About Fitted Finite-Difference Operators and Fitted Meshes for Solving Singularly Perturbed Linear Problems in one Dimension

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Abstract

Classical finite-difference operators on uniform meshes are not appropriate for solving singularly perturbed differential equations due to the effect of the boundary layers.If we want uniformly convergent difference schemes (with respect to some discrete norm), we need to adequate the numerical operator or the mesh, or both, the operator and the mesh. We summarize some recent results about uniform convergent methods for a general linear two point boundary value problem and present two algorithms related with the numerical method, which give indication about the transition points of the solution in the given domain. They are transition point indicators.

Key words:
Singularly perturbed problems, ordinary differential equations, finite-differences.
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1 Introduction

Differential equations (ordinary or partial) with a small paramater ϵ multiplying the highest derivative , as

$$-\epsilon u''(x) + b(x)u' + c(x)u(x) = f(x),\ x \in (0,1) \tag{1}$$

$$u(0) = u(1) = 0,\ 0 < \epsilon \ll 1,$$

are models for physical processes in fluid flows like pollution, convective heat or mass transport problems and others. Solution $u(x;\epsilon)$ of (1) and its derivatives approach a discontinuous limit as ϵ approches zero. These problems are characterized by the property that the solution has different asymptotic expansions in distinguished subdomains of the entire given domain. They present layers where the solution changes abruptly. See the references [4, 6, 8],etc.

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Let us take the example of J. Kevorkian (1990), pg. 478,

$$\epsilon u'' + u' = 1/2, \;\; 0 < x < 1, \;\; 0 < \epsilon << 1 \tag{2}$$

$$u(0) = 0; u(1) = 1.$$

The exact solution is:

$$u(x;\epsilon) = \frac{1 - \exp^{-\frac{x}{\epsilon}}}{2(1 - \exp^{-\frac{1}{\epsilon}})} + \frac{x}{2}.$$

From that, for any fixed $x \in (0,1)$ and taking $\epsilon \longrightarrow 0$, we see that

$$\lim_{\epsilon \longrightarrow 0_{x\,fixed \neq 0}} u(x;\epsilon) = \frac{1+x}{2} = u_0(x), \tag{3}$$

and $u_0(x)$ does not satisfy the boundary condition at $x = 0$.

The problem comes because we ignored $\exp^{-\frac{x}{\epsilon}}$ when we obtained (3), but this term is important if x is small, $x = \mathcal{O}(\epsilon)$. In a thin boundary layer over the interval

$$0 \leq x \leq \epsilon\eta$$

(η an arbitrary finite positive Ct.) the term $\exp^{-\frac{x}{\epsilon}}$ equals $\exp^{-\eta} = \mathcal{O}(1)$.

Other kind of layers may appear,not only boundary layers, see Miller(1996). The construction of asymptotic expansions must be appropriated to the behaviour of the solution which means that we need to localize transition points of such solution, in the domain, or transition regions,and then to obtain one asymptotic expansion for each subdomain or subregion. It may be not an easy task. For the example above, with one layer at $x = 0$, we must divide the domain as

$$[0,1] = [0,\sigma] \bigcup [\sigma, 1]$$

and to choose σ according to the problem.

So, if we do'nt know the exact solution of the problem, how can we find the transition point σ? In section 5 we present two ways of getting " transition point indicators". In section 2 we give a summary of the main difficulties and anomalies that appear when solving this kind of problems with finite-difference methods and in sections 3 and 4 we present recent results obtained with fitted meshes, called Shishkin meshes.

2 Numerical Approximations

Numerical approximations for Singularly Perturbed Problems (SP) are often the only option we have. Many methods have been proposed in the literature and the most frequent discretization methods are the "upwind finite-differences and the finite-element methods". A survey about difference schemes is given in Farrell (1987).More recently, two excellent books appear: J. Miller, O'Riordan and Shishkin(1996) and Roos,Stynes and Tobiska(1996).

Classical convergence theory for finite-difference methods is based on the complementary concepts of consistency and stability

$$consistency + stability \Longrightarrow convergence,$$

but for (SP) problems, if any discretization technique is applied, we need to analyse carefully the dependence on the parameter ϵ of those constants that arise in consistency, stability and error estimates. Truncation error may depend on ϵ.

The most common anomalous behaviour that appear when finite-difference schemes are used are:

1. Central-difference operators on uniform meshes (h=const.) applied to (SP) of the type (1) can produce approximate solutions presenting oscillations that are unbounded when $\epsilon \longrightarrow 0$;

2. Upwind difference operators on uniform meshes does not necessarily give satisfactory numerical solutions. Usually the pointwise error of such solutions increases as the mesh is refined, to a stage where the mesh paramater (h) is of the same order of magnitude as the singular perturbation paramater ϵ.

One obvious requirement for a numerical method being applied to these kind of problems, is that the pointwise errors of its solutions be bounded independently of ϵ and that they decrease as the mesh is refined, at the rate which should also be independent of ϵ.

Such requirements are not special,and for problem (1) we know some results about the analytical behaviour of the solution, that justify such requirements.

The maximum principle provides a simple proof of the stability inequality, showing that the solution is bounded

$$\| u \|_\infty \leq C \| f \|_\infty, \tag{4}$$

C independent of ϵ , with problem (1) satisfying $b(x) \geq b_0 > 0$.

Also the maximum principle with some techniques help us to prove that $u(x;\epsilon)$, solution of (1), satisfies

$$|u^{(i)}(x)| \leq C(1 + \epsilon^{-i} \exp{(-b_0 \frac{1-x}{\epsilon})}), 0 < x < 1$$

$i = 1, 2, \ldots$. From (4) we have an estimate for the exact solution that tell us that u is bounded uniformly with respect to ϵ, in the maximum norm. See Tobiska[7] for the above results.

Such results for problem (1) and for others, lead to the concept of ϵ- uniform convergence.

3 ϵ-uniform finite-difference methods

A numerical method is said $\epsilon - uniform$ if

$$\sup_{0<\epsilon<<1} \parallel u(x;\epsilon) - u \parallel \leq Ch^p,$$

where C,h,p are independent of ϵ. Obviously the following question appear:-Is it possible to construct numerical methods that behave uniformly well for all values of ϵ? The book of Miller, O'Riordan and Shishkin analyse and give affirmative answer for some (SP), not for all.

Considering finite-difference methods, two approaches have been taken, in order to construct $\epsilon - uniform$ numerical schemes:

(A) - FITTED OPERATORS - they replace the standard finite-difference operator by a different finite-difference operator which reflects the singular perturbation nature of the differential operator;

(B) - FITTED MESHES-they adapt meshes to the nature of the differential operator.

(A)- For linear problems, such operators may be obtained by choosing the coefficients of the difference operator so that some or all the exponential functions in the null-space of the differential operator, are also in the null space of the finite-difference operator. See Miller [1996].

Such fitted operators have been developed by many authors and usually work with uniform meshes. An important class of such methods is the class of " exponentially fitted methods ". The implementation of these methods is not straighforward and usually they introduce artifitial diffusion.

(B)-In this group of numerical schemes, meshes are taken such that are not uniform: highly nonequidistant grids, logarithmic grids as that of Bakhvalov. The convergence analyses for these schemes, is not well clarified, at the moment.

We may say that some problems are solved accuratly with numerical methods of the group (A), some with the methods of the group (B) and other problems (even linear ones) need both approaches.

We must remark: to know the localization and the width of the layers is a crucial question for asymptotic methods and for discretization methods.

For a general linear problem

$$-\epsilon u''(x) + b(x)u' + c(x)u(x) = f(x), \quad x \in (0,1) \tag{5}$$

$$u(0) = u(1) = 0,\ 0 < \epsilon \ll 1,$$

where b, c, f are sufficiently smooth functions, we know that the boundary layer exist at $x = 1$ ($x = 0$) according to $b(x) \geq 0, (b(x) \leq 0)$ and its width is of order Pe^{-1}, where

$$Pe \approx \frac{b(x)(average)}{\epsilon}$$

is the P eclet number.See Tobiska (1996).

4 Piecewise Uniform Meshes

As the solution of a (SP) is expressed by different asymptotic expansions on different subdomains of the entire domain, Shishkin (1990) raised the question: is it possible to prove nodal uniform convergence with a mesh that is uniform by sections? Such meshes are called Shishkin-meshes. For a problem with one boundary layer at $x = 0$, a mesh with N nodal points is as follows: $\frac{N}{2}$ grid points, equidistantly, in each of the subintervals $(0, \sigma)$ and $(\sigma, 1)$.

For a problem with two boundary layers, one at $x = 0$ and other at $x = 1$, a mesh with N nodal points is organized with $\frac{N}{4}$ equidistant points on $(0, \sigma)$; $\frac{N}{2}$ equidistant points on $(\sigma, 1 - \sigma)$ and again $\frac{N}{4}$ points on $(1 - \sigma, 1)$.

The parameter σ is chosen according to the problem and Shishkin gives some information about that question.For the first case (one boundary layer at $x = 0$), σ is given by

$$\sigma = \min\{\frac{1}{2}, K\epsilon\log(N)\}$$

and for the second case (two boundary layers, one at $x = 0$ and other at $x = 1$) by

$$\sigma = \min\{\frac{1}{4}, C\epsilon\log(N)\},$$

where $K$and C are constants independent of ϵ. The meshes go from fine to coarse and for the first case we have the stepsizes h_1 and h_2 given from:

$$h_1 = \frac{2\sigma}{N}\ \ on[0, \sigma]$$

$$h_2 = \frac{2(1 - \sigma)}{N}\ \ on[\sigma, 1]$$

For the second case we will have 3 stepsizes h_1, h_2 and h_3 as follows: $h_1 = \frac{4\sigma}{N}$ on $[0, \sigma]$; $h_2 = \frac{2(1-2\sigma)}{N}$ on $[\sigma, 1 - \sigma]$ and $h_3 = \frac{4\sigma}{N}$ on $[1 - \sigma, 1]$.

A piecewise uniform mesh Ω^* is then constructed and the following results have been proved by Shishkin:

Theorem 1 *-The standard upwind difference scheme together with the piecewise Shishkin mesh Ω^* is ϵ-uniformly convergent and satisfies*

$$||u - z||_\infty \leq MN^{-1} \log N,$$

where M is independent of ϵ.

Here we denoted by u the exact solution and by z the approximated solution.

Theorem 2 *-A necessary condition for the standard upwind difference scheme on the Ω^* mesh be ϵ-uniformly convergent is that*

$$\sigma = \min\{m, \Gamma(\epsilon, N)\}, \qquad 0 < m < 1$$

$$\Gamma(\epsilon, N) = \epsilon\tau(N) \qquad \textit{with} \qquad \tau(N) \to \infty \qquad \textit{when} N \to \infty.$$

These results are from Shishkin and Miller.

Unfortunatly Shishkin piecewise uniform meshes together with classical finite-difference or finite- element operators do not solve all kind of layers. Again, is important to have

1. layers indicators
2. width layer indicators
3. rate of increase or decrease of such width.

Theorem 2 gives a noncomplete answer to these questions. It would be very useful to have some kind of indicators of the local irregularity of the solution of the problem. In the next section we present two algorithms that give an approach for the analysis of such local irregularities.

5 Indicators of Local Irregularities of the Solution

In order to solve numerically (SP) different strategies can be taken within the two main ideas: fitted operators or fitted meshes. For all of them it will be important to designe an algorithm such that the localization of the transition points of the solution be easy. Such indicators may be associated with the numerical method being used. Here we assume that we will use a finite-difference method.

1-FIRST INDICATOR

It takes information from the local truncation error associated with the numerical method at each nodal point x_i, assuming that we use a uniform mesh (h=conts.).

Then, the local truncation error τ_i^h at the nodal point x_i satisfies

$$|\tau_i^h| = |L_h u(x_i) - f(x_i)|$$

where L_h is the finite-difference operator. For example, if L_h is the upwind operator, we know for the problem (1) some bounds as

$$|\tau_i^h| \leq C \int_{x_{i-1}}^{x_{i+1}} (\epsilon|u'''| + |u''|) dt. \tag{6}$$

Other bounds will be useful.

With $N = 2^r,\ r > 2$ *and* $h_1, h_2 > \epsilon$: take

$$h_1 = \frac{1}{2^r}, \quad h_2 = \frac{1}{2^{r+1}}.$$

From $L_{h_1} - f$ and from $L_{h_2} - f$ it is easy to prove that in (x_{i-2}, x_{i+2}) we have

$$|\tau_i^{h_2}| \approx \mathcal{O}(\frac{1}{2}|\tau_i^{h_1}|). \tag{7}$$

Assuming that we are solving problem (1) with the standard upwind difference scheme on a uniform mesh, we propose the following algorithm for getting information about irregularities of the solution:

1. Compute the numerical solutions $U_i^{h_1}$ and $U_i^{h_2}$;
2. With such solutions approximate the second member of (6), obtaining approximations for $|\tau_i^{h_1}|$ and for $|\tau_i^{h_2}|$;
3. If $|\tau_i^{h_1}|$ and $|\tau_i^{h_2}|$ do not satisfy relation (7) we have an indication of an irregular point for the solution $u(x;\epsilon)$ of problem (1) in (x_{i-2}, x_{i+2}).

2- SECOND INDICATOR

With the same assumptions as before we also assume that $u(x;\epsilon)$ is smooth enough in $I_j = [x_{j-2}, x_j]$. Then there is a $\delta > 0$ such that

$$|U_j - 2U_{j-1} + U_{j-2}| \leq \delta h^2, \tag{8}$$

where U_j is the numerical approximation of $u(x_j;\epsilon)$ given by the finite difference method. By the mean value theorem we have for $u(x;\epsilon)$

$$u(x_j;\epsilon) = u(x_{j-1}) + u'(\alpha)h, \quad \alpha \in (x_{j-1}, x_j)$$

$$u(x_j;\epsilon) = u(x_{j-2}) + 2u'(\beta)h, \quad \beta \in (x_{j-2}, x_j)$$

Multiplying the first by two and subtracting to the second equation we have (8).

The local regularity of $u(x;\epsilon)$ can be studied by the indicators s_j defined as:

$$s_0 = U_0 - 2U_1 + U_2, \quad s_1 = U_1 - 2U_2 + U_3$$

$$s_j = U_j - 2U_{j-1} + U_{j-2}.$$

Let $\theta = \mathcal{O}(h^2)$. If $|s_j| \gg \theta$ and $s_k < \theta,\ k = j-1, j+2$ then $u(x;\epsilon)$ has a transition point in (x_{j-1}, x_j). This is an adaptation of the results of C. Cunha and S. Gomes [1997].

Both algorithms are very simple and easy to be implemented toghether with the numerical method proposed.

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