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UNIVERSIDADE DE COIMBRA

Nuclear matter equation of state with light clusters

Dissertação submetida para a obtenção do Grau de Mestre em Física Nuclear e de Partículas

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Resumo

Neste projecto foi estudada a equação de estado da matéria nuclear assimétrica a baixas densidades e temperatura zero com a introdução de núcleos leves (hélio, trítio, alfa e deuterão). A equação de estado foi construída utilizando o modelo não linear de Walecka na aproximação de campo médio.

Foram estudadas as propriedades da matéria nuclear para vários valores de assimetria com e sem núcleos. A dependência da densidade de dissolução de cada núcleo em função do valor das suas constantes de acoplamento foi estudada utilizando duas parametrizações da equação de estado da matéria nuclear.

Abstract

In this project the equation of state of asymmetric nuclear matter at low densities and zero temperature introducing light nuclei (helion, triton, deuteron and alpha particles) was studied. The equation of state was built using the non-linear Walecka model in the mean-field approximation.

The properties of nuclear matter for various asymmetry values with and without clusters were studied. The dissolution density dependence for each nucleus as a function of its coupling constants was also determined using two parametrizations of the equation of state of nuclear matter.

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Chapter 1

Introduction

Supernova explosion is a stellar phenomenon that happens when massive stars ($M \geq 8M_{\odot}$) have burned up all the nuclear fuel. Through fusion reactions, hydrogen is converted to heavier elements by the heat produced by gravitational compression. The fusion reactions stop when iron, the most bound nuclear species, is reached. Iron is the heaviest element on the fusion chain. After the iron is formed in the core, the core collapses due to gravitational force, creating a supernova explosion.

A supernova explosion may lead to the formation of a neutron star or a black hole. A maximum mass exists, called the Oppenheimer-Volkoff mass limit [12], that can be sustained against gravitational collapse. Therefore, if the mass of the hot collapsed core of the supernova explosion is not high enough to form a black hole, a neutron star is born.

The initial temperature of the neutron star is of the order of tens of MeV and in a few seconds cools to ~ 1 MeV due to the diffusion of neutrinos and photons [5]. The mass and radii are typically $1.5M_{\odot}$ and 10 km, respectively. Thus, neutron stars are the smallest and densest stars known. Neutron stars are 10^{14} times denser than Earth [5].

Neutron stars can be detected by their periodic radio emission of pulsars. This periodic radio emission is due to its intense magnetic field and high rotation. The observed pulsars, the name given to neutron stars that emit periodic pulses, range from milliseconds to seconds with the mean period of ~ 0.7 ms [5]. A realistic description of the structure of neutron stars demands the use of Einstein's equations of General Relativity.

Neutron stars are composed by nucleons (protons and neutrons) and leptons (electrons, muons and neutrinos). Exotic matter such as hyperons, boson condensates and quark matter may also exist in their interior.

The nuclear short-range repulsion and Pauli exclusion principle give rise to an outward pressure that makes neutron stars stable against gravitational collapse [5].

The gravitational energy per nucleon in a neutron star can be estimated as ~ 157 MeV and the binding energy per nucleon in a limiting neutron star mass as ~ 100 MeV [5].
Therefore, the nuclear force contributes negatively to the binding of neutron stars.

This is due to the fact that the average density of a neutron star lies above the saturation density of nuclear matter and, consequently, the nucleons do not feel attraction from their neighbors, but repulsion. In fact, if we turned off the gravitational interaction, the neutron star would explode.

In a long time scale neutron stars will become invisible because, as the stars cool, their magnetic field decays and, therefore, its rotation decreases until the magnetic field pulse is no longer detectable.

One global constraint that neutron stars must satisfy is the charge neutrality. The repulsive Coulomb force for an infinitesimal charge per baryon in the star will expel particles of like charge [5].

The mass and radii of a neutron star is obtained solving the Equation of hydrostatic equilibrium of General Relativity called the Tolmann-Oppenheimer-Volkoff equation. For solving the Tolmann-Oppenheimer-Volkoff equation we need the equation of state $p = p(\mathcal{E})$ of the nuclear matter [5]. It is through the equation of state that the properties of dense matter enter the equations of stellar structure.

The density range of a neutron star falls from its high central value to approximatly fifteen orders of magnitude to the surface. At each density the matter is in beta equilibrium and consistent with charge neutrality.

The pressure is zero at the surface of a neutron star and the density is low, therefore, it is composed by atoms of the most strongly bounded nuclei, ^{56}Fe. As the density increases, the atoms become progressively more ionized, the electrons become increasingly relativistic and a lower energy state can be achieved through the capture of energetic electrons by nuclei. Thus, the nuclei will become increasingly neutron-rich. This process is called neutralization. All the neutrinos and photons created in these processes, diffuse out of the star lowering their energy.
As the density increases, the neutron drip is reached, where the bound neutrons leave the nuclei. At densities above the saturation density of nuclear matter, the superdense regime is reached, nuclei dissociate into a uniform charge-neutral mixture of baryons and leptons.

At superdense matter hyperons can be formed and constitute an important part of the population or a kaon condensate may occur [5]. The superdense matter may be composed of quark matter at high density. The density at which the confined hadronic matter converts into its quarks constituints (deconfined matter) is experimentally and theoretically unknown.

As we have seen, the neutron stars span a wide density range that go from ordinary nuclei at the surface to superdense quark matter in the core.

One of the primary goals of nuclear physics is the determination of the equation of state of nuclear matter. Astrophysical predictions are very difficult due to the large extrapolations involved: isospin extrapolations from stable nuclei to the very neutron-rich systems present in neutron stars and the extrapolation from density of normal nuclei to the very high densities or low densities that are present in the core or in the crust of a neutron star, respectively.

In stable nuclei, the asymmetry is $\sim 0-0.2$, in neutron star it ranges from ~ 0.9 in the dense interior to ~ 0.1 near the crust at densities $\rho \sim 10^{-5} - 10^{-1}\,\mathrm{fm}^{-3}$ [13].

To determine the equation of state of the dense matter we want to start from the known properties of bulk nuclear matter. A relativistic quantum field theory in a mean-field approximation that is based on nuclear effective interactions is used [11]. These effective interactions provide an efficient description of the structure of finite nuclei and nuclear matter and are fitted to reproduce well-determined nuclear matter properties.

In this work the inclusion of light clusters (^{2}H,^{3}H,^{3}He,^{4}He) at zero temperature and low densities in the EOS is studied. Below saturation density $\sim 0.16\,\mathrm{fm}^{-3}$, the system can minimize its free energy by forming clusters [6]. Therefore the study of low-density nuclear matter must take into account clusters. The challenging question is figuring out how can these clusters be included in the model and how do they affect the low density equation of state. Due to clustering, the physics of nuclear matter will be very different from neutron matter, mainly we expect that transport properties of the crust will be affected.

The second chapter is devoted to introducing the properties of nuclear matter and the nuclear field theory. On the third chapter, we discuss the model used in this work. It is a relativistic mean-field model with a omega-rho term, which allows changing the density dependence of the symmetry energy, while keeping the isoscalar channel unchanged

Finally, in the fourth chapter, the results obtained are presented. In particular we are interested in determining the dissolution density of the different types of clusters.

Chapter 2

Relativistic Nuclear Field Theory

Until the mid 1970s, nearly all dense nuclear matter studies were based on non-relativistic static potentials for describing the nucleon-nucleon interaction.

The relativistic, field-theoretical approach to nuclear matter was introduced in 1974 by J. D. Walecka [18]. It is a Lorentz covariant theory of nuclear matter describing the interaction between nucleons in matter through two meson fields, the scalar σ and the vector V^μ [5]. This theory is known as QHD-I (quantum hadrodynamics) or $\sigma - \omega$ model.

This theoretical approach has the following advantages:

- It is automatically causal.
- The properties of nuclear matter at the saturation density are built-in in the theory.

The second point is very important since, as we will see, the input parameters of the theory can be algebrically related with the properties of nuclear matter at the saturation density. Therefore, before introducing the model, we need to study the properties of nuclear matter.

In this whole chapter we follow Glendenning's book closely [5].

2.1 Properties of Nuclear Matter

Presently, it is unknown any point of the equation of state (EOS) of nuclear matter, $P(\mathcal{E})$, above nuclear density with precision. We use some properties of nuclear matter at the saturation density to normalize the EOS at one point in the energy-density plane and others to assure that the extrapolations to higher densities near the saturation density are valid.

We use the bulk aproximation that consists in considering uniform, infinite and symmetric nuclear matter. The properties of nuclear matter in the bulk approximation are inferred from experimentally observed properties of finite nuclei.

Some properties of nuclear matter cannot be directly measured and there is some uncertainty on their exact values.

All the empirical values used were taken from [5].

2.1.1 Saturation density

The short-ranged, strong nuclear interaction is the dominant interaction between nucleons. It is essentially attractive but repulsive at short distance ($< 0.4\,\mathrm{fm}$). Due to the properties of nuclear force, at a certain density, the central density of the system will not increase any further, even if more nucleons are added to the system. The density at which this occurs is referred to as the saturation density.

Thus we have the following relation

$$\rho = \frac{A}{V} \approx constant \tag{2.1}$$

where A is the number of nucleons (number of protons plus the number of neutrons). Considering, for simplicity, a spheric nuclei we have

$$\rho = \frac{A}{V} = A\left(\frac{4\pi R^3}{3}\right)^{-1} \approx constant \tag{2.2}$$

therefore we find the relation $R = r_0 A^{1/3}$. This relation is satisfied in good accuracy for finite nuclei.

For the saturation density, ρ_0, the observed value used is

$$\rho_0 = 0.153\,\mathrm{fm}^{-3}.$$

2.1.2 Binding energy

The parametrization of nuclear masses as a function of the number of neutrons, N, and protons, Z, is known as the Weizsäcker formula or the semi-empirical mass formula given by

$$M(Z,N) = Zm_p + Nm_n - B(Z,N)$$

where m_p and m_n are, respectively, the mass of proton and neutron in MeV, the $B(Z,N)$ is the binding energy of the nucleus given by

$$B(Z,N) = a_v A - a_s A^{2/3} - a_C Z^2 A^{-1/3} - a_{sym}\frac{(N-Z)^2}{A} \tag{2.3}$$

where a_v, a_s, a_C and a_{sym} correspond to volume, surface, Coulomb and asymmetry coefficients, respectively. The physical interpretation of the individual terms of (2.3) can be found in [9]. The exact values of the coefficients depend on the range of masses for which they are optimized. Several coefficient sets can be found in the literature.

From the equation (2.3) we get the binding energy per nucleon

$$\frac{B(Z,N)}{A} = a_v - \frac{a_s}{A^{1/3}} - a_C\frac{Z^2}{A^{4/3}} - a_{sym}\left(\frac{N-Z}{A}\right)^2. \tag{2.4}$$

In the bulk approximation, the Coulomb interaction is ignored. Letting $A \to \infty$ on (2.4), we get

$$\frac{B}{A} = a_v = -16.3\,\mathrm{MeV}.$$

Thus, in the bulk aproximation, the binding energy per nucleon is given by the volume term.

2.1.3 Symmetry energy

From the valley of the beta stability we know that stable nuclei with a low proton number prefer a nearly equivalent neutron number. This preference will diverge for nuclei with a large proton number due to the repulsive Coulomb interaction between the protons. This is exactly what the last term of (2.4) says. We call the difference between the neutron and protons numbers, the isospin symmetry, difined by $t \equiv (\rho_n - \rho_p)/\rho$.

The equation of state is related to the binding energy per nucleon by

$$\left(\frac{E}{A}\right)_0 = \left(\frac{\mathcal{E}}{\rho}\right)_0 = \frac{B}{A} + M \tag{2.5}$$

where $M = 938.93\,\mathrm{MeV}$ is the average of the neutron and proton masses, called nucleon mass. So, taking into account (2.4), we can calculate the asymmetry energy coefficient from (2.5), by

$$a_{sym} = \frac{1}{2}\left(\frac{\partial^2 (\mathcal{E}/\rho)}{\partial t^2}\right)_{t=0}. \tag{2.6}$$

A neutron star should be electrically neutral[1], therefore in a first approximation the neutron stars are mainly populated by neutrons. Thus, neutron stars are systems highly isospin asymmetric unlike nuclear matter that prefers isospin symmetry. So, for a good description of neutron stars, the theory should correctly reflect this fact.

If we assume charge symmetry of nuclear interaction, the energy per baryon of asymmetric nuclear matter can be expanded on a series in powers of t^2

$$\frac{E}{A}(\rho, t) = \frac{E}{A}(\rho, 0) + a_{sym}(\rho)t^2 + a^2_{sym}(\rho)t^4 + \mathcal{O}(t^6)$$

where $\frac{E}{A}(\rho, 0)$ is the energy per nucleon of symmetric nuclear matter.
Several theoretical studies show that the dominance dependence of the energy density of asymmetric nuclear matter on t, is essentially quadratic [16].
Thus, in good approximation, the symmetry energy is given by

$$a_{sym} \approx \frac{E}{A}(\rho, 1) - \frac{E}{A}(\rho, 0).$$

Therefore the symmetry energy could be defined as the difference between the energy per nucleon of the pure neutron matter ($t = 1$) and the symmetric nuclear matter ($t = 0$).

The nuclear symmetry energy is well constrained at the saturation density, $a_{sym}(\rho_0) = 32 \pm 4\,\mathrm{MeV}$ but its value at higher and lower densities are extremely diverse. An interesting fact is that most empirical models coincide around $\rho \approx 0.6\rho_0$ where $a_{sym} = 24\,\mathrm{MeV}$ [4]. This shows that constraints on finite nuclei are active for

[1]the net charge per nucleon in a neutron star must be very small, of the order $Z_{net}/A < 10^{-36}$. For details see [5].

densities $\approx 0.6\rho_0$.

In this work, we use the empirical value $a_{sym} = 32.5\,\mathrm{MeV}$ [5].

2.1.4 Compression modulus

The compression modulus K defines the curvature of de EOS $\mathcal{E}/\rho$ at ρ_0 and is related to the high density behavior of the EOS.
We normally use the term stiff and soft to characterize the behavior of the EOS, i.e., if the energy density rapidly increases with an increase in pressure, the EOS is referred to as stiff and if it increases slowly, it is referred to as soft. The maximum mass that a neutron star can take depends on the stiffness of the EOS, thus, the maximum mass of a neutron star is related to the value that we choose for the K.

The compression modulus is deffined by

$$K = 9\left[\rho^2 \frac{d^2}{d\rho^2}\left(\frac{\mathcal{E}}{\rho}\right)\right]_{\rho=\rho_0} . \tag{2.7}$$

The value of K is quite uncertain and lies in the range of 200 to $300\,\mathrm{MeV}$.

The compression modulus can not be measured directly and that is one of the reasons for the uncertainty on the experimental values. The value of K can be extracted from experimental energies of isoscalar monopole vibrations (GMR) in nuclei but its value can not be constraint better than 50%. Other approach is determining K based on microscopic calculations of GMR, giving $210 - 220\,\mathrm{MeV}$ using Skyrme effective interaction and $231 \pm 5\,\mathrm{MeV}$ based on Gogny effective interaction. Details of these values can be seen in [17] and references therein.

2.1.5 Effective mass

The effective mass also referred as relativistic Dirac mass is defined through the scalar part of the nucleon self-energy in the Dirac field equation, which is absorved into the effective mass $M^* = M + \Sigma(\rho)$ [4]. Various values for the effective mass can be found in [7] and references therein.

The experimental values for the effective mass lie in the range

$$\frac{M^*}{M} \approx 0.7 \quad \text{to} \quad 0.8.$$

As K, the effective mass will influence the high-density behavior of the EOS.

2.2 QHD-I Model

The Quantum Hadro-Dynamics I was introduced by J. D. Walecka [18]. It is based on four fields: the barionic field ψ, that represents the nucleons, the neutral scalar meson σ and the neutral vector meson V^μ, that represent, respectively, the σ and ω particles. The properties of these mesons are in Table 2.1.

Table 2.1: Meson properties: spin (s), isospin (I) and charge (q).

	m (MeV)	s	I	q
σ	~ 500	0	0	0
ω	782	1	0	0
ρ	770	1	1	$-1, 0, 1$

The scalar meson gives rise to a strong attractive force in the nucleon-nucleon interaction, while the vector meson gives rise to a strong repulsive force.

The model assumes that neutral scalar meson couples to the scalar density of baryons through $g_s\bar{\psi}\psi\sigma$ and the neutral vector meson couples to the conserved baryon current through $g_v\bar{\psi}\gamma_\mu\psi V^\mu$.

In the limit of heavy, static baryons, one-meson exchange gives rise to an effective nucleon-nucleon potencial of the form [11]

$$V_{eff}(r) = \frac{g_v^2}{4\pi}\frac{e^{-m_v r}}{r} - \frac{g_s^2}{4\pi}\frac{e^{-m_s r}}{r}.$$

Choosing the right coupling constants of the mesons, the V_{eff} reproduces the potencial of the nuclear force.

2.2.1 Lagrangian density

The lagrangian interaction of this model can be written as[2]

$$\mathcal{L}_{int} = g_s\sigma(x)\bar{\psi}(x)\psi(x) - g_v V_\mu(x)\bar{\psi}(x)\gamma^\mu\psi(x). \tag{2.8}$$

Including the free Lagrangians of the mesons and the nucleons we obtain

$$\begin{aligned}\mathcal{L} &= \bar{\psi}(x)\left[\gamma_\mu\left(i\partial^\mu - g_v\omega^\mu(x)\right) - \left(M - g_s\sigma(x)\right)\right]\psi(x) \\ &+ \frac{1}{2}\left(\partial_\mu\sigma(x)\partial^\mu\sigma(x) - m_\sigma^2\sigma^2(x)\right) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m_v^2 V_\mu(x)V^\mu(x),\end{aligned} \tag{2.9}$$

where $F_{\mu\nu} = \partial_\mu V_\nu(x) - \partial_\nu V_\mu(x)$.

As expected, the nucleons are described by the Dirac Lagrangian, the scalar meson by the Klein-Gordon Lagrangian and the vector meson by the massive vector meson Lagrangian that gives the Proca equation.

[2]hereafter we use the shorthand notation $x \equiv (t, x, y, z)$

The nucleon spinor is a eight-component spinor given by

$$\psi \equiv \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix}.$$

The coupling constans g_s and g_v are dimensionless constants that will be fitted to the bulk properties of nuclear matter.

2.2.2 Equations of motion

The equations of motion for the fields are calculated solving the Euler-Lagrange equation

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \tag{2.10}$$

As we will see in the next chapter, (2.9) is a special case of the Lagrangian that we use in this work. The calculation details of the equations of motion can be found in Appendix 1.

Using (2.10) and (2.9), we get

$$\left(\partial_\mu \partial^\mu + m_s^2\right) \sigma(x) = g_s \bar{\psi}(x) \psi(x) \tag{2.11}$$

$$\left(\partial_\mu \partial^\mu + m_v^2\right) V_\mu(x) - \partial_\mu \partial^\nu V_\nu(x) = g_v \bar{\psi} \gamma_\mu(x) \psi(x) \tag{2.12}$$

$$\left[\gamma_\mu \left(i\partial^\mu - g_v V^\mu(x)\right) - \left(M - g_s \sigma(x)\right)\right] \psi(x) = 0. \tag{2.13}$$

From the Proca equation for a massive vector field we know that $\partial^\mu V_\mu = 0$ due to current conservation. So, (2.12) gives

$$\left(\partial_\mu \partial^\mu + m_v^2\right) V_\mu(x) = g_v \bar{\psi}(x) \psi(x) \tag{2.14}$$

The three equations of motion constitute a system of coupled nonlinear differencial equations that are complicated to solve. Perturbative approaches to solve the equations are not useful due to the large values of the coupling constants. Therefore we need a new approach for solving the equations.

2.2.3 Mean-field approximation

The Relativistic Mean-Field (RMF) approximation consists on considering that the system is composed by static and uniform matter in its ground state. Then, we replace the meson fields by their mean values on this state

$$\sigma(x) \quad \rightarrow \quad \langle \sigma \rangle \tag{2.15}$$

$$V_\mu(x) \quad \rightarrow \quad \langle V_\mu \rangle. \tag{2.16}$$

In the ground state of static and uniform matter quantities, $\langle \sigma \rangle$ and $\langle V_0 \rangle$ are independent of $x \equiv (t, x, y, z)$, so are the source terms $\langle \bar{\psi}(x)\psi(x) \rangle$ and $\langle \bar{\psi}\gamma_\mu(x)\psi(x) \rangle$. Due to rotational invariance, the expectation value of $\langle V_i \rangle$ vanishes [11].

Thus, in the RMF approximation, the equations of motion for the mesons simplifiy to

$$m_\sigma^2\langle\sigma\rangle = g_s\langle\bar{\psi}\psi\rangle \tag{2.17}$$

$$m_v^2\langle V_0\rangle = g_v\langle\psi^\dagger\psi\rangle. \tag{2.18}$$

In the RMF, the Lagragian density (2.9) reduces to

$$\mathcal{L}_{RMF} = -\frac{1}{2}m_\sigma^2\sigma^2 + \frac{1}{2}m_v^2 V_\mu V^\mu. \tag{2.19}$$

2.2.4 Scalar and vector density

To calculate the energy spectrum of (2.13) we follow the method used in [5]. Setting the wave equation of the nucleons equal to the free single-particle solution for a Dirac particle,

$$\psi(x) = \psi(\mathbf{k})e^{-ikx} \tag{2.20}$$

where $kx = k_\mu x^\mu = E_0 t - \mathbf{k}.\mathbf{x}$ and using the wave function on (2.13) we obtain

$$\left[\gamma_\mu\left(k^\mu - g_v V^\mu\right) - \left(M - g_s\sigma\right)\right]\psi(\mathbf{k}) = 0. \tag{2.21}$$

Defining

$$K^\mu = k^\mu - g_v V^\mu \tag{2.22}$$

$$M^*(\sigma) = M - g_s\sigma, \tag{2.23}$$

where M^* is the effective mass, (2.21) takes the form

$$\left[\gamma_\mu K^\mu - M^*\right]\psi(\mathbf{k}) = 0. \tag{2.24}$$

By rationalizing de Dirac operator we find the eigenvalues. Multiplying (2.24) by $(\gamma_\mu K^\mu + M^*)$ on the left we get

$$\left(\gamma_\mu K^\mu + M^*\right)\left(\gamma_\mu K^\mu - M^*\right) = \gamma_\mu K^\mu\gamma_\nu K^\nu + M^*\gamma_\nu K^\nu - M^*\gamma_\mu K^\mu - \left(M^*\right)^2 \tag{2.25}$$

$$= \gamma_\mu K^\mu\gamma_\nu K^\nu - \left(M^*\right)^2 = K^\mu K^\nu\frac{\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu}{2} - \left(M^*\right)^2. \tag{2.26}$$

Taking into account the following property of the gamma matrices,

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu},$$

we obtain

$$\left(\gamma_\mu K^\mu + M^*\right)\left(\gamma_\mu K^\mu - M^*\right) = K^\mu K_\mu - \left(M^*\right)^2. \tag{2.27}$$

The final result is

$$\left[K_\mu K^\mu - M^*\right]\psi(\mathbf{k}) = 0. \tag{2.28}$$

The coefficient of $\psi(\mathbf{k})$ is no longer an operator, therefore

$$\begin{aligned} &K_\mu K^\mu - (M^*)^2 = 0 \\ (=)&K_0 K^0 + K_i K^i - (M^*)^2 = 0 \\ (=)&\left(k_0 - g_v V_0\right)\left(k^0 - g_v V^0\right) + \left(k_i - g_v V_i\right)\left(k^i - g_v V^i\right) - (M^*)^2 = 0, \end{aligned}$$

by the RMF approximation $V^i = 0$, then

$$\left(E - g_v V^0\right)^2 = k^2 + (M^*)^2 .$$

Finally, we find that

$$E^{\pm}(k) = g_v V^0 \pm \sqrt{k^2 + (M^*)^2} \tag{2.29}$$

are the eigenvalues for the particle and anti-particle, respectively.

We see that the scalar meson reduces the energy eigenvalue, decreasing the effective mass. The time component of the vector meson shifts the energy eigenvalue of a given k to higher values. The shift of the energy eigenvalue due to the vector meson is proportional to the baryon vector density as result of its equation of motion.

To obtain the energy spectrum we need the mesons' fields that are given by (2.17) and (2.18). For that we need the baryon currents whose eigenvalues are a function of the fields themselves.

To proceed we need to calculate the baryon currents $\langle\bar{\psi}\psi\rangle$ and $\langle\psi^\dagger\psi\rangle$. For that, we can follow [18] or the method used by [5]. We chose to use the method in [5]. It states that the expectation value of an operator Γ in the ground state can be given in terms of the expectation value of the single-particle state $\left(\bar{\psi}\Gamma\psi\right)_{\mathbf{k},\kappa}$, where $\mathbf{k}$ and κ denote, respectively, the momentum and spin-isospin state of the single-particle.

The expectation value in the many-nucleon system is given by

$$\langle\bar{\psi}\Gamma\psi\rangle = \sum_\kappa \int \frac{d\mathbf{k}}{(2\pi)^3} \left(\bar{\psi}\Gamma\psi\right)_{\mathbf{k},\kappa} \Theta\left(\mu - E(k)\right), \tag{2.30}$$

where μ is the chemical potencial and $\Theta\left(\mu - E(\mathbf{k})\right)$ is a step function defined by

$$\Theta\left(\mu - E(k)\right) = \begin{cases} 1 & \text{if } |\mathbf{k}| \leq k_f \\ 0 & \text{if } |\mathbf{k}| > k_f. \end{cases}$$

All the possibilities needed for Γ will in general appear in the Dirac Hamiltonian. The Dirac Hamiltonian H_D can be constructed from the nucleon equation of motion isolating the $k_0 = E$ as follows

$$\begin{aligned} &\left[\gamma_\mu\left(k^\mu - g_v\omega^\mu\right) - \left(M - g_s\sigma\right)\right]\psi(\mathbf{k}) = 0 \\ (=)\ &\left[\gamma_0\left(k^0 - g_v\omega^0\right) - \gamma_i k^i - M^*\right]\psi(\mathbf{k}) = 0 \\ (=)\ &\left[\gamma_0 g_v\omega^0 + \gamma_i k^i + M^*\right]\psi(\mathbf{k}) = \gamma_0 k^0\psi(\mathbf{k}) \\ (=)\ &\gamma_0\left[\gamma_0 g_v\omega^0 + \gamma_i k^i + M^*\right]\psi(\mathbf{k}) = E(k)\psi(\mathbf{k}) \\ (=)\ &H_D\psi(\mathbf{k}) = E(k)\psi(\mathbf{k}), \end{aligned}$$

where

$$H_D = \gamma_0 \left[\gamma_0 g_v \omega^0 + \gamma_i k^i + M^*\right]. \tag{2.31}$$

Taking the single-particle expectation value of H_D, we have

$$\left(\psi^\dagger H_D \psi\right)_{\mathbf{k},\kappa} = \left(\psi^\dagger E(k) \psi\right)_{\mathbf{k},\kappa} = E(k) \left(\psi^\dagger \psi\right)_{\mathbf{k},\kappa} \tag{2.32}$$

where $\left(\psi^\dagger \psi\right)_{\mathbf{k},\kappa}$ is the spinors normalization.
Taking the derivative on the left side of (2.32) with respect to any variable χ, yields

$$\begin{aligned}
\frac{\partial}{\partial \chi} \left(\psi^\dagger H_D \psi\right)_{\mathbf{k},\kappa} &= \left(\psi^\dagger \frac{\partial H_D}{\partial \chi} \psi\right)_{\mathbf{k},\kappa} + \left(\frac{\partial \psi^\dagger}{\partial \chi} H_D \psi\right)_{\mathbf{k},\kappa} + \left(\psi^\dagger H_D \frac{\partial \psi}{\partial \chi}\right)_{\mathbf{k},\kappa} \\
&= \left(\psi^\dagger \frac{\partial H_D}{\partial \chi} \psi\right)_{\mathbf{k},\kappa} + E(k) \left(\frac{\partial \psi^\dagger}{\partial \chi} \psi\right)_{\mathbf{k},\kappa} + E(k) \left(\psi^\dagger \frac{\partial \psi}{\partial \chi}\right)_{\mathbf{k},\kappa} \\
&= \left(\psi^\dagger \frac{\partial H_D}{\partial \chi} \psi\right)_{\mathbf{k},\kappa} + E(k) \frac{\partial}{\partial \chi} \left(\psi^\dagger \psi\right)_{\mathbf{k},\kappa} \\
&= \left(\psi^\dagger \frac{\partial H_D}{\partial \chi} \psi\right)_{\mathbf{k},\kappa}
\end{aligned}$$

where we use $H_D^\dagger = H_D$ and the last step is due to $\psi(\mathbf{k})$ be an eigenfunction. So we obtain the following relation

$$\left(\psi^\dagger \frac{\partial H_D}{\partial \chi} \psi\right)_{\mathbf{k},\kappa} = \frac{\partial}{\partial \chi} E(k). \tag{2.33}$$

We can take the normalization condition of the wave function by the above condition. Taking the derivative of (2.31) with respect to V_0 we obtain

$$\frac{\partial H_D}{\partial V_0} = g_v \left(\gamma^0\right)^2 = g_v$$

where, $\left(\gamma^0\right)^2 = I$, by the properties of the gamma matrices. Using the relation (2.33) and the eigenvalues, we obtain

$$g_v \left(\psi^\dagger \psi\right)_{\mathbf{k},\kappa} = \frac{\partial}{\partial V_0} E(k) = \frac{\partial}{\partial V_0} \left(g_v V^0 + \sqrt{k^2 + (M^*)^2}\right) = g_v,$$

therefore, we obtain the normalization condition $\left(\psi^\dagger \psi\right)_{\mathbf{k},\kappa} = 1$.

The calculation of the vector baryon density $\left\langle \psi^\dagger \psi \right\rangle$ is straightforward using (2.30)

$$\begin{aligned}
< \psi^\dagger \psi > &= \sum_\kappa \int \frac{d\mathbf{k}}{(2\pi)^3} \left(\psi^\dagger \psi\right)_{\mathbf{k},\kappa} \Theta\left(\mu - E(k)\right) \\
&= \sum_\kappa \int \frac{d\mathbf{k}}{(2\pi)^3} \Theta\left(\mu - E(k)\right) \\
&= \gamma \int_0^{k_f} \frac{d\mathbf{k}}{(2\pi)^3} = \gamma \int_0^{k_f} \frac{4\pi k^2 dk}{(2\pi)^3} = \frac{\gamma}{2} \frac{k_f^3}{3\pi^2} \\
&= \frac{2k_f^3}{3\pi^2}
\end{aligned}$$

where $\gamma = 2_{spin} \times 2_{isospin} = 4$ is the degeneracy of the nucleons.

To calculate the scalar baryon density $\langle \bar{\psi}\psi \rangle$ we use the same procedure, having

$$\langle \bar{\psi}\psi \rangle = \langle \psi^\dagger \gamma_0 \psi \rangle,$$

and by the equation (2.32) we get $\partial H_D / \partial M^* = \gamma_0$, so

$$\begin{aligned} \left(\psi^\dagger \frac{\partial H_D}{\partial M^*} \psi \right)_{\mathbf{k},\kappa} &= \frac{\partial}{\partial M^*} E(k) \\ (=) \quad \left(\psi^\dagger \gamma_0 \psi \right)_{\mathbf{k},\kappa} &= \frac{M^*}{\sqrt{k^2 + (M^*)^2}}. \end{aligned}$$

Therefore we have

$$\begin{aligned} \langle \bar{\psi}\psi \rangle &= \sum_\kappa \int \frac{d\mathbf{k}}{(2\pi)^3} \left(\psi^\dagger \gamma_0 \psi \right)_{\mathbf{k},\kappa} \Theta\left(\mu - E(k)\right) \\ &= \sum_\kappa \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{M^*}{\sqrt{k^2 + (M^*)^2}} \Theta\left(\mu - E(k)\right) \\ &= \gamma \int_0^{k_f} \frac{4\pi k^2 dk}{(2\pi)^3} \frac{M^*}{\sqrt{k^2 + (M^*)^2}} \\ &= \frac{\gamma}{2\pi^2} \int_0^{k_f} k^2 dk \frac{M^*}{\sqrt{k^2 + (M^*)^2}} \\ &= \frac{2}{\pi^2} \int_0^{k_f} k^2 dk \frac{M - g_s\sigma}{\sqrt{k^2 + (M - g_s\sigma)^2}}. \end{aligned}$$

Summarizing, for the vector and scalar baryon densities we obtain, respectively,

$$\rho \equiv \langle \psi^\dagger \psi \rangle = \frac{2k_f^3}{3\pi^2} \tag{2.34}$$

$$\rho_s \equiv \langle \bar{\psi}\psi \rangle = \frac{2}{\pi^2} \int_0^{k_f} k^2 dk \frac{M - g_s\sigma}{\sqrt{k^2 + (M - g_s\sigma)^2}}. \tag{2.35}$$

The explicit forms of the meson fields become

$$m_s^2 \sigma = g_\sigma \frac{2}{\pi^2} \int_0^{k_f} k^2 dk \frac{M - g_s\sigma}{\sqrt{k^2 + (M - g_s\sigma)^2}} \tag{2.36}$$

$$m_v^2 V_0 = g_v \rho. \tag{2.37}$$

The first equation is a non-linear equation that has to be solved self-consistently.

2.2.5 Pressure and energy density

The tensor energy-momentum is given by

$$T^{\mu\nu} = \left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_i\right)}\right)\partial^\nu \phi_i - \eta^{\mu\nu}\mathcal{L},$$

which in the rest frame of isotropic matter is diagonal[3]

$$T^{\mu\nu} = \begin{pmatrix} \mathcal{E} & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}.$$

The energy density and pressure are given by

$$\mathcal{E} = T^{00} = \left(\frac{\partial \mathcal{L}}{\partial\left(\partial_0 \psi\right)}\right)\partial^0 \psi - \eta^{00}\mathcal{L} = \left(\frac{\partial \mathcal{L}}{\partial\left(\partial_0 \psi\right)}\right)\partial^0 \psi - \mathcal{L} \tag{2.38}$$

$$p = \frac{1}{3}T^{ii} = \frac{1}{3}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_i \psi\right)}\right)\partial^i \psi - \frac{\eta^{ii}}{3}\mathcal{L} = \frac{1}{3}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_i \psi\right)}\right)\partial^i \psi + \mathcal{L} \tag{2.39}$$

The expectation value of the energy density on the ground state, taking into account (2.9) and (2.19), is

$$\mathcal{E} = \left\langle \left(\frac{\partial \mathcal{L}}{\partial\left(\partial_0 \psi\right)}\right)\partial^0 \psi \right\rangle - \left\langle \mathcal{L}_{RMF} \right\rangle \tag{2.40}$$

$$= \left\langle \bar{\psi}\gamma_0 i \partial^0 \psi \right\rangle + \frac{1}{2}m_s^2\sigma^2 - \frac{1}{2}m_v^2 V_0^2 \tag{2.41}$$

$$= \left\langle \bar{\psi}\gamma_0 k_0 \psi \right\rangle + \frac{1}{2}m_s^2\sigma^2 - \frac{1}{2}m_v^2 V_0^2 \tag{2.42}$$

where in the last step we use the wave function of nucleons.
To calculate $\langle \bar{\psi}\gamma_0 k_0 \psi \rangle$, using the same procedure as in the section before

$$\left(\bar{\psi}\gamma_0 k_0 \psi\right)_{\mathbf{k},\kappa} = \left(\psi^\dagger k_0 \psi\right)_{\mathbf{k},\kappa} = E(k)\left(\psi^\dagger \psi\right)_{\mathbf{k},\kappa} = E(k).$$

Using (2.30) we obtain

$$\begin{aligned} \left\langle \bar{\psi}\gamma_0 k_0 \psi \right\rangle &= \sum_\kappa \int \frac{d\mathbf{k}}{(2\pi)^3}\left(\psi^\dagger k_0 \psi\right)_{\mathbf{k},\kappa}\Theta\left(\mu - E(k)\right) \\ &= \gamma \int_0^{k_f} \frac{d\mathbf{k}}{(2\pi)^3} E(k) \\ &= \frac{2}{\pi^2}\int_0^{k_f} k^2 dk \left(\sqrt{k^2 + (M - g_s\sigma)^2} + g_v V^0\right) \\ &= g_v V^0 \rho + \frac{2}{\pi^2}\int_0^{k_f} k^2 dk \sqrt{k^2 + (M - g_s\sigma)^2}. \end{aligned}$$

[3] see [5] for details.

So the energy density is

$$\mathcal{E} = g_v V^0 \rho + \frac{2}{\pi^2}\int_0^{k_f} k^2 dk \sqrt{k^2 + (M - g_s\sigma)^2} + \frac{1}{2}m_s^2\sigma^2 - \frac{1}{2}m_v^2 V_0^2 \tag{2.43}$$

$$\mathcal{E} = \frac{1}{2}m_s^2\sigma^2 + \frac{1}{2}m_v^2 V_0^2 + \frac{2}{\pi^2}\int_0^{k_f} k^2 dk \sqrt{k^2 + (M - g_s\sigma)^2} \tag{2.44}$$

where, in the last step the equation of motion of the vector meson $m_v^2 V_0 = g_v \rho$, is used. The expectation value for the pressure is

$$\begin{aligned} p &= \frac{1}{3}\left\langle \left(\frac{\partial \mathcal{L}}{\partial\left(\partial_i \psi\right)}\right)\partial^i \psi \right\rangle + \langle \mathcal{L}_{RMF}\rangle \\ p &= \frac{1}{3}\langle \bar{\psi} i \gamma_i \partial^i \psi\rangle - \frac{1}{2}m_s^2\sigma^2 + \frac{1}{2}m_v^2 V_0^2 \\ p &= \frac{1}{3}\langle \bar{\psi}\gamma_i k^i \psi\rangle - \frac{1}{2}m_s^2\sigma^2 + \frac{1}{2}m_v^2 V_0^2. \end{aligned}$$

Using the same procedure

$$\begin{aligned} \left(\bar{\psi}\gamma_i k^i \psi\right)_{\mathbf{k},\kappa} &= \left(\bar{\psi}\,\boldsymbol{\gamma}.\mathbf{k}\,\psi\right)_{\mathbf{k},\kappa} = \left(\bar{\psi}\,\boldsymbol{\gamma}\,\psi\right)_{\mathbf{k},\kappa}.\mathbf{k} \\ &= \left(\bar{\psi}\frac{\partial H_D}{\partial \mathbf{k}}\psi\right)_{\mathbf{k},\kappa}.\mathbf{k} = \left(\frac{\partial}{\partial \mathbf{k}}E(k)\right).\mathbf{k} \\ &= \frac{\mathbf{k}.\mathbf{k}}{\sqrt{\mathbf{k}^2 + (M - g_s\sigma)^2}}. \end{aligned}$$

Through (2.30) we obtain

$$\begin{aligned} \langle \bar{\psi}\gamma_i k^i \psi\rangle &= \sum_\kappa \int \frac{d\mathbf{k}}{(2\pi)^3}\left(\frac{\partial E(k)}{\partial \mathbf{k}}\right)\Theta\left(\mu - E(k)\right) \\ &= \gamma \int_0^{k_f} \frac{d\mathbf{k}}{(2\pi)^3}\frac{k^2}{\sqrt{k^2 + (M - g_s\sigma)^2}} \\ &= \frac{2}{\pi^2}\int_0^{k_f} \frac{k^4 dk}{\sqrt{k^2 + (M - g_s\sigma)^2}}. \end{aligned}$$

The final expression for the pressure is

$$p = \frac{1}{3}\frac{2}{\pi^2}\int_0^{k_f} \frac{k^4 dk}{\sqrt{k^2 + (M - g_s\sigma)^2}} - \frac{1}{2}m_s^2\sigma^2 + \frac{1}{2}m_v^2 V_0^2. \tag{2.45}$$

The two coupling constants of the model can be fitted to reproduce the saturation density of nuclear matter and the energy per nucleon at saturation.
The other bulk properties are not reproduced so well. The compression modulus computed from the EOS at saturation gives $K \approx 550\,\mathrm{MeV}$, which is a very high value. The Dirac effective mass at saturation is small $M^*/M \approx 0.5$ comparing with the empirical range of ≈ 0.7 to 0.8.

2.3 Introduction of scalar self-interactions

Boguta and Bodmer [2] introduced an extension of the $\sigma - \omega$ model aimed to bring the compression modulus and nucleon effective mass under control.

They introduced scalar self-interactions by the following Lagrangian density:

$$\mathcal{L}_{BB} = \frac{1}{2} bM \left(g_s\sigma\right)^3 + \frac{1}{4} c \left(g_s\sigma\right)^4 . \tag{2.46}$$

So the lagrangian of the theory becomes

$$\mathcal{L} = \mathcal{L}_{(\sigma-\omega)} + \frac{1}{2} bM \left(g_s\sigma\right)^3 + \frac{1}{4} c \left(g_s\sigma\right)^4 . \tag{2.47}$$

It is easy to see from the Lagrangian density, that only the equation of motion of the scalar meson changes

$$g_s\sigma = \left(\frac{g_s}{m_s}\right)^2 \left[\rho_s - bM \left(g_s\sigma\right)^2 - c \left(g_s\sigma\right)^3\right] \tag{2.48}$$

where ρ_s is the scalar density given by (2.34).

The energy density and the pressure with the addition of the self-interactions become

$$\begin{aligned}
\mathcal{E} &= \frac{1}{2} bM \left(g_s\sigma\right)^3 + \frac{1}{4} c \left(g_s\sigma\right)^4 + \frac{1}{2} m_s^2 \sigma^2 + \frac{1}{2} m_v^2 V_0^2 + \frac{2}{\pi^2} \int_0^{k_f} k^2 dk \sqrt{k^2 + (M - g_s\sigma)^2} \\
p &= -\frac{1}{2} bM \left(g_s\sigma\right)^3 - \frac{1}{4} c \left(g_s\sigma\right)^4 - \frac{1}{2} m_s^2 \sigma^2 + \frac{1}{2} m_v^2 V_0^2 + \frac{1}{3}\frac{2}{\pi^2} \int_0^{k_f} \frac{k^4 dk}{\sqrt{k^2 + (M - g_s\sigma)^2}} .
\end{aligned}$$

In the last section of this chapter we will see how to fit the coupling constants in order to reproduce the experimental values of the compression modulus and the effective nucleon mass.

2.4 Introduction of ρ meson

To study the properties of the neutron rich matter of a neutron star, we need to introduce a new field to reproduce the isospin restoring interaction that are present in (2.4).

This is done by introducing the isospin triplet of the rho meson, denoted by $\mathbf{b}_\nu$, that has as source the 3-component of the isospin density $I_3 = \frac{1}{2}(\rho_p - \rho_n)$.

The interaction Lagrangian density is given by

$$\mathcal{L}_{int} = -g_\rho \mathbf{b}_\nu . \mathbf{I}^\nu .$$

On the ground state, only the isospin 3-component of the $\mathbf{b}_\mu$ has finite mean value. In the mean field approximation, the spatial component of $\mathbf{b}_\mu$ vanishes like in the

V^μ field. Thus, the only non-vanishing component of the rho meson is $\rho_3^{(0)}$. For details see [5].

The equation of motion for the rho meson is

$$g_\rho \rho_3^{(0)} = \frac{1}{2}\left(\frac{g_\rho}{m_\rho}\right)^2 \langle \bar{\psi}\gamma^0\tau^3\psi\rangle = \left(\frac{g_\rho}{m_\rho}\right)^2 \frac{1}{2}(\rho_p - \rho_n), \tag{2.49}$$

and the Dirac equation changes to

$$\left[\gamma_\mu\left(k^\mu - g_v\omega^\mu - \frac{1}{2}g_\rho\tau_3 b_3^\mu\right) - (M - g_s\sigma)\right]\psi(\mathbf{k}) = 0. \tag{2.50}$$

Its eigenvalues are

$$E(k) = g_v V^0 + g_\rho \rho_3^{(0)} I_3 + \sqrt{k^2 + (M - g_s\sigma)^2}, \tag{2.51}$$

where I_3 is the isospin of the nucleon, $I_3 = 1/2$ for protons and $I_3 = -1/2$ for neutrons.

The pressure and energy density becomes

$$\begin{aligned}
\mathcal{E} &= \frac{1}{2}bM\left(g_s\sigma\right)^3 + \frac{1}{4}c\left(g_s\sigma\right)^4 + \frac{1}{2}m_s^2\sigma^2 + \frac{1}{2}m_v^2 V_0^2 + \frac{1}{2}m_\rho^2\left(\rho_3^{(0)}\right)^2 \\
&+ \frac{1}{\pi^2}\left(\int_0^{k_p^f} k_p^2 dk_p\sqrt{k_p^2 + (M - g_s\sigma)^2} + \int_0^{k_n^f} k_n^2 dk_n\sqrt{k_n^2 + (M - g_s\sigma)^2}\right) \\
p &= -\frac{1}{2}bM\left(g_s\sigma\right)^3 - \frac{1}{4}c\left(g_s\sigma\right)^4 - \frac{1}{2}m_s^2\sigma^2 + \frac{1}{2}m_v^2 V_0^2 + \frac{1}{2}m_\rho^2\left(\rho_3^{(0)}\right)^2 \\
&+ \frac{1}{3\pi^2}\left(\int_0^{k_p^f} \frac{k_p^4 dk_p}{\sqrt{k_p^2 + (M - g_s\sigma)^2}} + \int_0^{k_n^f} \frac{k_n^4 dk_n}{\sqrt{k_n^2 + (M - g_s\sigma)^2}}\right).
\end{aligned}$$

The energy density can be separated in two. One part that depends on the asymmetry between protons and neutrons that takes the form

$$\mathcal{E}_s = \frac{1}{2}m_\rho^2\left(\rho_3^{(0)}\right)^2 + \frac{1}{\pi^2}\left(\int_0^{k_p^f} k_p^2 dk_p\sqrt{k_p^2 + (M - g_s\sigma)^2} + \int_0^{k_n^f} k_n^2 dk_n\sqrt{k_n^2 + (M - g_s\sigma)^2}\right).$$

Using the equation of motion of the $\rho_3^{(0)}$,

$$\begin{aligned}
\frac{\mathcal{E}_s}{\rho} =& \frac{1}{8}\left(\frac{g_\rho}{m_\rho}\right)^2 \rho\left(\frac{\rho_n - \rho_p}{\rho}\right)^2 + \frac{1}{\rho\pi^2}\left(\int_0^{k_p^f} k_p^2 dk_p\sqrt{k_p^2 + (M - g_s\sigma)^2}\right. \\
&\left. + \int_0^{k_n^f} k_n^2 dk_n\sqrt{k_n^2 + (M - g_s\sigma)^2}\right).
\end{aligned} \tag{2.52}$$

This way we reproduced the asymmetry term that favors symmetric systems.

The coupling constant of the rho meson is chosen to reproduce the empirical value of the symmetry energy. In the next section we show how that is made.

2.5 Coupling constants

One of the merits of the relativistic nuclear field theory is that the coupling constants of the theory can be algebrically related to the properties of matter at saturation density.

The expressions that relate the values of b, c and g_s/m_s are too extense to reproduce here and can be seen in [5]. The key point here is to understand that it is possible to algebrically relate the coupling constants in terms of the empirical quantities ρ, B/A, K, M^* and a_{sym} of symmetric nuclear matter at saturation.

For g_v/m_v and g_ρ/m_ρ the relations are

$$\left(\frac{g_v}{m_v}\right)^2 = \frac{M + B/A - \sqrt{k^2 + (M^*)^2}}{\rho} \tag{2.53}$$

$$\left(\frac{g_\rho}{m_\rho}\right)^2 = \frac{8}{\rho}\left(a_{sym} - \frac{k^2}{6\sqrt{k^2 + (M^*)^2}}\right). \tag{2.54}$$

Chapter 3

Model with light clusters

In this work we apply the nonlinear RMF model to study the dissolution of light clusters at zero temperature.

3.1 Relativistic mean field theory

The nonlinear RMF model [15, 13] for a system of nucleons, electrons and the following light elements: deuteron ($d = {}^2H$), triton ($t = {}^3H$), helion ($h = {}^3He$) and α particles (${}^4He$) is given by the Lagrangian density

$$
\begin{aligned}
\mathcal{L} = & \sum_{i=p,n,t,h} \bar{\psi}_i \left[\gamma_\mu i D_i^\mu - M_i^*\right] \psi_i + \frac{1}{2}\left(\partial_\mu \sigma \partial^\mu \sigma - m_s^2 \sigma^2 - \frac{1}{3} k \sigma^3 - \frac{1}{12}\lambda \sigma^4\right) \\
& + \frac{1}{2}\left(-\frac{1}{2}\Omega_{\mu\nu}\Omega^{\mu\nu} + m_v^2 V_\mu V^\mu + \frac{1}{12}\xi g_v^4 \left(V_\mu V^\mu\right)^2\right) + \frac{1}{2}\left(-\frac{1}{2}\mathbf{B}_{\mu\nu}.\mathbf{B}^{\mu\nu} + m_\rho^2 \mathbf{b}_\mu.\mathbf{b}^\mu\right) \\
& + \Lambda_v g_v^2 g_\rho^2 V_\mu V^\mu \mathbf{b}_\mu.\mathbf{b}^\mu + \frac{1}{2}\left(i\mathcal{D}_\alpha^\mu \phi_\alpha\right)^* \left(i\mathcal{D}_{\mu\alpha}\phi_\alpha\right) - \frac{1}{2}(M_\alpha^*)^2 \phi_\alpha^* \phi_\alpha + \bar{\psi}_e \left[\gamma_\mu i\partial^\mu - m_e\right]\psi_e \\
& + \frac{1}{4}\left(i\mathcal{D}_d^\mu \phi_d^\nu - i\mathcal{D}_d^\nu \phi_d^\mu\right)^* \left(i\mathcal{D}_{\mu d}\phi_{d\nu} - i\mathcal{D}_{\nu d}\phi_{d\mu}\right) - \frac{1}{2}(M_d^*)^2 \left(\phi_d^\mu\right)^* \phi_{d\mu}
\end{aligned}
\tag{3.1}
$$

where

$$
\begin{aligned}
iD_i^\mu &= i\partial^\mu - g_v^i V^\mu - \frac{g_\rho^i}{2}\boldsymbol{\tau}.\boldsymbol{b}^\mu \\
M_i^* &= M_i - g_s^i \sigma \\
i\mathcal{D}_i^\mu &= i\partial^\mu - g_v^i V^\mu
\end{aligned}
$$

and

$$
\begin{aligned}
\Omega_{\mu\nu} &= \partial_\mu V_\nu - \partial_\nu V_\mu \\
\mathbf{B}_{\mu\nu} &= \partial_\mu \mathbf{b}_\nu - \partial_\nu \mathbf{b}_\mu.
\end{aligned}
$$

The spin 0 field ϕ_α represents the α particles, the spin 1 field ϕ_d^μ represents the deuteron and the spin $1/2$ fields ψ_h and ψ_t represent the helion and triton, respectively. The electrons are represented by the spin $1/2$ field ψ_e. They are included when stellar matter is described, otherwise their contribution is neglected.

The clusters are treated as point-like particles and their internal structure is not taken into account.

The omega-meson self-interaction softens the equation of state at high density, tuning the parameter ξ. Instead, the nonlinear coupling constant Λ_v was included to modify the density dependence of the symmetry energy [3].

3.2 Equations of motion

The equations of motion of the fields are obtained applying the Euler-Lagrange equation to each field.
The calculation details are shown is Appendix A. The equations of motion for the meson fields in the RMF approximation are

$$m_s^2\sigma + \frac{\kappa}{2}\sigma^2 + \frac{\lambda}{6}\sigma^3 = \sum_{i=p,n,t,h} g_s^i\rho_s^i + \sum_{i=d,\alpha} g_s^i\rho_i \tag{3.2}$$

$$m_v^2V^0 + \frac{1}{6}\xi g_v^4\left(V^0\right)^3 + 2\Lambda_v g_v^2 g_\rho^2 V^0\left(b_3^{(0)}\right)^2 = \sum_{i=p,n,t,h,d,\alpha} g_v^i\rho_i \tag{3.3}$$

$$m_\rho^2 b_3^{(0)}3 + 2\Lambda_v g_v^2 g_\rho^2\left(V^0\right)^2 b_3^{(0)} = \sum_{i=p,n,t,h} g_\rho^i I_3^i\rho_i \tag{3.4}$$

where, ρ_s^i is the scalar density given by

$$\rho_s^i \equiv \langle\bar{\psi}_i\psi_i\rangle = \frac{1}{\pi^2}\int_0^{k_f^i} k_i^2 dk_i \frac{M_i - g_s^i\sigma}{\sqrt{k_i^2 + \left(M_i - g_s^i\sigma\right)^2}} \tag{3.5}$$

and ρ_i is the vector density for $i = n, p, t, h$. At zero temperature all the α and deuteron populations will condense in a state of zero momentum, therefore ρ_α and ρ_d correspond to the condensate density of each species. However since the α particles have a larger binding energy at $T = 0$ all nucleon of symmetric nuclear matter will form an α condensate. If matter is asymmetric only the more abundant species (protons or neutrons) will not completely condensate into α particles.

I_3^i is the isospin of the particle i, whose values are: $I_3(p) = 1/2$, $I_3(n) = -1/2$, $I_3(t) = -1/2$ and $I_3(h) = 1/2$.

3.3 Parameter sets

There are many relativistic effective interactions. In this work we use NL3 [8] and FSU [14], two well known and widely used. We use a new relativistic effective interaction named IUFSU introduced in [3]. NL3 has been fitted to the groundstate properties of both stable and unstable nuclei. FSU was fitted to static properties of nuclei and collective giant resonances.
IUFSU was built in a way close to FSU but it was refined to describe neutron stars

with mass $\sim 2M_{\odot}$.

The values of the parameters to each relativistic effective interaction are in Table 3.1 and the corresponding properties of infinite nuclear matter at saturation density are shown in Table 3.2.

Table 3.1: Parameter sets for the three models used in this work.

	NL3 [8]	FSU [14]	IUFSU [3]
m_s (MeV)	508.194	491.500	491.500
m_v (MeV)	782.501	782.500	782.500
m_ρ (MeV)	763.000	763.000	763.000
g_s^2	104.3871	112.1996	99.4266
g_v^2	165.5854	205.5469	169.8349
g_ρ^2	79.6000	138.4701	184.6877
κ (MeV)	3.8599	1.4203	3.3808
λ	−0.015905	+0.023762	+0.000296
ξ	0.00	0.06	0.03
Λ_v	0.000	0.030	0.046

Table 3.2: Bulk parameters characterizing the behavior of infinite nuclear matter at saturation density.

	NL3 [8]	FSU [14]	IUFSU [3]
ρ_0 $\left(\mathrm{fm}^{-3}\right)$	0.148	0.148	0.155
E/A (MeV)	−16.24	−16.30	−16.40
K (MeV)	271.5	230.0	231.2
a_{sym} (MeV)	37.29	32.59	31.30
L (MeV)	118.2	60.5	47.2
M (MeV)	939	939	939

Where L is the symmetry energy derivative. It is another property that can characterize nuclear matter on saturation density, defined as

$$L = 3\rho_0 \left(\frac{\partial a_{sym}}{\partial \rho} \right)_{\rho_0} .$$

This quantity is related to the stiffness of the symmetry energy at high densities. Recently, a combination of several data restricted its range to $62\ \mathrm{MeV} < L < 107\ \mathrm{MeV}$ [7].

3.4 Clusters binding energy

The total binding energy of a cluster i is given by $B_i = B_i^0 + \Delta B_i$, where B_i^0 is the experimental binding energy in vacuum and ΔB_i is the medium-dependent binding energy shift.

In [15], for the RMF, the empirical quadratic form is used

$$\Delta B_i(\rho, T) = -\tilde{\rho}_i \left[\delta B_i(T) + \frac{\tilde{\rho}_i}{2B_i^0} \delta B_i^2(T) \right] \tag{3.6}$$

with

$$\tilde{\rho}_i = \frac{2}{A_i} \left[Z_i \rho_p^{tot} + N_i \rho_n^{tot} \right]. \tag{3.7}$$

where

$$\rho = \rho_p^{tot} + \rho_n^{tot}.$$

is the total nucleon density. This expansion was obtained from a quantum statistical approach [10].

In the limit of zero temperature we obtain

$$\Delta B_i(\rho, 0) = -\tilde{\rho}_i \left[\delta B_i(0) + \frac{\tilde{\rho}_i}{2B_i^0} \delta B_i^2(0) \right] \tag{3.8}$$

where

$$\delta B_i(0) = \frac{a_{i,1}}{a_{i,2}^{3/2}}. \tag{3.9}$$

The parameters $a_{i,1}$ and $a_{i,2}$ are listed for symmetrical nuclear matter ($Y_p = 0.5$) in Table I of [15] that is partially reproduced in Table 3.3.

Table 3.3: Parameters for the cluster binding energy shifts.

Cluster i	$a_{i,1}$ $\left(\mathrm{MeV}^{5/2}\,\mathrm{fm}^3\right)$	$a_{i,2}$ (MeV)	B_i^0 [1] (MeV)
t	69516.2	7.49232	8.481798
h	58442.5	6.07718	7.718043
α	164371	10.6701	28.29566
d	38386.4	22.5204	2.224566

The density where a cluster becomes unbound, ρ_d, that we call dissolution density, is given by

$$B_i(\rho_d) = B_i^0 + \Delta B_i = 0. \tag{3.10}$$

In Figure 3.1 we represent the binding energy as a function of the density for all clusters.

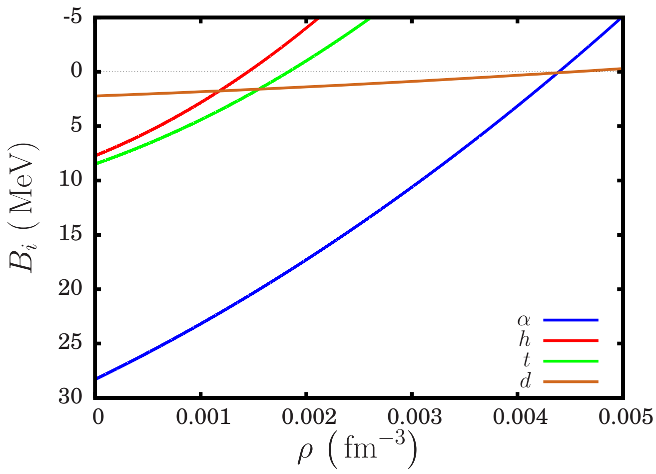

Figure 3.1: Clusters binding energy for symmetric nuclear matter at zero temperature as a function of the density.

The dissolution density obtained for the clusters is shown in the table 3.4.

Table 3.4: Densities at which the clusters become unbound given by equation (3.10)

	t	h	α	d
ρ_d $\left(\mathrm{fm}^{-3}\right)$	0.00183175	0.00144835	0.00439226	0.00453391

In this work, the values in of Table 3.4 are used as a reference for symmetric nuclear matter and the coupling constants of clusters that reproduce this dissolution densities were calculated. The results are shown in the next chapter.

3.5 Density energy and Pressure

The result of applying the RMF approximation to the density lagrangian (3.1), ignoring the terms of the α and deuteron, is

$$\mathcal{L}_{RMF} = -\frac{1}{2}m_s^2\sigma^2 - \frac{1}{6}\kappa\sigma^3 - \frac{1}{24}\lambda\sigma^4 + \frac{1}{2}m_v^2 V_0 V^0 + \frac{1}{24}\xi g_v^4\left(V_0V^0\right)^2 + \frac{1}{2}m_\rho^2\left(b_3^{(0)}\right)^2 + \Lambda_v g_v^2 g_\rho^2\left(V^0\right)^2\left(b_3^{(0)}\right)^2.$$

The density energy is

$$\mathcal{E} = \sum_{i=p,n,t,h}\left\langle\bar{\psi}_i\gamma_0 i\partial^0\psi_i\right\rangle - \left\langle\mathcal{L}_{RMF}\right\rangle + \sum_{i=\alpha,d}\mathcal{E}_i.$$

We know that

$$
\begin{aligned}
\langle\bar{\psi}_i\gamma_0 i\partial^0\psi_i\rangle &= \gamma_i\int_0^{k_f^i}\frac{d\mathbf{k}_i}{(2\pi)^3}E_i(k_i)\\
&= \frac{1}{\pi^2}\int_0^{k_f^i}k_i^2dk_i\left(g_v^iV^0+g_\rho^ib_3^{(0)}I_3^i+\sqrt{k_i^2+(M_i-g_s^i\sigma)^2}\right)\\
&= g_v^iV^0\rho_i+g_\rho^ib_3^{(0)}I_3^i\rho_i+\frac{1}{\pi^2}\int_0^{k_f^i}k_i^2dk_i\left(\sqrt{k_i^2+(M_i-g_s^i\sigma)^2}\right)
\end{aligned}
$$

where $\gamma_i = 2_{\text{spin}}$ is the degeneracy of each particle $i = p, n, t, h$. Therefore

$$
\begin{aligned}
\mathcal{E} = &\sum_{i=p,n,t,h}\left(g_v^iV^0\rho_i+g_\rho^ib_3^{(0)}I_3^i\rho_i+\frac{1}{\pi^2}\int_0^{k_f^i}k_i^2dk_i\left(\sqrt{k_i^2+(M_i-g_s^i\sigma)^2}\right)\right)\\
&+\frac{1}{2}m_s^2\sigma^2+\frac{1}{6}\kappa\sigma^3+\frac{1}{24}\lambda\sigma^4-\frac{1}{2}m_v^2V_0V^0-\frac{1}{24}\xi g_v^4\left(V_0V^0\right)^2-\frac{1}{2}m_\rho^2\left(b_3^{(0)}\right)^2\\
&-\Lambda_vg_v^2g_\rho^2V_0V^0\left(b_3^{(0)}\right)^2+\sum_{i=\alpha,d}\mathcal{E}_i.
\end{aligned}
$$

The contribution to the density energy of the α and d is calculated in the Appendix 1. The total density energy is

$$
\begin{aligned}
\mathcal{E} = &\sum_{i=p,n,t,h}\left(g_v^iV^0\rho_i+g_\rho^ib_3^{(0)}I_3^i\rho_i+\frac{1}{\pi^2}\int_0^{k_f^i}k_i^2dk_i\left(\sqrt{k_i^2+(M_i-g_s^i\sigma)^2}\right)\right)\\
&+\frac{1}{2}m_s^2\sigma^2+\frac{1}{6}\kappa\sigma^3+\frac{1}{24}\lambda\sigma^4-\frac{1}{2}m_v^2V_0V^0-\frac{1}{24}\xi g_v^4\left(V_0V^0\right)^2-\frac{1}{2}m_\rho^2\left(b_3^{(0)}\right)^2\\
&-\Lambda_vg_v^2g_\rho^2V_0V^0\left(b_3^{(0)}\right)^2+\sum_{i=\alpha,d}\left(M_i-g_s^i\sigma+g_v^iV^0\right)\rho_i.
\end{aligned}
\tag{3.11}
$$

The pressure is given by

$$
p = \frac{1}{3}\sum_{i=p,n,t,h}\langle\bar{\psi}_i i\gamma_i\partial^i\psi_i\rangle+\langle\mathcal{L}_{RMF}\rangle.
$$

We know that

$$
\langle\bar{\psi}_i i\gamma_i\partial^i\psi_i\rangle = \frac{1}{\pi^2}\sum_{i=p,n,t,h}\int_0^{k_f^i}\frac{k_i^4dk_i}{\sqrt{k_i^2+(M_i-g_s^i\sigma)^2}}.
$$

The final expression to the pressure is

$$
\begin{aligned}
p = &\frac{1}{3}\frac{1}{\pi^2}\sum_{i=p,n,t,h}\int_0^{k_f^i}\frac{k_i^4dk_i}{\sqrt{k_i^2+(M_i-g_s^i\sigma)^2}}-\frac{1}{2}m_s^2\sigma^2-\frac{1}{6}\kappa\sigma^3-\frac{1}{24}\lambda\sigma^4\\
&+\frac{1}{2}m_v^2V_0V^0+\frac{1}{24}\xi g_v^4\left(V_0V^0\right)^2+\frac{1}{2}m_\rho^2\left(b_3^{(0)}\right)+\Lambda_vg_v^2g_\rho^2V_0V^0\left(b_3^{(0)}\right)^2.
\end{aligned}
$$

There is no explicit contribution of the α and deuteron to the pressure. This was expected due to the fact that, at zero temperature, the α and deuteron particles will condensate on zero momentum state and do not contribute for the pressure. There is of course an implicit contribution in the meson fields, a presence of α or deuteron particles changes the meson fields and, consequently, the total pressure.

3.6 Free energy density

The free energy density is given by

$$\mathcal{F} = \mathcal{E} - T\mathcal{S}.$$

where $\mathcal{S}$ is the entropy density. At zero temperature the free energy is equal to the energy density.

We define the global proton fraction Y_{PG} by

$$Y_{PG} = \frac{\rho_p + 2\rho_\alpha + \rho_d + 2\rho_h + \rho_t}{\rho} = \frac{\rho_p}{\rho} + 2\frac{\rho_\alpha}{\rho} + \frac{\rho_d}{\rho} + 2\frac{\rho_h}{\rho} + \frac{\rho_t}{\rho}.$$

Defining the mass fraction $Y_i = A_i(\rho_i/\rho)$ of the various species we obtain

$$Y_{PG} = Y_p + \frac{1}{2}Y_\alpha + \frac{1}{2}Y_d + \frac{2}{3}Y_h + \frac{1}{3}Y_t.$$

Using the same procedure to obtain the global neutron fraction Y_{NG}

$$Y_{NG} = Y_n + \frac{1}{2}Y_\alpha + \frac{1}{2}Y_d + \frac{1}{3}Y_h + \frac{2}{3}Y_t.$$

With these definitions we obtain the relation $Y_{PG} + Y_{NG} = 1$.

The method used to find the density of dissolution of a cluster i is the following: we fix the Y_{PG} and compare the free energy of the EOS composed by neutrons and protons with the EOS with neutrons, protons and the specie i. Therefore, we can study the density regions where it is energetically favorable to form a cluster.

We define dissolution density ρ_d as the density where the free energy of both EOS is the same and study how this dissolution density depends on the coupling constants of the various cluster species.

Chapter 4

Results and Discussion

In this chapter we present the results obtained. In the first section we apply the nonlinear RMF model developed in the last chapter to study the equations of state of nuclear matter composed by nucleons. With this study, the dependence of the meson fields on the baryonic density as well as other important quantities for several proton fractions is shown. As we saw, the major population of a neutron star is neutrons. Therefore, it is interesting to study the properties of the equation of state of nuclear matter for low proton fractions.

In the second section we study the EOS at low densities where the true ground state of nuclear matter at zero temperature is composed mostly by light clusters rather than only by nucleons. This is an important study since, as it is referred in the introduction, in a neutron star crust, where the density is lower than the saturation density, it is believed that light clusters are present.

4.1 Nuclear matter

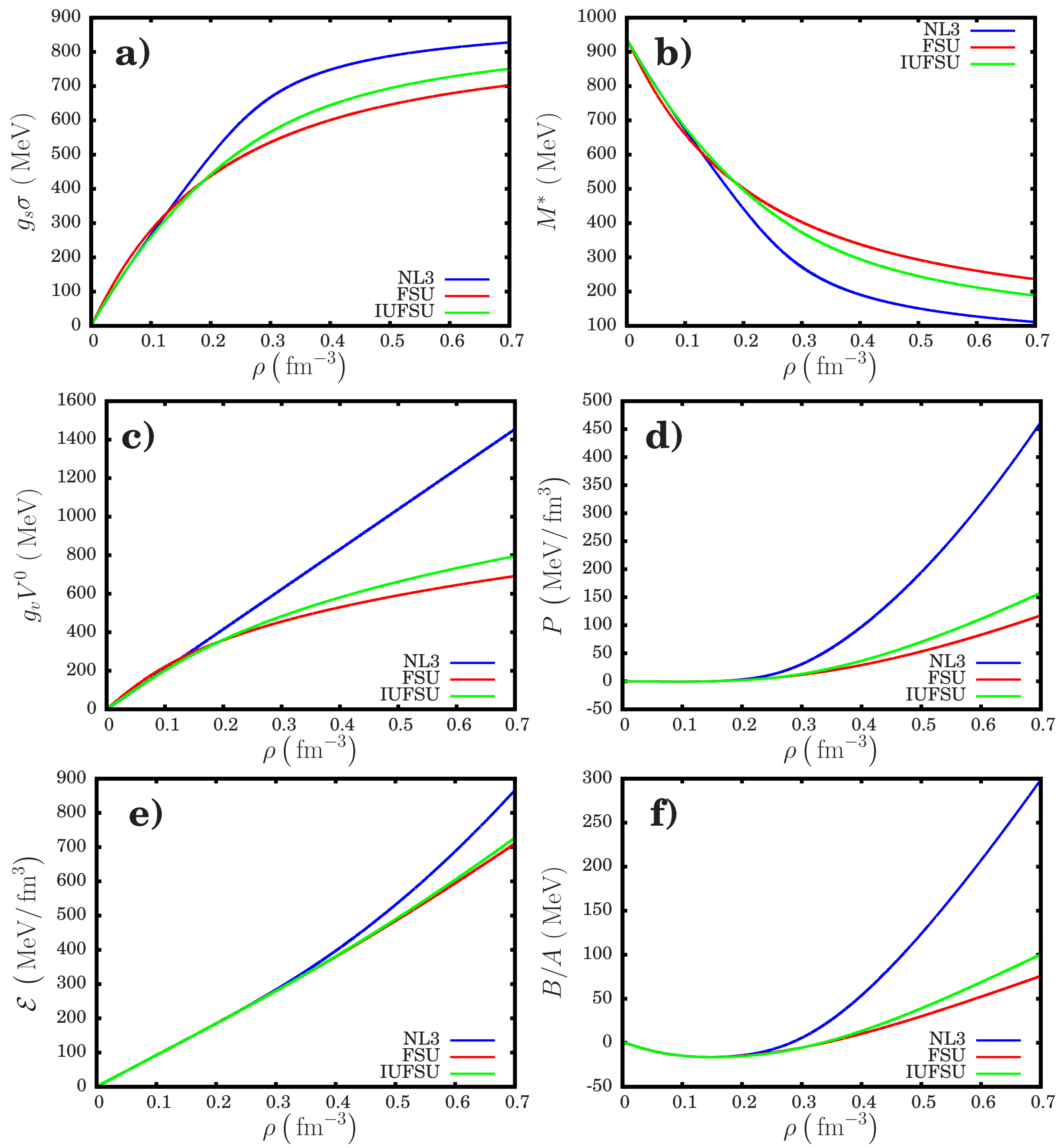

Figure 4.1: Results obtained to symmetric nuclear matter ($Y_p = 0.5$) as a function of baryon density: (a) and (c) are the meson fields, (b) is the nucleon effective mass, (d) is the pressure, (e) and (f) are, respectively, the energy density and the binding energy per nucleon.

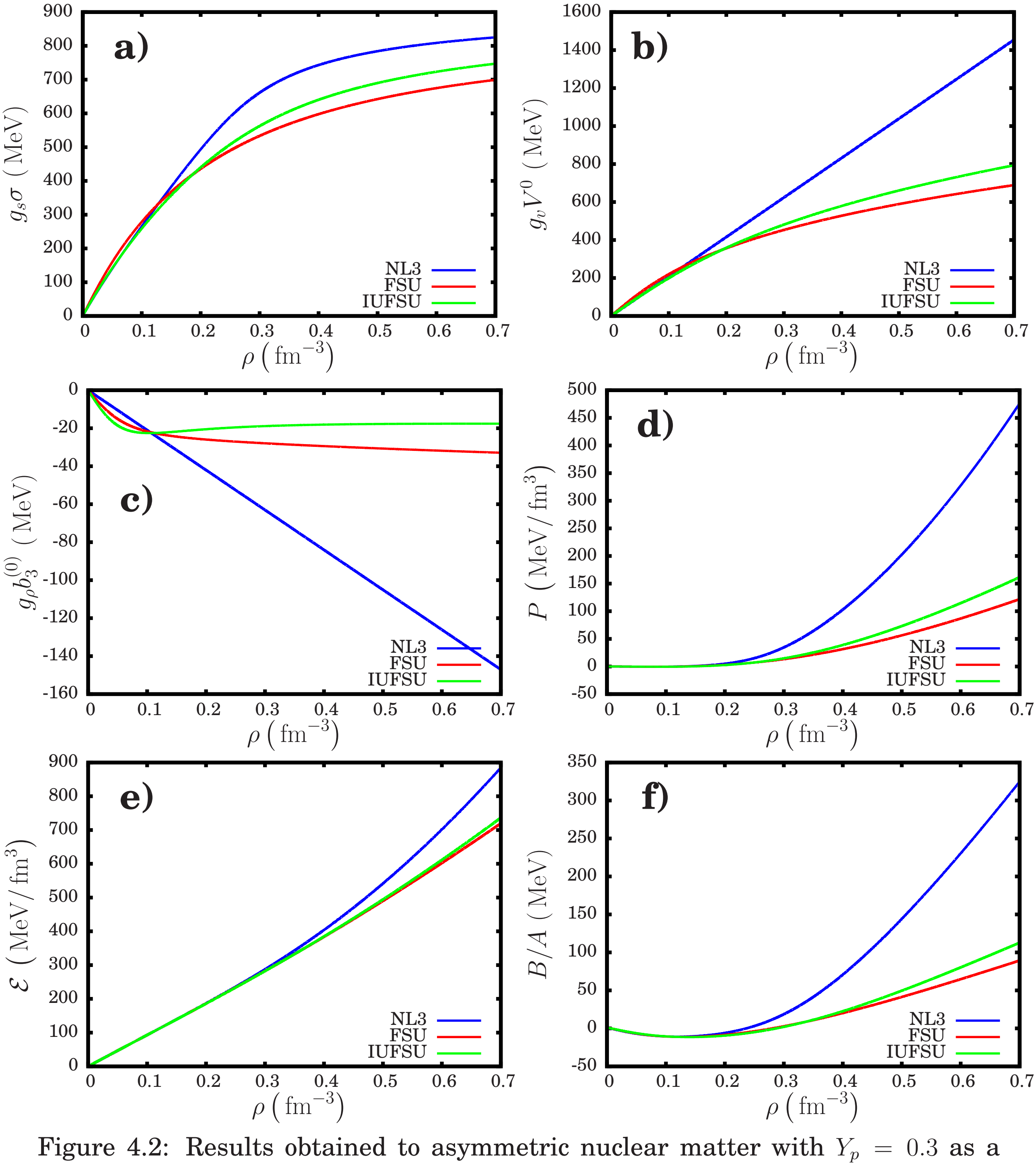

Figure 4.2: Results obtained to asymmetric nuclear matter with $Y_p = 0.3$ as a function of baryon density: (a), (b) and (c) are the meson fields, (d) is the pressure, (e) and (f) are, respectively, the energy density and the binding energy per nucleon.

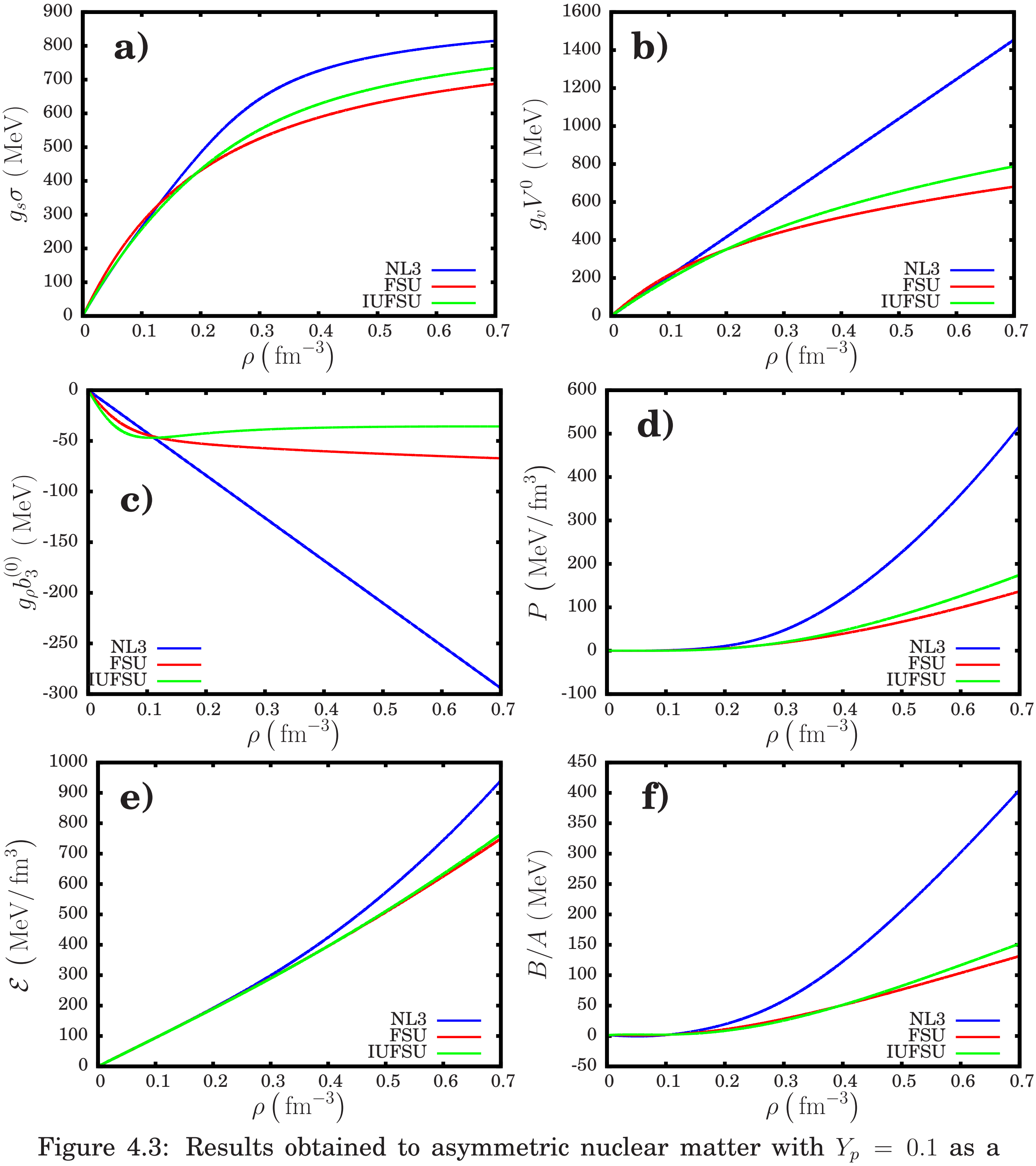

Figure 4.3: Results obtained to asymmetric nuclear matter with $Y_p = 0.1$ as a function of baryon density: (a), (b) and (c) are the meson fields, (d) is the pressure, (e) and (f) are, respectively, the energy density and the binding energy per nucleon.

In figures 4.1-4.2-4.3(a) the scalar meson field value grows as the baryonic density increases but, at high densities, it saturates. Therefore, figure 4.1(b) shows that the value of the Dirac effective mass decreases with the increase of the scalar meson field.
The expression of the Dirac effective mass is

$$M^*(\rho) = M - g_s\sigma(\rho)$$

taking the limit of infinite baryonic density we obtain $g_s\sigma(\rho) \longrightarrow M$ and therefore $M^* \longrightarrow 0$, as figure 4.1(b) shows.

From figures 4.1-4.2-4.3(a) we see that the scalar meson in NL3 saturates faster than in FSU and IUFSU. In figures 4.1(c) and 4.2-4.3(b), the vector meson field value in NL3 increases linearly with the baryonic density, due to the fact that the parameters ξ and Λ_v are zero. From figures 4.1-4.2-4.3(d) and 4.1-4.2-4.3(e) we conclude that the equation of state of NL3 is the stiffer and the equations of state of FSU and IUFSU are similar and substantially softer.

The behavior of the binding energy per nucleon is shown in figures 4.1-4.2-4.3(f). It is very similar for all the parametrizations at subsaturation densities but, for suprasaturation densities, the NL3 provides very different values comparatively to FSU and IUFSU.

The rho meson field value is shown in figures 4.2-4.3(c). In the NL3 parametrization the field value is proportional to the isopsin asymmetry due to $\Lambda_v = 0$.

It is interesting to note, from figures 4.1-4.2-4.3, that as the asymmetry decreases from $Y_p = 0.1$ to 0.5 (symmetric nuclear matter) the pressure and the energy density decrease, as expected, due to the decrease of the rho meson field which is zero at $Y_p = 0.5$. Thus, the higher the asymmetry, the stiffer is the equation of state.

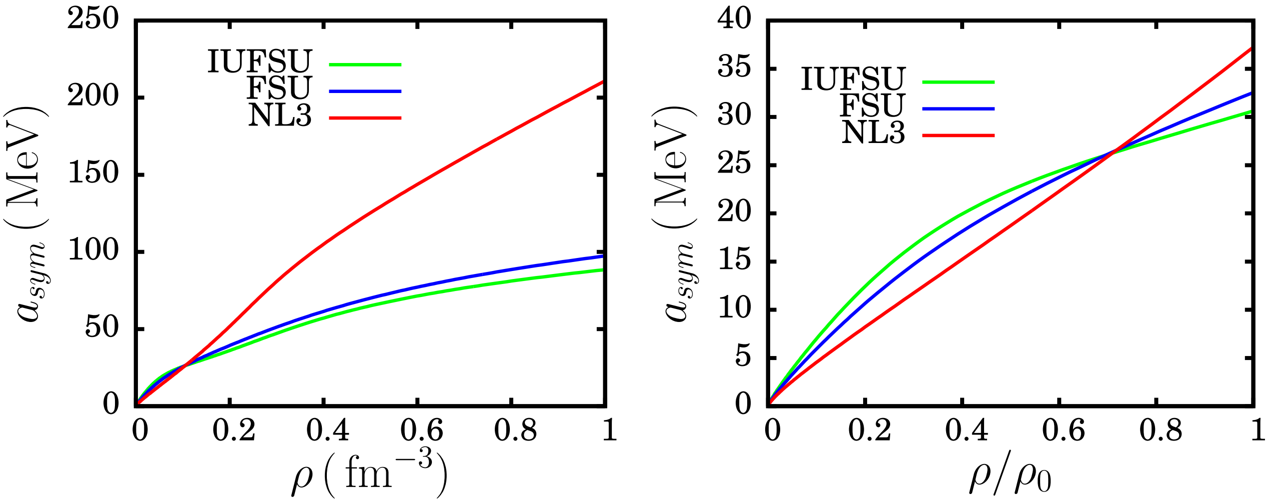

Figure 4.4: The symmetry coefficient as a function of baryonic density on the left and on the right for subsaturation densities.

In figure 4.4 the symmetry energy as a function of the baryonic density is represented. The symmetry energy for the Lagrangian density used is calculated in

Appendix C and is given by

$$a_{sym} = \frac{k_f^2}{6\sqrt{k_f^2 + (m^*)^2}} + \frac{g_\rho^2}{12\pi^2} \frac{k_f^3}{\left(m_\rho^*\right)^2}$$

where $\left(m_\rho^*\right)^2 = m_\rho^2 + 2g_\rho^2 g_v^2 \Lambda_v \left(V^0\right)^2$.
In the NL3 parametrization it is reduced to

$$a_{sym} = \frac{k_f^2}{6\sqrt{k_f^2 + (m^*)^2}} + \frac{g_\rho^2}{12\pi^2} \frac{k_f^3}{m_\rho^2}.$$

Taking the limit of the high baryonic density $\rho \to +\infty$ or $k_f \to +\infty$ we obtain

$$a_{sym}(\rho) = \frac{g_\rho^2}{12\pi^2} \frac{k_f^3}{m_\rho^2} \quad \text{in NL3,} \tag{4.1}$$

$$a_{sym}(\rho) = \frac{g_\rho^2}{12\pi^2} \frac{k_f^3}{m_\rho^2 + 2g_\rho^2 g_v^2 \Lambda_v \left(V^0\right)^2} \quad \text{in FSU and IUFSU.} \tag{4.2}$$

This behavior can be seen in figure 4.4 where, at a given density, the $a_{sym}(\rho)$ for the NL3 is almost linear as a function of the baryonic density. The coupling constant Λ_v can be used to change the baryonic density dependence of the symmetry energy as we see in figure 4.4 for FSU and IUFSU. The larger Λ_v, the larger is the denominator in (4.2), therefore reducing the a_{sym} at large densities.

4.2 Nuclear matter with clusters

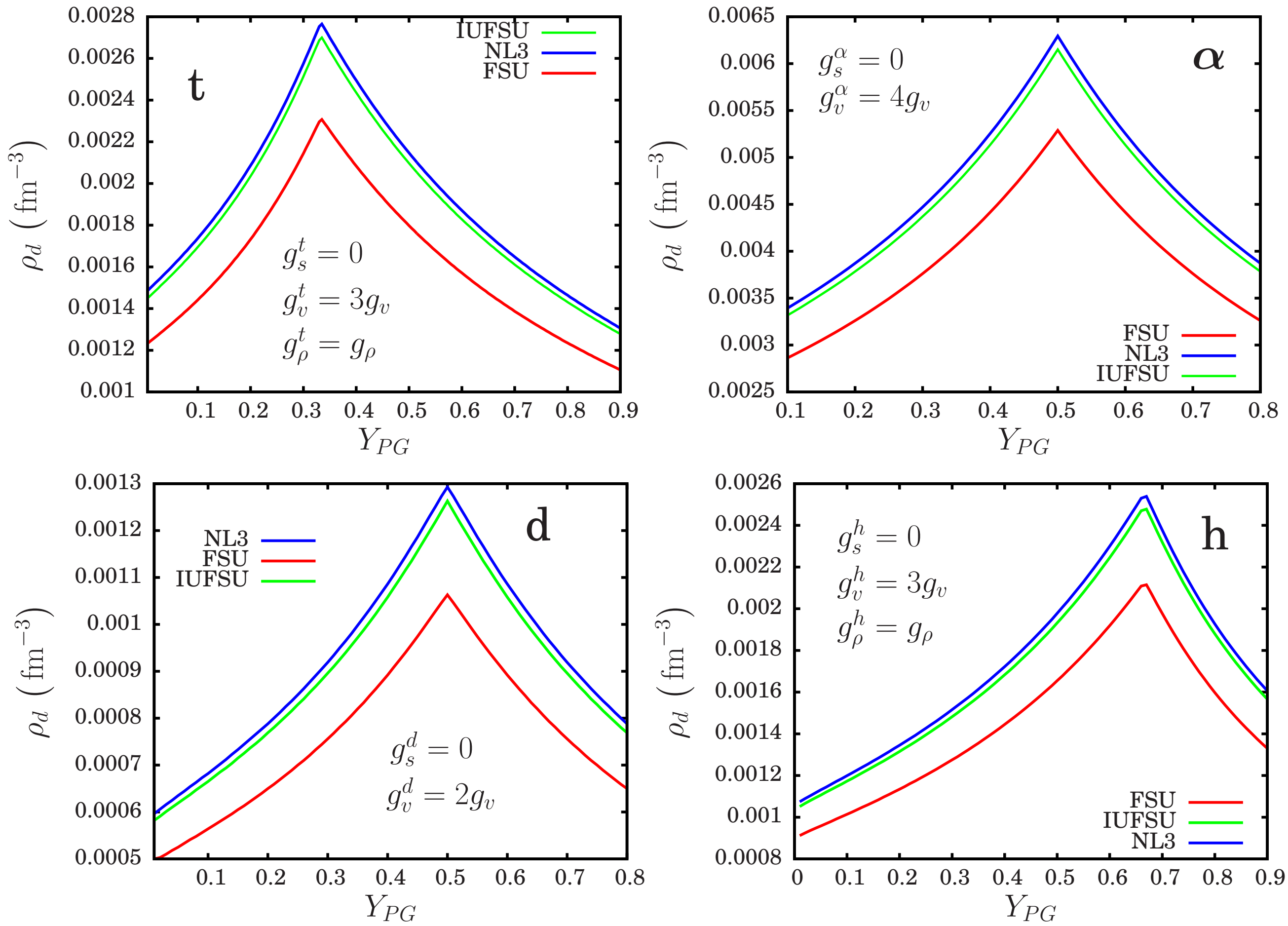

Figure 4.5: The dissolution density as a function of the total proton fraction Y_{PG} for the various clusters: deuteron ($d \equiv {}^2$H), triton ($t \equiv {}^3$H), helion ($h \equiv {}^3$He) and α particles (^{4}He).

In figure 4.5, for each cluster, the dissolution density is maximum for the Y_{PG} that allows the conversion of all nucleons into clusters. The proton fraction for the α particles is $y_p = \frac{Z}{A} = \frac{2}{4} = \frac{1}{2}$ and for the deuteron is $y_p = \frac{1}{2}$. Thus, at $Y_{PG} = 0.5$, all the neutrons and protons can be converted into deuterons or α particles, if it is energetically favorable. The proton fraction for helion is $y_p = \frac{2}{3} \approx 0.667$ and for triton is $y_p = \frac{1}{3} \approx 0.333$. Therefore, at $Y_{PG} = \frac{2}{3}$ and $Y_{PG} = \frac{1}{3}$, all the nucleons are convertible into helion and triton, respectively.

From figure 4.5 we see that the dissolution density for all the global proton fractions is bigger for NL3 and smaller for FSU. The dissolution density values of IUFSU and the NL3 are very close. Therefore, hereafter we ignore IUFSU in our studies and only use NL3 and FSU.

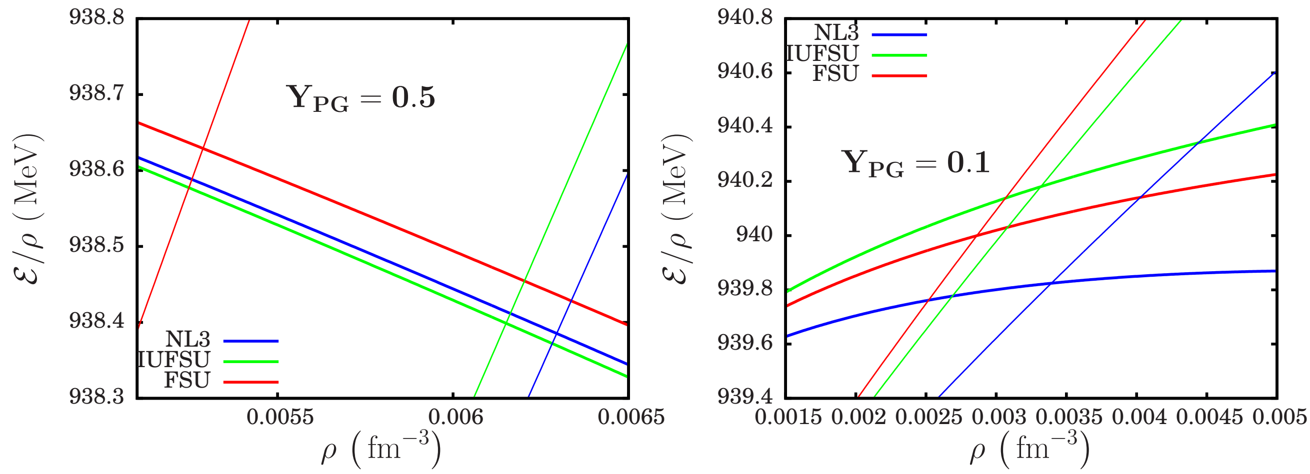

Figure 4.6: Energy per nucleon as a funtion of baryonic density for nuclear matter EOS, in thick lines, and nuclear matter EOS with α particles, in thin lines, with $Y_{PG} = 0.5$ and 0.1 for all parametrizations.

To understand why the dissolution density in the IUFSU and the NL3 are close and bigger than in FSU we show in figure 4.6, as an example, the energy density for global proton fractions of 0.5 and 0.1 for the α EOS and the nuclear matter EOS and we see that the intersection of both equations of state for the IUFSU and NL3 occurs near the same density. From 4.6 we see that the energy density from the α EOS increases when we change from NL3 to FSU. This is due to the fact that initialy we set $g_s^i = 0$ and $g_v^i = A_i g_v$; so this increase is related to the value of g_v in each parametrization. In fact, if we look at table 3.1, the FSU has the bigger g_v and the NL3 and IUFSU have approximately the same value but NL3 has the smaller value. This fact explains why the NL3 α EOS increases slower than the FSU α EOS.

In the following we test the effect of the σ-cluster coupling constant on the dissolution density. We fix $g_v^{cluster} = A_{cluster} g_v$, $g_\rho^h = g_\rho^t = g_\rho$, $g_\rho^d = g_\rho^\alpha = 0$ and parametrize the $g_s^{cluster}$ as $g_s^{cluster} = \eta g_s$.

In figures 4.7-4.8-4.9 the dissolution density grows with the increase of g_s^i for all clusters. As we saw, the σ meson gives the attraction between particles. This is due, as referred, to the fact that the sigma meson reduces the effective mass of the particle and, consequently, its energy eigenvalue. Therefore, as expected, the dissolution density is larger as the σ coupling increases because its effective mass decreases.

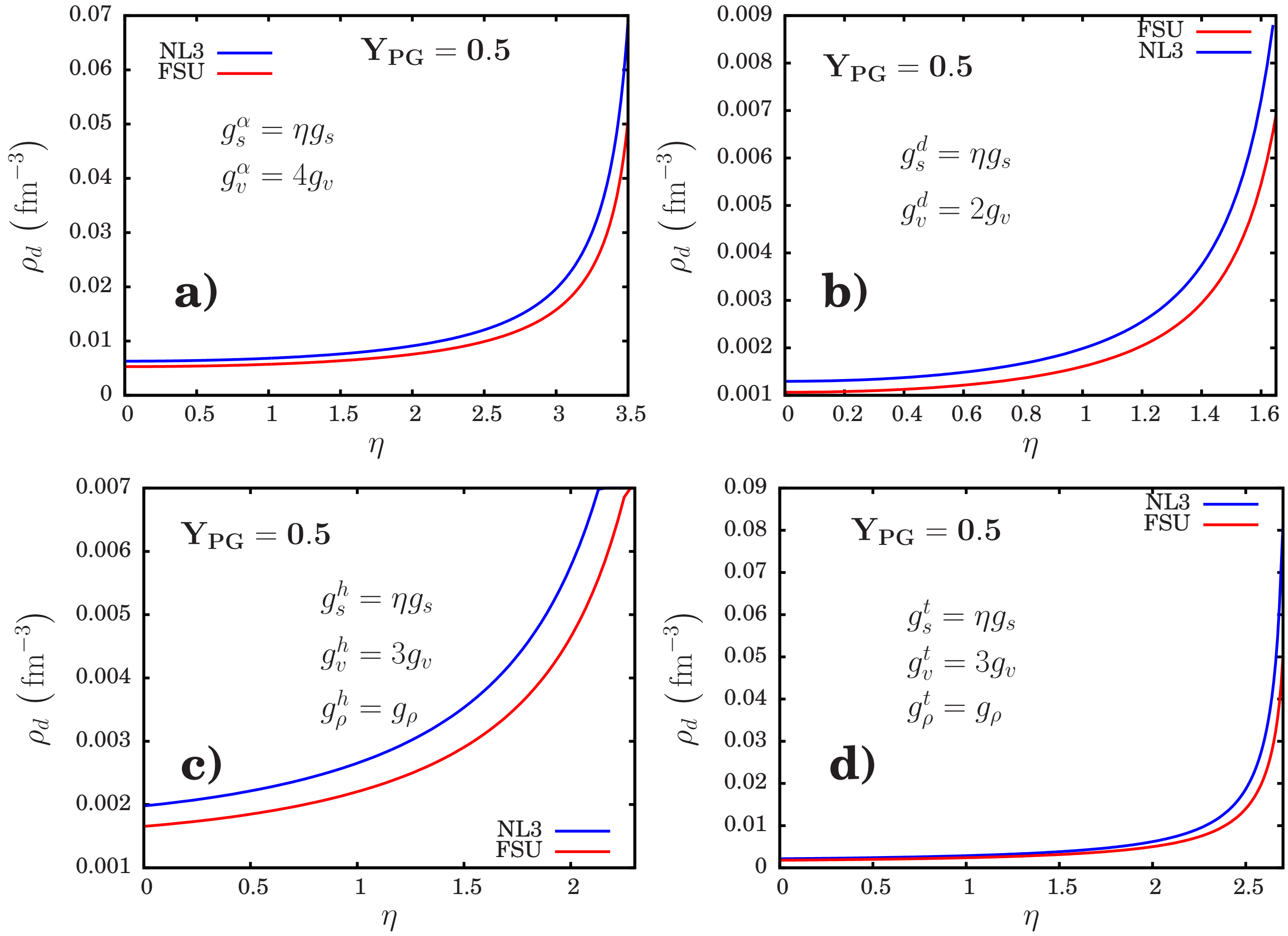

Figure 4.7: The dissolution density of each cluster as a function of the g_s^i for a global proton fraction of 0.5 for: α in a), deuteron in b), helion in c) and triton in d).

Now we will test the effect of the $\rho-$cluster coupling constant on the dissolution density of triton and helion. We fix $g_s^i = 0$, $g_v^i = A_i g_v$ and parametrize the g_ρ^i as $g_\rho^i = \eta g_\rho$.

In figure 4.10 we conclude that the dissolution density for triton and helion depends weakly on its rho coupling constants. Even in the wide range of g_ρ^t and g_ρ^h studied, the dissolution density variation was very small compared with the variation observed for the sigma and omega clusters' coupling constants.

To understand this behavior, we write only the energy density part of (3.11) that depends on the asymmetry

$$\begin{aligned}\mathcal{E}_{sym} &= \sum_{i=p,n,t,h} g_\rho^i b_3^{(0)} I_3^i \rho_i - \frac{1}{2} m_\rho^2 \left(b_3^{(0)}\right)^2 - \Lambda_v g_v^2 g_\rho^2 V_0 V^0 \left(b_3^{(0)}\right)^2 \\ &= \sum_{i=p,n,t,h} g_\rho^i b_3^{(0)} I_3^i \rho_i - \frac{1}{2}\left(m_\rho^2 b_3^{(0)} + 2\Lambda_v g_v^2 g_\rho^2 V_0 V^0 \left(b_3^{(0)}\right)\right) b_3^{(0)}\end{aligned}$$

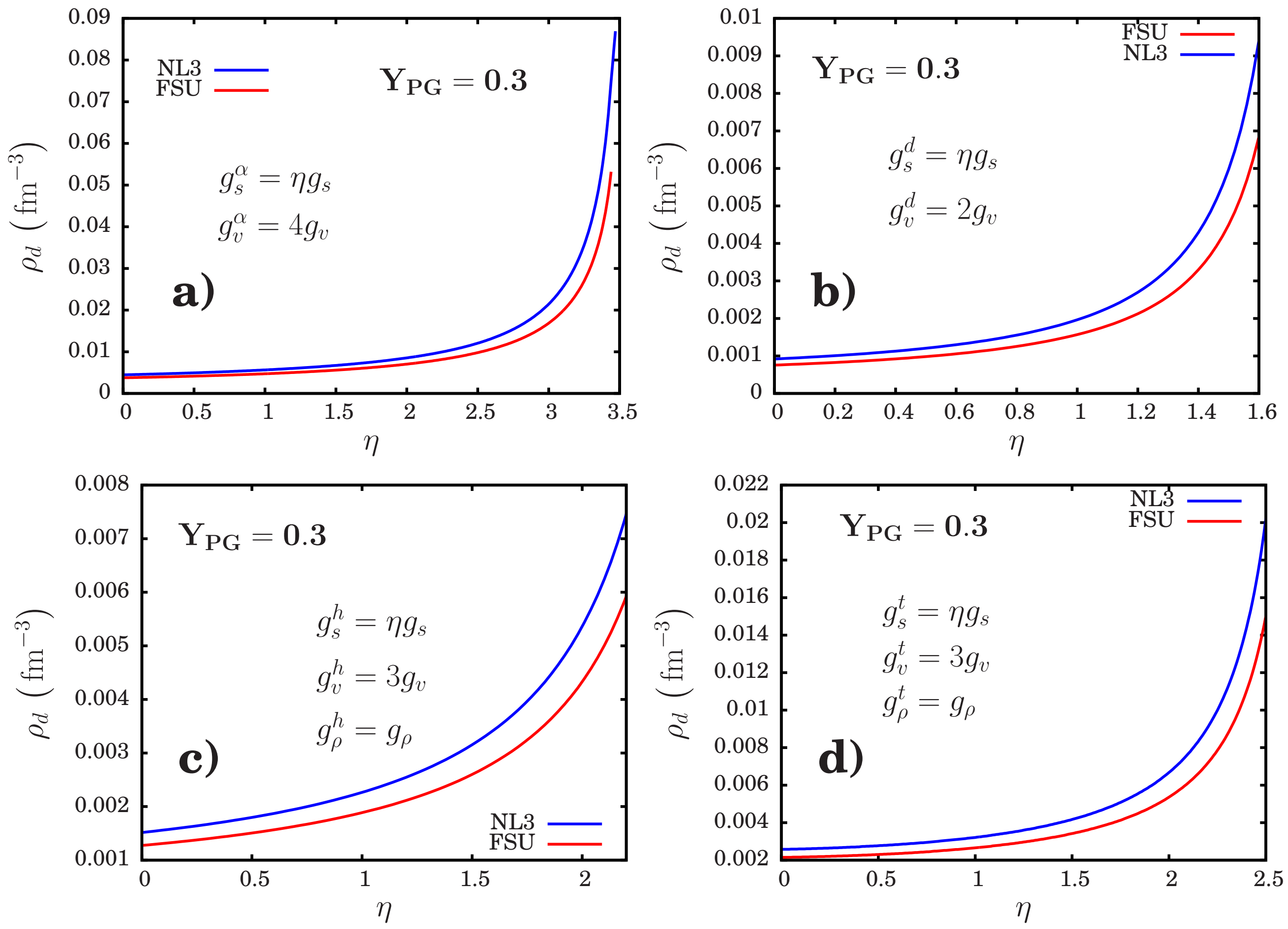

Figure 4.8: The dissolution density of each cluster as a function of the g_s^i for a global proton fraction of 0.3 for: α in a), deuteron in b), helion in c) and triton in d).

Using the equation of motion of the rho meson (3.4) we obtain

$$\begin{aligned}\mathcal{E}_{sym} &= \sum_{i=p,n,t,h} g_\rho^i b_3^{(0)} I_3^i \rho_i - \frac{1}{2} \sum_{i=p,n,t,h} g_\rho^i I_3^i b_3^{(0)} \rho_i \\ &= \frac{1}{2} \sum_{i=p,n,t,h} g_\rho^i I_3^i b_3^{(0)} \rho_i .\end{aligned}$$

We rewrite the equation of motion of the rho meson (3.4) as

$$b_0^3 = \frac{1}{\left(m_\rho^*\right)^2} \sum_{i=p,n,t,h} g_\rho^i I_3^i \rho_i . \tag{4.3}$$

where $\left(m_\rho^*\right)^2 = m_\rho^2 + 2\Lambda_v g_v^2 g_\rho^2 (V^0)^2$. Inserting this expression in $\mathcal{E}_{sym}$ we obtain

$$\begin{aligned}\mathcal{E}_{sym} &= \frac{1}{2} \frac{1}{\left(m_\rho^*\right)^2} \left(\sum_{i=p,n,t,h} g_\rho^i I_3^i \rho_i \right)^2 \\ &= \frac{1}{2} \frac{1}{\left(m_\rho^*\right)^2} \left(\frac{g_\rho^h}{2} \rho_h - \frac{g_\rho^t}{2} \rho_t + \frac{g_\rho}{2} \left(\rho_p - \rho_n\right) \right)^2\end{aligned}$$

The densities of all the species present are defined by fixing the global proton fraction. When only triton is present in nuclear matter we have

$$\mathcal{E}_{sym} = \frac{1}{2}\frac{1}{\left(m_\rho^*\right)^2}\left(-\frac{g_\rho^t}{2}\rho_t + \frac{g_\rho}{2}\left(\rho_p - \rho_n\right)\right)^2 .$$

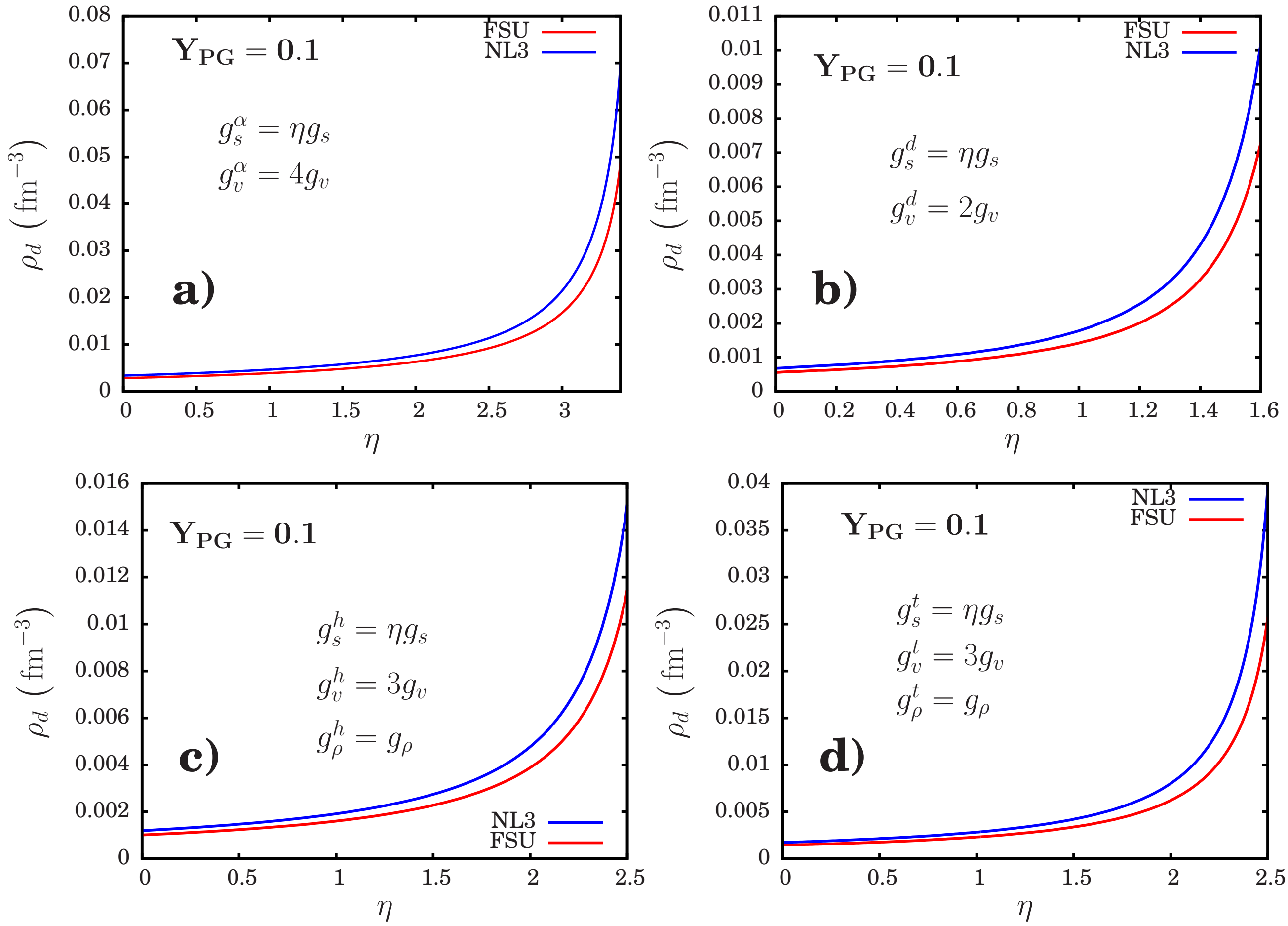

Figure 4.9: The dissolution density of each cluster as a function of the g_s^i for a global proton fraction of 0.1 for: α in a), deuteron in b), helion in c) and triton in d).

Let us discuss the case in which the global proton fraction is 0.1. If the EOS is only composed by nucleons, we have 10% of protons and 90% of neutrons. If we add the triton in the EOS we would get 0% of protons, 70% of neutrons and 30% of tritons. Therefore, we have $\rho_p = 0$, $\rho_n \neq 0$ and $\rho_t \neq 0$. So, the $\mathcal{E}_{sym}$ reads

$$\mathcal{E}_{sym} = \frac{1}{2}\frac{1}{\left(m_\rho^*\right)^2}\left(-\frac{g_\rho^t}{2}\rho_t - \frac{g_\rho}{2}\rho_n\right)^2$$
$$= \frac{1}{8}\frac{1}{\left(m_\rho^*\right)^2}\left(g_\rho^t\rho_t + g_\rho\rho_n\right)^2$$

Using the $g_\rho^t = \eta g_\rho$ dependence we obtain

$$\mathcal{E}_{sym} = \frac{1}{8}\frac{1}{(m_\rho^*)^2} g_\rho^2 \left(\eta\rho_t + \rho_n\right)^2 .$$

Therefore as the g_ρ^t increases by the parameter η in figure 4.10, the $\mathcal{E}_{sym}$ increases and also the total density energy $\mathcal{E}$. The intersection of the nucleons EOS and triton EOS free energy densities is at lower densities. Consequently, as the g_ρ^t increases, the dissolution density ρ_d decreases.

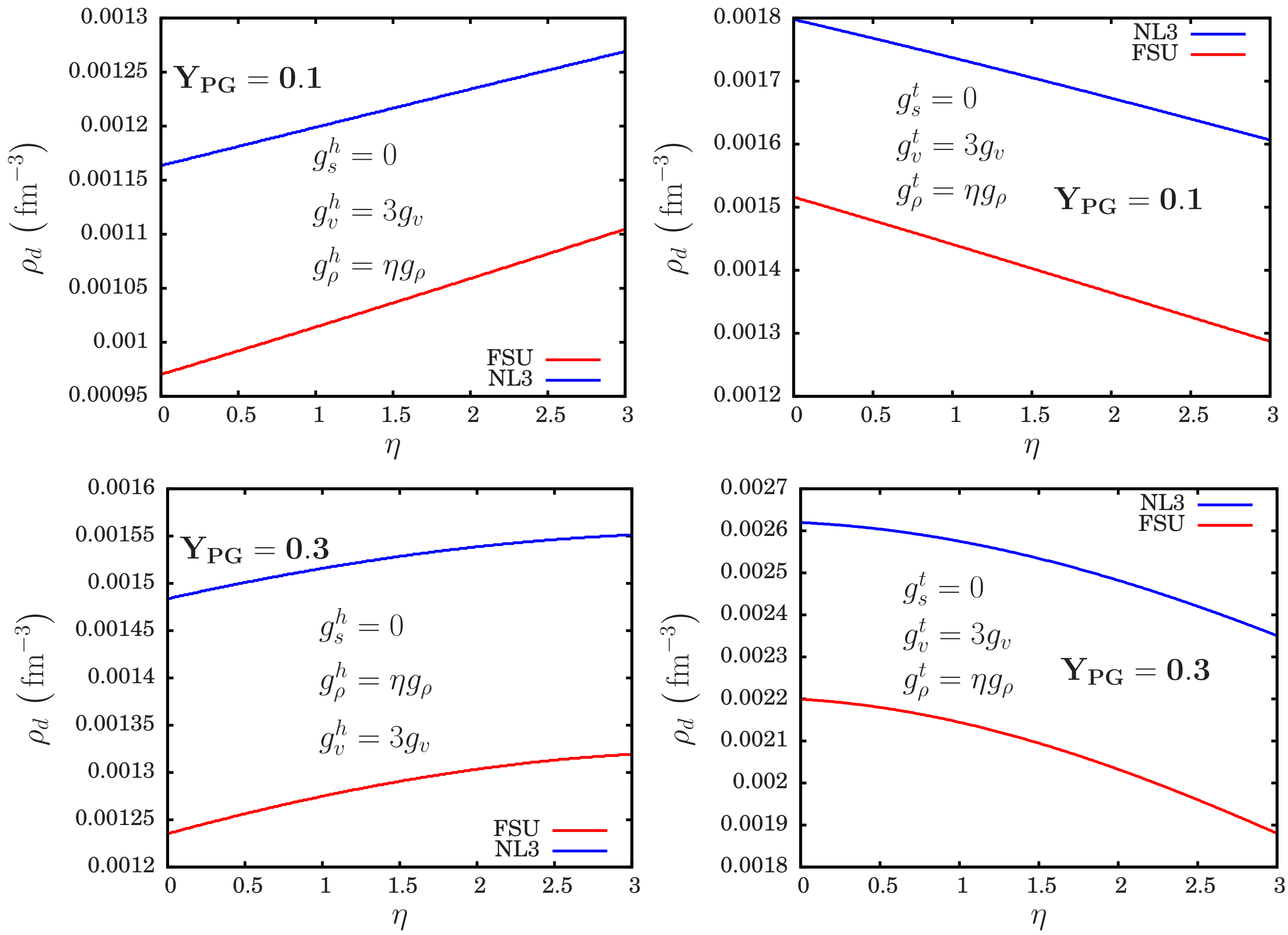

Figure 4.10: The dissolution density of each cluster as a function of the g_ρ^i for a global proton fraction of 0.3 and 0.1 for: helion on the left and triton on the right.

It is easier to understand if we think only in terms of the isospin of the present species. The EOS with neutrons and tritons is highly isospin asymmetric, both having $I_3 = -1/2$ and, so, the total energy density is higher. This is why, if besides protons, neutrons and tritons, we added helions to the EOS as a new degree of freedom, the system would decrease its energy density by creating, not only tritons, but also helions if it is energetically favorable. This way the system would decrease its isospin asymmetry. For the case where $Y_{PG} = 0.3$ we would have 90% of tritons and 10% of neutrons. Therefore, we would have $\rho_p = 0$, $\rho_n \neq 0$, $\rho_t \neq 0$ and the consequence is exactly the same as in the case of $Y_{PG} = 0.1$.

For the helion case in figure 4.10, we apply the same procedure as for triton. For $Y_{PG} = 0.1$ we would have 0% of protons, 85% of neutrons and 15% of helions. Therefore we have $\rho_p = 0$, $\rho_n \neq 0$ and $\rho_h \neq 0$. So, the resulting $\mathcal{E}_{sym}$ is

$$\begin{aligned}\mathcal{E}_{sym} &= \frac{1}{2}\frac{1}{\left(m_\rho^*\right)^2}\left(\frac{g_\rho^h}{2}\rho_h - \frac{g_\rho}{2}\rho_n\right)^2 \\ &= \frac{1}{8}\frac{1}{\left(m_\rho^*\right)^2}g_\rho^2\left(\eta\rho_h - \rho_n\right)^2\end{aligned}$$

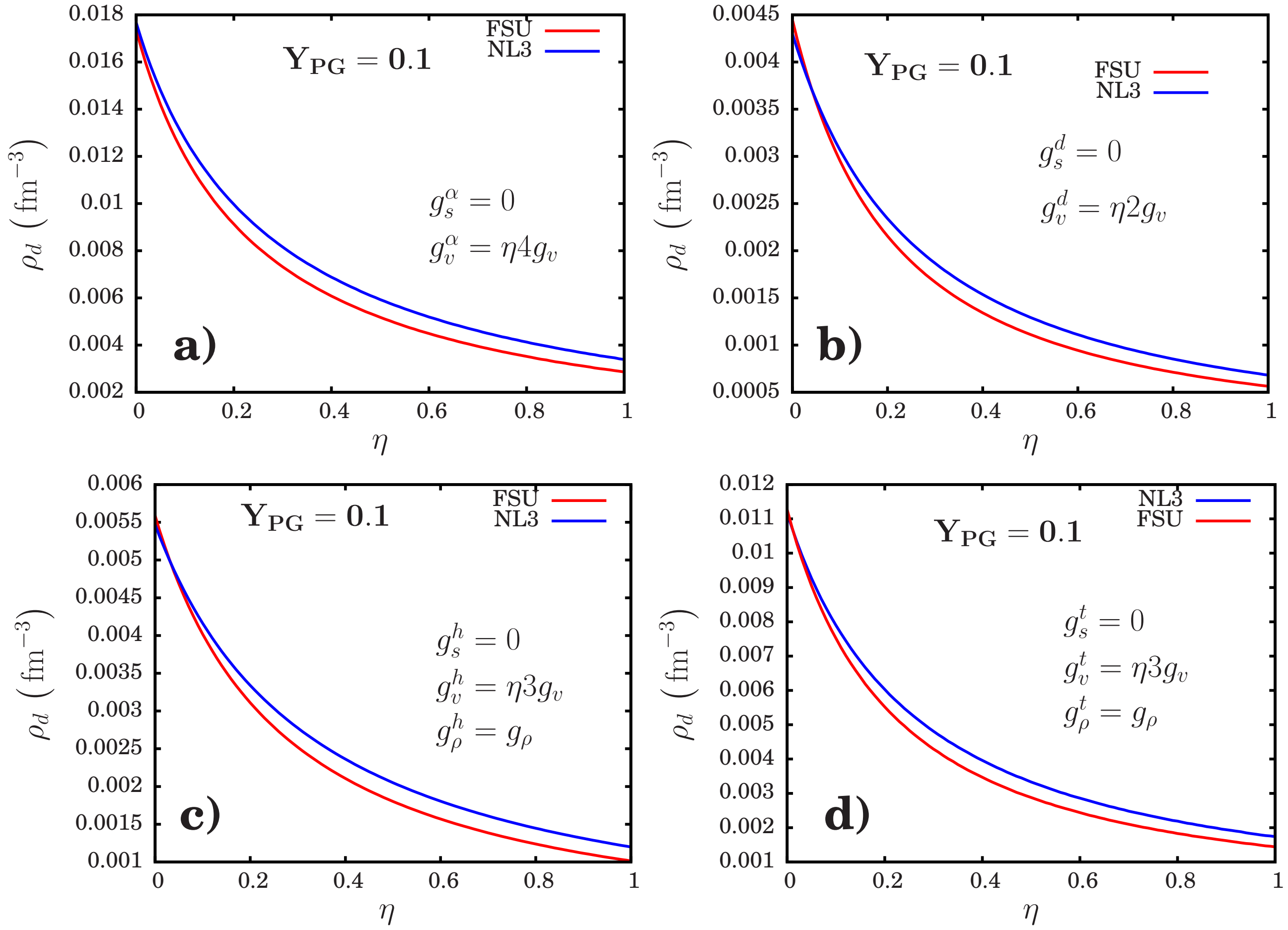

Figure 4.11: The dissolution density of each cluster as a function of the g_v^i for a global proton fraction of 0.1 for: α in a), deuteron in b), helion in c) and triton in d).

Thus, as η increases, so does the g_ρ^h and in this way, we are reducing the asymmetry of the system and, consequently, $\mathcal{E}_{sym}$ decreases. Thus, the dissolution density will increase with the increase of g_ρ^h.

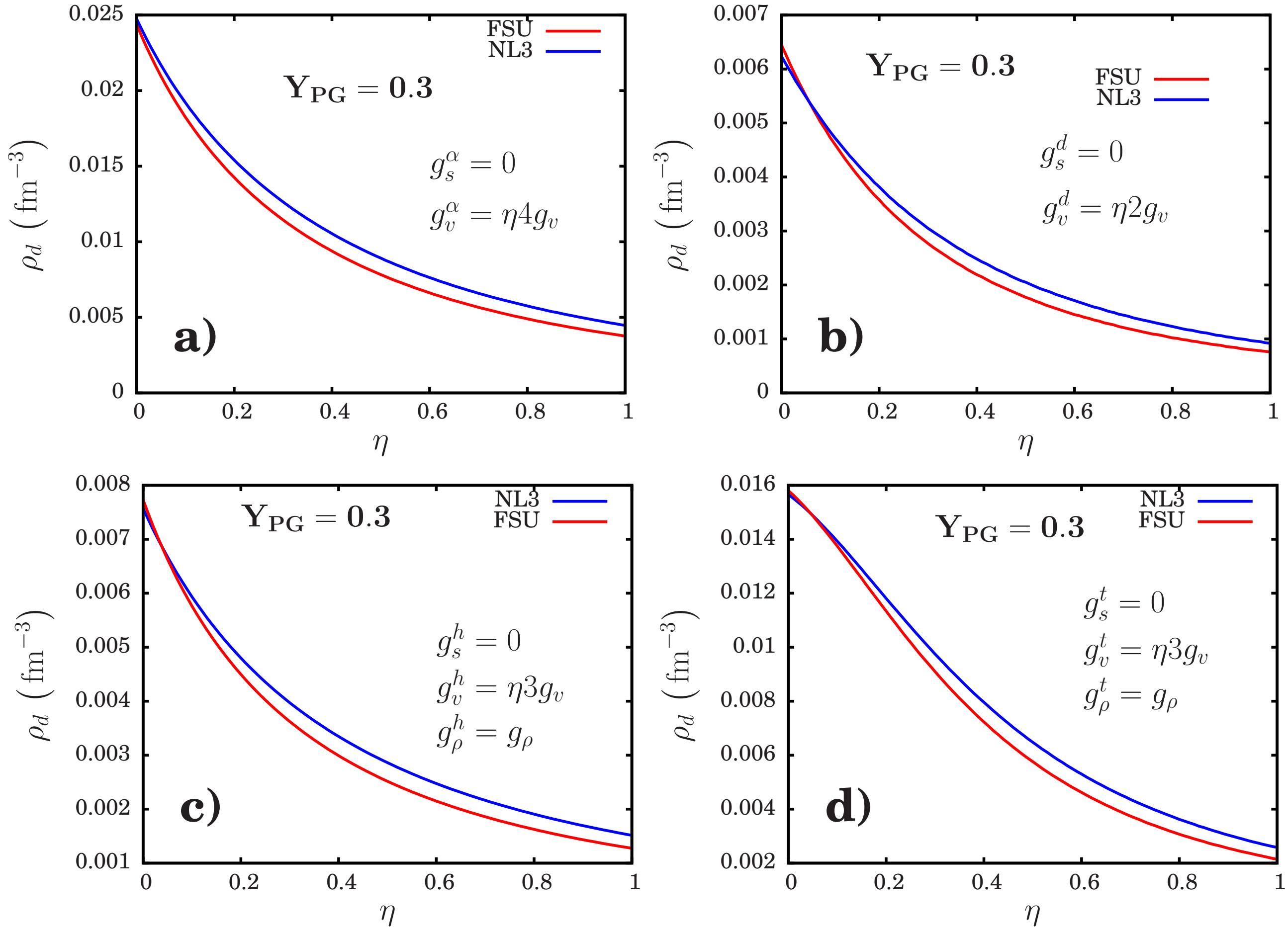

Figure 4.12: The dissolution density of each cluster as a function of the g_v^i for a global proton fraction of 0.3 for: α in a), deuteron in b), helion in c) and triton in d).

We next discuss the effect of the ω-cluster coupling on the dissolution density. In the figures 4.11, 4.12 and 4.13 the dissolution density for all clusters is represented to $Y_{PG} = 0.1$, 0.3 and 0.5. We see that the dissolution density becomes smaller as the g_v^i increases. This is due to the fact that the V^μ meson shifts the energy eigenvalues to higher values, giving the repulsive feature to the vector meson.

In order to determine which clusters are present we must analyze the free energy density of the system. In equilibrium, this quantity has a minimun.

In chapter two, we saw that the binding energy per baryon can be written as

$$\frac{B}{A} = \frac{\mathcal{E}}{\rho} - M.$$

Multiplying the equation by ρ we obtain

$$\frac{B}{A}\rho = \mathcal{E} - M\rho.$$

This is shown in figure 4.14. From figure 4.14 we see that the binding energy per baryon is the lowest in the α particle, being followed by triton, helion and deuteron.

The range in which the binding energy per baryon is lower than nuclear matter only with nucleons, decreases with the same order. The conclusion from figure 4.14 is that, at low densities, it is energetically favorable for nucleons to convert to light clusters, lowering the binding energy per baryon. Therefore, at zero temperature and at very low densities, the ground state of nuclear matter is composed by light clusters.

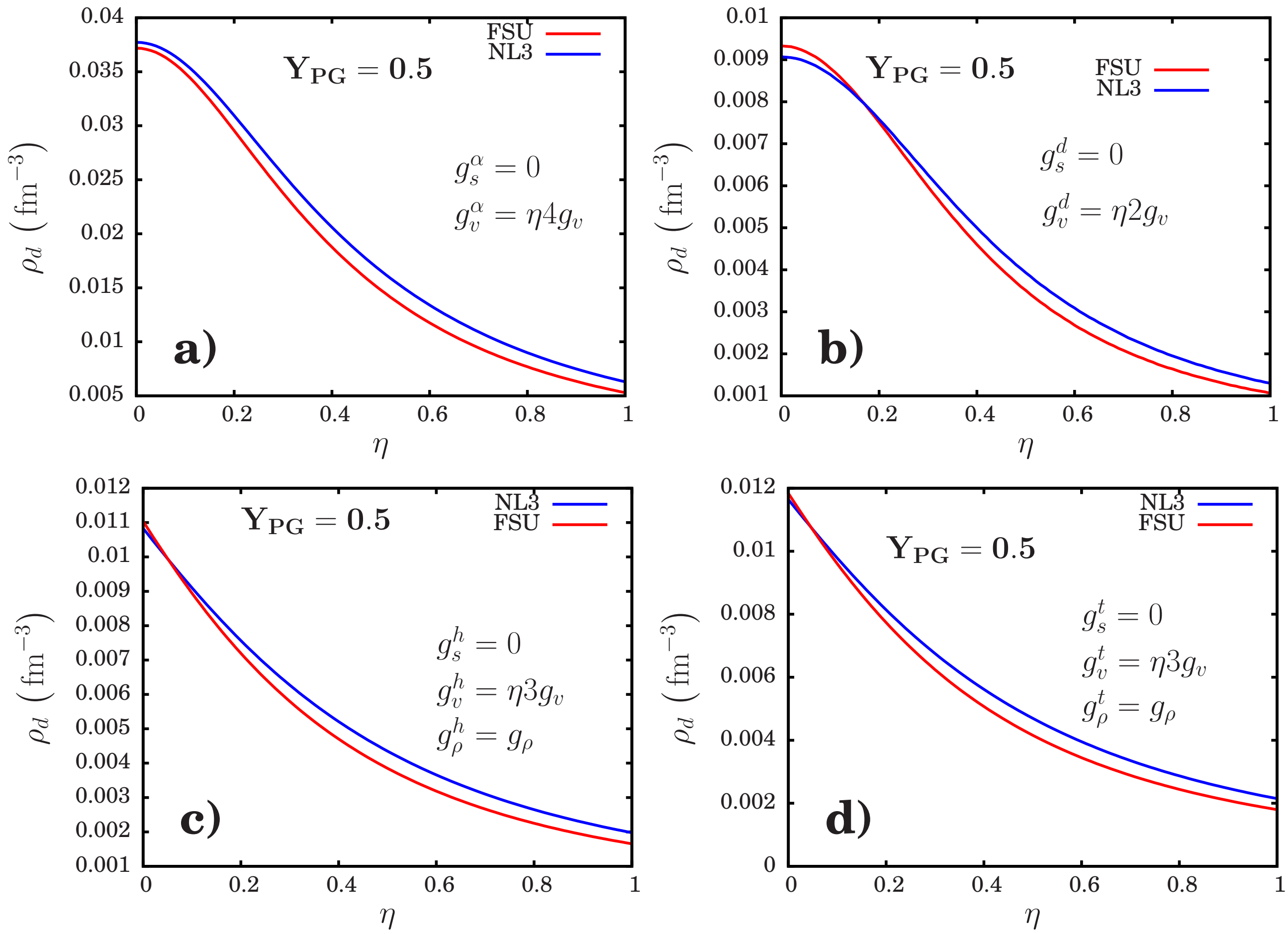

Figure 4.13: The dissolution density of each cluster as a function of the g_v^i for a global proton fraction of 0.5 for: α in a), deuteron in b), helion in c) and triton in d).

At $T = 0$ only α particles are formed because they minimize the free energy. However, at finite temperature, chemical equilibrium will determine the fraction of each cluster present in the system. Large temperatures will favour the light clusters like the deuteron.

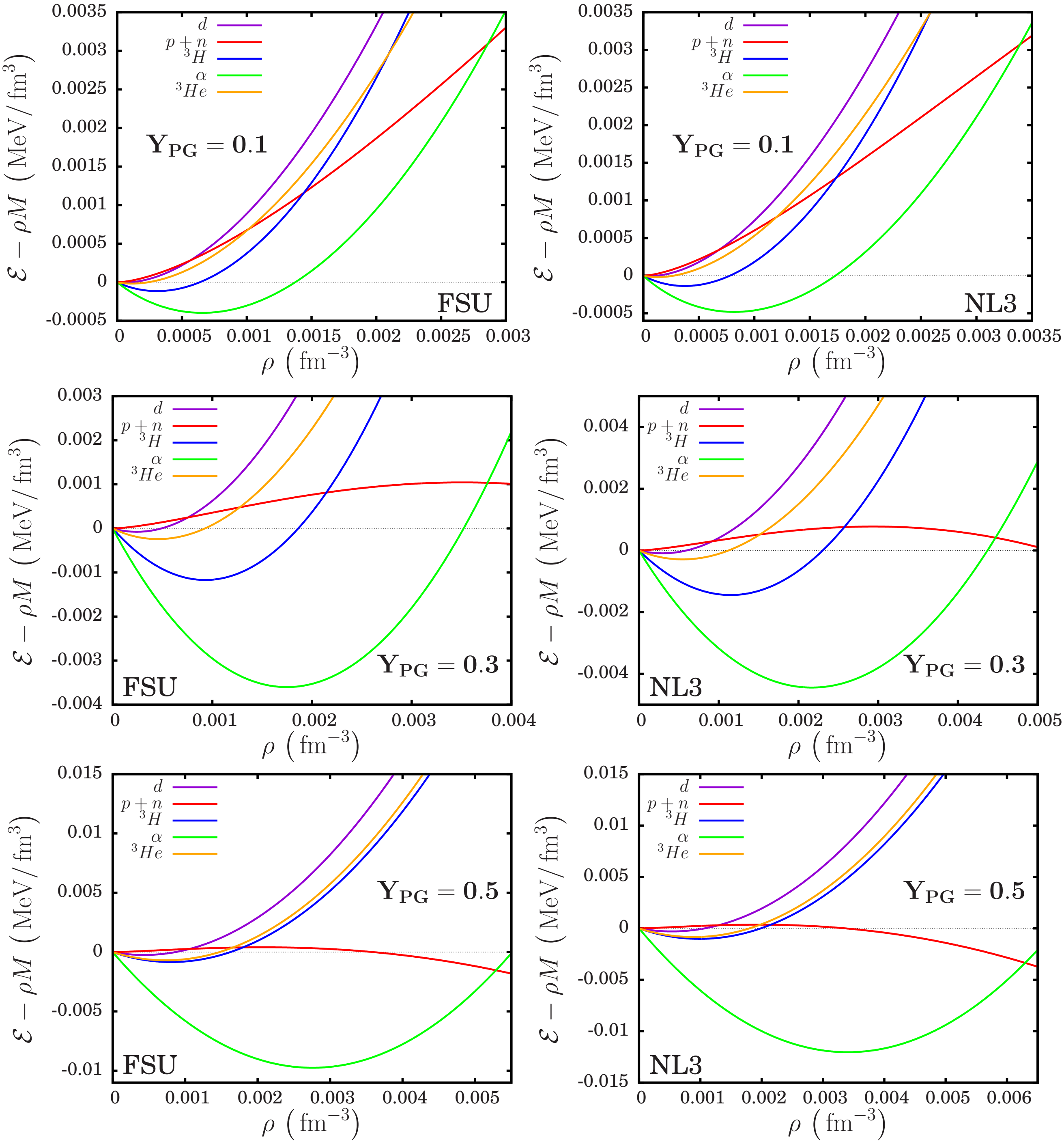

Figure 4.14: The binding energy per baryon multiplied by the baryonic density $(B/A)\rho$ as a funtion of baryonic density for global proton fractions of 0.1, 0.3 and 0.5 for FSU on the left and NL3 on the right.

4.3 Clusters coupling constants

To fit the coupling constants of the clusters, the only information we have is the dissolution densities for symmetrical nuclear matter shown in table 3.4. Therefore, we can find several sets of coupling constants that reproduce the dissolution densities or, as we did, fix two coupling constants for each cluster and fit the other to reproduce the dissolution density. Namely, we fix $g_s^i = 0$ for all clusters and fix $g_\rho^i = g_\rho$ for triton and helion. Then, we fit for all clusters the g_v^i that reproduces the dissolution density. The obtained values are shown in table 4.1.

Table 4.1: Clusters coupling constants obtained for symmetric matter ($Y_p = 0.5$) that reproduce the dissolution densities of Table 3.4

	NL3			FSU		
Cluster i	g_s^i/g_s	g_v^i/g_v	g_ρ^i/g_ρ	g_s^i/g_s	g_v^i/g_v	g_ρ^i/g_ρ
t (^3H)	0.0	3.35637	g_ρ	0.0	2.96154	g_ρ
h (^3He)	0.0	3.71748	g_ρ	0.0	3.28334	g_ρ
d (^2H)	0.0	0.88224	0.0	0.0	0.81178	0.0
α (^4He)	0.0	4.91988	0.0	0.0	4.43779	0.0

More experimental information is needed in order to determine the coupling constants of the phenomenological model we propose.

Chapter 5

Conclusions

The aim of this work was to study the EOS of nuclear matter at low densities and at zero temperature. As expected, the ground state of nuclear matter at low densities and at zero temperature is not only composed by neutrons and protons, as our results show. We conclude that the true ground state of the nuclear matter at low densities and zero temperature is mainly composed by the light clusters studied. This happens because, at low temperatures, the binding energy per nucleon can be reduced through the formation of light clusters.

The RMF model used has the coupling constants of the various clusters to the mesons of the model that mediate the interactions between the particles as the free parameters. Because there are no experimental values that can be used to fit the coupling constans, we fitted them to the values taken from [15] for nuclear symmetric matter. The coupling constants obtained can be used in further studies at finite temperature. In particular, we expect that the presence of light clusters in the crust of a compact star will affect properties such as thermal and electrical conductivity.

Appendix A

Equations of motion

The Lagrangian density we use reads

$$\mathcal{L} = \sum_{i=p,n,t,h} \mathcal{L}_i + \mathcal{L}_\sigma + \mathcal{L}_\omega + \mathcal{L}_\rho + \mathcal{L}_{\omega\rho} + \mathcal{L}_e + \mathcal{L}_d + \mathcal{L}_\alpha \tag{A.1}$$

or writting explicitly all terms

$$\begin{aligned}
\mathcal{L} &= \sum_{i=p,n,t,h} \bar{\psi}_i \left[\gamma_\mu \left(i\partial^\mu - g_v^i V^\mu - \frac{g_\rho^i}{2}\boldsymbol{\tau}.\mathbf{b}^\mu \right) - (M_i - g_s^i \sigma) \right] \psi \\
&+ \frac{1}{2}\left(\partial_\mu \sigma \partial^\mu \sigma - m_s^2 \sigma^2 - \frac{1}{3}k\sigma^3 - \frac{1}{12}\lambda\sigma^4 \right) - \frac{1}{4}\Omega_{\mu\nu}\Omega^{\mu\nu} + \frac{1}{2}m_v^2 V_\mu V^\mu + \frac{1}{24}\xi g_v^4 \left(V_\mu V^\mu\right)^2 \\
&+ \frac{1}{2}\left(-\frac{1}{2}\mathbf{B}_{\mu\nu}.\mathbf{B}_{\mu\nu} + m_\rho^2 \mathbf{b}_\mu.\mathbf{b}^\mu \right) + \Lambda_v g_v^2 g_\rho^2 V_\mu V^\mu \mathbf{b}_\mu.\mathbf{b}^\mu + \bar{\psi}_e \left[\gamma_\mu i\partial^\mu - m_e\right] \psi_e \\
&+ \frac{1}{2}\Big(\partial^\mu \left(\phi_d^\nu\right)^* \partial_\mu \phi_{d\nu} + i\partial^\mu \left(\phi_d^\nu\right)^* g_v^d V_\mu \phi_{d\nu} - g_v^d V^\mu \left(\phi_d^\nu\right)^* i\partial_\mu \phi_{d\nu} + \left(g_v^d\right)^2 V^\mu \left(\phi_d^\nu\right)^* V_\mu \phi_{d\nu} \\
&- (M_d^*)^2 \left(\phi_d^\mu\right)^* \phi_{d\mu} \Big) + \frac{1}{2}\Big(\partial^\mu \phi^* \partial_\mu \phi + i\left(\partial^\mu \phi^*\right) g_v^\alpha V_\mu \phi - i g_v^\alpha V^\mu \phi^* \partial_\mu \phi + (g_v^\alpha)^2 V_\mu V^\mu \phi\phi^* \\
&- (M_\alpha^*)^2 \phi^* \phi \Big)
\end{aligned}$$

The $\mathcal{L}_d$ and $\mathcal{L}_\alpha$ are simplified and the details are shown in the next sections.
The equation of motion for a field ϕ is calculated by solving the Euler-Lagrange equation

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = \frac{\partial \mathcal{L}}{\partial \phi} \tag{A.2}$$

A.1 Deuteron field

Starting from its Lagrangian density,

$$\begin{aligned}
\mathcal{L}_d &= \frac{1}{4}\left(i\mathcal{D}_d^{\mu}\phi_d^{\nu} - i\mathcal{D}_d^{\nu}\phi_d^{\mu}\right)^* \left(i\mathcal{D}_{\mu d}\phi_{d\nu} - i\mathcal{D}_{d\nu}\phi_{d\mu}\right) - \frac{1}{2}(M_d^*)^2 \left(\phi_d^{\mu}\right)^* \phi_{d\mu} \\
i\mathcal{D}_d^{\mu} &= i\partial^{\mu} - g_v^d V^{\mu} \\
M_d^* &= M_d - g_s^d \sigma
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_d &= \frac{1}{4}\left(i\partial^{\mu}\phi_d^{\nu} - g_v^d V^{\mu}\phi_d^{\nu} - i\partial^{\nu}\phi_d^{\mu} + g_v^d V^{\nu}\phi_d^{\mu}\right)^* \left(i\partial_{\mu}\phi_{d\nu} - g_v^d V_{\mu}\phi_{d\nu} - i\partial_{\nu}\phi_{d\mu} + g_v^d V_{\nu}\phi_{d\mu}\right) \\
&\quad - \frac{1}{2}(M_d^*)^2 \left(\phi_d^{\mu}\right)^* \phi_{d\mu} \\
&= \frac{1}{4}\left(-i\partial^{\mu}\left(\phi_d^{\nu}\right)^* - g_v^d V^{\mu}\left(\phi_d^{\nu}\right)^* + i\partial^{\nu}\left(\phi_d^{\mu}\right)^* + g_v^d V^{\nu}\left(\phi_d^{\mu}\right)^*\right) \left(i\partial_{\mu}\phi_{d\nu} - g_v^d V_{\mu}\phi_{d\nu} - i\partial_{\nu}\phi_{d\mu} + g_v^d V_{\nu}\phi_{d\mu}\right) \\
&\quad - \frac{1}{2}(M_d^*)^2 \left(\phi_d^{\mu}\right)^* \phi_{d\mu} \\
&= \frac{1}{4}\Big[\partial^{\mu}\left(\phi_d^{\nu}\right)^* \partial_{\mu}\phi_{d\nu} + i\partial^{\mu}\left(\phi_d^{\nu}\right)^* g_v^d V_{\mu}\phi_{d\nu} - \partial^{\mu}\left(\phi_d^{\nu}\right)^* \partial_{\nu}\phi_{d\mu} - i\partial^{\mu}\left(\phi_d^{\nu}\right)^* g_v^d V_{\nu}\phi_{d\mu} \\
&\quad - g_v^d V^{\mu}\left(\phi_d^{\nu}\right)^* i\partial_{\mu}\phi_{d\nu} + \left(g_v^d\right)^2 V^{\mu}\left(\phi_d^{\nu}\right)^* V_{\mu}\phi_{d\nu} + g_v^d V^{\mu}\left(\phi_d^{\nu}\right)^* i\partial_{\nu}\phi_{d\mu} - \left(g_v^d\right)^2 V^{\mu}\left(\phi_d^{\nu}\right)^* V_{\nu}\phi_{d\mu} \\
&\quad - \partial^{\nu}\left(\phi_d^{\mu}\right)^* \partial_{\mu}\phi_{d\nu} - i\partial^{\nu}\left(\phi_d^{\mu}\right)^* g_v^d V_{\mu}\phi_{d\nu} + \partial^{\nu}\left(\phi_d^{\mu}\right)^* \partial_{\nu}\phi_{d\mu} + i\partial^{\nu}\left(\phi_d^{\mu}\right)^* g_v^d V_{\nu}\phi_{d\mu} \\
&\quad + g_v^d V^{\nu}\left(\phi_d^{\mu}\right)^* i\partial_{\mu}\phi_{d\nu} - \left(g_v^d\right)^2 V^{\nu}\left(\phi_d^{\mu}\right)^* V_{\mu}\phi_{d\nu} - g_v^d V^{\nu}\left(\phi_d^{\mu}\right)^* i\partial_{\nu}\phi_{d\mu} + \left(g_v^d\right)^2 V^{\nu}\left(\phi_d^{\mu}\right)^* V_{\nu}\phi_{d\mu}\Big] \\
&\quad - \frac{1}{2}(M_d^*)^2 \left(\phi_d^{\mu}\right)^* \phi_{d\mu}
\end{aligned}$$

By the Euler-Lagrange equation,

$$\partial_{\theta}\left(\frac{\partial\mathcal{L}_{\alpha}}{\partial(\partial_{\theta}\phi_{\lambda}^*)}\right) = \frac{\partial\mathcal{L}_{\alpha}}{\partial\phi_{\lambda}^*}$$

$$\begin{aligned}
(=)&\frac{1}{4}\partial_{\theta}\left[\partial^{\theta}\phi^{\lambda} + ig_v^d V^{\theta}\phi^{\lambda} - \partial^{\lambda}\phi^{\theta} - ig_v^d V^{\lambda}\phi^{\theta} - \partial^{\lambda}\phi^{\theta} - ig_v^d V^{\lambda}\phi^{\theta} + \partial^{\theta}\phi^{\lambda} + ig_v^d V^{\theta}\phi^{\lambda}\right] \\
=&\frac{1}{4}\Big[- g_v^d V^{\mu} i\partial_{\mu}\phi^{\lambda} + \left(g_v^d\right)^2 V^{\mu}V_{\mu}\phi^{\lambda} + ig_v^d V^{\mu}\partial^{\lambda}\phi_{\mu} - \left(g_v^d\right)^2 V^{\mu}V^{\lambda}\phi_{\mu} + g_v^d V^{\nu} i\partial^{\lambda}\phi_{\nu} \\
&- \left(g_v^d\right)^2 V^{\nu}V^{\lambda}\phi_{\nu} - g_v^d V^{\nu} i\partial_{\nu}\phi^{\lambda} + \left(g_v^d\right)^2 V^{\nu}V_{\nu}\phi^{\lambda}\Big] - \frac{1}{2}\left(M_d^*\right)^2\phi^{\lambda} \\
(=)&\partial_{\theta}\left[\partial^{\theta}\phi^{\lambda} + ig_v^d V^{\theta}\phi^{\lambda} - \partial^{\lambda}\phi^{\theta} - ig_v^d V^{\lambda}\phi^{\theta}\right] = \Big[- g_v^d V^{\mu} i\partial_{\mu}\phi^{\lambda} + \left(g_v^d\right)^2 V^{\mu}V_{\mu}\phi^{\lambda} \\
&+ ig_v^d V^{\mu}\partial^{\lambda}\phi_{\mu} - \left(g_v^d\right)^2 V^{\mu}V^{\lambda}\phi_{\mu}\Big] - \left(M_d^*\right)^2\phi^{\lambda} \\
(=)&\partial_{\theta}\partial^{\theta}\phi^{\lambda} + ig_v^d V^{\theta}\partial_{\theta}\phi^{\lambda} - \partial_{\theta}\partial^{\lambda}\phi^{\theta} - ig_v^d V^{\lambda}\partial_{\theta}\phi^{\theta} + g_v^d V^{\mu} i\partial_{\mu}\phi^{\lambda} - \left(g_v^d\right)^2 V^{\mu}V_{\mu}\phi^{\lambda} \\
&- ig_v^d V^{\mu}\partial^{\lambda}\phi_{\mu} + \left(g_v^d\right)^2 V^{\mu}V^{\lambda}\phi_{\mu} + \left(M_d^*\right)^2\phi^{\lambda} = 0
\end{aligned}$$

$$
\begin{aligned}
\mathcal{D}_\mu\left(\mathcal{D}^\mu\phi^\nu - \mathcal{D}^\nu\phi^\mu\right) &= \mathcal{D}_\mu\left(\partial^\mu\phi^\nu + ig_v^dV^\mu\phi^\nu - \partial^\nu\phi^\mu - ig_v^dV^\nu\phi^\mu\right) \\
&= \left(i\partial_\mu - g_v^dV_\mu\right)\left(\partial^\mu\phi^\nu + ig_v^dV^\mu\phi^\nu - \partial^\nu\phi^\mu - ig_v^dV^\nu\phi^\mu\right) \\
&= \partial_\mu\partial^\mu\phi^\nu + ig_v^dV^\mu\partial_\mu\phi^\nu - \partial_\mu\partial^\nu\phi^\mu - ig_v^dV^\nu\partial_\mu\phi^\mu + g_v^dV^\mu i\partial_\mu\phi^\nu \\
&- \left(g_v^d\right)^2 V^\mu V_\mu\phi^\nu - ig_v^dV^\mu\partial^\nu\phi_\mu + \left(g_v^d\right)^2 V^\mu V^\nu\phi_\mu
\end{aligned}
$$

$$
\mathcal{D}_\mu\left(\mathcal{D}^\mu\phi^\nu - \mathcal{D}^\nu\phi^\mu\right) + \left(M_d^*\right)^2\phi^\nu = 0
$$

$$
\mathcal{D}_\nu \times \left(\mathcal{D}_\mu\left(\mathcal{D}^\mu\phi^\nu - \mathcal{D}^\nu\phi^\mu\right) + \left(M_d^*\right)^2\phi^\nu = 0\right) \Rightarrow \mathcal{D}_\nu\phi^\nu = 0.
$$

Using the Lorentz gauge, $\mathcal{D}_\nu\phi^\nu = 0$, we obtain the equation of motion for the deuteron field

$$
\boxed{\left[\mathcal{D}_\mu\mathcal{D}^\mu + \left(M_d^*\right)^2\right]\phi^\nu = 0.}
$$

As expected, the equation of motion of the deuteron is the Klein-Gordon equation. Using the Lorentz gauge, $D_\mu\phi_d^\mu = 0(=)\partial^\mu\phi_{d\mu} = -ig_v^dV^\mu\phi_{d\mu}$, we could simplify the Lagrangian density

$$
\begin{aligned}
\mathcal{L}_d = \frac{1}{2}\Big[&\partial^\mu\left(\phi_d^\nu\right)^*\partial_\mu\phi_{d\nu} + i\partial^\mu\left(\phi_d^\nu\right)^* g_v^dV_\mu\phi_{d\nu} - \partial^\mu\left(\phi_d^\nu\right)^*\partial_\nu\phi_{d\mu} - i\partial^\mu\left(\phi_d^\nu\right)^* g_v^dV_\nu\phi_{d\mu} \\
&- g_v^dV^\mu\left(\phi_d^\nu\right)^* i\partial_\mu\phi_{d\nu} + \left(g_v^d\right)^2 V^\mu\left(\phi_d^\nu\right)^* V_\mu\phi_{d\nu} + g_v^dV^\mu\left(\phi_d^\nu\right)^* i\partial_\nu\phi_{d\mu} \\
&- \left(g_v^d\right)^2 V^\mu\left(\phi_d^\nu\right)^* V_\nu\phi_{d\mu} - \left(M_d^*\right)^2\left(\phi_d^\mu\right)^*\phi_{d\mu}\Big],
\end{aligned}
$$

notting the following relations

a)
$$
\begin{aligned}
-\partial^\mu\left(\phi_d^\nu\right)^*\partial_\nu\phi_{d\mu} &= -\partial^\mu\left(\left(\phi_d^\nu\right)^*\partial_\nu\phi_{d\mu}\right) + \left(\phi_d^\nu\right)^*\partial_\nu\partial^\mu\phi_{d\mu} \\
&= -\partial^\mu\left(\left(\phi_d^\nu\right)^*\partial_\nu\phi_{d\mu}\right) - ig_v^dV^\mu\left(\phi_d^\nu\right)^*\partial_\nu\phi_{d\mu} \\
&= -\partial^\mu\left(\left(\phi_d^\nu\right)^*\partial_\nu\phi_{d\mu}\right) - \partial_\nu\left(ig_v^dV^\mu\left(\phi_d^\nu\right)^*\phi_{d\mu}\right) + ig_v^dV^\mu\partial_\nu\left(\phi_d^\nu\right)^*\phi_{d\mu} \\
&= -\partial^\mu\left(\left(\phi_d^\nu\right)^*\partial_\nu\phi_{d\mu}\right) - \partial_\nu\left(ig_v^dV^\mu\left(\phi_d^\nu\right)^*\phi_{d\mu}\right) + ig_v^dV^\mu\left(+ig_v^dV^\nu\left(\phi_{d\nu}\right)^*\right)\phi_{d\mu} \\
&= -\partial^\mu\left(\left(\phi_d^\nu\right)^*\partial_\nu\phi_{d\mu}\right) - \partial_\nu\left(ig_v^dV^\mu\left(\phi_d^\nu\right)^*\phi_{d\mu}\right) - \left(g_v^d\right)^2 V^\mu V^\nu\left(\phi_{d\nu}\right)^*\phi_{d\mu}
\end{aligned}
$$

b)
$$
\begin{aligned}
-i\partial^\mu\left(\phi_d^\nu\right)^* g_v^dV_\nu\phi_{d\mu} &= -\partial^\mu\left(i\left(\phi_d^\nu\right)^* g_v^dV_\nu\phi_{d\mu}\right) + i\left(\phi_d^\nu\right)^* g_v^dV_\nu\partial^\mu\phi_{d\mu} \\
&= -\partial^\mu\left(i\left(\phi_d^\nu\right)^* g_v^dV_\nu\phi_{d\mu}\right) + \left(g_v^d\right)^2 V_\nu V^\mu\left(\phi_d^\nu\right)^*\phi_{d\mu}
\end{aligned}
$$

c)
$$
\begin{aligned}
g_v^dV^\mu\left(\phi_d^\nu\right)^* i\partial_\nu\phi_{d\mu} &= \partial^\mu\left(g_v^dV^\mu\left(\phi_d^\nu\right)^* i\phi_{d\mu}\right) - g_v^dV^\mu\partial_\nu\left(\phi_d^\nu\right)^* i\phi_{d\mu} \\
&= \partial^\mu\left(g_v^dV^\mu\left(\phi_d^\nu\right)^* i\phi_{d\mu}\right) + \left(g_v^d\right)^2 V^\mu V^\nu\left(\phi_{d\nu}\right)^*\phi_{d\mu}.
\end{aligned}
$$

Defining the following Lagrangian density

$$
\begin{aligned}
\mathcal{L}_d^2 &= \frac{1}{2}\Big[-\partial^\mu \left(\phi_d^\nu\right)^* \partial_\nu \phi_{d\mu} - i\partial^\mu \left(\phi_d^\nu\right)^* g_v^d V_\nu \phi_{d\mu} + g_v^d V^\mu \left(\phi_d^\nu\right)^* i\partial_\nu \phi_{d\mu} - \left(g_v^d\right)^2 V^\mu \left(\phi_d^\nu\right)^* V_\nu \phi_{d\mu}\Big] \\
&= \frac{1}{2}\Big[-\partial^\mu \left(\left(\phi_d^\nu\right)^* \partial_\nu \phi_{d\mu}\right) - \partial_\nu \left(i g_v^d V^\mu \left(\phi_d^\nu\right)^* \phi_{d\mu}\right) - \left(g_v^d\right)^2 V^\mu V^\nu \left(\phi_{d\nu}\right)^* \phi_{d\mu} \\
&\quad - \partial^\mu \left(i \left(\phi_d^\nu\right)^* g_v^d V_\nu \phi_{d\mu}\right) + \left(g_v^d\right)^2 V_\nu V^\mu \left(\phi_d^\nu\right)^* \phi_{d\mu} + \partial^\mu \left(g_v^d V^\mu \left(\phi_d^\nu\right)^* i\phi_{d\mu}\right) \\
&\quad + \left(g_v^d\right)^2 V^\mu V^\nu \left(\phi_{d\nu}\right)^* \phi_{d\mu} - \left(g_v^d\right)^2 V^\mu \left(\phi_d^\nu\right)^* V_\nu \phi_{d\mu}\Big] \\
&= \frac{1}{2}\Big[-\partial^\mu \left(\left(\phi_d^\nu\right)^* \partial_\nu \phi_{d\mu}\right) - \partial_\nu \left(i g_v^d V^\mu \left(\phi_d^\nu\right)^* \phi_{d\mu}\right) - \partial^\mu \left(i \left(\phi_d^\nu\right)^* g_v^d V_\nu \phi_{d\mu}\right) + \partial^\mu \left(g_v^d V^\mu \left(\phi_d^\nu\right)^* i\phi_{d\mu}\right)\Big] \\
&= \partial^\mu \frac{1}{2}\Big[-\left(\left(\phi_d^\nu\right)^* \partial_\nu \phi_{d\mu}\right) - \left(i g_v^d V^\nu \left(\phi_{d\mu}\right)^* \phi_{d\nu}\right) - \left(i \left(\phi_d^\nu\right)^* g_v^d V_\nu \phi_{d\mu}\right) + \left(g_v^d V^\mu \left(\phi_d^\nu\right)^* i\phi_{d\mu}\right)\Big] \\
&= \partial^\mu f_\mu .
\end{aligned}
$$

Looking at the original Lagrangian density we can see that it can be written in the following manner

$$
\begin{aligned}
\mathcal{L}_d &= \frac{1}{2}\Big[\partial^\mu \left(\phi_d^\nu\right)^* \partial_\mu \phi_{d\nu} + i\partial^\mu \left(\phi_d^\nu\right)^* g_v^d V_\mu \phi_{d\nu} - g_v^d V^\mu \left(\phi_d^\nu\right)^* i\partial_\mu \phi_{d\nu} \\
&\quad + \left(g_v^d\right)^2 V^\mu \left(\phi_d^\nu\right)^* V_\mu \phi_{d\nu} - (M_d^*)^2 \left(\phi_d^\mu\right)^* \phi_{d\mu}\Big] + \partial^\mu f_\mu \\
&= \mathcal{L} + \partial^\mu f_\mu .
\end{aligned}
$$

By the action properties we obtain

$$
\begin{aligned}
\mathcal{S} = \int \mathcal{L}_d d^4x &= \int \mathcal{L} d^4x + \int \partial^\mu f_\mu d^4x \\
&= \int \mathcal{L} d^4x + \int f_\mu n^\mu d^4S \\
\delta\mathcal{S} = \delta \int \mathcal{L}_d d^4x &= \delta \int \mathcal{L} d^4x + \delta \int f_\mu n^\mu d^4S \\
\to \delta \int f_\mu n^\mu d^4S &= 0 \\
\delta\mathcal{S} = \delta \int \mathcal{L}_d d^4x &= \delta \int \mathcal{L} d^4x
\end{aligned}
$$

Thus, the original Lagrangian density in the Lorentz gauge can be written as

$$
\begin{aligned}
\mathcal{L}_d &= \frac{1}{2}\Big[\partial^\mu \left(\phi_d^\nu\right)^* \partial_\mu \phi_{d\nu} + i\partial^\mu \left(\phi_d^\nu\right)^* g_v^d V_\mu \phi_{d\nu} - g_v^d V^\mu \left(\phi_d^\nu\right)^* i\partial_\mu \phi_{d\nu} \\
&\quad + \left(g_v^d\right)^2 V^\mu \left(\phi_d^\nu\right)^* V_\mu \phi_{d\nu} - (M_d^*)^2 \left(\phi_d^\mu\right)^* \phi_{d\mu}\Big] .
\end{aligned} \tag{A.3}
$$

In the mean-field approximation, the deuteron takes the following form [5]

$$
\phi^0 = \phi(k) e^{-ik_\mu x^\mu} .
$$

Inserting this expression in the equation of motion we get

$$
\begin{aligned}
0 &= \partial^\mu \partial_\mu \phi + i g_v^\alpha V^\mu \partial_\mu \phi + i g_v^d V^\mu \partial_\mu \phi - (g_v^d)^2 V_\mu V^\mu \phi + (M_d^*)^2 \phi \\
0 &= (-ik^\mu)(-ik_\mu)\phi + 2 i g_v^d V^\mu (-ik_\mu)\phi - (g_v^d)^2 V_\mu V^\mu \phi + (M_d^*)^2 \phi \\
0 &= \phi \left[-k^\mu k_\mu + 2 g_v^d V^\mu k_\mu - (g_v^d)^2 V_\mu V^\mu + (M_d^*)^2 \right] \\
0 &= -\omega^2 + \vec{k} + 2 g_v^d V^\mu k_\mu - (g_v^d)^2 V_\mu V^\mu + (M_d^*)^2 .
\end{aligned}
$$

For zero temperature the deuteron particles will condensate in a state with $\vec{k} = 0$ and by the mean-field approximation $V^i = 0$. Thus, we obtain

$$
\begin{aligned}
-\omega^2 + 2 g_v^d V^0 \omega - (g_v^d)^2 V_0 V^0 + (M_d^*)^2 = 0 \\
(=)\, (\omega - g_v^d V^0)^2 = (M_d^*)^2 \rightarrow \boxed{\omega = M_d^* + g_v^d V^0}
\end{aligned}
$$

A.2 α field

Starting from $\mathcal{L}_\alpha$

$$
\begin{aligned}
\mathcal{L}_\alpha &= \frac{1}{2} \left(i \mathcal{D}^\mu \phi \right)^* \left(i \mathcal{D}_\mu \phi \right) - \frac{1}{2} (M_\alpha^*)^2 \phi^* \phi \\
i \mathcal{D}^\mu &= i \partial^\mu - g_v^\alpha V^\mu \\
M_\alpha^* &= M_\alpha - g_s^\alpha \sigma
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_\alpha &= \frac{1}{2} \left(-i \partial^\mu \phi^* - g_v^\alpha V^\mu \phi^* \right) \left(i \partial_\mu \phi - g_v^\alpha V_\mu \phi \right) - \frac{1}{2} (M_\alpha^*)^2 \phi^* \phi \\
&= \frac{1}{2} \left(-i \partial^\mu \phi^* i \partial_\mu \phi - i \left(\partial^\mu \phi^* \right) \left(-g_v^\alpha V_\mu \phi \right) - i g_v^\alpha V^\mu \phi^* \partial_\mu \phi + (g_v^\alpha)^2 V_\mu V^\mu \phi \phi^* \right) - \frac{1}{2} (M_\alpha^*)^2 \phi^* \phi \\
&= \frac{1}{2} \left(\partial^\mu \phi^* \partial_\mu \phi + i \left(\partial^\mu \phi^* \right) g_v^\alpha V_\mu \phi - i g_v^\alpha V^\mu \phi^* \partial_\mu \phi + (g_v^\alpha)^2 V_\mu V^\mu \phi \phi^* \right) - \frac{1}{2} (M_\alpha^*)^2 \phi^* \phi .
\end{aligned}
$$

Applying the Euler-Lagrange equation

$$
\partial_\mu \left(\frac{\partial \mathcal{L}_\alpha}{\partial (\partial_\mu \phi^*)} \right) = \frac{\partial \mathcal{L}_\alpha}{\partial \phi^*}
$$

$$
\begin{aligned}
(=)\, \partial_\mu \left[\partial^\mu \phi + i g_v^\alpha V^\mu \phi \right] &= -i g_v^\alpha V^\mu \partial_\mu \phi + (g_v^\alpha)^2 V_\mu V^\mu \phi - (M_\alpha^*)^2 \phi \\
(=)\, \partial^\mu \partial_\mu \phi + i g_v^\alpha V^\mu \partial_\mu \phi + i g_v^\alpha V^\mu \partial_\mu \phi - (g_v^\alpha)^2 V_\mu V^\mu \phi + (M_\alpha^*)^2 \phi &= 0
\end{aligned}
$$

$$
\begin{aligned}
i \mathcal{D}^\mu i \mathcal{D}_\mu \phi &= \left(i \partial^\mu - g_v^\alpha V^\mu \right) \left(i \partial^\mu - g_v^\alpha V^\mu \right) \phi \\
&= -\partial^\mu \partial_\mu \phi - i g_v^\alpha V^\mu \partial_\mu \phi - i g_v^\alpha V^\mu \partial_\mu \phi + (g_v^\alpha)^2 V_\mu V^\mu \phi \\
\rightarrow \mathcal{D}^\mu \mathcal{D}_\mu \phi &= \partial^\mu \partial_\mu \phi + i g_v^\alpha V^\mu \partial_\mu \phi + i g_v^\alpha V^\mu \partial_\mu \phi - (g_v^\alpha)^2 V_\mu V^\mu \phi
\end{aligned}
$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}_\alpha}{\partial(\partial_\mu \phi^*)} \right) = \frac{\partial \mathcal{L}_\alpha}{\partial \phi^*} (=) \boxed{\left[\mathcal{D}^\mu \mathcal{D}_\mu + (M_\alpha^*)^2\right] \phi = 0}.$$

As expected, the equation of motion for the α particle is the Klein-Gordon equation.

Taking $\phi = \phi(k)e^{-ik_\mu x^\mu}$ as in the deuteron field we obtain

$$\begin{aligned}
0 &= \partial^\mu \partial_\mu \phi + ig_v^\alpha V^\mu \partial_\mu \phi + ig_v^\alpha V^\mu \partial_\mu \phi - (g_v^\alpha)^2 V_\mu V^\mu \phi + (M_\alpha^*)^2 \phi \\
0 &= (-ik^\mu)(-ik_\mu)\phi + 2ig_v^\alpha V^\mu(-ik_\mu)\phi - (g_v^\alpha)^2 V_\mu V^\mu \phi + (M_\alpha^*)^2 \phi \\
0 &= \phi \left[-k^\mu k_\mu + 2g_v^\alpha V^\mu k_\mu - (g_v^\alpha)^2 V_\mu V^\mu + (M_\alpha^*)^2\right] \\
0 &= -\omega^2 + \vec{k} + 2g_v^\alpha V^\mu k_\mu - (g_v^\alpha)^2 V_\mu V^\mu + (M_\alpha^*)^2
\end{aligned}$$

for $\vec{k} = 0$ and since $V^i = 0$ we obtain

$$\begin{aligned}
&- \omega^2 + 2g_v^\alpha V^0 \omega - (g_v^\alpha)^2 V_0 V^0 + (M_\alpha^*)^2 = 0 \\
&(=) (\omega - g_v^\alpha V^0)^2 = (M_\alpha^*)^2 \to \boxed{\omega = M_\alpha^* + g_v^\alpha V^0}
\end{aligned}$$

A.3 σ field

Applying (A.2) for the σ field in (A.1) we obtain for each term

$$\begin{aligned}
\bullet \partial_\theta \left(\frac{\partial \mathcal{L}}{\partial(\partial_\theta \sigma)} \right) &= \frac{1}{2} \partial_\theta \left(\frac{\partial}{\partial_\theta \sigma} (\partial_\mu \sigma) \partial^\mu \sigma + \partial_\mu \sigma \frac{\partial}{\partial_\theta \sigma} (\partial^\mu \sigma) \right) \\
&= \frac{1}{2} \partial_\theta \left(\delta_\theta^\mu \partial^\mu \sigma + \partial_\mu \sigma g^{\mu\theta} \frac{\partial}{\partial_\theta \sigma} (\partial_\theta \sigma) \right) \\
&= \frac{1}{2} \partial_\theta \left(\partial^\theta \sigma + \partial_\theta \sigma \right) = \partial_\theta \partial^\theta \sigma.
\end{aligned}$$

$$\begin{aligned}
\bullet \left(\frac{\partial \mathcal{L}_i}{\partial \sigma} \right) &= \frac{\partial}{\partial \sigma} \left(-\frac{1}{2} \left(M_i^*\right)^2 \phi_i^* \phi_i \right) \\
&= -\frac{1}{2} \frac{\partial}{\partial \sigma} \left(\left(M_i - g_s^i\right)^2 \phi_i^* \phi_i \right) \\
&= -\frac{1}{2} \frac{\partial}{\partial \sigma} \left(M_i^2 + \left(g_s^i \sigma\right)^2 - 2M_i g_s^i \sigma \right) \phi_i^* \phi_i \\
&= -\frac{1}{2} \left(2 \left(g_s^i\right)^2 \sigma - 2M_i g_s^i \right) \phi_i^* \phi_i \\
&= \left(2M_i g_s^i - \left(g_s^i\right)^2 \sigma \right) \phi_i^* \phi_i \\
&= g_s^i M_i^* \phi_i^* \phi_i = g_s^i \rho_i
\end{aligned}$$

where in the last step we use the relation $\rho_i = M_i^* \phi_i \left(\phi^i\right)^*$ deduced in the Appendix 2.

$$\frac{\partial \mathcal{L}}{\partial_\theta \sigma} = -m_s^2 \sigma - \frac{k}{2} \sigma^2 - \frac{\lambda}{6} \sigma^3 + \sum_{i=p,n,t,h} g_s^i \bar{\psi}_i \psi_i + \sum_{i=\alpha,d} g_s^i \rho_i.$$

The equation of motion for the σ field is

$$\partial_\mu \partial^\mu \sigma + m_s^2 \sigma + \frac{k}{2}\sigma^2 + \frac{\lambda}{6}\sigma^3 = \sum_{i=p,n,t,h} g_s^i \bar{\psi}_i \psi_i + \sum_{i=\alpha,d} g_s^i \rho_i \tag{A.4}$$

A.4 V^μ field

Applying (A.2) for the ω^μ field to (A.1) we obtain for each term

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial\left(\partial_\alpha V_\beta\right)} &= -\frac{1}{4} g^{\mu\mu'} g^{\nu\nu'} \left(\frac{\partial \Omega_{\mu\nu}}{\partial\left(\partial_\alpha V_\beta\right)} \Omega_{\mu'\nu'} + \Omega_{\mu\nu} \frac{\partial \Omega_{\mu'\nu'}}{\partial\left(\partial_\alpha V_\beta\right)} \right) \\
&= -\frac{1}{2} g^{\mu\mu'} g^{\nu\nu'} \frac{\partial \Omega_{\mu\nu}}{\partial\left(\partial_\alpha V_\beta\right)} \Omega_{\mu'\nu'} \\
&= -\frac{1}{2} g^{\mu\mu'} g^{\nu\nu'} \frac{\partial}{\partial\left(\partial_\alpha V_\beta\right)} \left(\partial_\mu V_\nu - \partial_\nu V_\mu\right) \Omega_{\mu'\nu'} \\
&= -\frac{1}{2} g^{\mu\mu'} g^{\nu\nu'} \left(\delta_\mu^\alpha \delta_\nu^\beta - \delta_\nu^\alpha \delta_\mu^\beta\right) \Omega_{\mu'\nu'} \\
&= -\frac{1}{2} \left(g^{\alpha\mu'} g^{\beta\nu'} \Omega_{\mu'\nu'} - g^{\beta\mu'} g^{\alpha\nu'} \Omega_{\mu'\nu'} \right) \\
&= -\frac{1}{2} \left(\Omega^{\alpha\beta} - \Omega_{\beta\alpha}\right) = -\Omega^{\alpha\beta} \\
\rightarrow \partial_\alpha \left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\alpha V_\beta\right)} \right) &= -\partial_\alpha \Omega^{\alpha\beta}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial V_\beta} = \frac{\partial}{\partial V_\beta} &\left(\frac{1}{2} m_v^2 V_\mu V^\mu + \frac{1}{24} \xi g_v^4 \left(V_\mu V^\mu\right)^2 + \Lambda g_v^2 g_\rho^2 V_\mu V^\mu \mathbf{b}_\mu . \mathbf{b}^\mu - \sum_{i=p,n,t,h} g_v^i \bar{\psi}_i \gamma_\mu V^\mu \psi_i \right) \\
&+ \frac{\partial \mathcal{L}_d}{\partial V_\beta} + \frac{\partial \mathcal{L}_\alpha}{\partial V_\beta}.
\end{aligned}$$

For the deuteron and α particles

$$\begin{aligned}
\frac{\partial \mathcal{L}_i}{\partial V_\mu} &= \frac{1}{2} \left(i(\partial^\mu \phi_i^*) g_v^i \phi_i - i g_v^i \phi_i^* \partial^\mu \phi_i + 2(g_v^i)^2 V^\mu \phi_i \phi_i^* \right) \\
&= \frac{1}{2} \left(i(ik^\mu) g_v^i - i g_v^i (-ik^\mu) + 2(g_v^i)^2 V^\mu \right) \phi_i \phi_i^* \\
&= \frac{1}{2} \left(-k^\mu g_v^i - g_v^i k^\mu + 2(g_v^i)^2 V^\mu \right) \phi_i \phi_i^* \\
\frac{\partial \mathcal{L}_i}{\partial V_0} &= \frac{1}{2} \left(-2k^0 g_v^i + 2(g_v^i)^2 V^0 \right) \phi_i \phi_i^* = g_v^i \left(-\omega + g_v^i V^0 \right) \phi_i \phi_i^* \\
&= -g_v^i \left(\omega - g_v^i V^0 \right) \phi_i \phi_i^* = -g_v^i \rho_i.
\end{aligned}$$

where $i = \alpha, d$. The other terms are

- $$\frac{\partial}{\partial V_\beta}\left(\frac{1}{2}m_v^2 V_\mu V^\mu\right) = \frac{1}{2}m_v^2 g^{\mu\mu'}\left(\frac{\partial V_\mu}{\partial V_\beta}V'_\mu + V_\mu \frac{\partial V_{\mu'}}{\partial V_\beta}\right) = \frac{1}{2}m_v^2 g^{\mu\mu'}\left(\delta^\mu_\beta V_{\mu'} + V_\mu \delta^{\mu'}_\beta\right)$$
$$= \frac{1}{2}m_v^2\left(V^\beta + V^\beta\right) = m_v^2 V^\beta$$

- $$\frac{\partial}{\partial V_\beta}\left(\frac{1}{24}\xi g_v^4 \left(V_\mu V^\mu\right)^2\right) = \frac{1}{24}\xi g_v^4\left(\frac{\partial}{\partial V_\beta}\left(V_\mu V^\mu\right)V_\nu V^\nu + \frac{\partial}{\partial V_\beta}\left(V_\nu V^\nu\right)V_\mu V^\mu\right)$$
$$= \frac{1}{12}\xi g_v^4\left(\frac{\partial}{\partial V_\beta}\left(V_\mu V^\mu\right)V_\nu V^\nu\right) = \frac{1}{12}\xi g_v^4 V_\nu V^\nu\left(\frac{\partial}{\partial V_\beta}\left(V_\mu V^\mu\right)\right)$$
$$= \frac{1}{6}\xi g_v^4 V_\nu V^\nu V^\beta$$

- $$\frac{\partial}{\partial V_\beta}\left(-\sum_{i=p,n,t,h} g_v^i \bar{\psi}_i \gamma_\mu V^\mu \psi_i\right) = -\sum_{i=p,n,t,h} g_v^i \bar{\psi}_i \gamma^\beta \psi_i$$

- $$\frac{\partial}{\partial V_\beta}\left(\Lambda g_v^2 g_\rho^2 V_\mu V^\mu \mathbf{b}_\mu . \mathbf{b}^\mu\right) = 2\Lambda g_v^2 g_\rho^2 V^\beta \mathbf{b}_\mu . \mathbf{b}^\mu$$

The equation of motion for V^μ is

$$\partial_\alpha \Omega^{\alpha\beta} + m_v^2 V^\beta + \frac{1}{6}\xi g_v^4 V_\nu V^\nu V^\beta + 2\Lambda g_v^2 g_\rho^2 V^\beta \mathbf{b}_\mu . \mathbf{b}^\mu = \sum_{i=p,n,t,h} g_v^i \bar{\psi}_i \gamma^\beta \psi_i + \sum_{i=\alpha,d} g_v^i \rho_i$$

A.5 $\mathbf{b}^\mu$ field

Applying (A.2) for the $\mathbf{b}^\mu$ field using (A.1) we obtain for each term

- $$\frac{\partial \mathcal{L}}{\partial b^\mu_\lambda} = \frac{\partial}{\partial b^\mu_\lambda}\left(-\sum_{i=p,n,t,h}\frac{g_\rho^i}{2}\bar{\psi}_i \gamma_\mu \tau_\lambda b^\mu_\lambda \psi_i + \frac{1}{2}m_\rho^2 b_{\lambda\mu} b^\mu_\lambda + \Lambda_v g_v^2 g_\rho^2 V_\mu V^\mu b_{\lambda\mu} b^\mu_\lambda\right)$$
$$= -\sum_{i=p,n,t,h}\frac{g_\rho^i}{2}\bar{\psi}_i \gamma_\mu \tau_\lambda \psi_i + \Lambda_v g_v^2 g_\rho^2 V_\mu V^\mu b_{\lambda\mu} + \frac{\partial}{\partial b^\mu_\lambda}\left(\frac{1}{2}m_\rho^2 b_{\lambda\mu} b^\mu_\lambda\right)$$
$$= -\sum_{i=p,n,t,h}\frac{g_\rho^i}{2}\bar{\psi}_i \gamma_\mu \tau_\lambda \psi_i + \Lambda_v g_v^2 g_\rho^2 V_\mu V^\mu b_{\lambda\mu} + m_\rho^2 b_{\lambda\mu}$$

- $$\partial^\nu\left(\frac{\partial \mathcal{L}}{\partial\left(\partial^\nu b^\mu_\lambda\right)}\right) = \partial^\nu\left(B_{\nu\mu,\lambda}\right) = -\partial^\nu B_{\nu\mu,\lambda}.$$

The equation of motion for $\mathbf{b}^\mu$ is

$$\partial^\nu \mathbf{B}_{\nu\mu} + m_\rho^2 \mathbf{b}_\mu + \Lambda_v g_v^2 g_\rho^2 V_\mu V^\mu \mathbf{b}_\mu = \sum_{i=p,n,t,h} \frac{g_\rho^i}{2} \bar{\psi}_i \gamma_\mu \boldsymbol{\tau} \psi_i \tag{A.5}$$

A.6 ψ_n,ψ_p, ψ_h and ψ_t fields

Applying (A.2) for the ψ_i $(i = p, n.h, t)$ fields using (A.1) is straighfoward

$$\left[\gamma_\mu \left(i\partial^\mu - g_v^i V^\mu - \frac{g_\rho^i}{2}\boldsymbol{\tau}.\boldsymbol{b}^\mu \right) - \left(M_i - g_s^i \sigma \right) \right] \psi = 0 \tag{A.6}$$

for $i = n, p, t, h$.

Appendix B

Energy density for the α and deuteron

The Langrangian density of the deuteron and α are

$$\begin{aligned}
\mathcal{L}_d =& \frac{1}{2}\Bigg[\partial^\mu\left(\phi_d^\nu\right)^*\partial_\mu\phi_{d\nu} + i\partial^\mu\left(\phi_d^\nu\right)^* g_v^d V_\mu\phi_{d\nu} - g_v^d V^\mu\left(\phi_d^\nu\right)^* i\partial_\mu\phi_{d\nu} + \left(g_v^d\right)^2 V^\mu\left(\phi_d^\nu\right)^* V_\mu\phi_{d\nu} \\
&- \left(M_d^*\right)^2\left(\phi_d^\mu\right)^*\phi_{d\mu}\Bigg] \\
\mathcal{L}_\alpha =& \frac{1}{2}\left(\partial^\mu\phi^*\partial_\mu\phi + i\left(\partial^\mu\phi^*\right)g_v^\alpha V_\mu\phi - ig_v^\alpha V^\mu\phi^*\partial_\mu\phi + (g_v^\alpha)^2 V_\mu V^\mu\phi\phi^* - (M_\alpha^*)^2\phi^*\phi\right).
\end{aligned}$$

The deuteron Lagrangian density for each component of the deuteron field in the Lorentz gauge is equal to the α Lagrangian density.

In the mean-field approximation, ϕ_d^i is zero. Therefore, the energy density of both is the same.

The energy-momentum tensor is given by

$$\begin{aligned}
T^{\mu\nu} &= \sum_i \frac{\partial\mathcal{L}_\rangle}{\partial(\partial_\mu\phi_i)}\partial^\nu\phi_i - \eta^{\mu\nu}\mathcal{L}_\rangle = \frac{\partial\mathcal{L}_\rangle}{\partial(\partial_\mu\phi_i^*)}\partial^\nu(\phi_i)^* + \frac{\partial\mathcal{L}_\rangle}{\partial(\partial_\mu\phi)}\partial^\nu\phi_i - \eta^{\mu\nu}\mathcal{L}_i \\
&= \frac{1}{2}\left(\partial^\mu\phi_i\partial^\nu\phi_i^* + ig_v^i V^\mu\phi\partial^\nu\phi_i^* + \partial^\mu\phi_i^*\partial^\nu\phi_i - ig_v^i V^\mu\phi_i^*\partial^\nu\phi_i\right) \\
&- \frac{\eta^{\mu\nu}}{2}\left(\partial^\theta\phi_i^*\partial_\theta\phi_i + i\left(\partial^\theta\phi_i^*\right)g_v^i V_\theta\phi_i - ig_v^i V^\theta\phi_i^*\partial_\theta\phi_i + (g_v^i)^2 V_\theta V^\theta\phi_i\phi_i^* - (M_i^*)^2\phi_i^*\phi_i\right)
\end{aligned}$$

where $i = \alpha, d$. Denoting $\phi_\alpha = \phi_d = \phi$ we get

$$
\begin{aligned}
\mathcal{E} = T^{00} &= \frac{1}{2}\left(\partial^0\phi\partial^0\phi^* + ig_v^\alpha V^0\phi\partial^0\phi^* + \partial^0\phi^*\partial^0\phi - ig_v^\alpha V^0\phi^*\partial^0\phi\right) \\
&- \frac{\eta^{00}}{2}\Big(\partial^0\phi^*\partial_0\phi + \partial^j\phi^*\partial_j\phi + i\left(\partial^0\phi^*\right)g_v^iV_0\phi + i\left(\partial^j\phi^*\right)g_v^iV_j\phi - ig_v^iV^0\phi^*\partial_0\phi \\
&- ig_v^iV^j\phi^*\partial_j\phi + (g_v^i)^2V_0V^0\phi\phi^* + (g_v^i)^2V_jV^j\phi\phi^* - (M_i^*)^2\phi^*\phi\Big) \\
&= \frac{1}{2}\left(\partial^0\phi\partial^0\phi^* + ig_v^\alpha V^0\phi\partial^0\phi^* + \partial^0\phi^*\partial^0\phi - ig_v^iV^0\phi^*\partial^0\phi\right) \\
&- \frac{1}{2}\Big(\partial^0\phi^*\partial_0\phi + \partial^j\phi^*\partial_j\phi + i\left(\partial^0\phi^*\right)g_v^iV_0\phi + i\left(\partial^j\phi^*\right)g_v^iV_j\phi - ig_v^iV^0\phi^*\partial_0\phi \\
&- ig_v^iV^j\phi^*\partial_j\phi + (g_v^i)^2V_0V^0\phi\phi^* + (g_v^i)^2V_jV^j\phi\phi^* - (M_\alpha^*)^2\phi^*\phi\Big) \\
&= \frac{1}{2}\Big(\partial^0\phi^*\partial^0\phi - \partial^j\phi^*\partial_j\phi - i\left(\partial^j\phi^*\right)g_v^iV_j\phi + ig_v^iV^j\phi^*\partial_j\phi - (g_v^i)^2V^0V^0\phi\phi^* \\
&- (g_v^\alpha)^2V_jV^j\phi\phi^* + (M_\alpha^*)^2\phi^*\phi\Big) \\
&= \frac{1}{2}\left(ik^0(-ik^0) - (ik^j)(-ik_j) - (g_v^i)^2V^0V^0 + (M_i^*)^2\right)\phi\phi^* \\
&= \frac{1}{2}\left(\omega^2 - \mathbf{k}^2 - (g_v^iV^0)^2 + (M_i^*)^2\right)\phi\phi^* = \frac{1}{2}\left(\omega^2 - (g_v^iV^0)^2 + \left(\omega - g_v^iV^0\right)^2\right)\phi\phi^*.
\end{aligned}
$$

For a state with $\mathbf{k} = 0$, we have

$$\mathcal{E}_i = \frac{1}{2}\left(2\omega^2 - 2g_v^i\omega V^0\right)\phi\phi^* = \left(\omega^2 - g_v^i\omega V^0\right)\phi\phi^* = \omega\left(\omega - g_v^iV^0\right)\phi\phi^*$$

To go further we need to calculate the current density that is given by

$$J^\mu = i\left((\phi_i)^*\frac{\partial\mathcal{L}_i}{\partial(\partial_\mu(\phi_i)^*)} - \frac{\partial\mathcal{L}_i}{\partial(\partial_\mu\phi_i)}\phi_i\right)$$

$$J^\mu = \frac{i}{2}\left[\phi^*\partial^\mu\phi - \partial^\mu(\phi)^*\phi + i(\phi)^*g_v^iV^\mu\phi + ig_v^iV^\mu(\phi)^*\phi\right]$$

$$
\begin{aligned}
J^0 = \rho_d &= \frac{i}{2}\left[\phi^*\partial^0\phi - \partial^0\phi^*\phi + i\phi^*g_v^iV^0\phi + ig_v^iV^0\phi^*\phi\right] \\
&= \frac{i}{2}\left(-ik_\mu - ik_\mu + ig_v^iV_\mu + ig_v^iV_\mu\right)\phi\phi^* \\
&= \left(k_\mu - g_v^dV_\mu\right)\phi\phi^*
\end{aligned}
$$

For a state of $\mathbf{k} = 0$, we obtain

$$\rho_d = \left(k_0 - g_v^iV_0\right)\phi_0\left(\phi^0\right)^* = \left(\omega - g_v^iV_0\right)\phi\phi^* = M_i^*\phi\phi^*.$$

Therefore,

$$\boxed{\rho_i = M_i^*\phi_i\phi_i^*}$$

Consequently, we have

$$\mathcal{E}_i = \omega\rho_i = \left(M_i^* + g_v^iV^0\right)\rho_i = \left(M_i - g_s^i\sigma + g_v^iV^0\right)\rho_i.$$

Appendix C

Asymmetry energy

Defining the symmetry energy part of expression (3.11) as

$$\begin{aligned}\mathcal{E}_{sym} &= \sum_{i=p,n} g_\rho b_3^{(0)} I_3^i \rho_i - \frac{1}{2} m_\rho^2 (b_3^{(0)})^2 - \Lambda_v g_v^2 g_\rho^2 (V^0)^2 (b_3^{(0)})^2 + \frac{1}{\pi^2} \sum_{i=p,n} \int_0^{k_f^i} k_i^2 dk_i \sqrt{k_i^2 + (M^*)^2} \\ &= \sum_{i=p,n} g_\rho b_3^{(0)} I_3^i \rho_i - \frac{1}{2} \left(m_\rho^2 b_3^{(0)} + 2\Lambda_v g_v^2 g_\rho^2 (v^0)^2 b_3^{(0)} \right) b_3^{(0)} + \frac{1}{\pi^2} \sum_{i=p,n} \int_0^{k_f^i} k_i^2 dk_i \sqrt{k_i^2 + (M^*)^2}.\end{aligned}$$

Using the rho meson equation of motion (3.4) we get

$$\begin{aligned}\mathcal{E}_{sym} &= \sum_{i=p,n} g_\rho b_3^{(0)} I_3^i \rho_i - \frac{1}{2} \sum_{i=p,n} g_\rho I_3^i \rho_i b_3^{(0)} + \frac{1}{\pi^2} \sum_{i=p,n} \int_0^{k_f^i} k_i^2 dk_i \sqrt{k_i^2 + (M^*)^2} \\ &= \frac{1}{2} \sum_{i=p,n} g_\rho I_3^i \rho_i b_3^{(0)} + \frac{1}{\pi^2} \sum_{i=p,n} \int_0^{k_f^i} k_i^2 dk_i \sqrt{k_i^2 + (M^*)^2}\end{aligned}$$

Rewriting the rho meson equations of motion (3.4) as

$$b_3^{(0)} = \frac{1}{\left(m_\rho^*\right)^2} \sum_{i=p,n} g_\rho I_3^i \rho_i$$

where $\left(m_\rho^*\right)^2 = m_\rho^2 + 2\Lambda_v g_v^2 g_\rho^2 (V^0)^2$, we obtain

$$\begin{aligned}\mathcal{E}_{sym} &= \frac{1}{2} \frac{1}{\left(m_\rho^*\right)^2} \left(\sum_{i=p,n} g_\rho I_3^i \rho_i \right)^2 + \frac{1}{\pi^2} \sum_{i=p,n} \int_0^{k_f^i} k_i^2 dk_i \sqrt{k_i^2 + (M^*)^2} \\ &= \frac{1}{8} \frac{g_\rho^2}{\left(m_\rho^*\right)^2} \left(\rho_p - \rho_n\right)^2 + \frac{1}{\pi^2} \sum_{i=p,n} \int_0^{k_f^i} k_i^2 dk_i \sqrt{k_i^2 + (M^*)^2} \\ &= \frac{1}{8} \frac{g_\rho^2 \rho^2}{\left(m_\rho^*\right)^2} t^2 + \frac{1}{\pi^2} \sum_{i=p,n} \int_0^{k_f^i} k_i^2 dk_i \sqrt{k_i^2 + (M^*)^2}\end{aligned}$$

where $t \equiv (\rho_n - \rho_p)/\rho$. We know that the symmetry energy coefficient is given by

$$a_{sym} = \frac{1}{2} \left(\frac{\partial^2 (\mathcal{E}/\rho)}{\partial t^2} \right)_{t=0}.$$

To calculate the a_{sym} first we need to define the Fermi momentum of neutron and proton as a function of $t \equiv (\rho_n - \rho_p)/\rho$ and the Fermi momentum at saturation density [5]

$$k_f \equiv \left(\frac{3\pi^2\rho_0}{2}\right)^{1/3}.$$

Is it straightforward to obtain the following relations

$$\begin{aligned} k_n &= k_F(1+t)^{1/3} \\ k_p &= k_F(1-t)^{1/3}. \end{aligned}$$

Therefore the a_{sym} is given by

$$a_{sym} = \frac{1}{2}\left(\frac{\partial^2}{\partial t^2}\left[\frac{1}{8}\frac{g_\rho^2\rho}{\left(m_\rho^*\right)^2}t^2 + \frac{1}{\rho\pi^2}\sum_{i=p,n}\int_0^{k_f^i} k_i^2 dk_i\sqrt{k_i^2 + (M^*)^2}\right]\right)_{t=0}$$

The first term gives

$$\begin{aligned} a_{sym} &= \frac{1}{2}\left(\frac{\partial^2}{\partial t^2}\left[\frac{1}{8}\frac{g_\rho^2\rho}{\left(m_\rho^*\right)^2}t^2\right]\right)_{t=0} \\ &= \frac{1}{8}\frac{g_\rho^2\rho_0}{\left(m_\rho^*\right)^2} \\ &= \frac{1}{12\pi^2}\frac{g_\rho^2 k_F^3}{\left(m_\rho^*\right)^2}. \end{aligned}$$

The second term gives

$$\begin{aligned} a_{sym} &= \frac{1}{2}\left(\frac{1}{\rho\pi^2}\sum_{i=p,n}\int_0^{k_f^i} k_i^2 dk_i\sqrt{k_i^2 + (M^*)^2}\right) \\ &= \frac{1}{2}\left(\frac{\partial^2}{\partial t^2}\left[\frac{1}{\rho\pi^2}\int_0^{k_p} k^2 dk\sqrt{k^2 + (M^*)^2} + \frac{1}{\rho\pi^2}\int_0^{k_n} k^2 dk\sqrt{k^2 + (M^*)^2}\right]\right) \end{aligned}$$

For the proton's part is

- $$\left(\frac{\partial}{\partial t}\left[\frac{1}{\rho\pi^2}\int_0^{k_p} k^2dk\sqrt{k^2+(M^*)^2}\right]\right) = \frac{1}{\rho\pi^2}\frac{\partial k_p}{\partial t}\frac{\partial}{\partial k_p}\int_0^{k_p} k^2dk\sqrt{k^2+(M^*)^2}$$
$$= \frac{1}{\rho\pi^2}\left(-\frac{1}{3}k_f(1-t)^{-2/3}\right)k_p^2\sqrt{k_p^2+(M^*)^2}$$
$$= -\frac{1}{\rho\pi^2}\frac{1}{3}k_f^3\sqrt{k_F^2(1-t)^{2/3}+(M^*)^2}$$

- $$\left(\frac{\partial^2}{\partial t^2}\left[\frac{1}{\rho\pi^2}\int_0^{k_p} k^2dk\sqrt{k^2+(M^*)^2}\right]\right) = -\frac{\partial}{\partial t}\left(\frac{1}{\rho\pi^2}\frac{1}{3}k_f^3\sqrt{k_F^2(1-t)^{2/3}+(M^*)^2}\right)$$
$$= \frac{1}{3\rho\pi^2}k_F^3\frac{1}{2}\left(k_F^2(1-t)^{2/3}+(M^*)^2\right)^{-1/2}\frac{2}{3}k_F^2(1-t)^{-1/3}$$
$$= \frac{k_F^5}{9\rho\pi^2}\frac{(1-t)^{-1/2}}{\sqrt{k_F^2(1-t)^{2/3}+(M^*)^2}}$$
$$= \frac{3\pi^2}{2k_F^3}\frac{k_F^5}{9\pi^2}\frac{(1-t)^{-1/2}}{\sqrt{k_F^2(1-t)^{2/3}+(M^*)^2}}$$
$$= \frac{k_F^2}{6}\frac{(1-t)^{-1/2}}{\sqrt{k_F^2(1-t)^{2/3}+(M^*)^2}}.$$

For the neutron's part is

- $$\left(\frac{\partial}{\partial t}\left[\frac{1}{\rho\pi^2}\int_0^{k_n} k^2dk\sqrt{k^2+(M^*)^2}\right]\right) = \frac{1}{\rho\pi^2}\frac{\partial k_n}{\partial t}\frac{\partial}{\partial k_n}\int_0^{k_n} k^2dk\sqrt{k^2+(M^*)^2}$$
$$= \frac{1}{\rho\pi^2}\left(\frac{1}{3}k_f(1+t)^{-2/3}\right)k_n^2\sqrt{k_p^2+(M^*)^2}$$
$$= \frac{1}{3\rho\pi^2}k_f^3\sqrt{k_F^2(1+t)^{2/3}+(M^*)^2}$$

- $$\left(\frac{\partial^2}{\partial t^2}\left[\frac{1}{\rho\pi^2}\int_0^{k_n} k^2dk\sqrt{k^2+(M^*)^2}\right]\right) = \frac{\partial}{\partial t}\left(\frac{1}{\rho\pi^2}\frac{1}{3}k_F^3\sqrt{k_F^2(1+t)^{2/3}+(M^*)^2}\right)$$
$$= \frac{1}{3\rho\pi^2}k_F^3\frac{1}{2}\left(k_F^2(1+t)^{2/3}+(M^*)^2\right)^{-1/2}\frac{2}{3}k_F^2(1+t)^{-1/3}$$
$$= \frac{k_F^5}{9\rho\pi^2}\frac{(1+t)^{-1/2}}{\sqrt{k_F^2(1+t)^{2/3}+(M^*)^2}}$$
$$= \frac{3\pi^2}{2k_F^3}\frac{k_F^5}{9\pi^2}\frac{(1+t)^{-1/2}}{\sqrt{k_F^2(1+t)^{2/3}+(M^*)^2}}$$
$$= \frac{k_F^2}{6}\frac{(1+t)^{-1/2}}{\sqrt{k_F^2(1+t)^{2/3}+(M^*)^2}}.$$

The final result is

$$
\begin{aligned}
a_{sym} &= \frac{1}{2}\left(\frac{\partial^2}{\partial t^2}\left[\frac{1}{8}\frac{g_\rho^2\rho}{\left(m_\rho^*\right)^2}t^2 + \frac{1}{\rho\pi^2}\sum_{i=p,n}\int_0^{k_f^i} k_i^2 dk_i\sqrt{k_i^2+\left(M^*\right)^2}\right]\right)_{t=0} \\
&= \frac{1}{12\pi^2}\frac{g_\rho^2 k_F^3}{\left(m_\rho^*\right)^2} + \frac{1}{2}\left\{\frac{k_F^2}{6}\frac{(1-t)^{-1/2}}{\sqrt{k_F^2(1-t)^{2/3}+\left(M^*\right)^2}} + \frac{k_F^2}{6}\frac{(1+t)^{-1/2}}{\sqrt{k_F^2(1+t)^{2/3}+\left(M^*\right)^2}}\right\}_{t=0} \\
&= \frac{1}{12\pi^2}\frac{g_\rho^2 k_F^3}{\left(m_\rho^*\right)^2} + \frac{1}{12}\left\{\frac{k_F^2}{\sqrt{k_F^2+\left(M^*\right)^2}} + \frac{k_F^2}{\sqrt{k_F^2+\left(M^*\right)^2}}\right\} \\
&= \frac{1}{12\pi^2}\frac{g_\rho^2 k_F^3}{\left(m_\rho^*\right)^2} + \frac{1}{6}\left\{\frac{k_F^2}{\sqrt{k_F^2+\left(M^*\right)^2}}\right\}.
\end{aligned}
$$

Therefore the symmetry energy is givin by

$$
a_{sym} = \frac{1}{6}\left(\frac{k_F^2}{\sqrt{k_F^2+\left(M^*\right)^2}}\right) + \frac{g_\rho^2}{12\pi^2}\frac{k_F^3}{\left(m_\rho^2 + 2\Lambda_v g_v^2 g_\rho^2 (V^0)^2\right)}
$$

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