures and the large number of constraints (stresses and displacements) under control, the analytical discrete direct method was used for the sake of sensitivity analysis. The expressions for this method are obtained by differentiating the equilibrium equations

$$\boldsymbol{K}\cdot\boldsymbol{u} = \boldsymbol{P}\,. \tag{5}$$

The following expression is obtained:

$$\frac{d\boldsymbol{K}}{dx_i}\boldsymbol{u} + \boldsymbol{K}\frac{d\boldsymbol{u}}{dx_i} = \frac{d\boldsymbol{P}}{dx_i}\,, \tag{6}$$

which can be rewritten in the form

$$\boldsymbol{K}\frac{d\boldsymbol{u}}{dx_i} = \frac{d\boldsymbol{P}}{dx_i} - \frac{d\boldsymbol{K}}{dx_i}\boldsymbol{u} = \boldsymbol{Q}_{vi}\,, \tag{7}$$

where $\boldsymbol{Q}_{vi}$ is the virtual pseudo-load vector of the system with respect to the i^{th} design variable.

The stress derivatives are accurately determined from the chain derivation of the finite element stress matrix

$$\boldsymbol{\sigma} = \boldsymbol{D}\cdot\boldsymbol{B}^e\cdot\boldsymbol{u}^e \tag{8}$$

$$\frac{d\boldsymbol{\sigma}}{dx_i} = \frac{d\left(\boldsymbol{D}\cdot\boldsymbol{B}^e\right)}{dx_i}\cdot\boldsymbol{u}^e + \boldsymbol{D}\cdot\boldsymbol{B}^e\cdot\frac{d\boldsymbol{u}^e}{dx_i} \tag{9}$$

The first term of right-hand side may be directly computed during the computation of element contribution for the global system, on the condition that derivative expressions are pre-programmed and called on that stage. The second term on the right-hand side is somewhat more difficult to compute because an explicit relation between displacement vector and design variable set does not exist. Pre-programming and storing the stiffness matrix and right-hand side derivatives in the same way as described for the stress matrix, the displacement derivatives may be computed by the solution of N pseudo-load right hand sides. The stress derivatives are then computed in a straightforward way. The explicit form of matrix derivatives depends on the type of element. For two-dimensional bar and beam elements their calculation is a straightforward task.

7. Segmental Optimum Design

The segmental method assumes that each member of the structure is composed of a total of D segments, each with geometrical properties equal to one of the discrete sizes t_d, $d = 1,..., D$, such that all sizes are represented among the segments. Let l_{id} be the unknown length of the segment of member i which belongs to the discrete set t_d, $d = 1,..., D$. This is shown in Figure 5 for a member which has three discrete sizes. The geometry of all segments are known, but the segment lengths are unknown. The ordering of the segments along a member is immaterial.

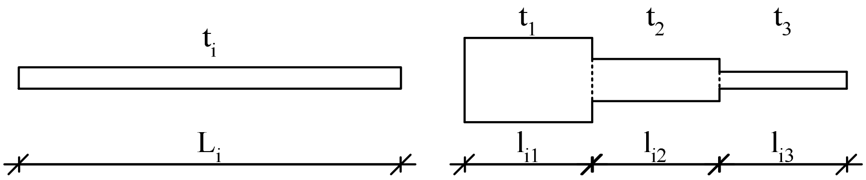

Figure 5: Conventional member and segmental member

t_d represents discrete thicknesses for the plates of the deck and pylons cross-sections. Considering that, the structure volume minimization problem using the segmental method can be formulated as:

$$\min Z = \boldsymbol{C} \cdot \boldsymbol{X} \tag{10}$$

$$\min Z = l_{11} \cdot t_1 + l_{12} \cdot t_2 + l_{13} \cdot t_3 + ... + l_{id} \cdot t_d = \sum_{i=1}^{N} \sum_{d=1}^{D} l_{id} \cdot t_d \ , \tag{11}$$

Subject to

$$\sum_{d=1}^{D} l_{id} = L_i ; i = 1,..., N \ , \tag{12}$$

Eqn (12) means that the sum of the segment length of each bar must total the bar length;

$$\sum_{d=1}^{D} \sigma_{id} \cdot l_{id} \leq \sigma_{ad} \cdot L_i \ , \tag{13}$$

Moreover the stress in each element must be less than an admissible value. Also displacement constraints could be formulated

$$\sum_{d=1}^{D} \delta_{id} \cdot l_{id} \leq \delta_0 \cdot L_i \ . \tag{14}$$

The cost vector, $\boldsymbol{C}$, contains the values of the D discrete sections available and vector $\boldsymbol{X}$ contains the design variables that are the lengths l_{id} of all the segments of all the members.

To enable a computer solution of the design optimization problem it is first necessary to formulate the stress σ in each member as an explicit function of the design variables. Since stresses vary inversely with the section properties a good quality explicit approximation of each stress, σ , is provided by the first-order Taylor series:

$$\sigma = \sigma_0 + \sum_{i=1}^{n} \frac{d\sigma}{dx_i} \Delta x_i \ , \tag{15}$$

where the subscript zero (0) defines known quantities for the current structure, while x_i are the design variables and Δx_i denotes the variation in that design variables [7]. Problem (11) is an LP problem which may be solved by any LP algorithm, and will yield what can be termed a segmental optimum design.

8. Numerical Example

A numerical example with sizing design variables only and without erection stages will be presented next. This example is composed by a symmetrical steel cable-stayed bridge with a total length of 290 m, with a central span of 160 m and lateral spans of 65 m. Pylons total height is of 75 m with the deck placed 30 m above the foundation.

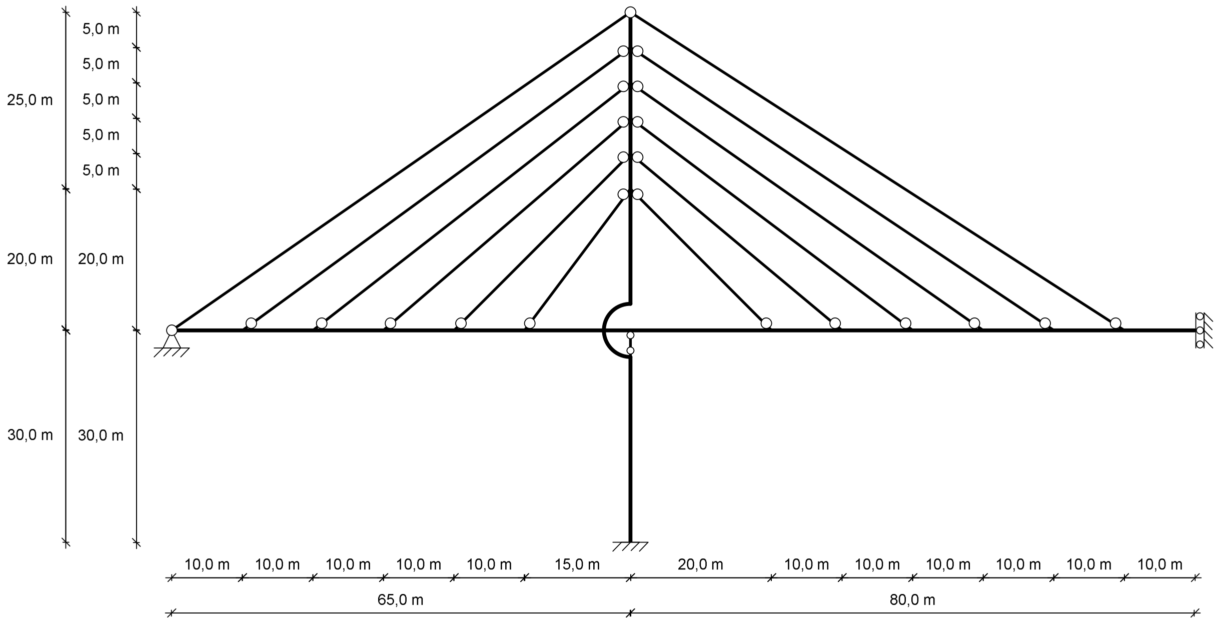

Figure 6: Bridge geometry

Half of the bridge was modelled, by using bridge and loading symmetry. Three loading cases were considered, combining uniformly distributed dead and live loads. The live load was applied on the whole deck, only in lateral spans and only on central span, respectively for first, second and third load cases. The non structural dead load (relative to the deck slab and the bitumen covering) was considered 6 kN/m^2 and for the road traffic live load a value of 4 kN/m^2 was adopted. For the structural steel self-weigh it was considered a value of 77 kN/m^3. The allowable stresses were set at 200 MPa for deck and pylon elements and 500 MPa for cable elements. Minimum stress in stays is prescribed as 10% of allowable stress.

A continuous design was made to obtain a starting point for the discrete design. Table 1 shows the initial and optimized values of the decision variables obtained with the continuous optimization procedure taking into account the stress envelope for the three loading cases described.

Table 1: Initial and final (optimized) values of decision variables

Decision variable	Initial value	Final value
x_1 [m]	$1{,}00\times10^{-2}$	$8{,}87\times10^{-3}$
x_2 [m]	$1{,}00\times10^{-2}$	$8{,}00\times10^{-3}$
x_3 [m]	$8{,}00\times10^{-3}$	$7{,}98\times10^{-3}$
x_4 [m]	$1{,}00\times10^{-2}$	$9{,}73\times10^{-3}$
x_5 [m]	$3{,}50\times10^{-2}$	$3{,}12\times10^{-2}$
x_6 [m]	$3{,}00\times10^{-2}$	$3{,}00\times10^{-2}$
x_7 [m]	$3{,}00\times10^{-2}$	$3{,}00\times10^{-2}$
x_8 [m^2]	$3{,}00\times10^{-2}$	$2{,}22\times10^{-2}$
x_9 [m^2]	$3{,}00\times10^{-2}$	$9{,}24\times10^{-3}$
x_{10} [m^2]	$3{,}00\times10^{-2}$	$2{,}75\times10^{-2}$
x_{11} [m^2]	$3{,}00\times10^{-2}$	$2{,}70\times10^{-2}$
x_{12} [m^2]	$2{,}00\times10^{-2}$	$2{,}00\times10^{-2}$
x_{13} [m^2]	$2{,}00\times10^{-2}$	$5{,}00\times10^{-2}$
x_{14} [m^2]	$2{,}00\times10^{-2}$	$1{,}33\times10^{-2}$
x_{15} [m^2]	$2{,}00\times10^{-2}$	$1{,}77\times10^{-2}$
x_{16} [m^2]	$3{,}00\times10^{-2}$	$2{,}37\times10^{-2}$
x_{17} [m^2]	$3{,}00\times10^{-2}$	$1{,}50\times10^{-2}$
x_{18} [m^2]	$3{,}00\times10^{-2}$	$2{,}29\times10^{-2}$
x_{19} [m^2]	$3{,}00\times10^{-2}$	$1{,}22\times10^{-2}$
Volume of the structure [m^3]	115,875	102,847

Table 2 presents the volume variation of the structure components obtained with the continuous optimization.

Table 2: Volume variation obtained with the continuous optimization

Volume	Initial value [m^3]	Final value [m^3]	ΔVolume [%]
Cables	18,050	12,998	-28,0%
Pylons	28,432	27,142	-4,5%
Deck	69,393	62,707	-9,6%
Total structure	115,875	102,847	-11,2%

For this solution it can be stated that the deck's volume represents 61,0% of the total structure volume, while pylons and stays represents only 26,4% and 12,6% respectively. Considering an equal volume cost for the steel of the cable and the remaining structural elements, the deck and the pylons are the most important components for the structure total cost. For that the discrete optimization was only applied to the bridge deck and pylons sections.

The solution of the segmental method was obtained by linear programming. In this work the Revised Simplex Algorithm was used.
In order to obtain a discrete optimum design two different approaches were made. In the first one the optimization of the pylons cross-sections was made independently from the discrete solution of the deck cross-sections. In the second one the discrete solution for the pylons cross-sections was obtained taking into account the discrete solution achieved for the deck cross-sections.
According to the values obtained for the decision variables of the deck sections (x_1, x_2, x_3 and x_4) five discrete sections were defined for the flanges plate thickness, as presented in Table 3.

Table 3: Discrete sections considered for the deck in the segmental method

Section	t_g [m]
1	$8{,}00\times10^{-3}$
2	$9{,}00\times10^{-3}$
3	$1{,}00\times10^{-2}$
4	$1{,}10\times10^{-2}$
5	$1{,}20\times10^{-2}$

The results obtained with the segmental method procedure to the optimization of the bridge deck volume are presented in Table 4.

Table 4: Segmental method results for the deck

Decision variable	$L_{Section\ 1}$ [m]	$L_{Section\ 2}$ [m]	$L_{Section\ 3}$ [m]	$L_{Section\ 4}$ [m]	$L_{Section\ 5}$ [m]
x_1	27,451	0,0	0,0	0,0	22,549
x_2	35,0	0,0	0,0	0,0	0,0
x_3	50,0	0,0	0,0	0,0	0,0
x_4	10,0	0,0	0,0	0,0	0,0

In Table 5 is presented the volume variation obtained for the bridge deck using the segmental method for optimum discrete design.

Table 5: Deck volume variation with the discrete optimum design

Continuous value [m^3]	Discrete value [m^3]	ΔVolume [%]
62,707	64,268	2,5%

Considering the continuous optimum results obtained for the pylons (x_5, x_6, and x_7) the following discrete sections were defined, see Table 6.

Table 6: Discrete sections considered for the pylons in the segmental method

Section	t_t [m]
1	$3,00\times10^{-2}$
2	$3,10\times10^{-2}$
3	$3,20\times10^{-2}$
4	$3,40\times10^{-2}$
5	$3,50\times10^{-2}$

In Table 7 are presented the results achieved with the segmental method for the discrete optimization of the pylons cross-sections.

Table 7: Segmental method results for the pylons – 1st approach

Decision variable	$L_{Section\ 1}$ [m]	$L_{Section\ 2}$ [m]	$L_{Section\ 3}$ [m]	$L_{Section\ 4}$ [m]	$L_{Section\ 5}$ [m]
x_5	4,913	0,0	0,0	0,0	25,087
x_6	20,0	0,0	0,0	0,0	0,0
x_7	25,0	0,0	0,0	0,0	0,0

The second approach was made considering sensitivities and stress values computed taking into account the discrete solution for the deck elements, shown in Table 4. The solution obtained for this approach is presented in Table 8.

Table 8: Segmental method results for the pylons – 2nd approach

Decision variable	$L_{Section\ 1}$ [m]	$L_{Section\ 2}$ [m]	$L_{Section\ 3}$ [m]	$L_{Section\ 4}$ [m]	$L_{Section\ 5}$ [m]
x_5	10,184	0,0	0,0	0,0	19,816
x_6	20,0	0,0	0,0	0,0	0,0
x_7	25,0	0,0	0,0	0,0	0,0

Table 9 summarizes the results obtained for the pylons volume with both approaches.

Table 9: Pylons volume variation with the discrete optimum design attempts

	Continuous value [m^3]	Discrete value [m^3]	ΔVolume [%]
1st approach	27,142	28,203	3,9%
2nd approach	27,142	27,894	2,8%

An increase of 6,4% and 5,3% in the structure total volume was verified for the first and second approaches respectively. This upper bound in the discrete optimum solution can be improved by adopting other strategies. These heuristic procedures consist of adding fictitious additional costs to the segmental members.
The discrete solutions obtained were checked to ensure the allowable stresses are not exceeded.

9. Conclusion

The work presented here shows the application of the segmental method for the optimum discrete design of a cable-stayed bridge. A linear explicit approximation was used to compute the stress in each element. This approximation was found to be able to predict the stresses accurately.
Two approaches used in the discrete design lead to different volume increases. The independent optimization of the deck and pylons cross-sections gave poorer results than optimizing the pylons using the results of the deck discrete optimization. The choice of thicker flanges in the second approach allows the redistribution of stresses, and therefore, thinner flanges for the pylons sections are required.
The procedure used in this work seems to be adequate to the discrete optimization of a cable-stayed bridge revealing to be more efficient than combinatorial or genetic algorithms.
The segmental method employed here extends the original concept of displacement based design to accommodate stresses by employing an explicit approximation of these constraints.

10. References

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