ON INDICATRICES

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1. Introduction

We start by considering the space H of half-lines in R^{n+1} which can be identified with $S^n \times R^{n+1}$ in a natural way. Let $f : M \to R^{n+1}$ be a smooth, bounded map and assume that $g : M \to H$ is a smooth map such that, for $x \in M$, $g(x)$ is a half-line starting at $f(x)$. Now surround $f(M)$ by a round n-sphere S and define $F : M \to S$ by taking $F(x)$ to be the intersection point of $g(x)$ with S. We will say that F is a *Spherical Indicatrix*. The purpose of the present note is to study the map F in some particular cases. Maps of this type have cropped up in some of our work [3],[4].

2. Normal Indicatrices of a Codimension 1 Immersion

In what follows M will denote a boundaryless, compact, connected, oriented, n-dimensional manifold. Let $f : M \to R^{n+1}$ be a smooth immersion, smooth here meaning C^∞ . Assume that S is a round n-sphere surrounding $f(M)$. For what follows there is no loss of generality in assuming that the sphere is centred at the origin. Let now $g : M \to S^n \times R^{n+1}$ be of the form (U, f). The *Indicatrix* F_U is defined as follows. For $x \in M$, consider the half-line $f(x) + \alpha U(x), \alpha \geq 0$. Then $F_U(x)$ is the intersection of the half-line with S. We can write $F_U = f + \lambda U$, where λ is a positive, smooth map.

Let us now denote by $N : M \to R^{n+1}$ the normal unit vector field determined in the following way. If, for $x \in M$, θ_x is the orientation for the tangent space T_xM, then $[f_{*x}(\theta_x), N(x)]$ is the usual orientation of R^{n+1}. Here f_{*x} denotes the induced linear map and the tangent space to R^{n+1} at x will be identified with R^{n+1} itself. The maps F_N and F_{-N} are the *Normal Indicatrices.*

Maps of the type $f_\xi : M \to R^{n+k}$, with $f_\xi(x) = f(x)+\xi(x)$, were ξ is a parallel normal field where studied by Carter and Şentürk [1] among other people.

If M is not diffeomorphic to S^n then any indicatrix will have critical points. We shall see next what happens for S^n and the normal indicatrices.

Proposition 1: Let $f : S^n \to R^{n+1}$ be a smooth immersion such that its Gaussian curvature does not vanish. Then one of the maps F_N,F_{-N} is an immersion. Moreover, if the radius of S is sufficiently large then both maps are immersions.

Proof: Let $F_N = f + \lambda N$. Then $F_{N*x} = f_{*x} + \lambda(x)N_{*x} + N(x)\lambda_{*x}$, where $f_{*x}, N_{*x} : T_xS^n \to f_{*x}(T_xS^n)$ are linear isomorphisms and ${f_{*x}}^{-1} \circ N_{*x}$ is symmetric.

Let $\alpha_i, i = 1 \ldots, n$, be continuous maps such that at every point they give the principal curvatures of f with respect to N. Since we are assuming nonvanishing Gaussian curvature they are all positive or all negative maps. For $x \in S^n$, let $(e_1, \ldots, e_n)$ be a basis for T_xS^n formed by eigenvectors. Then $F_{N*x}(e_i) = (1+\lambda(x)\alpha_i(x))f_{*x}(e_i)+\lambda_{*x}(e_i)N(x), i = 1, \ldots, n$. If the α_i's are positive then F_N is an immersion. If they are negative then F_{-N} is an immersion.

Let now, for each i, m_i be a positive real number such that $m_i <\mid \alpha_i(x) \mid$, for $x \in S^n$. Take $a = \max\limits_{i=1,\ldots,n} \dfrac{1}{m_i}$. If the radius R of the surrounding sphere S is such that $R \geq a+r$, with $r = \max\limits_{x \in S^n} \| f(x) \|$. Then F_N, F_{-N} are immersions. In fact in that case we have $d(f(S^n), S) \geq \frac{1}{m_i} > \frac{1}{|\alpha_i(x)|}, i = 1, \ldots, n$, and consequently the scalars $1 + \lambda(x)\alpha_i(x), i = 1, \ldots, n$, are different from zero. $\bowtie$

It is important to observe from the proof above that, for every immersion $f : M \to R^{n+1}$, if $F_N(x)$ (respectively $F_{-N}(x)$) is not a focal point of f with x as base point then x is not a critical point of F_N (respectively F_{-N}).

2. Degree of an Indicatrix

Let us have the same assumptions and notations as in §2.

We start with a result on $mod 2$ degrees.

Proposition 1: Let $f : M \to R^{n+1}$ be an immersion. Then $degree_2\ F_N + degree_2\ F_{-N} \equiv e(M)\ mod 2$, where $e(M)$ stands for the Euler number of M.

Proof: Choose $p \in S$ such that p is a regular value for both F_N and F_{-N}. Consider $L_p : M \to R$ given by $L_p(x) =\| f(x) - p \|^2$. The result follows easily from the fact that L_p is a Morse function and the number of its critical points is congruent with $e(M)\ mod 2$. $\bowtie$

Proposition 2: Let M be even-dimensional. If F_U is an indicatrix such that, for $x \in M$, $\angle(U(x), N(x)) \leq \frac{\pi}{2}$ then *degree* $F_U = \frac{1}{2}\ e(M)$.

Proof: This follows from an old result of Heinz Hopf [5] and the fact that there is a homotopy between F_U and the map rN, where r is the radius of S. In fact, for $t \in [0,1], x \in M$, $H(x,t) = rN(x) + t(F_U(x) - rN(x))$ is different from zero. If, for some $t \neq 0, 1, x \in M$, $H(t,x)$ were zero we would have $F_U(x) = -rN(x)$ and consequently $f(x) = -\lambda(x)U(x) - rN(x)$. This would imply $\| f(x) \| > r$. We can then use H to define a homotopy between F_U and the map rN. $\bowtie$

If M is odd-dimensional there is still a homotopy but the result is no longer true. For instance, for $M = S^n$, odd n, we can have arbitrary odd degree. We refer the reader to the results in [5].

Assume now that $M = S^1$.

Proposition 3 : Let $F_U : S^1 \to S$ be an indicatrix such that, for $s \in S^1$, $\angle(U(s), N(s)) < \pi$. Then F_U is homotopic to F_N.

Proof: Let $\widetilde{T} : S^1 \times [0,1] \to S^1$ be such that $\widetilde{T}(s,t) = \frac{U(s)+t(N(s)-U(s))}{\|U(s)+t(N(s)-U(s))\|}$. Use now $\widetilde{T}$ to obtain $H : S^1 \times [0,1] \to S$ given by $H(s,t) = f(s) + \mu(s,t)\widetilde{T}(s,t)$, where $\mu(s,t)$ is obtained after finding the intersection of the half-line $f(s) + \alpha\widetilde{T}(s,t), \alpha \geq 0$, with S. $\bowtie$

We see that the *Tangential Indicatrix*, F_T, with $T(x)$ the tangent vector to the curve at X, is homotopic to F_N. Using a rotation of angle πt in R^2 we can show that F_T and F_N are homotopic to F_{-T} and F_{-N} respectively. That does not necessarily happen in higher dimensions.

Proposition 4: Let $f : S^1 \to R^2$ be such that no tangent line passes through 0. Then, for U as in proposition 2, *degree* F_U is the winding number of f with respect to O.

Proof: It is enough to consider the case $U = T$. Now define $H : S^1 \times [0,1] \to S$ by $H(x,t) = r\frac{f(x)+t\lambda(x)T(x)}{\|f(x)+t\lambda(x)T(x)\|}$, where r is the radius of S. $\bowtie$

Proposition 5: Let $f : S^1 \to R^2$ be such that its curvature does not vanish. Then, for U as in proposition 2, 2π *degree* $F_U = rot\ f$, where $rot\ f$ stands for the rotation number of f.

Proof: Again we consider he case $U = T$. From $F_T = f + \lambda T$ it is clear that f and F_T are regularly homotopic and consequently $rot\ f = rot\ F_T$. Since $rot\ F_T = 2\pi\ degree\ F_T$ the result follows. ⋈

3. Applications

There is no reason to consider just immersions with codimension 1. An interesting situation occurs with curves in 3-space.

A. **Curves with Small Total Torsion**

In[2] it was convenient at some stage to indicate how curves with small total torsion could be obtained. There we used a convenient non-degenerate homotopy as suggested by [6]. Here we will use another type of homotopy for a similar purpose.

Let $f : S^1 \to R^3$ be a closed curve with nonvanishing torsion. Consider $F_T : S^1 \to S$ given by $F_T(x) = f(x) + \lambda(x)T(x)$, where $T(x)$ is the unit tangent vector at x. For $0 \leq t \leq 1$, $g_t = f + t\lambda T$ gives rise to a non-degenerate homotopy, that is one that at every stage t the corresponding curve g_t has curvature which vanishes nowhere. Since under a non-degenerate deformation the total torsion varies continuously and the total torsion of a spherical curve is zero it follows that curves with very small, nonzero total torsion can be obtained.

B. **Linking Numbers**

Proposition 1: Let $f, g : S^1 \to R^3$ be curves such that the image of one of them does not intersect any tangent line to the other. Then the linking number $L(f, g)$ is zero.

Proof: Assume that no tangent line to f meets $g(S^1)$. Consider $F_T = f + \lambda T$. Then f is homotopic to F_T and the homotopy induces a homotopy between $\phi : S^1 \times S^1 \to S^2$, given by $\phi(x, y) = \frac{f(x)-g(y)}{||f(x)-g(y)||}$ and $\psi : S^1 \times S^1 \to S^2$, given by $\psi(x, y) = \frac{F_T(x)-g(y)}{||F_T(x)-g(y)||}$. Therefore $L(f, g) = L(F_T, g)$. If we choose the 2-sphere S for the definition of F_T sufficiently big it follows that $L(F_T, g) = 0$.

⋈

Athwart curves [4] are examples of curves in the conditions of Proposition 1. It is known [4] that there are curves which cannot be athwart to any other curve.

We are going to show that on the other hand given a curve we can always find another one such that the conditions of Proposition 1 are satisfied.

Proposition 2: Let $f : S^1 \to R^3$ be a curve. Then there is $g : S^1 \to R^3$ such that no tangent line to f meets $g(S^1)$.

Proof: We follow [3] where we showed that the tangent lines to f do not fill R^3. Consider F_T, F_{-T}. Then $X = F_T(S^1) \cup F_{-T}(S^1)$ is a set of which the complement in S^2 is open and nonempty. Any curve $g : S^1 \to R^3$ with image in $S^2 \setminus X$ will do. $\bowtie$

Obviously similar results can be obtained replacing tangent by principal normal or binormal if the extra assumption of nonvanishing curvature is imposed on f.

References

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