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ON GABOR FRAMES WITH HERMITE FUNCTIONS: POLYANALYTIC SPACES FROM THE HEISENBERG GROUP

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Dedicated to Paul L. Butzer on the occasion of his 80th birthday

ABSTRACT: Gabor frames with Hermite functions are equivalent to Fock frames with monomials windows and to sampling sequences in true poly-Fock spaces. In the L^2 case, such an equivalence results from the unitarity of the so-called true poly-Bargmann transform. We will extend the equivalence to Banach spaces, applying Feichtinger-Gröchenig coorbit theory to the Fock representation of the Heisenberg group. This task requires L^p estimates for the true poly-Bargmann transform which are obtained using the theory of modulation spaces. In the L^2 case we will also revisit the complex variables approach and obtain an explicit formula for the interpolation problem in true poly-Fock spaces, which yields Gabor frames with Hermite functions by a duality argument.

KEYWORDS: Gabor frames, Heisenberg group, Hermite functions, polyanalytic functions, coorbit theory, Fock spaces.
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1. Introduction

In this note we will be concerned with *Gabor expansions* of the form

$$f(t) = \sum_{l,k\in\mathbb{Z}^d} c_{k,l} e^{2\pi i\omega_l t}\Phi_n(t - x_k), \tag{1}$$

where Φ_n are the *d-dimensional Hermite functions*

$$\Phi_n(x) = \prod_{j=1}^{d} h_{n_j}(x_j), \quad \text{with } h_n(t) = c_n e^{\pi t^2}\left(\frac{d}{dt}\right)^n\left(e^{-2\pi t^2}\right)$$

(c_n the orthonormalizing constant). Expansions (1) are useful in image processing [15] and in multiplexing of signals [3], [20], [1].

Gabor expansions with Hermite functions enjoy rich "soft" (functional/group theoretical) and "hard" (complex variables) analytic structures. They have

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been recently introduced in mathematical time-frequency analysis by Gröchenig and Lyubarskii [19] and studied further in [13], [20] and [1], with an emphasis on vector-valued Gabor frames (the so called Gabor super-frame).

The nice properties of Gabor expansions with Hermite functions are the result of an interplay between classical (orthogonal polynomials, Weierstrass sigma functions) and modern (frame and group theory, modulation spaces) mathematical topics. In the present work, we will combine complex variables methods with the theory of expansions through integrable group representations (Feichtinger-Gröchenig *coorbit theory* [10], [9], [16]).

It has been discovered recently [1] that expansions (1) are equivalent to sampling problems in spaces of functions which satisfy generalized Cauchy-Riemann equations of the form

$$\left(\frac{d}{d\overline{z}}\right)^n F(z) = \frac{1}{2^n}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial \omega}\right)^n F(x+i\omega) = 0.$$

Those functions are century-old objects known as *polyanalytic functions.* They have been investigated thoroughly, notably by the russian school led by Balk [4] and they provide extensions of classical operators from complex analysis [5]. The connection to Gabor expansions seems to be yet another instance of how, as epigraphed by Folland [12], "the abstruse meets the applicable" in time-frequency analysis. Indeed, time-frequency analysis is prone to reveal unexpected relations to other fields of mathematics. Two recent examples are the associations with Banach algebras [18] and with noncommutative geometry [14].

Our ideas are organized in three sections. In section 2, we explain the connection between Gabor transforms with Hermite functions and the true polyanalytic Bargmann transform. The L^2 theory is mostly from [1]. Moreover, we extend the L^2 theory of the true poly-Bargmann transform to the appropriated Banach spaces, using the theory of Modulation spaces. In section 3 we study Gabor frames with Hermite functions in $L^2(\mathbb{R})$, using those generalized transforms. We provide a different proof of the sufficient condition given in [19] for the the lattice parameters which generate those frames. In the last section we study Gabor Banach frames with Hermite functions in higher dimensions, by using the representations of the Heisenberg group and Feichtinger-Gröchenig coorbit theory. Using these results we prove a sampling theorem for certain Fock spaces of polyanalytic functions (true Banach poly-Fock spaces). The results can also be seen as Banach Gabor frames with Hermite functions in Modulation spaces.

2. Gabor transforms with Hermite functions

2.1. The Bargmann transform. Expansions of the type (1) are closely related to the samples of the Gabor transform of f with respect to the windows Φ_n:

$$V_{\Phi_n} f(x,\omega) = \int_{\mathbb{R}^d} f(t)\overline{\Phi_n(t-x)}e^{-2\pi i\omega t}dt. \tag{2}$$

The Hermite function $\Phi_0(t) = 2^{\frac{d}{4}}e^{-\pi t^2}$ is the d-dimensional Gaussian. Writing $z = x + i\omega$ a simple calculation shows that

$$e^{-i\pi x\omega+\pi\frac{|z|^2}{2}} V_{\Phi_0} f(x,-\omega) = \int_{\mathbb{R}^d} f(t)e^{2\pi tz-\pi z^2-\frac{\pi}{2}t^2}dt = (\mathcal{B}f)(z), \tag{3}$$

where $(\mathcal{B}f)(z)$ is the Bargmann transform of f. The function $\mathcal{B}f$ satisfies

$$\frac{d}{d\overline{z}}\mathcal{B}f = 0.$$

Thus, it is an analytic function and we can use powerful complex analysis tools (at least in the case $d = 1$) to study its properties. Moreover, it is an isomorphism $\mathcal{B} : L^2(\mathbb{R}^d) \to \mathcal{F}(\mathbb{C}^d)$, where $\mathcal{F}(\mathbb{C}^d)$ stands for the Bargmann-Fock space of analytic functions in $\mathbb{C}^d$ with the norm

$$\|F\|^2_{\mathcal{F}(\mathbb{C}^d)} = \int_{\mathbb{C}^d} |F(z)|^2 e^{-\pi|z|^2}dz. \tag{4}$$

2.2. The true poly-Bargmann transform. Let us go back to the window Φ_n for general n. A calculation (see [19] for details) shows that

$$e^{-i\pi x\omega+\frac{\pi}{2}|z|^2} V_{\Phi_n} f(x,-\omega) = (\pi^{|n|}n!)^{-\frac{1}{2}} \sum_{0\leq k\leq n} \binom{n}{k}(-\pi\overline{z})^k \left(\frac{d}{dz}\right)^{n-k} [\mathcal{B}f](z). \tag{5}$$

Now we have a serious obstruction regarding the possibility of using complex analysis tools: The function on the right hand side of (5) is analytic no more. However, differentiating (5) $n+1$ times with respect to $\overline{z}$, one realizes that it satisfies the equation

$$\left(\frac{d}{d\overline{z}}\right)^{n+1} f(z) = 0. \tag{6}$$

Functions satisfying (6) are called polyanalytic of order $n+1$. Moreover, we can use Leibnitz formula in order to write (5) as

$$e^{-i\pi x\omega+\frac{\pi}{2}|z|^2}V_{\Phi_n}f(x,-\omega)=(\pi^{|n|}n!)^{-\frac{1}{2}}e^{\pi|z|^2}\frac{d^n}{dz^n}\left[e^{-\pi|z|^2}\mathcal{B}f(z)\right].$$

Now, integration by parts shows that the Fock norm (4) of these functions is finite. Therefore, it is natural to define a transform as

$$\mathcal{B}^n f(z)=e^{-i\pi x\omega+\frac{\pi}{2}|z|^2}V_{\Phi_n}f(x,-\omega). \tag{7}$$

This transform has been studied in [1] and [2], where it is shown that $\|\mathcal{B}^n f\|_{\mathcal{F}(\mathbb{C}^d)}=\|f\|_{L^2(\mathbb{R}^d)}$. Thus the image of $L^2(\mathbb{R}^d)$ under $\mathcal{B}^n$ is a function with membership in a space equiped with the same norm as $\mathcal{F}(\mathbb{C}^d)$, whose elements are not analytic functions but satisfy the equation (6). These spaces are denoted by $\mathcal{F}^{n-1}(\mathbb{C}^d)$ and called *true poly-Fock spaces of order* $n-1$ (since their elements are polyanalytic of order n but not polyanalytic of any other lower order, see [1], [2] and [26]). The prefix "true" has been used by Vasilevski [26] to distinguish them from the polyanalytic Fock space, $\mathbf{F}^n(\mathbb{C}^d)$, constituted by *all* polyanalytic functions up to order n. The relation between the spaces is given by the orthogonal decomposition:

$$\mathbf{F}^n(\mathbb{C}^d)=\mathcal{F}^0(\mathbb{C}^d)\oplus\ldots\oplus\mathcal{F}^{n-1}(\mathbb{C}^d), \tag{8}$$

the transform $\mathcal{B}^n$ is the *true poly-Bargmann transform* of f and its "multiplexed version" $\mathbf{B}^n: L^2(\mathbb{R}^d,\mathbb{C}^n)\to\mathbf{F}(\mathbb{C}^d)$ defined, for $\mathbf{f}=(f_0+\ldots f_{n-1})\in L^2(\mathbb{R}^d,\mathbb{C}^n)$ by

$$\mathbf{B}^n\mathbf{f}=\mathcal{B}^0 f_0+\ldots+\mathcal{B}^{n-1}f_{n-1},$$

is also a Hilbert space isomorphism called the *poly-Bargmann transform* [1]. The applications in multiplexing are a result of (8), since we can "encode" every individual signal into a space $\mathcal{F}^k(\mathbb{C}^d)$, multiplex the n signals and then, when required, we can recover each signal from the orthogonal projection of the multiplexed signal over $\mathcal{F}^k(\mathbb{C}^d)$.

2.3. The true poly-Bargmann transform and the spaces $\mathcal{F}_p(\mathbb{C}^d)$. Let $p\in[1,\infty[$ and consider the norm

$$\|F\|^p_{\mathcal{F}_p(\mathbb{C}^d)}=\int_{\mathbb{C}^d}|F(z)|^p\,e^{-\pi p\frac{|z|^2}{2}}dz.$$

Consider among the functions with finite $\mathcal{F}_p$ norm, those such that

$$\left(\frac{d}{d\overline{z}}\right)^{n+1} F(z) = 0, \text{ but } \left(\frac{d}{d\overline{z}}\right)^{k} F(z) \neq 0, k = 0, ..., n.$$

We denote these (true poly-Fock) Banach spaces by $\mathcal{F}_p^n(\mathbb{C}^d)$. Clearly, $\mathcal{F}_p^0(\mathbb{C}^d) = \mathcal{F}_p(\mathbb{C}^d)$ is the standard analytic Fock space. The space $\mathcal{F}_1(\mathbb{C}^d)$ is the complex version of the Feichtinger algebra ([7],[8]) and it will play an important role in last section of the paper.

Now we will need the concept of modulation space. Follow notations of [17] and set $M^p(\mathbb{R}^{2d}) = M^{p,p}(\mathbb{R}^{2d})$. The modulation space $M^p(\mathbb{R}^{2d})$ consists of all tempered distributions such that $V_g f \in L^p(\mathbb{R}^{2d})$ equipped with the norm

$$\|f\|_{M^p(\mathbb{R}^{2d})} = \|V_g f\|_{L^p(\mathbb{R}^{2d})}.$$

Modulation spaces are ubiquitous in time-frequency analysis. They were introduced by Feichtinger in [6].

With a view to studying sampling sequences in poly-Fock spaces $\mathcal{F}_p^n(\mathbb{C}^d)$ for general p, we prove some statements concerning the properties of the true poly-Fock transform which may have independent interest.

In the proof of the next proposition, a property of Modulation spaces (the definition of Modulation space is independent of the particular window chosen) comes in handy. We wonder whether one can prove it directly from the complex variables setting.

Proposition 1. *For every $f \in L^2(\mathbb{R}^{2d})$, there exist constants C, D, such that*

$$C \|\mathcal{B}^n f\|_{\mathcal{F}_p^n(\mathbb{C}^d)} \leq \|\mathcal{B} f\|_{\mathcal{F}_p(\mathbb{C}^d)} \leq D \|\mathcal{B}^n f\|_{\mathcal{F}_p^n(\mathbb{C}^d)}. \tag{9}$$

Proof: This follows from the theory of modulation spaces: since the definition of Modulation space is independent of the particular window chosen [17, Proposition 11.3.1], then the norms

$$\|f\|_{M^p(\mathbb{R}^{2d})} = \|V_{\Phi_n} f\|_{L^p(\mathbb{R}^{2d})}$$

and

$$\|f\|_{M^p(\mathbb{R}^{2d})} = \|V_{\Phi_0} f\|_{L^p(\mathbb{R}^{2d})},$$

must be equivalent. Therefore, there exist constants C, D, such that

$$C \|V_{\Phi_n} f\|_{L^p(\mathbb{R}^{2d})} \leq \|V_{\Phi_0} f\|_{\mathcal{F}_p(\mathbb{C}^d)} \leq D \|V_{\Phi_n} f\|_{L^p(\mathbb{R}^{2d})}.$$

By definition of $\mathcal{B}^n$ and $\mathcal{B}$, this yields (9). ∎

Corollary 1. *For every* $F \in \mathcal{F}_p^n(\mathbb{C}^d)$ *there exists* $f \in L^2(\mathbb{R}^d)$ *such that* $F = \mathcal{B}^n f$ *and*

$$C \left\| F \right\|_{\mathcal{F}_p^n(\mathbb{C}^d)} \leq \left\| \mathcal{B} f \right\|_{\mathcal{F}_p(\mathbb{C}^d)} \leq D \left\| F \right\|_{\mathcal{F}_p^n(\mathbb{C}^d)} . \tag{10}$$

Proof: Let $F \in \mathcal{F}_p^n(\mathbb{C}^d) \cap \mathcal{F}_2^n(\mathbb{C}^d)$. Since $\mathcal{B}^n$ maps $L^2(\mathbb{R}^d)$ onto $\mathcal{F}_2^n(\mathbb{C}^d)$, it is possible to choose $f \in L^2(\mathbb{R}^d)$ such that $F = \mathcal{B}^n f$. Thus, the range of $\mathcal{B}^n$ contains a set which is dense in $\mathcal{F}_p^n(\mathbb{C}^d)$. Thus, $\mathcal{B}^n : L^2(\mathbb{R}^d) \rightarrow \mathcal{F}_p^n(\mathbb{C}^d)$ is onto. Then (9) is equivalent to (10). ■

From the norm equivalences it is possible to adapt the arguments of [23] in order to prove that the orthogonal decomposition (8) extends as a decomposition in direct sums of Banach spaces.

Proposition 2. *The following decomposition holds for* $1 < p < \infty$*:*

$$\mathbf{F}_p^n(\mathbb{C}^d) = \mathcal{F}_p^0(\mathbb{C}^d) \oplus ... \oplus \mathcal{F}_p^{n-1}(\mathbb{C}^d).$$

Proof: Define an integral operator P^n acting on $L^p(\mathbb{C}^d, e^{-\pi|z|^2})$ by

$$(P^n F)(w) = \int_{\mathbb{C}^d} F(z) K^n(w,z) e^{-\pi|z|^2} dz,$$

where

$$K^n(w,z) = \frac{1}{n!} e^{\pi|w|^2} \left(\frac{d}{dw} \right)^n \left[e^{\pi\overline{z}w - \pi|w|^2} (w-z)^n \right]$$

is the reproducing kernel of $\mathcal{F}_2^n(\mathbb{C}^d)$ [2]. It follows that, if $F \in \mathcal{F}_p^n(\mathbb{C}^d) \cap \mathcal{F}_2^n(\mathbb{C}^d)$, then

$$(P^n F)(w) = F(w), \tag{11}$$

and if $F \in \mathcal{F}_2^k(\mathbb{C}^d)$, with $k \neq n$, then $(P^n F)(w) = 0$. By density, (11) is valid for every $F \in \mathcal{F}_p^n(\mathbb{C}^d)$ and $P^n F = 0$ if $F \in \mathcal{F}_p^k(\mathbb{C}^d)$, with $k \neq n$. A similar argument can be used to extend the reproducing equation to the space $\mathbf{F}_p^n(\mathbb{C}^d)$, using the operator $\mathbf{P}^n = P^0 + ... + P^{n-1}$. Thus, every $\mathbf{F} \in \mathbf{F}_p^n(\mathbb{C}^d)$, can be written as

$$\mathbf{P}^n \mathbf{F} = P^0 \mathbf{F} + ... + P^{n-1} \mathbf{F} = F_0 + ... + F_{n-1},$$

with $F_k \in \mathcal{F}_p^k(\mathbb{C}^d)$. Moreover, the intersection of the spaces is $\{0\}$: if $F \in \mathcal{F}_2^k(\mathbb{C}^d)$, with $k \neq n$, then $(P^n F)(w) = 0$; again by density, this extends to $\mathcal{F}_p^k(\mathbb{C}^d)$. ■

3. Gabor frames in L^2

Stable Gabor expansions (1) can be obtained from frame theory. Given a point $\lambda = (x, \omega)$ in phase-space $\mathbb{R}^{2d}$, the corresponding time-frequency shift is

$$\pi_\lambda f(t) = e^{2\pi i \omega t} f(t - \omega), \quad t \in \mathbb{R}^d.$$

Using this notation, the Gabor transform with respect to the window g can be written as

$$V_g f(x, \omega) = \langle f, \pi_\lambda g \rangle_{L^2(\mathbb{R}^d)}.$$

Let $\Gamma = \{\lambda_{k,l} = (x_k, \omega_l) : l, k \in \mathbb{Z}^d\}$. The Gabor system $\mathcal{G}(\Phi_n, \Gamma) = \{\pi_{\lambda_{k,l}} \Phi_n : l, k \in \mathbb{Z}^d\}$ is a *Gabor frame* or *Weyl-Heisenberg frame* in $L^2(\mathbb{R}^d)$, whenever there exist constants $A, B > 0$ such that, for all $f \in L^2(\mathbb{R}^d)$,

$$A \|f\|^2_{L^2(\mathbb{R}^d)} \leq \sum_{l,k \in \mathbb{Z}^d} \left| \left\langle f, \pi_{\lambda_{k,l}} \Phi_n \right\rangle_{L^2(\mathbb{R}^d)} \right|^2 \leq B \|f\|^2_{L^2(\mathbb{R}^d)}. \tag{12}$$

3.1. A polyanalytic interpolation formula for $\mathcal{F}^n(\mathbb{C})$. In this section we restrict to $d = 1$. It has been proved recently ([19], [20]) that, if $\Lambda = M\mathbb{Z}^2 \subset \mathbb{R}^2$ is a lattice in $\mathbb{R}^2$, where M is a 2×2 invertible real valued matrix with $s(\Lambda) = |\det M| < 1/(n+1)$, then $\mathcal{G}(h_n, \Gamma)$ is a Gabor frame. In this section we will obtain this result from an interpolation formula in $\mathcal{F}^n(\mathbb{C})$, which is a polyanalytic version of the one used in [25].

Considering the Weierstrass sigma function associated with the lattice $\Lambda = \{\lambda_{k,l}\}_{k,l \in \mathbb{Z}}$,

$$\sigma_\Lambda(z) = z \prod_{\lambda_{k,l} \in \Lambda \setminus \{0\}} \left(1 - \frac{z}{\lambda_{k,l}}\right) e^{\frac{z}{\lambda_{k,l}} + \frac{z^2}{2\lambda_{k,l}^2}},$$

we have

$$|\sigma_\Lambda(z)| \leq C e^{\frac{\pi}{2s(\Lambda)}|z|^2}. \tag{13}$$

The lattice Λ is *an interpolating sequence* for $\mathcal{F}^n(\mathbb{C})$ if, for every sequence $\{\alpha_{k,l}\} \in l^2$, there exists $F \in \mathcal{F}^n(\mathbb{C})$ such that

$$e^{-\frac{\pi}{2}|\lambda_{k,l}|^2} F(\lambda_{k,l}) = \alpha_{k,l},$$

for every $\lambda_{k,l} \in \Lambda$.

The lattice Λ is required to be separated: $\inf_{n,m,k,j \in \mathbb{Z}^d} |\lambda_{k,l} - \lambda_{n,m}| = \delta > 0$.

Theorem 1. *If $s(\Lambda) > n+1$, then the lattice Λ is an interpolating sequence for $\mathcal{F}^n(\mathbb{C})$. The interpolation problem is solved explicitly by*

$$F(z) = \sum_{k,l\in\mathbb{Z}^d} a_{k,l} e^{\pi\overline{\lambda_{k,l}}z-\pi|\lambda_{k,l}|^2} \frac{G_{\Lambda-\lambda_{k,l}}(z-\lambda_{k,l})}{(z-\lambda_{k,l})}. \tag{14}$$

Proof: Estimate (13) gives

$$\left|(\sigma_\Lambda)^{s(\Lambda)}(z)\right| \leq Ce^{\frac{\pi}{2}|z|^2}.$$

Thus, since $s(\Lambda) > n+1$, then $(\sigma_\Lambda)^{n+1} \in \mathcal{F}_2(\mathbb{C})$. As a result, there exists a $f_\Lambda \in L^2(\mathbb{R})$ such that $(\sigma_\Lambda(z))^{n+1} = \mathcal{B}f_\Lambda(z)$. Now, let

$$G_\Lambda(z) = (\mathcal{B}^n f_\Lambda)(z) = (\pi^{|n|}n!)^{-\frac{1}{2}} e^{\pi|z|^2} \frac{d^n}{dz^n}\left[e^{-\pi|z|^2}(\sigma_\Lambda(z))^{n+1}\right].$$

Then $G_\Lambda(z) \in \mathcal{F}_2^n(\mathbb{C})$ and $G_\Lambda(\lambda_{k,l}) = 0$ for every $\lambda_{k,l} \in \Lambda$. Similarly, define $\left(\sigma_{\Lambda-\lambda_{k,l}}\right)^{n+1}(z) = \mathcal{B}f_{\Lambda-\lambda_{k,l}}(z)$ and $G_{\Lambda-\lambda_{k,l}}(z) = (\mathcal{B}^n f_{\Lambda-\lambda_{k,l}})(z)$. Since $G_{\Lambda-\lambda_{k,l}}(z) \in \mathcal{F}_2^n(\mathbb{C})$, then

$$\int_{\mathbb{C}^d} \left|G_{\Lambda-\lambda_{k,l}}(z-\lambda_{k,l})\right|^2 e^{-\pi|z-\lambda_{k,l}|^2} dz < \infty. \tag{15}$$

Now,

$$\left|e^{-\frac{\pi}{2}|z|^2}F(z)\right| = \sum_{k,l\in\mathbb{Z}^d} \left|a_{k,l}e^{-\frac{\pi}{2}|\lambda_{k,l}|^2}\right| e^{-\frac{\pi}{2}|z-\lambda_{k,l}|^2} \left|\frac{G_{\Lambda-\lambda_{k,l}}(z-\lambda_{k,l})}{(z-\lambda_{k,l})}\right|.$$

This estimate shows that the series converges uniformly to a polyanalytic function F such that $F(\lambda_{k,l}) = a_{k,l}$. By Cauchy-Schwarz,

$$e^{-\pi|z|^2}|F(z)|^2 \leq \sum_{k,l\in\mathbb{Z}^d} e^{-\pi|\lambda_{k,l}|^2}|a_{k,l}|^2 \sum_{k,l\in\mathbb{Z}^d} e^{-\pi|z-\lambda_{k,l}|^2} \left|\frac{G_{\Lambda-\lambda_{k,l}}(z-\lambda_{k,l})}{(z-\lambda_{k,l})}\right|^2.$$

Integrating with respect to area measure in the plane and using (15), we see that $F \in \mathcal{F}^n(\mathbb{C})$. ■

3.2. Gabor frames with Hermite functions in $L^2(\mathbb{R})$. Following Feichtinger and Kozek [11], the adjoint lattice Λ^0 is defined by the commuting property as

$$\Lambda^0 = \{\mu \in \mathbb{R}^{2d} : \pi_z\pi_\mu = \pi_\mu\pi_z, \text{ for all } z \in \Lambda\}.$$

If $\Lambda = \alpha\mathbb{Z} \times \beta\mathbb{Z}$, then $\Lambda^0 = \beta^{-1}\mathbb{Z} \times \alpha^{-1}\mathbb{Z}$. There exists a remarkable duality between the Gabor system with respect to Λ^0 and those with respect to Λ. This is often refered to as the *Janssen-Ron-Shen duality principle* [24], [21].

Theorem A (*duality principle*)**.** The Gabor system $\mathcal{G}(g, \Lambda)$ is a *frame* for $L^2(\mathbb{R}^d)$ if and only if the Gabor system $\mathcal{G}(g, \Lambda^0)$ is a Riesz basis for its linear span inside $L^2(\mathbb{R}^d)$.

Combining the duality principle with Theorem 1, one recovers the theorem of Gröchenig and Lyubarskii [19]:

Theorem 2. *If $s(\Lambda) < \frac{1}{n+1}$, then the Gabor system $\mathcal{G}(h_n, \Lambda)$ is a* frame *for $L^2(\mathbb{R})$.*

Proof: First observe that $s\left(\Lambda^0\right) = \left|\det M^{-1}\right| = \frac{1}{s(\Lambda)}$. Thus, if $s(\Lambda) < \frac{1}{n+1}$, then $s\left(\Lambda^0\right) > n+1$. It follows from Theorem 1 that the lattice Λ^0 is an interpolating sequence for $\mathcal{F}^n(\mathbb{C})$. Since

$$\langle f, \pi_\lambda h_n \rangle_{L^2(\mathbb{R}^d)} = V_{h_n} f(x, \omega) = e^{i\pi x\omega - \frac{\pi}{2}|z|^2} \mathcal{B}^n f(z),$$

then it is clear that $\mathcal{G}(h_n, \Lambda^0)$ is a Riesz basis for its linear span inside $L^2(\mathbb{R})$. By the duality principle, the Gabor system $\mathcal{G}(h_n, \Lambda)$ is a *frame* for $L^2(\mathbb{R})$. ■

4. Banach frames

4.1. Banach frames in the Fock space $\mathcal{F}_p(\mathbb{C}^d)$. Define a "translation" β_z on $\mathcal{F}(\mathbb{C}^d)$ by

$$\beta_z F(\zeta) = e^{-\pi \frac{|z|^2}{2}} e^{\pi \overline{z} \zeta} F(\zeta - z). \tag{16}$$

The action of the Bargmann-Fock representation of the Heisenberg group (which can be identified with $\mathbb{C}^d \times \mathbb{R}$) in the space $\mathcal{F}(\mathbb{C}^d)$ is given by the operator β_z is (modulo the action on $\mathbb{R}$). The operator β_z satisfies the intertwining property

$$\beta_z \mathcal{B} = \mathcal{B}\left(e^{i\pi x\omega} \pi_z\right), \qquad z = x + i\omega, \tag{17}$$

which provides the equivalence between the Bargmann-Fock and the Schrödinger representations of the Heisenberg group (which is the operator $e^{i\pi x\omega}\pi_z$, after identifying with $\mathbb{R}^{2d} \times \mathbf{T}$ and factoring the action on the torus-see [17, Chapter 9] for details).

Now we collect some facts from Feichtinger-Gröchenig coorbit theory in the context of $\mathcal{F}_p(\mathbb{C}^d)$:

Definition [16]. The set $\{\beta_{\lambda_{k,l}} g : l,k \in \mathbb{Z}^d\}$ is a Banach frame for $\mathcal{F}_p(\mathbb{C}^d)$ if the following three conditions hold:

i)$F \in \mathcal{F}_p(\mathbb{C}^d)$ if and only if $\left\langle f, \beta_{\lambda_{k,l}} g \right\rangle_{\mathcal{F}_2(\mathbb{C}^d)} \in l^p$.

ii)There exist two constants A, B depending only on g such that

$$A \left\| F \right\|_{\mathcal{F}_p(\mathbb{C}^d)} \leq \left[\sum_{l,k \in \mathbb{Z}^d} \left| \left\langle F, \beta_{\lambda_{k,l}} g \right\rangle_{\mathcal{F}_2(\mathbb{C}^d)} \right|^p \right]^{\frac{1}{p}} \leq B \left\| F \right\|_{\mathcal{F}_p(\mathbb{C}^d)}.$$

iii)$f \in \mathcal{F}_p(\mathbb{C}^d)$ can be unambiguously reconstructed from the coefficients $\left\langle F, \beta_{\lambda_{k,l}} g \right\rangle_{\mathcal{F}_2(\mathbb{C}^d)}$.

Let U a neighborhood of 0 in the complex plane. A set $X = (\lambda_{k,l})$ of points in $\mathbb{C}^d$ is U*-dense* if $\bigcup_{l,k \in \mathbb{Z}^d} \beta_{\lambda_{k,l}} U = \mathbb{C}^d$ and it is separated if for some compact neighborhood V of 0 we have $\beta_{\lambda_{k,l}} V \cap \beta_{\lambda_{i,j}} V = \{\}$, $(k,l) \neq (i,j)$ and relatively separated if X is a finite union of separated sets. Now set $G(z) = \langle g, \beta_z g \rangle_{\mathcal{F}_2(\mathbb{C}^d)}$ and define the oscillation function of G as

$$osc_U G(z) = \sup_{u \in U} \left| \beta_u G(z) - G(z) \right|.$$

The application of Theorem T and Theorem S in [16] gives the following result.

Theorem B [16] *Assume that* $g \in F_1(\mathbb{C}^d)$. *Let* U *be so small that*

$$\left\| osc_U G \right\|_{\mathcal{F}_1(\mathbb{C}^d)} < 1.$$

Then for any U*-dense and relatively separated set* $\Gamma = \{\xi_{k,l} = x_k + i\omega_l) : l,k \in \mathbb{Z}^d\}$, *every* $f \in F_p(\mathbb{C}^d)$ *has the atomic decomposition*

$$F(z) = \sum_{l,k \in \mathbb{Z}^d} c_{k,l} \beta_{\xi_{k,l}} g,$$

for some numbers $\lambda_{m,l}$, *with convergence in* $F_p(\mathbb{C}^d)$ *and* $\left\| F \right\|_{\mathcal{F}_p(\mathbb{C}^d)}$ *equivalent to* $\left(\sum_{k,l \in I} \left| c_{k,l} \right|^p \right)^{\frac{1}{p}}$. *Moreover, the system* $(g, \Gamma) = \{\beta_{\lambda_{k,l}} g : l,k \in \mathbb{Z}^d\}$ *is a Banach frame for* $F_p(\mathbb{C}^d)$.

If one takes $g = 1 \in \mathcal{F}_1(\mathbb{C}^d)$, we are led to the sampling theorem 8.4 of [22]. However, we can also choose as windows any element of the canonical orthonormal basis of $\mathcal{F}(\mathbb{C}^d)$,

$$e_n(z) = \left(\frac{\pi^{|n|}}{n!}\right)^{\frac{1}{2}} z^n = \prod_{j=1}^{d} \frac{\pi^{n_j}}{\sqrt{n_j!}} z^{n_j}, \tag{18}$$

since $e_n \in \mathcal{F}_1(\mathbb{C}^d)$. Thus, under the conditions of the above theorem, $\mathcal{F}(e_n, \Gamma)$ is a Banach frame for $\mathcal{F}_p(\mathbb{C}^d)$. In the next section we will use the fact that $(\mathcal{B}\Phi_n)(z) = e_n(z)$.

4.2. Sampling sequences for the poly-Fock space $\mathcal{F}_p^n(\mathbb{C}^d)$. As remarked in [16], combining the intertwining property (17) of $\mathcal{B}$ with Theorem 4.8 in [9] (about automatic extension of intertwining operators from Hilbert to Banach settings), $\mathcal{B}$ extends to an isomorphism between $M^p(\mathbb{R}^{2d})$ and $\mathcal{F}_p(\mathbb{C}^d)$. However, such a direct argument does not apply to the transforms $\mathcal{B}^n$, since $\mathcal{B}^n$ does not satisfy (17). Thus, we will require the results of section 2.3.

Theorem 3. *Set $G_n(z) = \langle e_n, \beta_z e_n\rangle_{\mathcal{F}_2(\mathbb{C}^d)}$ and let U be so small that*

$$\|osc_U G_n\|_{\mathcal{F}_1(\mathbb{C}^d)} < 1.$$

Then the following holds:

(1) $F \in \mathcal{F}_p^n(\mathbb{C}^d)$ if and only if $e^{-\pi\frac{\left|\lambda_{k,l}\right|^2}{2}} F(\lambda_{k,l}) \in l^p$.
(2) There exist two constants A, B depending only on g such that

$$A\,\|F\|_{\mathcal{F}_p^n(\mathbb{C}^d)} \leq \left[\sum_{l,k\in\mathbb{Z}^d} |F(\lambda_{k,l})|^p\, e^{-\pi p\frac{\left|\lambda_{k,l}\right|^2}{2}}\right]^{\frac{1}{p}} \leq B\,\|F\|_{\mathcal{F}_p^n(\mathbb{C}^d)}.$$

(3) $F \in \mathcal{F}_p(\mathbb{C}^d)$ can be unambiguously reconstructed from the samples $\{F(\lambda_{k,l})\}_{l,k\in\mathbb{Z}^d}$.

Proof: Since $(\mathcal{B}\Phi_n)(z) = e_n(z)$, the unitarity of the Bargmann transform together with the intertwining property (17), gives:

$$\begin{aligned}
\left\langle \mathcal{B}f, \beta_{\lambda_{k,l}} e_n\right\rangle_{\mathcal{F}_2(\mathbb{C}^d)} &= \left\langle \mathcal{B}f, \beta_{\lambda_{k,l}}\mathcal{B}\Phi_n, \right\rangle_{\mathcal{F}_2(\mathbb{C}^d)} \\
&= \left\langle \mathcal{B}f, \mathcal{B}\left(e^{-i\pi x_k\omega_l}\pi_{\lambda_{k,l}}\Phi_n\right), \right\rangle_{\mathcal{F}_2(\mathbb{C}^d)} \\
&= \left\langle f, e^{-i\pi x_k\omega_l}\pi_{\lambda_{k,l}}\Phi_n, \right\rangle_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

As a result,

$$\left\langle \mathcal{B}f, \beta_{\lambda_{k,l}} e_n \right\rangle_{\mathcal{F}_2(\mathbb{C}^d)} = e^{-i\pi x_k \omega_l} \left\langle f, \pi_{\lambda_{k,l}} \Phi_n \right\rangle_{L^2(\mathbb{R}^d)} = e^{-\pi \frac{\left|\lambda_{k,l}\right|^2}{2}} (\mathcal{B}^n f)(\lambda_{k,l}). \quad (19)$$

It follows immediately that $\mathcal{F}(e_n, \Gamma)$ is a Banach frame for $\mathcal{F}_2(\mathbb{C}^d)$ if and only if $\Gamma = \{\xi_{k,l} = x_k + i\omega_l) : l, k \in \mathbb{Z}^d\}$ is a sampling sequence for $\mathcal{F}_2^n(\mathbb{C}^d)$. We can extend this equivalence to the Banach frame setting: given $F \in \mathcal{F}_p^n(\mathbb{C}^d)$, we choose $f \in L^2(\mathbb{R}^d)$ such that $F = \mathcal{B}^n f$. Since Theorem B applies, the first assertion follows immediately from i) in the definition of Banach frames, and the identity (19). To prove 2., observe that, combining ii) in the definition of Banach frames with (19), it follows that there exist two constants A, B depending only on g and such that

$$A \left\|\mathcal{B}f\right\|_{\mathcal{F}_p(\mathbb{C}^d)} \leq \left[\sum_{l,k \in \mathbb{Z}^d} \left|(\mathcal{B}^n f)(\lambda_{k,l})\right|^p e^{-\pi p \frac{\left|\lambda_{k,l}\right|^2}{2}} \right]^{\frac{1}{p}} \leq B \left\|\mathcal{B}f\right\|_{\mathcal{F}_p(\mathbb{C}^d)}. \quad (20)$$

The statement then follows from (9). ■

4.3. Gabor frames with Hermite functions in modulation spaces. Using again identity (19), Theorem B and

$$\left\|f\right\|_{M^p(\mathbb{R}^{2d})} = \left\|V_{\Phi_n} f\right\|_{L^p(\mathbb{R}^{2d})} = \left\|\mathcal{B}f\right\|_{\mathcal{F}_p(\mathbb{C}^d)},$$

we can rewrite (20) as an inequality yielding Banach Gabor frames with Hermite windows in modulation spaces.

Theorem 4. *In the conditions of the theorems in the previous subsections, we have that every $f \in M^p(\mathbb{R}^{2d})$ has an expansion of the form 1. Moreover,*

(1) $f \in M^p(\mathbb{R}^{2d})$ if and only if $\left\langle f, \pi_{\lambda_{k,l}} \Phi_n \right\rangle_{L^2(\mathbb{R}^{2d})} \in l^p$.
(2) There exist two constants A, B depending only on g such that

$$A \left\|f\right\|_{M^p(\mathbb{R}^{2d})} \leq \left[\sum_{l,k \in \mathbb{Z}^d} \left| \left\langle f, \pi_{\lambda_{k,l}} \Phi_n \right\rangle_{L^2(\mathbb{R}^{2d})} \right|^p \right]^{\frac{1}{p}} \leq B \left\|f\right\|_{M^p(\mathbb{R}^{2d})}.$$

(3) $f \in \mathcal{F}_p(\mathbb{C}^d)$ can be unambiguously reconstructed from the coefficients $\left\langle f, \pi_{\lambda_{k,l}} \Phi_n \right\rangle_{L^2(\mathbb{R}^{2d})}$.

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