# CONVEXITY OF THE KREIN SPACE TRACIAL NUMERICAL RANGE AND MORSE THEORY 

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Abstract: In this paper we present a Krein space convexity theorem on the tracialnumerical range of a matrix. This theorem is the analogue of Westwick's theorem. The proof is an application of Morse theory.

Keywords: Numerical range, indefinite inner product, $J$-Hermitian matrix, noninterlacing.

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## 1. Introduction

Let $J$ denote a Hermitian involutive matrix, that is, $J^{*}=J, J^{2}=I_{n}$, with signature ( $r, n-r$ ), $0<r \leq n$ (i.e. with $r$ positive and $n-r$ negative eigenvalues). We consider $\mathbf{C}^{n}$ endowed with the indefinite inner product induced by $J$ :

$$
[x, y]=y^{*} J x, \quad x, y \in \mathbf{C}^{n} .
$$

Let $M_{n}$ be the algebra of $n \times n$ complex matrices. A matrix $H \in M_{n}$ is $J$-Hermitian if $H^{\#}=H$, where $H^{\#}=J H^{*} J$ is the $J$-adjoint of $H$. A matrix $U \in M_{n}$ is $J$-unitary if $U^{\#} U=U U^{\#}=I_{n}$, the identity matrix of size $n$. The $J$-unitary matrices form a locally compact connected group denoted by $G=\mathcal{U}(r, n-r)$ and called the $J$-unitary group. For $C, T \in M_{n}$, the $J$-tracial numerical range of $T$ is denoted and defined as

$$
\begin{equation*}
W_{C}^{J}(T)=\left\{\operatorname{Tr}\left(C U T U^{-1}\right): U \in \mathcal{U}(r, n-r)\right\} . \tag{1.1}
\end{equation*}
$$

This set is a connected set in the Gaussian plane $\mathbf{C}$ and satisfies the symmetry property $W_{C}^{J}(T)=W_{T}^{J}(C)$ (see [1], [2], [4] for other properties). In the case $r=n$, the matrix $J$ reduces to the identity matrix and the group $\mathcal{U}(n, 0)=\mathcal{U}(n)$ is the unitary group, which is a compact group. In this case $W_{C}^{J}(T)$ is simply denoted by $W_{C}(T)$ and called the $C$-numerical range of $T$. Hausdorff proved the convexity of $W_{C}(T)$ in the case $C$ is a rank one orthogonal projection. Using Morse theory, Westwick [8] proved the
convexity of $W_{C}(T)$ for $C$ Hermitian and in [5] Poon obtained an alternative proof by majorization techniques. In [7], Tam generalized Westwick's result for certain compact Lie groups. In this paper we obtain a non-compact group analogue of Westwick's convexity theorem applying Morse theory (cf. Lemma 4.1). The eigenvalues of a $J$-unitarily diagonalizable $J$-Hermitian matrix $T$ are real and will be denoted as:

$$
\begin{aligned}
& \sigma_{+}(T)=\left\{\lambda \in \mathbf{C}: T \xi=\lambda \xi \text { for some } \xi \in \mathbf{C}^{n},[\xi, \xi]>0\right\} \\
& \sigma_{-}(T)=\left\{\lambda \in \mathbf{C}: T \xi=\lambda \xi \text { for some } \xi \in \mathbf{C}^{n},[\xi, \xi]<0\right\}
\end{aligned}
$$

The eigenvalues are said to be noninterlacing if

$$
\begin{equation*}
\left.\max \sigma_{-}(T)<\min \sigma_{+}(T) \quad \text { or } \quad \max \sigma_{+}(T)\right)<\min \sigma_{-}(T) \tag{1.2}
\end{equation*}
$$

Replacing $<$ in (1.2) by $\leq$ the eigenvalues are said to be weakly noninterlacing. Noninterlacing property plays a key role in the investigation of numerical ranges associated with non-compact groups.

In the sequel we consider $J=I_{r} \oplus\left(-I_{n-r}\right)$. Our main result is the following.

Theorem 1.1. Let $C$ be a J-unitarily diagonalizable J-Hermitian matrix with noninterlacing eigenvalues and let $T \in M_{n}$ be a matrix such that for some $\theta \in \mathbf{R}$

$$
T_{\theta}=\cos \theta\left(T+T^{\#}\right) / 2+\sin \theta\left(T-T^{\#}\right) /(2 i)
$$

is a J-unitarily diagonalizable J-Hermitian matrix with noninterlacing eigenvalues:

$$
\begin{aligned}
& \sigma_{+}(C)=\left\{c_{1} \geq \cdots \geq c_{r}\right\}, \sigma_{-}(C)=\left\{c_{r+1} \geq \cdots \geq c_{n}\right\} \\
& \sigma_{+}\left(T_{\theta}\right)=\left\{\lambda_{1} \geq \cdots \geq \lambda_{r}\right\}, \sigma_{-}\left(T_{\theta}\right)=\left\{\lambda_{r+1} \geq \cdots \geq \lambda_{n}\right\} \\
& c_{n}>c_{1} \text { or } c_{r}>c_{r+1}, \quad \lambda_{n}>\lambda_{1} \text { or } \lambda_{r}>\lambda_{r+1}
\end{aligned}
$$

Then $W_{C}^{J}(T)$ is a closed convex set in $\mathbf{C}$ contained in a closed cone $\left\{z_{0}+r \mathrm{e}^{i \eta}\right.$ : $\left.r \geq 0, \theta_{1} \leq \eta \leq \theta_{2}\right\}$ with $0 \leq \theta_{2}-\theta_{1}<\pi$.

Denote by $C^{\infty}(G)$ the commutative algebra of all $C^{\infty}$-differentiable complexvalued functions on $G=\mathcal{U}(r, n-r)$. We consider the even dimensional coset space $M=G / D$, where

$$
D=\left\{\operatorname{diag}\left(d_{1}, \ldots, d_{n}\right): d_{1}, \ldots, d_{n} \in \mathbf{C},\left|d_{1}\right|=\cdots=\left|d_{n}\right|=1\right\}
$$

Let the real valued function $f_{0}$ on $M$ be defined by

$$
\begin{equation*}
f_{0}(g)=\sum_{h, j=1}^{n} c_{h} \lambda_{j}\left|u_{h j}\right|^{2} \epsilon_{h} \epsilon_{j}=\operatorname{Tr}\left(C g T_{\theta} g^{-1}\right), \tag{1.3}
\end{equation*}
$$

where $\epsilon_{h}=1$ for $1 \leq h \leq r$ and $\epsilon_{h}=-1$ for $r+1 \leq h \leq n$.
This paper is organized as follows. The critical points and the Morse indices of the function $f_{0}$ are investigated in Section 2. Global properties of $f_{0}$ and of $W_{C}^{J}(T)$ are studied in Section 3. The proof of Theorem 1.1 is presented in Section 4 using the results in Sections 2, 3.

## 2. Local properties of the function $f_{0}$

To treat local properties of the function $f_{0}$, we introduce some notation and prerequisites. The Lie algebra of $G=\mathcal{U}(r, n-r)$ is given as

$$
\begin{equation*}
\mathfrak{G}=\left\{X \in M_{n}: X^{\#}=-X\right\} . \tag{2.1}
\end{equation*}
$$

For $1 \leq k_{0}<l_{0} \leq r$ or $r+1 \leq k_{0}<l_{0} \leq n$, let

$$
X_{k_{0}, l_{0}}=\left(\delta_{k k_{0}} \delta_{l l_{0}}-\delta_{k l_{0}} \delta_{l k_{0}}\right), \quad Y_{k_{0}, l_{0}}=i\left(\delta_{k k_{0}} \delta_{l l_{0}}+\delta_{k l_{0}} \delta_{l k_{0}}\right),
$$

where $\delta_{k l}$ is the Kronecker symbol. For $1 \leq k_{0} \leq r, r+1 \leq l_{0} \leq n$, let

$$
V_{k_{0}, l_{0}}=\left(\delta_{k k_{0}} \delta_{l l_{0}}+\delta_{k l_{0}} \delta_{l k_{0}}\right), \quad W_{k_{0}, l_{0}}=i\left(\delta_{k k_{0}} \delta_{l l_{0}}-\delta_{k l_{0}} \delta_{l k_{0}}\right)
$$

Throughout, we consider the basis of the Lie algebra $\mathfrak{G}$ constituted by the matrices $X_{k_{0}, l_{0}}, Y_{k_{0}, l_{0}}, V_{k_{0}, l_{0}}, W_{k_{0}, l_{0}}$ and the diagonal matrices $D_{1}=$ $\operatorname{diag}(i, 0, \ldots, 0), \cdots, D_{n}=\operatorname{diag}(0, \ldots, 0, i)$, which generate the Lie algebra $\mathfrak{D}$. We shall use a canonical coordinate system of second kind

$$
\exp \left(t_{1} X_{1}\right) \exp \left(t_{2} X_{2}\right) \cdots \exp \left(t_{n^{2}} X_{n^{2}}\right)
$$

in a neighborhood of the identity $I_{n}$, where $\left\{X_{n^{2}-n+1}, \ldots, X_{n^{2}}\right\}$ is the above basis of $\mathfrak{D}$, and $\left\{X_{1}, X_{2}, \ldots, X_{n^{2}-n}\right\}$ is the remaining system of vectors of the basis of $\mathfrak{G}$.
For $f \in C^{\infty}(G)$, consider a representation $\alpha$ of $\mathfrak{G}$ on $C^{\infty}(G)$ given by

$$
\begin{equation*}
\alpha(X)(f)(g)=-\lim _{t \rightarrow 0} \frac{f\left(\mathrm{e}^{t X} g\right)-f(g)}{t}=\lim _{t \rightarrow 0} \frac{f\left(\mathrm{e}^{-t X} g\right)-f(g)}{t} . \tag{2.2}
\end{equation*}
$$

The function space $C^{\infty}(G / D)$ is identified with the space

$$
\begin{equation*}
\left\{f \in C^{\infty}(G): f(g d)=f(g) \text { for } d \in D\right\} . \tag{2.3}
\end{equation*}
$$

For $S_{n}$ the symmetric group of degree $n$, let

$$
S_{r, n-r}=\left\{\sigma_{1} \sigma_{2}: \sigma_{1}(j)=j(r+1 \leq j \leq n), \sigma_{2}(j)=j(1 \leq j \leq r)\right\} .
$$

Proposition 2.1. Let $H=\operatorname{diag}\left(\lambda_{1}, \cdots, \lambda_{n}\right)$ and $C=\operatorname{diag}\left(c_{1}, \cdots, c_{n}\right)$ be such that $c_{h} \neq c_{j}, \lambda_{h} \neq \lambda_{j}$ for $1 \leq h \neq j \leq n$. A point $g=\left(u_{h j}\right)$ of $G$ is a critical point of $f_{0}$ if and only if $\left(u_{h j}\right)$ is a permutation matrix associated with a permutation in $S_{r, n-r}$.
Proof. We take an arbitrary critical point $g$. A straightforward computation shows that

$$
-\alpha(X)\left(f_{0}\right)(g)=\operatorname{Tr}\left(X\left[g H g^{-1}, C\right]\right)=0
$$

where $[X, Y]=X Y-Y X$ stands for the commutator. Thus, $\mathrm{gHg}^{-1}$ commutes with the diagonal matrix $C$, whose diagonal entries are pairwise distinct, and so

$$
g H g^{-1}=\operatorname{diag}\left(\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(n)}\right),
$$

for some $\sigma \in S_{n}$. For $g=\left(u_{h j}\right)$, we have $u_{h j} \lambda_{j}=\lambda_{\sigma(h)} u_{h j}$ and so $u_{h j}=0$ unless $\lambda_{j}=\lambda_{\sigma(h)}$. Therefore, $u_{h j}=\eta_{j} \delta_{j \sigma(h)}$, where $\left|\eta_{j}\right|=1$. We notice that $g \mathrm{Hg}^{-1}=d g \mathrm{Hg}^{-1} d^{-1}$ for any $d \in D$. Moreover, $\sigma$ belongs to $S_{r, n-r}$ because $g^{-1}=J g^{*} J$. The converse is clear.

Proposition 2.2. Let $H=\operatorname{diag}\left(\lambda_{1}, \cdots, \lambda_{n}\right)$ and $C=\operatorname{diag}\left(c_{1}, \cdots, c_{n}\right)$ be such that $c_{h} \neq c_{j}, \lambda_{h} \neq \lambda_{j}$ for $1 \leq h \neq j \leq n$. The numbers of positive and negative eigenvalues of the Hessian matrix of $f_{0}$ at each critical point are even.
Proof. We analyze the Hessian matrix of the function $f_{0}$ at a critical point. For $f \in C^{\infty}(G)$, the representation $\alpha$ of $\mathfrak{G}$ satisfies

$$
\begin{equation*}
\alpha(X) \alpha(Y)(f)(g)=\left.\frac{d}{d t} \frac{d}{d s} f\left(\mathrm{e}^{-s Y} \mathrm{e}^{-t X} g\right)\right|_{s, t=0} \tag{2.4}
\end{equation*}
$$

For $X=-X^{\#}$, consider the expansion of $f\left(\mathrm{e}^{-t X} g\right)$ in powers of $t$
$f_{0}\left(\mathrm{e}^{-t X} g\right)=\operatorname{Tr}\left(C g H g^{-1}\right)-t \operatorname{Tr}\left(C\left[X, g H g^{-1}\right]\right)+\frac{t^{2}}{2} \operatorname{Tr}\left(C\left[X,\left[X, g H g^{-1}\right]\right]\right)+\mathcal{O}\left(t^{3}\right)$.
Assuming $g$ is critical, we get

$$
f_{0}\left(\mathrm{e}^{-t X} g\right)=\operatorname{Tr}\left(C g H g^{-1}\right)+\frac{t^{2}}{2} \operatorname{Tr}\left(C\left[X,\left[X, g H g^{-1}\right]\right]\right)+\mathcal{O}\left(t^{3}\right)
$$

Let $\mathcal{M}(X, Y)=\alpha(X) \alpha(Y)\left(f_{0}\right)(g)=\operatorname{Tr}\left(C\left[Y,\left[X, g H g^{-1}\right]\right]\right)$. Having in mind Jacobi identity and performing some computations, we find

$$
\begin{equation*}
\alpha(X) \alpha(Y)\left(f_{0}\right)(g)=\operatorname{Tr}\left(C\left[Y,\left[X, g H g^{-1}\right]\right]\right)=\operatorname{Tr}\left([C, Y]\left[X, g H g^{-1}\right]\right) \tag{2.5}
\end{equation*}
$$

and so $\mathcal{M}(X, Y)=\mathcal{M}(Y, X)$. For $X=\left(x_{k l}\right), Y=\left(y_{k l}\right)$, we have $[C, Y]=$ $\left(y_{h j}\left(c_{h}-c_{j}\right)\right)$ and $\left[X, g H g^{-1}\right]=\left(x_{h j}\left(\lambda_{\sigma(j)}-\lambda_{\sigma(h)}\right)\right)$. By straightforward computations we obtain

$$
\begin{aligned}
\mathcal{M}(X, Y) & =\operatorname{Tr}\left(C\left[Y,\left[X, g H g^{-1}\right]\right]\right)=\operatorname{Tr}\left(\left(y_{h j}\left(c_{h}-c_{j}\right)\right)\left(x_{h j}\left(\lambda_{\sigma(j)}-\lambda_{\sigma(h)}\right)\right)\right) \\
& =\sum_{k, l=1}^{n} y_{k l} x_{l k}\left(c_{k}-c_{l}\right)\left(\lambda_{\sigma(k)}-\lambda_{\sigma(l)}\right) .
\end{aligned}
$$

The matrix representation of $\mathcal{M}(X, Y)$, relative to the fixed basis in $\mathfrak{G} / \mathfrak{D}$, is diagonal. In fact, for $1 \leq k_{0}<l_{0} \leq r$ or $r+1 \leq k_{0}<l_{0} \leq n$, an easy computation leads to

$$
\mathcal{M}\left(X_{k_{0} l_{0}}, X_{k_{0}^{\prime} l_{0}^{\prime}}\right)=\mathcal{M}\left(Y_{k_{0} l_{0}}, Y_{k_{0}^{\prime} l_{0}^{\prime}}\right)=-2 \delta_{k_{0} k_{0}^{\prime} \delta_{l_{0}^{\prime} l_{0}}\left(c_{k_{0}}-c_{l_{0}}\right)\left(\lambda_{\sigma\left(k_{0}\right)}-\lambda_{\sigma\left(l_{0}\right)}\right) . . . . . . . . . ~}
$$

For $1 \leq k_{0}, k_{0}^{\prime} \leq r, r+1 \leq l_{0}, l_{0}^{\prime} \leq n$, we analogously have

$$
\mathcal{M}\left(V_{k_{0} l_{0}}, V_{k_{0}^{\prime} l_{0}^{\prime}}\right)=\mathcal{M}\left(W_{k_{0} l_{0}}, W_{k_{0}^{\prime} l_{0}^{\prime}}\right)=2 \delta_{k_{0} k_{0}^{\prime}} \delta_{l_{0} l_{0}^{\prime}}\left(c_{k_{0}}-c_{l_{0}}\right)\left(\lambda_{\sigma\left(k_{0}\right)}-\lambda_{\sigma\left(l_{0}\right)}\right) .
$$

The off-diagonal entries of $\mathcal{M}(X, Y)$ vanish. Thus, the $\left(n^{2}-n\right) \times\left(n^{2}-\right.$ $n$ )-Hessian matrix of the function $f_{0}$ at every critical point is nonsingular. Moreover, the numbers of positive and negative eigenvalues of the Hessian matrix are even.

We recall, in passing, that the number of negative eigenvalues of the Hessian is called the Morse index.

## 3. Global properties of the function $f_{0}$ and $W_{C}^{J}(H)$

Proposition 3.1. Let $C$ and $H$ be $J$-unitarily diagonalizable $J$-Hermitian matrices with respective eigenvalues $\sigma_{+}(C)=\left\{c_{1}, \ldots, c_{r}\right\}, \sigma_{-}(C)=\left\{c_{r+1}, \ldots, c_{n}\right\}$, $\sigma_{+}(H)=\left\{\lambda_{1}, \ldots, \lambda_{r}\right\}, \sigma_{-}(H)=\left\{\lambda_{r+1}, \ldots, \lambda_{n}\right\}$ satisfying the conditions

$$
\begin{gather*}
c_{r+1} \geq \cdots \geq c_{n}>c_{1} \geq \cdots \geq c_{r}  \tag{3.1}\\
\lambda_{1} \geq \cdots \geq \lambda_{r}>\lambda_{r+1} \geq \cdots \geq \lambda_{n} \tag{3.2}
\end{gather*}
$$

Then the superlevel set $\left\{U \in \mathcal{U}(r, n-r): a \leq f_{0}(U)\right\}$ is compact for an arbitrary real number $a$.

Proof. Analogously to (1.3) we define a function $g_{0}$ for the diagonal $J$ Hermitian matrices $C^{\prime}=\operatorname{diag}\left(c_{1}^{\prime}, \ldots, c_{n}^{\prime}\right), H^{\prime}=\operatorname{diag}\left(\lambda_{1}^{\prime}, \ldots, \lambda_{n}^{\prime}\right)$ with eigenvalues $c_{1}^{\prime}=\cdots=c_{r}^{\prime}=c_{1}, c_{r+1}^{\prime}=\cdots=c_{n}^{\prime}=c_{n}$ and $\lambda_{1}^{\prime}=\cdots=\lambda_{r}^{\prime}=$ $\lambda_{r}, \lambda_{r+1}^{\prime}=\cdots=\lambda_{n}^{\prime}=\lambda_{r+1}$. We show that the functions $f_{0}, g_{0}$ satisfy the inequality $f_{0}(g) \leq g_{0}(g)$ for $g \in G$. To prove this inequality, we may assume that $c_{n}>0>c_{1}, \lambda_{r}>0>\lambda_{r+1}$ by adding suitable scalar operators to $C$ and $H$. Observing that $c_{1} \lambda_{r}-c_{h} \lambda_{j}=\left(c_{1}-c_{h}\right) \lambda_{r}-c_{h}\left(\lambda_{j}-\lambda_{r}\right)(1 \leq h, j \leq r)$, the function $g_{0}(U)-f_{0}(U)$ can be expressed as a linear combination of $\left|u_{h, j}\right|^{2}$ ( $1 \leq h, j \leq n$ ) with nonnegative coefficients. Henceforth the inclusion holds

$$
\begin{equation*}
\left\{[U] \in M: f_{0}([U]) \geq a\right\} \subset\left\{[U] \in M: g_{0}(U) \geq a\right\} \tag{3.3}
\end{equation*}
$$

and the left-hand side of (3.3) is a closed subset of the set in the right-hand side. It can be easily checked that

$$
g_{0}(U)=r c_{1} \lambda_{r}+(n-r) c_{n} \lambda_{r+1}-\frac{c_{n}-c_{1}}{4}\left(\lambda_{r}-\lambda_{r+1}\right)\left(\sum_{h, j=1}^{n}\left|u_{h, j}\right|^{2}-n\right)
$$

so the right-hand side of (3.3) is compact for an arbitrary $a$ and so is the left-hand side.

Proposition 3.2. Let $C$ be a J-unitarily diagonalizable J-Hermitian matrix with noninterlacing eigenvalues $\sigma_{+}(C)=\left\{c_{1} \geq \cdots \geq c_{r}\right\}, \sigma_{-}(C)=\left\{c_{r+1} \geq\right.$ $\left.\ldots \geq c_{n}\right\}, c_{n}>c_{1}$. Let $H$ be a J-Hermitian matrix such that $W_{C}^{J}(H)=$ $\left(-\infty, a_{0}\right]$. Then the matrix $H$ is $J$-unitarily diagonalizable and its eigenvalues are weakly noninterlacing.

Proof. We assume that the maximum value $a_{0}$ is attained at $\operatorname{Tr}\left(\mathrm{CgHg}^{-1}\right)$. This assumption implies that $\operatorname{Tr}\left(X\left[g \mathrm{Hg}^{-1}, C\right]\right)=0$ for an arbitrary $X \in \mathfrak{G}$, so that $\left[\mathrm{gHg}^{-1}, C\right]=0$. The condition $c_{n}>c_{1}$ implies that $g \mathrm{Hg}^{-1}=H_{1} \oplus H_{2}$, where $H_{1}, H_{2}$ are Hermitian block matrices of sizes $r$ and $n-r$, respectively. Thus, $g H^{-1}$ is $J$-unitarily diagonalizable as $V g H g^{-1} V^{-1}=\operatorname{diag}\left(\lambda_{1}, \ldots, \lambda_{n}\right)$, where $V$ is a $J$-unitary matrix of the form $V=V_{1} \oplus V_{2}$, the blocks $V_{1}, V_{2}$ being unitary of sizes $r$ and $n-r$, respectively. Weak noninterlacing property of the eigenvalues follows from Theorem 1.1 (iii) in [1].

Proposition 3.3. Let $C=\operatorname{diag}\left(c_{1}, \ldots, c_{n}\right)$ with $\sigma_{+}(C)=\left\{c_{1} \geq \cdots \geq c_{r}\right\}$, $\sigma_{-}(C)=\left\{c_{r+1} \geq \cdots \geq c_{r}\right\}, c_{n}>c_{1}$, and let $H=\operatorname{diag}\left(\lambda_{1}, \cdots, \lambda_{n}\right)$ with

$$
\begin{gathered}
\sigma_{+}(H)=\left\{\lambda_{1} \geq \cdots \geq \lambda_{r}\right\}, \sigma_{-}(H)=\left\{\lambda_{r+1} \geq \cdots \geq \lambda_{n}\right\}, \lambda_{r} \geq \lambda_{r+1}, \text { so that } \\
\left\{\operatorname{Tr}\left(C g H g^{-1}\right): g \in \mathcal{U}(r, n-r)\right\}=\left(-\infty, \sum_{j=1}^{n} c_{j} \lambda_{j}\right] .
\end{gathered}
$$

Then, for $K$ J-Hermitian, the set

$$
\mathcal{W}=\left\{\operatorname{Tr}\left(C g K g^{-1}\right): g \in \mathcal{U}_{r, n-r}, \operatorname{Tr}\left(C g H g^{-1}\right)=\sum_{j=1}^{n} c_{j} \lambda_{j}\right\}
$$

is connected.

Proof. Let $H=\left(\lambda_{1}^{\prime} I_{n_{1}}\right) \oplus \cdots \oplus\left(\lambda_{s}^{\prime} I_{n_{s}}\right)$, where the $\lambda_{j}^{\prime}$ are all distinct. Consider $C=C_{1} \oplus \cdots \oplus C_{s}, J=J_{1} \oplus \cdots \oplus J_{s}$, and let $K$ be the $J$-Hermitian block matrix

$$
K=\left(\begin{array}{ccc}
K_{11} & \ldots & K_{1 s} \\
\ldots & \ldots & \ldots \\
K_{s 1} & \ldots & K_{s s}
\end{array}\right),
$$

where $C_{j}, J_{j}, K_{j j} \in M_{n_{j}}, j=1, \ldots, s$. If $\operatorname{Tr}\left(C g H g^{-1}\right)=\sum_{j=1}^{n} c_{j} \lambda_{j}$, then $g=W V$, where $W, V \in \mathcal{U}(r, n-r), W C W^{-1}=C$ and $V H V^{-1}=H$. Obviously, $V=V_{1} \oplus \cdots \oplus V_{s}, V_{j} \in M_{n_{j}}$, and $V_{j} J_{j} V_{j}^{*}=J_{j}$. Moreover,

$$
V K V^{-1}=\left(\begin{array}{ccc}
V_{1} K_{11} V_{1}^{-1} & \ldots & V_{1} K_{1 s} V_{s}^{-1} \\
\ldots & \ldots & \ldots \\
V_{s} K_{s 1} V_{1}^{-1} & \ldots & V_{s} K_{s s} V_{s}^{-1}
\end{array}\right)
$$

and

$$
\operatorname{Tr}\left(C g K g^{-1}\right)=\sum_{j=1}^{n} \operatorname{Tr}\left(C_{j} V_{j} K_{j j} V_{j}^{-1}\right)
$$

Thus

$$
\begin{equation*}
\mathcal{W}=W_{C}^{J_{1}}\left(K_{11}\right)+W_{C_{2}}^{J_{2}}\left(K_{22}\right)+\cdots+W_{C_{s}}^{J_{s}}\left(K_{s s}\right) . \tag{3.4}
\end{equation*}
$$

Since each summand in the Minkowski sum in the right hand side of (3.4) is connected, the result follows.

## 4. Proof of Theorem 1.1

Lemma 4.1. (cf. [3, 6]). Let $M$ be an m-dimensional connected $C^{\infty}$ manifold ( $m \geq 2$ ) and let $f$ be a real-valued $C^{\infty}$-function on $M$. Assume that the function $f$ satisfies the following conditions:
(i) The function $f$ attains a minimum value;
(ii) The set $\{x \in M: a \leq f(x) \leq b\}$ is compact for every $-\infty<a<b<$ $+\infty$;
(iii) The number of critical points $x_{1}, x_{2}, \ldots, x_{n}$ of $f$ is finite, and the critical values $f\left(x_{1}\right), f\left(x_{2}\right), \ldots, f\left(x_{n}\right)$ are all distinct;
(iv) The Hessian matrix of $f$ at each critical point $x_{j}$ is non-singular;
(v) The number of the negative eigenvalues of the Hessian matrix of $f$ at each critical point $x_{j}$ is neither 1 nor $m-1$.
Then the level set $\{x \in M: f(x)=a\}$ is connected for every $a \in \mathbf{R}$.
Some critical values of the function $f_{0}$ given by (1.3) at distinct critical points may coincide. Then we apply a perturbative method. We choose a vector $\left(y_{1}, \ldots, y_{n}\right)$ satisfying $\sum_{k=1}^{n} y_{k}\left(c_{\sigma(k)}-c_{\tau(k)}\right) \neq 0$ for any distinct pair $\sigma, \tau \in \mathcal{S}_{r, n-r}$. We can find $t_{0} \in \mathbf{R}$ with sufficiently small modulus such that $\lambda_{k}+y_{k} t_{0}$ satisfy the condition (iii) of Lemma 4.1.

Now, we prove Theorem 1.1. First we show the simple connectedness of $W_{C}^{J}(T)$ under the assumptions of the theorem. Consider the case $c_{n}>c_{1}$, $\lambda_{r}>\lambda_{r+1}$. The remaining cases reduce to this one by replacing $C$ by $-C$ and/or $T_{\theta}$ by $T_{\theta+\pi}$. By a rotation in the Gaussian plane $\mathbf{C}$, we assume that $\theta=0$ and so $H=T_{0}=\left(T+T^{\#}\right) / 2$ is a $J$-unitarily diagonalizable $J$-Hermitian matrix with noninterlacing eigenvalues $\lambda_{r}>\lambda_{r+1}$. Define the following sequences of $J$-Hermitian matrices

$$
C_{m}=\operatorname{diag}\left(c_{1}^{(m)}, \ldots, c_{n}^{(m)}\right), \quad H_{m}=\operatorname{diag}\left(\lambda_{1}^{(m)}, \ldots, \lambda_{n}^{(m)}\right),
$$

where $c_{n}^{(m)}=c_{n}>c_{1}=c_{1}^{(m)}, \lambda_{r}^{(m)}=\lambda_{r}>\lambda_{r+1}=\lambda_{r+1}^{(m)}$ for $m=1,2,3, \ldots$. Suppose that $\left|c_{h}^{(m)}-c_{h}\right| \rightarrow 0, \quad\left|\lambda_{h}^{(m)}-\lambda_{h}\right| \rightarrow 0$ as $m \rightarrow \infty ; c_{h}^{(m)} \neq c_{j}^{(m)}$, $\lambda_{h}^{(m)} \neq \lambda_{j}^{(m)}$ for each $m, 1 \leq h<j \leq n$ and assume that the critical values $\sum_{h=1}^{n} c_{h}^{(m)} \lambda_{\sigma(h)}^{(m)}$ for any distinct permutations $\sigma \in \mathcal{S}_{r, n-r}$ are distinct. By Proposition 3.1 the set
$\left\{g \in U(r, n-r): \operatorname{Tr}\left(C_{m} g H_{m} g^{-1}\right) \geq b\right\} \cup\left\{g \in U(r, n-r): \operatorname{Tr}\left(C g H g^{-1}\right) \geq b\right\}$
is contained in a compact set $B$ for any $b \in \mathbf{R}$. The pairs $\left\{C_{m}, H_{m}\right\}$ satisfy the conditions of Propositions 2.1 and 2.2. It follows that these pairs satisfy the conditions of Lemma 4.1. By Lemma 4.1, each set $\{g \in U(r, n-r)$ : $\left.\operatorname{Tr}\left(C_{m} g H_{m} g^{-1}\right)=a-1 / 2^{m}\right\}$ is a compact connected set. Hence for $T_{m}=$ $H_{m}+i\left(T-T^{\#}\right) /(2 i) \in M_{n}$ and $a \in \mathbf{R}, \mathcal{I}_{m}=\left\{z \in W_{C_{m}}^{J}\left(T_{m}\right): \Re(z)=\right.$ $\left.a-1 / 2^{m}\right\}$ is a closed interval. For every $a \leq \sum_{h=1}^{n} c_{h} \lambda_{h}$, the set $\{z \in$ $\left.W_{C}^{J}(T): \Re(z)=a\right\}$ is compact. Let

$$
\begin{aligned}
& b_{0}=\min \left\{y \in \mathbf{R}: a+i y \in W_{C}^{J}(T)\right\}, \quad c_{0}=\max \left\{y \in \mathbf{R}: a+i y \in W_{C}^{J}(T)\right\}, \\
& b_{m}=\min \left\{y \in \mathbf{R}: a-\frac{1}{2^{m}}+i y \in W_{C_{m}}^{J}\left(T_{m}\right)\right\} \\
& c_{m}=\max \left\{y \in \mathbf{R}: a-\frac{1}{2^{m}}+i y \in W_{C_{m}}^{J}\left(T_{m}\right)\right\}
\end{aligned}
$$

Let us take an arbitrary $0<t_{0}<1$ and choose a point $U_{m}$ in $B$ satisfying

$$
\operatorname{Tr}\left(C_{m} U_{m} T_{m} U_{m}^{-1}\right)=a-\frac{1}{2^{m}}+i\left(t_{0} b_{m}+\left(1-t_{0}\right) c_{m}\right)
$$

for each $m$. Since the set $B$ is compact, we consider a subsequence $U_{m_{k}}$ of $U_{m}$ converging to a point $U_{0}$ of $B$. Then as $k \rightarrow \infty$

$$
\operatorname{Tr}\left(C_{m_{k}} U_{m_{k}} T_{m_{k}} U_{m_{k}}^{-1}\right) \rightarrow \operatorname{Tr}\left(C U_{0} T U_{0}^{-1}\right)=a+i\left(t_{0} b_{0}+\left(1-t_{0}\right) c_{0}\right) .
$$

Hence the set

$$
\begin{equation*}
\left\{z \in W_{C}^{J}(T): \Re(z)=a\right\} \tag{4.1}
\end{equation*}
$$

is also a closed interval. We set $a_{0}=\sum_{h=1}^{n} c_{h} \lambda_{h}$. Since the set (4.1) depends continuously on $a$, then

$$
W_{C}^{J}(T)=\left\{x+i y:-\infty<x \leq a_{0}, \phi_{1}(x) \leq y \leq \phi_{2}(x)\right\}
$$

for some continuous real-valued functions $\phi_{1}(x) \leq \phi_{2}(x)$ defined on the halfline $\left(-\infty, a_{0}\right]$, and so it is simply connected.
Now we prove that the boundary of $W_{C}^{J}(T), \partial W_{C}^{J}(T)$, is convex. The set $W_{C}^{J}(T)$ has a point $z_{0}$ with $\Re\left(z_{0}\right)=a_{0}$ and the line $\Re(z)=a_{0}$ is a support line of $W_{C}^{J}(T)$. If $\partial W_{C}^{J}(T)$ is not convex, there exists a tangent line at some point $z_{1} \in \partial W_{C}^{J}(T)$ expressed as $\Re\left(z \mathrm{e}^{-i \theta_{1}}\right)=b_{0}$ for some $\theta_{1}, b_{0} \in \mathbf{R}$ satisfying $\Re\left(z_{2} \mathrm{e}^{-i \theta_{1}}\right)<b_{0}, \Re\left(z_{3} \mathrm{e}^{-i \theta_{1}}\right)>b_{0}$, where $z_{2}, z_{3} \in W_{C}^{J}(T)$. Then we find a point $z_{4}$ on the arc of $\partial W_{C}^{J}(T)$ joining $z_{0}, z_{1}$ and a support line $\ell_{\theta}$ passing through $z_{4}, \Re\left(z \mathrm{e}^{-i \theta_{0}}\right)=a_{\theta}$, such that $W_{C}^{J}(T) \cap \ell_{\theta_{0}}$ is not connected. However the last condition of $\ell_{\theta}$ contradicts Proposition 3.3. So $\partial W_{C}^{J}(T)$ is convex.

We take a point $x_{0}<a_{0}$ at which $\phi_{1}, \phi_{2}$ are differentiable. Then we have $\phi_{2}^{\prime}\left(x_{0}\right) \leq \phi_{1}^{\prime}\left(x_{0}\right)$ and the cone $\left\{x+i y \in \mathbf{C}: \phi_{1}^{\prime}\left(x_{0}\right)\left(x-x_{0}\right)+\phi_{1}\left(x_{0}\right) \leq y \leq\right.$ $\left.\phi_{2}^{\prime}\left(x_{0}\right)\left(x-x_{0}\right)+\phi_{2}\left(x_{0}\right)\right\}$ satisfies the last assertion of Theorem 1.1. This completes the proof of the theorem.

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