THE SEMICONTINUOUS QUASI-UNIFORMITY OF
A FRAME, REVISITED

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Abstract: In this note we present a new treatment of the pointfree version of the
semicontinuous quasi-uniformity based on the new tool of the ring of arbitrary (not
necessarily continuous) real-valued functions made available recently by J. Gutiérrez
García, T. Kubiak and J. Picado [Localic real functions: a general setting, Journal
of Pure and Applied Algebra 213 (2009) 1064-1074]. The purpose is to show how
the basic facts about the semicontinuous quasi-uniformity can be easily presented
and proved with that tool at hand.

Keywords: Frame, quasi-uniform frame, quasi-uniform biframe, quasi-metric quasi-
uniformity, totally bounded quasi-uniformity, semicontinuous real function, biframe
of reals, countably compact frame.

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1. Introduction

Let $X$ be a locale with corresponding frame $L = \mathcal{O}(X)$. The lattice of
sublocales of $X$ (that is, the subobject lattice of $X$ in the category of locales)
may be described in several equivalent ways. Here we use the following one
[18]:

a subset $S$ of $L$ is a sublocale of $X$ if, whenever $A \subseteq S$, $a \in L$
and $b \in S$, then $\bigwedge A \in S$ and $a \rightarrow b \in S$.

Any intersection of sublocales is again a sublocale, so that the set of all
sublocales is a complete lattice under inclusion. In fact, it is a co-frame.
We make it into a frame $S(L)$ by considering the dual ordering $S_1 \leq S_2$
iff $S_2 \subseteq S_1$. Among the important examples of sublocales are the closed sublocales

$$c(a) = \uparrow a = \{b \in L : a \leq b\}$$
and the open sublocales

\[ \mathfrak{o}(a) = \{ a \rightarrow b : b \in L \} \]

for every \( a \in L \) (which are complements of each other). The map \( a \mapsto c(a) \) is a frame embedding \( L \hookrightarrow S(L) \). The subframe of \( S(L) \) consisting of all closed sublocales will be denoted by \( cL \). It is isomorphic to \( L \). Denoting by \( oL \) the subframe of \( S(L) \) generated by all \( o(a) \), \( a \in L \), the triple \( (S(L), cL, oL) \) constitutes a biframe.

It is well-known that a quasi-uniformity \( \mathcal{E} \) on a set \( X \) may be described in several equivalent ways, most notably as a collection of ordered pairs of covers of \( X \) (the paircover approach) and as a collection of relations on \( X \) (the entourage approach). Associated with any quasi-uniformity \( \mathcal{E} \) on \( X \) there is the bitopological space \( (X, \mathcal{I}_\mathcal{E}, \mathcal{I}_{\mathcal{E}^{-1}}) \) induced by \( \mathcal{E} \).

In the pointfree setting, the theory of quasi-uniformities was first exploited using the paircover approach \([8, 9]\); the Weil entourages of \([15, 16, 17]\) provided then the direct analogue of entourages. The former is defined as a structure \( \mathcal{U} \) on a biframe \( (L_0, L_1, L_2) \) and the latter directly as a structure \( \mathcal{E} \) on a frame \( L \) which establishes two subframes \( L_1(\mathcal{E}) \) and \( L_2(\mathcal{E}) \) of \( L \) such that the triple \( (L, L_1(\mathcal{E}), L_2(\mathcal{E})) \) is a biframe (this is the pointfree version of the bitopological space \( (X, \mathcal{I}_\mathcal{E}, \mathcal{I}_{\mathcal{E}^{-1}}) \) above). The two approaches are equivalent \([15, 16]\).

While the approach via paircovers is most convenient for calculations (the entourage approach asks for a good knowledge of the construction of binary coproducts of frames), the entourage approach allows to formulate the theory directly on frames, in a way very similar to the spatial setting \([4, 5, 17]\). For instance, given a frame \( L \), there exists a (entourage) transitive quasi-uniformity \( \mathcal{E} \) on the sublocale frame \( S(L) \) which is compatible with \( L \), that is, \( L_1(\mathcal{E}) = cL \) (which means that \( L_1(\mathcal{E}) \) is an isomorphic copy of the given frame \( L \) inside \( S(L) \)) \([4, 5]\). This is the pointfree analogue of the well-known classical fact that for every topological space \( (X, \Phi) \) there exists a transitive quasi-uniformity \( \mathcal{E} \) on \( X \), compatible with \( (X, \Phi) \), that is, which induces as its first topology \( \Phi_{\mathcal{E}} \) the given topology \( \Phi \).

The semicontinuous quasi-uniformity \( USC(L) \) of \( L \) is a nice example of a transitive compatible quasi-uniformity \([5, 6]\). The purpose of this paper is to show how the basic facts about \( USC(L) \) can be nicely presented with the help of the ring of arbitrary (not necessarily continuous) real-valued functions made available recently by J. Gutiérrez García, T. Kubiak and J. Picado
To keep the background at the minimum possible we use the paircover approach [8, 10] to quasi-uniformities.

2. Background

For general information on locales and frames we refer to [13] and [18]. A biframe [2] is a triple $L = (L_0, L_1, L_2)$ in which $L_0$ is a frame, $L_1$ and $L_2$ are subframes of $L_0$ and $L_1 \cup L_2$ generates $L_0$ (by joins of finite meets). A biframe map $h : L \to M$ is a frame homomorphism from $L_0$ to $M_0$ such that the image of $L_i$ under $h$ is contained in $M_i$ for $i = 1, 2$. Biframes and biframe maps are the objects and arrows of the category $\text{BiFrm}$. For more details on biframes consult [2].

Let $L = (L_0, L_1, L_2)$ be a biframe. A subset $C$ of $L_1 \times L_2$ is a paircover [8, 10] of $L$ if $\bigvee \{c_1 \land c_2 \mid (c_1, c_2) \in C\} = 1$. A paircover $C$ of $L$ is strong if, for any $(c_1, c_2) \in C$, $c_1 \lor c_2 = 0$ whenever $c_1 \land c_2 = 0$ (that is, $(c_1, c_2) = (0, 0)$ whenever $c_1 \land c_2 = 0$).

For any paircovers $C$ and $D$ of $L$ we write $C \leq D$ (and say that $C$ refines $D$) if for any $(c_1, c_2) \in C$ there is $(d_1, d_2) \in D$ with $c_1 \leq d_1$ and $c_2 \leq d_2$. Further $C \land D = \{(c_1 \land d_1, c_2 \land d_2) \mid (c_1, c_2) \in C, (d_1, d_2) \in D\}$. It is obvious that $C \land D$ is a paircover of $L$. For $a \in L_0$ and $C, D$ paircovers of $L$, let

$$st_1(a, C) = \bigvee \{c_1 \mid (c_1, c_2) \in C \text{ and } c_2 \land a \neq 0\},$$

$$st_2(a, C) = \bigvee \{c_2 \mid (c_1, c_2) \in C \text{ and } c_1 \land a \neq 0\}$$

and

$$C \cdot D = \{(st_1(d_1, C), st_2(d_2, C)) \mid (d_1, d_2) \in D\}.$$ 

The particular case $C \cdot C$ is usually denoted by $C^*$. The paircover $C$ is said to star-refines $D$ if $C^* \leq D$.

The following lemma is easy to prove [8].

**Lemma 2.1.** For any paircovers $C, D$ of $(L_0, L_1, L_2)$ and any $a, b \in L_0$ we have:

1. $a \leq st_i(a, C)$ $(i = 1, 2)$.
2. $a \leq b \Rightarrow st_i(a, C) \leq st_i(b, C)$ $(i = 1, 2)$.
3. If $D^* \leq C$ then $st_i(st_i(a, D), D) \leq st_i(a, C)$ $(i = 1, 2)$.
4. For any biframe map $h : (L_0, L_1, L_2) \to (M_0, M_1, M_2)$, $st_i(h(a), h[C]) \leq h(st_i(a, C))$ $(i = 1, 2)$, where $h[C] = \{(h(c_1), h(c_2)) \mid (c_1, c_2) \in C\}$. 
A non-empty family $U$ of paircovers of $L = (L_0, L_1, L_2)$ is a quasi-uniformity on $L$ if:

(U1) The family of strong members of $U$ is a filter-base for $U$ with respect to $\land$ and $\leq$.

(U2) For any $C \in U$ there is $D \in U$ such that $D^* \leq C$.

(U3) For each $a \in L_i$, $a = \bigvee \{b \in L_i \mid \text{st}_i(b, C) \leq a \text{ for some } C \in U\}$, $(i = 1, 2)$.

The pair $(L, U)$ is called a quasi-uniform biframe [10]. $B \subseteq U$ is a base for $U$ if, for each $C \in U$, there is $B \in B$ such that $B \leq C$.

Let $(L, U)$ and $(M, V)$ be quasi-uniform biframes. A biframe map $h : L \to M$ is uniform if for every $C \in U$, $h[C] \in V$. Quasi-uniform biframes and uniform maps constitute a category that we denote by $\text{QUBiFrm}$.

The biframe of reals is the triple $(L(R), L_l(R), L_u(R))$ where $L(R)$ is the frame of reals [1] defined by generators $(p, q) \in \mathbb{Q} \times \mathbb{Q}$ and relations

(R1) $(p, q) \land (r, s) = (p \lor r, q \land s)$,
(R2) $(p, q) \lor (r, s) = (p, s)$ whenever $p \leq r < q \leq s$,
(R3) $(p, q) = \bigvee \{(r, s) : p < r < s < q\}$,
(R4) $\bigvee_{p,q\in\mathbb{Q}}(p, q) = 1$.

We shall use also the following notation:

$$(p, -) = \bigvee_{q\in\mathbb{Q}} (p, q) \quad \text{and} \quad (-, q) = \bigvee_{p\in\mathbb{Q}} (p, q);$$

note that $(p, -) \land (-, q) = (p, q)$.

Equivalently, $L(R)$ may be defined by taking $(p, -)$ and $(-, q)$ as primitive notions, with relations

(S1) $(p, -) \land (-, q) = 0$ whenever $p \geq q$,
(S2) $(p, -) \lor (-, q) = 1$ whenever $p < q$,
(S3) $(p, -) = \bigvee_{r>p}(r, -)$,
(S4) $(-, q) = \bigvee_{s<q}(-, s)$,
(S5) $\bigvee_{p\in\mathbb{Q}}(p, -) = 1$,
(S6) $\bigvee_{q\in\mathbb{Q}}(-, q) = 1$.

Then $L_u(R)$ and $L_l(R)$ are just the following subframes of $L(R)$:

$L_u(R) = \langle \{(p, -) : p \in \mathbb{Q}, (p, -) \text{ satisfy (R3) and (R5) for all } p \in \mathbb{Q}\} \rangle$,
$L_l(R) = \langle \{(-, q) : q \in \mathbb{Q}, (-, q) \text{ satisfy (R4) and (R6) for all } q \in \mathbb{Q}\} \rangle$. 

In general topology one sometimes deals with arbitrary (not necessarily continuous) real-valued functions on a topological space \(X\). This is also possible in the pointfree setting with the approach recently introduced in [12] (which extends the approach to pointfree continuous real functions of Banaschewski [1]). Let \(L\) be a frame. A *real-valued function* on \(L\) is a frame homomorphism \(f : \mathcal{L}(\mathbb{R}) \to \mathcal{S}(L)\). It is

1. **lower semicontinuous** if \(f(\mathcal{L}_u(\mathbb{R})) \subseteq cL\),
2. **upper semicontinuous** if \(f(\mathcal{L}_l(\mathbb{R})) \subseteq cL\),
3. **continuous** if \(f(\mathcal{L}(\mathbb{R})) \subseteq cL\).

The set \(F(L)\) of all real-valued functions on \(L\) is partially ordered by

\[
f \leq g \iff f(p, -) \leq g(p, -) \text{ for every } p \in \mathbb{Q}
\]

\[
g(-, q) \leq f(-, q) \text{ for every } q \in \mathbb{Q}.
\]

We denote by \(\text{LSC}(L)\), \(\text{USC}(L)\) and \(\text{C}(L)\) the collections of all lower semicontinuous, upper semicontinuous, and continuous members of \(F(L)\). Of course, one has

\[
\text{C}(L) = \text{LSC}(L) \cap \text{USC}(L).
\]

Note that \(\text{USC}(L) \simeq \text{BiFrm}((\mathcal{L}(\mathbb{R}), \mathcal{L}_l(\mathbb{R}), \mathcal{L}_u(\mathbb{R})), (\mathcal{S}(L), cL, oL))\).

A nice way of constructing real functions is with the help of the so-called *scales* [12]. A collection of sublocales \(\{S_r : r \in \mathbb{Q}\} \subseteq \mathcal{S}(L)\) is a *scale* on \(\mathcal{S}(L)\) if \(S_r \vee S_s^* = 1\) whenever \(r < s\) and \(\bigvee\{S_r : r \in \mathbb{Q}\} = 1 = \bigvee\{S_r^* : r \in \mathbb{Q}\}\) (here \(S^*\) denotes the pseudocomplement of \(S\)). For each scale \(\{S_r : r \in \mathbb{Q}\}\) in \(\mathcal{S}(L)\) the function \(f\) defined by

\[
f(p, -) = \bigvee_{r > p} S_r \quad \text{and} \quad f(-, q) = \bigvee_{r < q} S_r^* \quad (p, q \in \mathbb{Q})
\]

(2.1)

belongs to \(F(L)\). If, moreover, each \(S_r\) is an open sublocale then \(f \in \text{USC}(L)\).

For instance, given a complemented sublocale \(S\) of \(L\), with complement \(\neg S\), the *characteristic map* \(\chi_S : \mathcal{L}(\mathbb{R}) \to \mathcal{S}(L)\) is defined by

\[
\chi_S(p, -) = \begin{cases} 
1 & \text{if } p < 0, \\
\neg S & \text{if } 0 \leq p < 1, \\
0 & \text{if } p \geq 1,
\end{cases}
\]

and

\[
\chi_S(-, q) = \begin{cases} 
0 & \text{if } q \leq 0, \\
S & \text{if } 0 < q \leq 1, \\
1 & \text{if } q > 1,
\end{cases}
\]

for each \(p, q \in \mathbb{Q}\) [12]. Then, as in the classical context, we have:

(a) \(\chi_S \in \text{LSC}(L)\) if and only if \(S\) is open,
(b) \(\chi_S \in \text{USC}(L)\) if and only if \(S\) is closed,
(c) \( \chi_S \in C(L) \) if and only if \( S \) is clopen.

For any \( f \in F(L) \) the upper regularization \( f^- \in USC(L) \) of \( f \) is defined by

\[
f^-(p, -) = \bigvee_{q > p} -f(-, q) \quad \text{and} \quad f^-(-, p) = \bigvee_{q < p} f(-, q)
\]

(see [11] and [12] for more information). Of course, when \( f \in USC(L) \) then \( f^- = f \). Thus, for any \( f \in USC(L) \), we have

\[
f(p, -) = \bigvee_{q > p} -f(-, q) \in oL \quad \text{and} \quad f(-, p) = \bigvee_{q < p} f(-, q) \in cL. \quad (2.2)
\]

3. The semicontinuous quasi-uniformity \( USC(L) \)

For each \( n \in \mathbb{N} \),

\[
Q_n = \left\{((-q, (p, -)) \mid p, q \in \mathbb{Q}, 0 < q - p < \frac{1}{n}\right\}
\]

is a strong paircover of the biframe \((\mathcal{L}(\mathbb{R}), \mathcal{L}_l(\mathbb{R}), \mathcal{L}_u(\mathbb{R}))\). These paircovers satisfy the following (easy to check) properties:

**Lemma 3.1.** \( (1) \) For every \( n \in \mathbb{N} \) and \( p, q \in \mathbb{Q} \) with \( p < q, \frac{1}{q-p} < n \), we have:

(a) \( st_1((-p), Q_n) \leq (-, q) \).

(b) \( st_2((q, -), Q_n) \leq (p, -) \).

(2) For every \( p_i, q_i \in \mathbb{Q} \) with \( p_i < q_i \), we have:

(a) \( st_1(\bigvee_{i \in I}(p_i, q_i), Q_n) = st_1(\bigvee_{i \in I}(-, q_i), Q_n) \).

(b) \( st_2(\bigvee_{i \in I}(p_i, q_i), Q_n) = st_2(\bigvee_{i \in I}(p_i, -), Q_n) \).

(3) For each \( n \in \mathbb{N} \), \( Q_{n+1} \subseteq Q_n \) (thus \( Q_{n+1} \leq Q_n \)).

Moreover:

**Proposition 3.2.** For every \( n \in \mathbb{N} \) and \( p \in \mathbb{Q} \), we have:

(1) \( Q^*_{3n} \leq Q_n \).

(2) \( (-, p) = \bigvee\{(-, q) \in \mathcal{L}_l(\mathbb{R}) \mid st_1((-q, Q_n) \leq (-, q) \text{ for some } n \in \mathbb{N}\} \).

(3) \( (p, -) = \bigvee\{(q, -) \in \mathcal{L}_u(\mathbb{R}) \mid st_2((q, -), Q_n) \leq (p, -) \text{ for some } n \in \mathbb{N}\} \).
Proof: (1) Let $((-q), (p, -)) \in Q_{3n}$. We have to show that there is $((-\tilde{q}), (\tilde{p}, -)) \in Q_n$ such that $st_1((-q), Q_{3n}) \leq (-\tilde{q})$ and $st_2((p, -), Q_{3n}) \leq (\tilde{p}, -)$. But

$$st_1((-q), Q_{3n}) = \bigvee \{(-d_1) \mid ((-d_1), (d_2, -)) \in Q_{3n}, (d_2, -) \wedge (-q) \neq 0\} \leq (-q + \frac{1}{3n})$$

since $(d_2, -) \wedge (-q) \neq 0 \iff d_2 < q$ and $0 < d_1 - d_2 < \frac{1}{3n}$ (which implies $d_1 < d_2 + \frac{1}{3n} < q + \frac{1}{3n}$). Similarly,

$$st_2((p, -), Q_{3n}) = \bigvee \{(d_2, -) \mid ((-d_1), (d_2, -)) \in Q_{3n}, (-d_1) \wedge (p, -) \neq 0\} \leq (p - \frac{1}{3n}, -).$$

It suffices then to take $\tilde{q} = q + \frac{1}{3n}$ and $\tilde{p} = p - \frac{1}{3n}$. Indeed, $((-q + \frac{1}{3n}), (p - \frac{1}{3n}, -)) \in Q_n$, since $0 < q + \frac{1}{3n} - p + \frac{1}{3n} < \frac{1}{3n} + \frac{1}{3n} + \frac{1}{3n} = \frac{1}{n}$.

(2) By Lemma 3.1(1), for every $q < p$ there is some $n \in \mathbb{N}$ such that $st_1((-q), Q_n) \leq (-p)$. Thus, by Lemma 2.1(1),

$$(-p) = \bigvee_{q < p} (-q) \leq \bigvee_{q < p} \{(-q) \mid st_1((-q), Q_n) \leq (-p) \text{ for some } n \in \mathbb{N}\} \leq (-p).$$

(3) may be proved similarly.

In conclusion, the strong paircovers $Q_n (n \in \mathbb{N})$, generate a quasi-uniformity $Q$ on the biframe of reals $(\mathcal{L}(\mathbb{R}), \mathcal{L}_l(\mathbb{R}), \mathcal{L}_u(\mathbb{R}))$.

Corollary 3.3. The pair $((\mathcal{L}(\mathbb{R}), \mathcal{L}_l(\mathbb{R}), \mathcal{L}_u(\mathbb{R})), Q)$ is a quasi-uniform biframe.

We refer to it as the quasi-metric quasi-uniformity of the reals.

Now let $f \in \text{USC}(L)$. Then (recall (2.2))

$$f(p, -) = \bigvee_{q > p} f(-, q) \in \mathcal{O}L \quad \text{and} \quad f(-, p) = \bigvee_{q < p} f(-, q) \in \mathcal{C}L$$

so $f : \mathcal{L}(\mathbb{R}) \to \mathcal{S}(L)$ is a biframe map

$$f : (\mathcal{L}(\mathbb{R}), \mathcal{L}_l(\mathbb{R}), \mathcal{L}_u(\mathbb{R})) \to (\mathcal{S}(L), \mathcal{C}L, \mathcal{O}L).$$
Clearly, for each \( n \in \mathbb{N} \),

\[
C_{f,n} = \{(f(-, q), f(p, -)) \mid p, q \in \mathbb{Q}, f(p, q) \neq 0, 0 < q - p < \frac{1}{n}\}
\]
is a strong paircover of the sublocale lattice \((S(L), cL, oL)\). Further, we have [6]:

**Lemma 3.4.** (1) For any \( f_1, \ldots, f_k \in \text{USC}(L) \), \( n_1, \ldots, n_k \in \mathbb{N} \) and \( S \in S(L) \):

(a) \( st_1(S, \bigwedge_{i=1}^k C_{f_i, n_i}) \in cL \).

(b) \( st_2(S, \bigwedge_{i=1}^k C_{f_i, n_i}) \in oL \).

(2) For any \( a \in L \) and \( n \in \mathbb{N} \):

(a) \( st_1(c(a), C_{\chi_{c(a)}, n}) = c(a) \).

(b) \( st_2(o(a), C_{\chi_{c(a)}, n}) = o(a) \).

We have finally the required result that extends Proposition 1.1 of [14] (also Theorem 3.1 of [3]).

**Proposition 3.5.** \( \{C_{f,n} \mid f \in \text{USC}(L), n \in \mathbb{N}\} \) is a subbase for a quasi-uniformity \( \text{USC}(L) \) on the biframe \((S(L), cL, oL)\).

**Proof:** For each \( (f(-, q), f(p, -)) \in C_{f,3n} \) we have

\[
st_1(f(-, q), C_{f,3n}) \leq f(-, q + \frac{1}{3n})
\]
and

\[
st_2(f(p, -), C_{f,3n}) \leq f(p - \frac{1}{3n}, -)
\]
(the proof goes as in Proposition 3.2). Since \( f(p - \frac{1}{3n}, q + \frac{1}{3n}) \geq f(p, q) \neq 0 \), this shows that \( C_{f,3n}^* \leq C_{f,n}^* \).

Conditions (U1) and (U3) follow immediately from Lemma 3.4.

\( \text{USC}(L) \) is called the *semicontinuous quasi-uniformity* on \( L \). This can be immediately generalized to any collection \( C \) containing all characteristic functions \( \chi_S \) for a closed sublocale \( S \):

**Corollary 3.6.** Let \( C \) be a collection of upper semicontinuous real functions, containing all upper characteristic functions \( \chi_{c(a)} \) (\( a \in L \)). Then \( \{C_{f,n} \mid f \in C, n \in \mathbb{N}\} \) is a subbase for a quasi-uniformity \( \text{UC} \) on the biframe \((S(L), cL, oL)\).
4. Properties of \( USC(L) \)

Proposition 4.1. \( USC(L) \) is the coarsest quasi-uniformity \( U \) on \( (S(L), cL, oL) \) for which each biframe map \( h : (\mathcal{L}(\mathbb{R}), \mathcal{L}_l(\mathbb{R}), \mathcal{L}_u(\mathbb{R})) \rightarrow (S(L), cL, oL) \) is a uniform homomorphism \( h : ((\mathcal{L}(\mathbb{R}), \mathcal{L}_l(\mathbb{R}), \mathcal{L}_u(\mathbb{R})), \mathcal{Q}) \rightarrow ((S(L), cL, oL), USC(L)) \).

Proof: We begin by checking that any biframe map

\[
h : (\mathcal{L}(\mathbb{R}), \mathcal{L}_l(\mathbb{R}), \mathcal{L}_u(\mathbb{R})) \rightarrow (S(L), cL, oL)
\]

is a uniform homomorphism

\[
((\mathcal{L}(\mathbb{R}), \mathcal{L}_l(\mathbb{R}), \mathcal{L}_u(\mathbb{R})), \mathcal{Q}) \rightarrow ((S(L), cL, oL), USC(L)),
\]

that is, \( h[Q_n] \in USC(L) \) for every \( n \in \mathbb{N} \). Obviously, the frame map \( h : \mathcal{L}(\mathbb{R}) \rightarrow S(L) \) belongs to USC(L). It suffices then to show that \( C_{h,n} \leq h[Q_n] \), which is obvious since \( C_{h,n} \subseteq h[Q_n] \).

Now let \( U \) be a quasi-uniformity on \( (S(L), cL, oL) \) for which any biframe map

\[
h : (\mathcal{L}(\mathbb{R}), \mathcal{L}_l(\mathbb{R}), \mathcal{L}_u(\mathbb{R})) \rightarrow (S(L), cL, oL)
\]

is a uniform homomorphism

\[
h : ((\mathcal{L}(\mathbb{R}), \mathcal{L}_l(\mathbb{R}), \mathcal{L}_u(\mathbb{R})), \mathcal{Q}) \rightarrow ((S(L), cL, oL), U).
\]

In order to show that \( USC(L) \subseteq U \) it suffices to check that, for any \( f \in USC(L) \) and \( n \in \mathbb{N} \), \( C_{f,n} \in U \). By hypothesis,

\[
f[Q_n] = \{(f(-, q), f(p, -)) \mid p, q \in \mathbb{Q}, 0 < q - p < \frac{1}{n}\} \in U.
\]

So there is a strong paircover \( C \in U \) such that \( C \leq f[Q_n] \). Then \( C \leq C_{f,n} \).

Indeed, for any \( ((-, q), (p, -)) \in C \) there are \( \tilde{p}, \tilde{q} \in \mathbb{Q} \) with \( (-, q) \leq f(-, \tilde{q}) \), \( (p, -) \leq f(\tilde{p}, -) \) and \( 0 < \tilde{q} - \tilde{p} < \frac{1}{n} \); since \( (p, q) \neq 0 \), then \( f(\tilde{p}, \tilde{q}) \neq 0 \).

Hence \( C_{f,n} \in U \) as required.

For every frame \( L \),

\[
\{(c(a), 1), (1, o(a)) \mid a \in L\}
\]

is a subbase for a quasi-uniformity on \( (S(L), cL, oL) \) [8]. It is clearly a quasi-uniformity compatible with the given frame \( L \) since the first subframe \( cL \) is an isomorphic copy of \( L \). This is the pointfree analogue of the Császár-Pervin quasi-uniformity of a set \( X \). We refer to it as the Frith quasi-uniformity and denote it by \( \mathcal{F} \).
Since
\[ C_{\chi_{r(a)},n} = \{(\chi_{r(a)}(-, q), \chi_{r(a)}(p, -)) \mid p, q \in \mathbb{Q}, 0 < q - p < \frac{1}{n}, \chi_{r(a)}(p, q) \neq 0\}, \]
then it is straightforward to check the following.

Lemma 4.2. For each characteristic function \(\chi_{r(a)}, a \in L\),
\[ C_{\chi_{r(a)},n} = \{(c(a), 1), (1, o(a))\}. \]

Therefore, for \(C = \{\chi_{r(a)} \mid a \in L\}\), \(U_C\) and \(F\) have a common subbase and we have:

Corollary 4.3. Let \(C = \{\chi_{r(a)} \mid a \in L\}\). Then \(U_C = F\).

A real-valued function \(f \in F(L)\) is bounded [12] if there exist some \(p < q\) in \(\mathbb{Q}\) for which \(f(p, q) = 1\). More generally, \(f\) is upper bounded if \(f(-, q) = 1\) for some \(q \in \mathbb{Q}\). Since every upper characteristic function \(\chi_{r(a)}\) is bounded, the previous corollary leads immediately to the following result, which is the pointfree extension of Proposition 2.10 of [7].

Proposition 4.4. Let \(C\) be the collection of all bounded upper semicontinuous real functions on \(L\). Then \(\{C_{f,n} \mid f \in C, n \in \mathbb{N}\}\) is a subbase for \(F\).

Proposition 4.5. Let \(h : (S(L), cL, oL) \rightarrow (S(M), cM, oM)\) be a biframe map. Then \(h\) is a uniform homomorphism
\[ ((S(L), cL, oL), USC(L)) \rightarrow ((S(M), cM, oM), USC(M)). \]

Proof: Let \(C_{f,n} \in USC(L)\), for some \(f \in USC(L)\) and \(n \in \mathbb{N}\). Evidently, \(hf \in USC(M)\) and
\[ h[C_{f,n}] = \{(hf(-, q), hf(p, -)) \mid p, q \in \mathbb{Q}, 0 < q - p < \frac{1}{n}, f(p, q) \neq 0\} \geq C_{hf,n} \in SC(M) \]
because \(hf(p, q) \neq 0 \Rightarrow f(p, q) \neq 0\).

We say that a quasi-uniform biframe \((L, U)\) is totally bounded if \(U\) has a base of finite paircovers.

Lemma 4.6. If \((S(L), cL, oL), U\) is a totally bounded quasi-uniform biframe then every uniform homomorphism
\[ h : ((\mathcal{L}(\mathbb{R}), \mathcal{L}_l(\mathbb{R}), \mathcal{L}_u(\mathbb{R})), \mathcal{Q}) \rightarrow ((S(L), cL, oL), U) \]
is bounded.
Proposition 4.1. Let \( h : ((L(R), L_l(R), L_u(R)), Q) \to ((S(L), cL, oL), U) \) be a uniform homomorphism. For each \( n \in \mathbb{N} \), \( h[Q_n] \in U \), so there exists a finite paircover \( C = \{(c(a_1), o(b_1)), \ldots, (c(a_k), o(b_k))\} \) of \( S(L) \) such that \( C \leq h[Q_n] \). Therefore, for each \( i \in \{1, \ldots, k\} \), \( c(a_i) \leq h(\cdot, q_i) \) and \( o(b_i) \leq h(p_i, \cdot) \) for some \( p_i, q_i \in Q \) with \( 0 < q_i - p_i < \frac{1}{n} \). Hence
\[
1 = \bigvee_{i=1}^k c(a_i) \wedge o(b_i) \leq \bigvee_{i=1}^k h(p_i, q_i).
\]
Let \( q = \max_{i=1,\ldots,k} q_i \) and \( p = \min_{i=1,\ldots,k} p_i \). Immediately, \( h(p, q) = 1 \) and \( h \) is bounded.

**Proposition 4.7.** Let \(((S(L), cL, oL), U)\) be a totally bounded quasi-uniform frame. Then there exists a collection \( C \) of bounded \( f \in USC(L) \) such that \( \{C_{f,n} \mid f \in C, n \in \mathbb{N}\} \) is a subbase for \( U \).

**Proof:** Let \(((S(L), cL, oL), U)\) be a totally bounded quasi-uniform frame. Every uniform homomorphism
\[
h : ((L(R), L_l(R), L_u(R)), Q) \to ((S(L), cL, oL), U),
\]
which is bounded by Lemma 4.6, is upper semicontinuous. Let \( C \) be the collection of every such maps. Since \( C \) contains all characteristic functions \( \chi_{\varepsilon(a)} \) (\( a \in L \)), then, by Corollary 3.6, \( \{C_{h,n} \mid h \in C, n \in \mathbb{N}\} \) is a subbase for a quasi-uniformity \( U_C \) on \((S(L), cL, oL)\). Since \( h \) is uniform,
\[
h[Q_n] = \{(h(\cdot, q), h(p, \cdot)) \mid p, q \in Q, 0 < q - p < \frac{1}{n}\} \in U.
\]
So there is a strong paircover \( C \in U \) such that \( C \leq h[Q_n] \). Then \( C \leq C_{h,n} \) (the proof is similar to the proof at the end of 4.1 that \( C \leq C_{f,n} \)). Hence \( \{C_{h,n} \mid h \in C, n \in \mathbb{N}\} \) is also a subbase for \( U \).

**Theorem 4.8.** Let \( L \) be a frame. Then \( USC(L) \) is totally bounded if and only if every \( f \in USC(L) \) is bounded.

**Proof:** Assume that \( USC(L) \) is totally bounded and let \( f \in USC(L) \). Then we have a biframe map \( f : (L(R), L_l(R), L_u(R)) \to (S(L), cL, oL) \) which, by Proposition 4.1, is uniform. Then, by Lemma 4.6, \( f \) is bounded.

Conversely, let \( C = USC(L) = \{\text{bounded } f \in USC(L)\} \). Then \( U_C = USC(L) \) coincides by Proposition 4.4 with \( F \). Since \( F \) is totally bounded, then \( USC(L) \) is totally bounded.

Recall that a frame is **countably compact** if each countable cover has a finite subcover.
**Theorem 4.9.** Let $L$ be a frame. Then every $f \in \text{USC}(L)$ is upper bounded if and only if $L$ is countably compact.

**Proof:** Let $A = \{a_i \mid i \in \mathbb{N}\}$ be a countable cover of $L$. For each $q \in \mathbb{Q}$ let $m(q) = \min\{n \in \mathbb{N}_0 \mid n \geq q\}$. Further, let $a_0 = 0$ and define, for each $r \in \mathbb{Q}$,

$$S_r = o\left(\bigvee_{i=0}^{m(r)} a_i\right).$$

This is clearly a scale of open sublocales so, by (2.1), the function $f$ defined by

$$f(p, -) = \bigvee_{r>p}^{m(r)} o\left(\bigvee_{i=0}^r a_i\right) \quad \text{and} \quad f(-, q) = \bigvee_{r<q}^{m(r)} c\left(\bigvee_{i=0}^r a_i\right) \quad (p, q \in \mathbb{Q})$$

is in \text{USC}(L). By hypothesis, $f$ is bounded. Consequently, there is some $q \in \mathbb{Q}$ for which $f(-, q) = 1$. This means precisely that

$$1 = \bigvee_{r<q}^{m(r)} c\left(\bigvee_{i=0}^r a_i\right) = c\left(\bigvee_{r<q}^{m(r)} \bigvee_{i=0}^r a_i\right) = c\left(\bigvee_{i=0}^{m(q)} a_i\right),$$

that is, $\bigvee_{i=0}^{m(q)} a_i = 1$. Hence $\{a_1, \ldots, a_{m(q)}\}$ is a finite subcover of $A$. This shows that $L$ is countably compact.

Conversely, let $L$ be countably compact and let $f \in \text{USC}(L)$. Then $\{f(-, q) \mid q \in \mathbb{Q}\}$ is a countable cover of $cL \cong L$. By hypothesis, there exist $q_1, \ldots, q_k \in \mathbb{Q}$ such that $\bigvee_{i=1}^k f(-, q_i) = 1$, that is, $f(-, \bigvee_{i=1}^k q_i) = 1$, which shows that $f$ is upper bounded.

This is the pointfree counterpart of Lemma 3.2 of [3]. Our last result extends Corollary 3.3 of [3]. It asserts that every frame $L$ with a unique compatible quasi-uniform structure is countably compact.

**Corollary 4.10.** If $(\mathcal{S}(L), cL, oL)$ has a unique quasi-uniform structure then $L$ is countably compact.

**Proof:** If $\mathcal{U}$ is the unique quasi-uniform structure on $(\mathcal{S}(L), cL, oL)$ then $\mathcal{U}$ coincides with $\mathcal{F}$ which is totally bounded. But also $\mathcal{U} = \text{USC}(L)$ so, by the theorems above, $L$ is countably compact. 

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