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ON PRESHEAF SUBMONADS OF QUANTALE-ENRICHED CATEGORIES

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Abstract. This paper focuses on the presheaf monad, or the free cocompletion monad, and its submonads on the realm of V-categories, for a quantale V. First we present two characterisations of presheaf submonads, both using V-distributors: one based on admissible classes of V-distributors, and other using Beck-Chevalley conditions on V-distributors. Further we prove that lax idempotency for 2-monads on V-**Cat** can be characterized via such a Beck-Chevalley condition. Then we focus on the study of the Eilenberg-Moore categories of algebras for our monads, having as main examples the formal ball monad and the Lawvere-Cauchy completion monad.

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Introduction. Having as guideline Lawvere's point of view that it is worth to regard metric spaces as categories enriched in the extended real half-line $[0,\infty]_+$ (see [18]), we regard both the formal ball monad and the monad that identifies Cauchy complete spaces as its algebras – which we call here the *Lawvere monad* – as submonads of the presheaf monad on the category **Met** of $[0,\infty]_+$-enriched categories. This leads us to the study of general presheaf submonads, that is, submonads of the presheaf monad, on the category of V-enriched categories, for a given quantale V. Hence this applies not only to metric spaces but also to ordered sets, ultrametric spaces and probabilistic metric spaces, among others.

Here we expand on known general characterisations of presheaf submonads and their algebras, and introduce a new ingredient – conditions of Beck-Chevalley type – which allows us to identify properties of functors and natural transformations, and, most importantly, contribute to a new facet of the behaviour of presheaf submonads.

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In order to do that, after introducing the basic concepts needed to the study of V-categories in Section 1, Section 2 presents the presheaf monad and a characterisation of its submonads using admissible classes of V-distributors which is based on [2]. Next we introduce the already mentioned Beck-Chevalley conditions (BC*) which resemble those discussed in [5], with V-distributors playing the role of V-relations. In particular we show that lax idempotency of a monad $\mathbb{T}$ on V-**Cat** can be identified via a BC* condition, and that the presheaf monad satisfies fully BC*. This leads to the use of BC* to present a new characterisation of presheaf submonads in Section 4.

The remaining sections are devoted to the study of the Eilenberg-Moore category induced by presheaf submonads. In Section 5, based on [2], we detail the relationship between the algebras, (weighted) cocompleteness, and injectivity. Next we focus on the algebras and their morphisms, first for the formal ball monad, and later for a general presheaf submonad. We end by presenting the relevant example of the presheaf submonad whose algebras are the so-called Lawvere complete V-categories studied deeply in [3], which, when $V = [0, \infty]_+$, are exactly the Cauchy complete (generalised) metric spaces, while their morphisms are the V-functors which preserve the limits for Cauchy sequences.

1. Preliminaries.

Our work focuses on V-categories (or V-enriched categories, cf. [18, 15]) in the special case of V being a quantale.

Throughout V is a *commutative and unital quantale*; that is, V is a complete lattice endowed with a symmetric tensor product $\otimes$, with unit $k \neq \bot$, commuting with joins, so that it has a right adjoint hom; this means that, for $u, v, w \in V$,

$$u \otimes v \leq w \;\Leftrightarrow\; v \leq \hom(u, w).$$

As a category, V is a complete and cocomplete (thin) symmetric monoidal closed category.

DEFINITION 1.1. A *V-category* is a pair (X, a) where X is a set and $a\colon X \times X \to V$ is a map such that:

(R) for each $x \in X$, $k \leq a(x, x)$;

(T) for each $x, x', x'' \in X$, $a(x, x') \otimes a(x', x'') \leq a(x, x'')$.

If (X, a), (Y, b) are V-categories, a *V-functor* $f\colon (X, a) \to (Y, b)$ is a map $f\colon X \to Y$ such that

(C) for each $x, x' \in X$, $a(x, x') \leq b(f(x), f(x'))$.

The category of V-categories and V-functors will be denoted by V-**Cat**. Sometimes we will use the notation $X(x, y) = a(x, y)$ for a V-category (X, a) and $x, y \in X$.

We point out that V has itself a V-categorical structure, given by the right adjoint to $\otimes$, hom; indeed, $u \otimes k \leq u \;\Rightarrow\; k \leq \hom(u, u)$, and $u \otimes \hom(u, u') \otimes \hom(u', u'') \leq u' \otimes \hom(u', u'') \leq u''$ gives that $\hom(u, u') \otimes \hom(u', u'') \leq \hom(u, u'')$. Moreover, for every V-category (X, a), one can define its *opposite*

V-category $(X,a)^{\mathrm{op}}=(X,a^\circ)$, with $a^\circ(x,x')=a(x',x)$ for all $x,x'\in X$. Throughout we will make use of the V-category $E=(\{*\},k)$, with $k(*,*)=k$, which is both a generator of V-**Cat** and a unit for the tensor product of V as described in (1.i).

EXAMPLES 1.2. (1) For $V=\mathbf{2}=(\{0<1\},\wedge,1)$, a **2**-category is an *ordered set* (not necessarily antisymmetric) and a **2**-functor is a *monotone map.* We denote **2-Cat** by **Ord**.

(2) The lattice $V=[0,\infty]$ ordered by the "greater or equal" relation $\geq$ (so that $r\wedge s=\max\{r,s\}$, and the supremum of $S\subseteq[0,\infty]$ is given by $\inf S$) with tensor $\otimes=+$ will be denoted by $[0,\infty]_+$. A $[0,\infty]_+$-category is a *(generalised) metric space* and a $[0,\infty]_+$-functor is a *non-expansive map* (see [18]). We denote $[0,\infty]_+$-**Cat** by **Met**. We note that

$$\hom(u,v)=v\ominus u:=\max\{v-u,0\},$$

for all $u,v\in[0,\infty]$.

If instead of $+$ one considers the tensor product $\wedge$, then $[0,\infty]_\wedge$-**Cat** is the category **UMet** of *ultrametric spaces* and *non-expansive maps.*

(3) The complete lattice $[0,1]$ with the usual "less or equal" relation $\leq$ is isomorphic to $[0,\infty]$ via the map $[0,1]\to[0,\infty]$, $u\mapsto-\ln(u)$ where $-\ln(0)=\infty$. Under this isomorphism, the operation $+$ on $[0,\infty]$ corresponds to the multiplication $*$ on $[0,1]$. In other words, this is an isomorphism of quantales. Therefore, denoting this quantale by $[0,1]_*$, one has $[0,1]_*$-**Cat** isomorphic to the category **Met** $=[0,\infty]_+$-**Cat** of (generalised) metric spaces and non-expansive maps.

Since $[0,1]$ is a frame, so that finite meets commute with infinite joins, we can also consider it as a quantale with $\otimes=\wedge$. The category $[0,1]_\wedge$-**Cat** is isomorphic to the category **UMet**.

Another interesting tensor product in $[0,1]$ is given by the *Łukasiewicz tensor* $\odot$ where $u\odot v=\max(0,u+v-1)$; here $\hom(u,v)=\min(1,1-u+v)$. Then $[0,1]_\odot$-**Cat** is the category of *bounded-by-1 (generalised) metric spaces* and *non-expansive maps.*

(4) We consider now the set

$$\Delta=\{\varphi\colon[0,\infty]\to[0,1]\mid \text{for all }\alpha\in[0,\infty]\colon\ \varphi(\alpha)=\bigvee_{\beta<\alpha}\varphi(\beta)\},$$

of *distribution functions.* With the pointwise order, it is a complete lattice. For $\varphi,\psi\in\Delta$ and $\alpha\in[0,\infty]$, define $\varphi\otimes\psi\in\Delta$ by

$$(\varphi\otimes\psi)(\alpha)=\bigvee_{\beta+\gamma\leq\alpha}\varphi(\beta)*\psi(\gamma).$$

Then $\otimes:\Delta\times\Delta\to\Delta$ is associative and commutative, and

$$\kappa:[0,\infty]\to[0,1],\ \alpha\mapsto\begin{cases}0 & \text{if }\alpha=0,\\ 1 & \text{else}\end{cases}$$

is a unit for $\otimes$. Finally, $\psi \otimes - : \Delta \to \Delta$ preserves suprema since, for all $u \in [0,1]$, $u * -\colon [0,1] \to [0,1]$ preserves suprema. A Δ-category is a *(generalised) probabilistic metric space* and a Δ-functor is a *probabilistic non-expansive map* (see [13] and references there).

We will also make use of two additional categories we describe next, the category V-**Rel**, of sets and V-relations, and the category V-**Dist**, of V-categories and V-distributors.

Objects of V-**Rel** are sets, while morphisms are V-relations, i.e., if X and Y are sets, a *V-relation* $r\colon X \longrightarrow\!\!\!\!\!\!\mapsto Y$ is a map $r\colon X \times Y \to V$. Composition of V-relations is given by *relational composition*, so that the composite of $r\colon X \longrightarrow\!\!\!\!\!\!\mapsto Y$ and $s\colon Y \longrightarrow\!\!\!\!\!\!\mapsto Z$ is given by

$$(s \cdot r)(x,z) = \bigvee_{y \in Y} r(x,y) \otimes s(y,z),$$

for every $x \in X$, $z \in Z$. Identities in V-**Cat** are simply identity relations, with $1_X(x,x') = k$ if $x = x'$ and $1_X(x,x') = \bot$ otherwise. The category V-**Rel** has an involution $(\ )^\circ$, assigning to each V-relation $r\colon X \longrightarrow\!\!\!\!\!\!\mapsto Y$ the V-relation $r^\circ\colon Y \longrightarrow\!\!\!\!\!\!\mapsto X$ defined by $r^\circ(y,x) = r(x,y)$, for every $x \in X$, $y \in Y$.

Since every map $f\colon X \to Y$ can be thought as a V-relation through its graph $f_\circ\colon X \times Y \to V$, with $f_\circ(x,y) = k$ if $f(x) = y$ and $f_\circ(x,y) = \bot$ otherwise, there is an injective on objects and faithful functor **Set** $\to V$-**Rel**. When no confusion may arise, we use also f to denote the V-relation $f_\circ$.

The category V-**Rel** is a 2-category, when equipped with the 2-cells given by the pointwise order; that is, for $r, r'\colon X \longrightarrow\!\!\!\!\!\!\mapsto Y$, one defines $r \le r'$ if, for all $x \in X$, $y \in Y$, $r(x,y) \le r'(x,y)$. This gives us the possibility of studying adjointness between V-relations. We note in particular that, if $f\colon X \to Y$ is a map, then $f_\circ \cdot f^\circ \le 1_Y$ and $1_X \le f^\circ \cdot f_\circ$, so that $f_\circ \dashv f^\circ$.

Objects of V-**Dist** are V-categories, while morphisms are V-distributors (also called V-bimodules, or V-profunctors); i.e., if (X,a) and (Y,b) are V-categories, a *V-distributor* – or, simply, a *distributor* – $\varphi\colon (X,a) \multimap\!\!\!\rightarrow (Y,b)$ is a V-relation $\varphi\colon X \longrightarrow\!\!\!\!\!\!\mapsto Y$ such that $\varphi \cdot a \le \varphi$ and $b \cdot \varphi \le \varphi$ (in fact $\varphi \cdot a = \varphi$ and $b \cdot \varphi = \varphi$ since the other inequalities follow from (R)). Composition of distributors is again given by relational composition, while the identities are given by the V-categorical structures, i.e. $1_{(X,a)} = a$. Moreover, V-**Dist** inherits the 2-categorical structure from V-**Rel**.

Each V-functor $f\colon (X,a) \to (Y,b)$ induces two distributors, $f_*\colon (X,a) \multimap\!\!\!\rightarrow (Y,b)$ and
$f^*\colon (Y,b) \multimap\!\!\!\rightarrow (X,a)$, defined by $f_*(x,y) = Y(f(x),y)$ and $f^*(y,x) = Y(y,f(x))$, that is, $f_* = b \cdot f_\circ$ and $f^* = f^\circ \cdot b$. These assignments are functorial, as we explain below.

First we define 2-cells in V-**Cat**: for $f, f'\colon (X,a) \to (Y,b)$ V-functors, $f \le f'$ when $f^* \le (f')^*$ as distributors, so that

$$f \le f' \;\Leftrightarrow\; \forall x \in X,\, y \in Y,\; Y(y,f(x)) \le Y(y,f'(x)).$$

V-**Cat** is then a 2-category, and we can define two 2-functors

$$\begin{array}{rcl} (\ )_*\colon V\text{-}\mathbf{Cat}^{\mathrm{co}} & \longrightarrow & V\text{-}\mathbf{Dist} \\ X & \longmapsto & X \\ f & \longmapsto & f_* \end{array} \quad \text{and} \quad \begin{array}{rcl} (\ )^*\colon V\text{-}\mathbf{Cat}^{\mathrm{op}} & \longrightarrow & V\text{-}\mathbf{Dist} \\ X & \longmapsto & X \\ f & \longmapsto & f^* \end{array}$$

Note that, for any V-functor $f\colon (X,a)\to(Y,b)$,

$$f_*\cdot f^* = b\cdot f_\circ\cdot f^\circ\cdot b \le b\cdot b = b \text{ and } f^*\cdot f_* = f^\circ\cdot b\cdot b\cdot f_\circ \ge f^\circ\cdot f_\circ\cdot a \ge a;$$

hence every V-functor induces a pair of adjoint distributors, $f_*\dashv f^*$. A V-functor $f\colon X\to Y$ is said to be *fully faithful* if $f^*\cdot f_* = a$, i.e. $X(x,x') = Y(f(x),f(x'))$ for all $x,x'\in X$, while it is *fully dense* if $f_*\cdot f^* = b$, i.e. $Y(y,y') = \bigvee_{x\in X} Y(y,f(x))\otimes Y(f(x),y')$, for all $y,y'\in Y$. A fully faithful V-functor $f\colon X\to Y$ does not need to be an injective map; it is so in case X and Y are separated V-categories (as defined below).

REMARK 1.3. In V-**Cat** adjointness between V-functors

$$\begin{tikzcd} Y \arrow[r, bend left, "g"] \arrow[r, phantom, "\top"] & X \arrow[l, bend left, "f"] \end{tikzcd}$$

can be equivalently expressed as:

$$f\dashv g \iff f_* = g^* \iff g^*\dashv f^* \iff (\forall x\in X)\ (\forall y\in Y)\ X(x,g(y)) = Y(f(x),y).$$

In fact the latter condition encodes also V-functoriality of f and g; that is, if $f\colon X\to Y$ and $g\colon Y\to X$ are maps satisfying the condition

$$(\forall x\in X)\ (\forall y\in Y)\ \ X(x,g(y)) = Y(f(x),y),$$

then f and g are V-functors, with $f\dashv g$.

Furthermore, it is easy to check that, given V-categories X and Y, a map $f\colon X\to Y$ is a V-functor whenever f_* is a distributor (or whenever f^* is a distributor).

The order defined on V-**Cat** is in general not antisymmetric. For V-functors $f,g\colon X\to Y$, one says that $f\simeq g$ if $f\le g$ and $g\le f$ (or, equivalently, $f^*=g^*$). For elements x,y of a V-category X, one says that $x\le y$ if, considering the V-functors $x,y\colon E\to X$ defined by $x(*)=x$ and $y(*)=y$, one has $x\le y$; or, equivalently, $X(x,y)\ge k$. Then, for any V-functors $f,g\colon X\to Y$, $f\le g$ if, and only if, $f(x)\le g(x)$ for every $x\in X$.

DEFINITION 1.4. A V-category Y is said to be *separated* if, for $f,g\colon X\to Y$, $f=g$ whenever $f\simeq g$; equivalently, if, for all $x,y\in Y$, $x\simeq y$ implies $x=y$.

The tensor product $\otimes$ on V induces a tensor product on V-**Cat**, with $(X,a)\otimes(Y,b)=(X\times Y,a\otimes b)=X\otimes Y$, where

$$(X\otimes Y)((x,y),(x',y'))=X(x,x')\otimes Y(y,y'). \tag{1.i}$$

The V-category E is a $\otimes$-neutral element. With this tensor product, V-**Cat** becomes a monoidal closed category. Indeed, for each V-category X, the functor $X\otimes(\;)\colon V\text{-}\mathbf{Cat}\to V\text{-}\mathbf{Cat}$ has a right adjoint $(\;)^X$ defined by $Y^X=(V\text{-}\mathbf{Cat}(X,Y),[\![\;,\;]\!])$, with $[\![f,g]\!]=\bigwedge_{x\in X}Y(f(x),g(x))$ (see [7, 18, 15] for details).

It is interesting to note the following well-known result (see, for instance, [3, Theorem 2.5]).

Theorem 1.5. *For V-categories (X,a) and (Y,b), and a V-relation $\varphi\colon X\to\!\!\!\!\!\!\!\;\shortmid\;\, Y$, the following conditions are equivalent:*

(i) $\varphi\colon(X,a){-\!\!\circ\!\!\to}(Y,b)$ *is a distributor.*

(ii) $\varphi\colon(X,a)^{\mathrm{op}}\otimes(Y,b)\to(V,\hom)$ *is a V-functor.*

In particular, the V-categorical structure a of (X,a) is a V-distributor $a\colon(X,a){-\!\!\circ\!\!\to}(X,a)$, and therefore a V-functor $a\colon(X,a)^{\mathrm{op}}\otimes(X,a)\to(V,\hom)$, which induces, via the closed monoidal structure of V-**Cat**, the *Yoneda V-functor* $y_X\colon(X,a)\to(V,\hom)^{(X,a)^{\mathrm{op}}}$. Thanks to the theorem above, $V^{X^{\mathrm{op}}}$ can be equivalently described as

$$PX:=\{\varphi\colon X{-\!\!\circ\!\!\to}E\,|\,\varphi\ V\text{-distributor}\}.$$

Then the structure $\widetilde{a}$ on PX is given by

$$\widetilde{a}(\varphi,\psi)=[\![\varphi,\psi]\!]=\bigwedge_{x\in X}\hom(\varphi(x),\psi(x)),$$

for every $\varphi,\psi\colon X{-\!\!\circ\!\!\to}E$, where by $\varphi(x)$ we mean $\varphi(x,*)$, or, equivalently, we consider the associated V-functor $\varphi\colon X\to V$. The Yoneda functor $y_X\colon X\to PX$ assigns to each $x\in X$ the distributor $x^*\colon X{-\!\!\circ\!\!\to}E$, where we identify again $x\in X$ with the V-functor $x\colon E\to X$ assigning x to the (unique) element of E. Then, for every $\varphi\in PX$ and $x\in X$, we have that

$$[\![y_X(x),\varphi]\!]=\varphi(x),$$

as expected. In particular y_X is a fully faithful V-functor, being injective on objects (i.e. an injective map) when X is a separated V-category. We point out that $(V,\hom)$ is separated, and so is PX for every V-category X.

For more information on V-**Cat**, for a quantale V, we refer to [12, Appendix].

2. The presheaf monad and its submonads. The assignment $X\mapsto PX$ defines a functor $P\colon V\text{-}\mathbf{Cat}\to V\text{-}\mathbf{Cat}$: for each V-functor $f\colon X\to Y$, $Pf\colon PX\to PY$ assigns to each distributor $X\xrightarrow{\varphi}\!\!\!\!\circ\; E$ the distributor $Y\xrightarrow{f^*}\!\!\!\!\circ\; X\xrightarrow{\varphi}\!\!\!\!\circ\; E$.

It is easily checked that the Yoneda functors $(y_X\colon X \to PX)_X$ define a natural transformation $y\colon 1 \to P$. Moreover, since, for every V-functor f, the adjunction $f_* \dashv f^*$ yields an adjunction $Pf = (\ )\cdot f^* \dashv (\ )\cdot f_* =: Qf$, Py_X has a right adjoint, which we denote by $m_X\colon PPX \to PX$. It is straightforward to check that $\mathbb{P} = (P, m, y)$ is a 2-monad on V-**Cat** – the so-called *presheaf monad* or *free cocompletion monad*–, which, by construction of m_X as the right adjoint to Py_X, is lax idempotent.

We recall that (cf. [16, Definition 1.1], [8, Definition 4.1.2]):

DEFINITION 2.1. A 2-monad $\mathbb{T} = (T, \mu, \eta)$ on an **Ord**-enriched category is said to be *lax idempotent* or *Kock-Zöberlein* if it satisfies one of the following equivalent conditions:

(i) $T\eta \dashv \mu$;

(ii) $\mu \dashv \eta T$;

(iii) $T\eta \leq \eta T$.

Next we present a characterisation of the submonads of $\mathbb{P}$ which is partially in [2]. We recall that, given two monads $\mathbb{T} = (T, \mu, \eta)$, $\mathbb{T}' = (T', \mu', \eta')$ on a category $\mathbf{C}$, a monad morphism $\sigma\colon \mathbb{T} \to \mathbb{T}'$ is a natural transformation $\sigma\colon T \to T'$ such that

(2.i)
$$\begin{tikzcd} 1 \arrow[r, "\eta"] \arrow[dr, "\eta'"'] & T \arrow[d, "\sigma"] \\ & T' \end{tikzcd} \qquad \begin{tikzcd} TT \arrow[r, "\sigma_T"] \arrow[d, "\mu"'] & T'T \arrow[r, "T'\sigma"] & T'T' \arrow[d, "\mu'"] \\ T \arrow[rr, "\sigma"'] & & T' \end{tikzcd}$$

By *submonad of* $\mathbb{P}$ we mean a 2-monad $\mathbb{T} = (T, \mu, \eta)$ on V-**Cat** with a monad morphism $\sigma : \mathbb{T} \to \mathbb{P}$ such that σ_X is an embedding (i.e. both fully faithful and injective on objects) for every V-category X.

DEFINITION 2.2. Given a class Φ of V-distributors, for every V-category X let

$$\Phi X = \{\varphi\colon X \,{\to\!\!\!\!\!\!\circ}\, E \mid \varphi \in \Phi\}$$

have the V-category structure inherited from the one of PX. We say that Φ is *admissible* if, for every V-functor $f\colon X \to Y$ and V-distributors $\varphi\colon Z \,{\to\!\!\!\!\!\!\circ}\, Y$ and $\psi\colon X \,{\to\!\!\!\!\!\!\circ}\, Z$ in Φ,

(1) $f^* \in \Phi$;

(2) $\psi \cdot f^* \in \Phi$ and $f^* \cdot \varphi \in \Phi$;

(3) $\varphi \in \Phi \;\Leftrightarrow\; (\forall y \in Y)\ y^* \cdot \varphi \in \Phi$;

(4) for every V-distributor $\gamma\colon PX \,{\to\!\!\!\!\!\!\circ}\, E$, if the restriction of γ to ΦX belongs to Φ, then $\gamma \cdot (y_X)_* \in \Phi$.

LEMMA 2.3. *Every admissible class Φ of V-distributors induces a submonad $\Phi = (\Phi, m^\Phi, y^\Phi)$ of $\mathbb{P}$.*

Proof. For each V-category X, equip ΦX with the initial structure induced by the inclusion $\sigma_X : \Phi X \to PX$, that is, for every $\varphi, \psi \in \Phi X$, $\Phi X(\varphi, \psi) = PX(\varphi, \psi)$. For each V-functor $f : X \to Y$ and $\varphi \in \Phi X$, by condition (2), $\varphi \cdot f^* \in \Phi$, and so Pf (co)restricts to $\Phi f : \Phi X \to \Phi Y$.

Condition (1) guarantees that $y_X : X \to PX$ corestricts to $y_X^\Phi : X \to \Phi X$.

Finally, condition (4) guarantees that $m_X : PPX \to PX$ also (co)restricts to $m_X^\Phi : \Phi\Phi X \to \Phi X$: if $\gamma : \Phi X \rightarrow\!\!\!\!\!\circ\;\, E$ belongs to Φ, then $\widetilde{\gamma} := \gamma \cdot (\sigma_X)^* : PX \rightarrow\!\!\!\!\!\circ\;\, E$ belongs to Φ by (2), and then, since γ is the restriction of $\widetilde{\gamma}$ to ΦX, by (4) $m_X(\widetilde{\gamma}) = \gamma \cdot (\sigma_X)^* \cdot (y_X)_* = \gamma \cdot (\sigma_X)^* \cdot (\sigma_X)_* \cdot (y_X^\Phi)_* = \gamma \cdot (y_X^\Phi)_* \in \Phi$.

By construction, $(\sigma_X)_X$ is a natural transformation, each σ_X is an embedding, and σ makes diagrams (2.i) commute. □

Theorem 2.4. *For a 2-monad $\mathbb{T} = (T, \mu, \eta)$ on V-**Cat**, the following assertions are equivalent:*

(i) $\mathbb{T}$ *is isomorphic to Φ, for some admissible class of V-distributors Φ.*

(ii) $\mathbb{T}$ *is a submonad of $\mathbb{P}$.*

Proof. (i) $\Rightarrow$ (ii) follows from the lemma above.

(ii) $\Rightarrow$ (i): Let $\sigma : \mathbb{T} \to \mathbb{P}$ be a monad morphism, with σ_X an embedding for every V-category X, which, for simplicity, we assume to be an inclusion. First we show that

$$\Phi = \{\varphi : X \rightarrow\!\!\!\!\!\circ\;\, Y \mid \forall y \in Y \; y^* \cdot \varphi \in TX\} \tag{2.ii}$$

is admissible. In the sequel $f : X \to Y$ is a V-functor.

(1) For each $x \in X$, $x^* \cdot f^* = f(x)^* \in TY$, and so $f^* \in \Phi$.

(2) If $\psi : X \rightarrow\!\!\!\!\!\circ\;\, Z$ is a V-distributor in Φ, and $z \in Z$, since $z^* \cdot \psi \in TX$, $Tf(z^* \cdot \psi) = z^* \cdot \psi \cdot f^* \in TY$, and therefore $\psi \cdot f^* \in \Phi$ by definition of Φ. Now, if $\varphi : Z \rightarrow\!\!\!\!\!\circ\;\, Y \in \Phi$, then, for each $x \in X$, $x^* \cdot f^* \cdot \varphi = f(x)^* \cdot \varphi \in TZ$ because $\varphi \in \Phi$, and so $f^* \cdot \varphi \in \Phi$.

(3) follows from the definition of Φ.

(4) If the restriction of $\gamma : PX \rightarrow\!\!\!\!\!\circ\;\, E$ to TX, i.e., $\gamma \cdot (\sigma_X)_*$, belongs to Φ, then $\mu_X(\gamma \cdot (\sigma_X)_*) = \gamma \cdot (\sigma_X)_* \cdot (\eta_X)_* = \gamma \cdot (y_X)_*$ belongs to TX. □

We point out that, with $\mathbb{P}$, also $\mathbb{T}$ is lax idempotent. This assertion is shown at the end of next section, making use of the Beck-Chevalley conditions we study next. (We note that the arguments of [6, Proposition 16.2], which states conditions under which a submonad of a lax idempotent monad is still lax idempotent, cannot be used directly here.)

3. The presheaf monad and Beck-Chevalley conditions. In this section our aim is to show that $\mathbb{P}$ verifies some interesting conditions of Beck-Chevalley type, that resemble the BC conditions studied in [5] as we outline below. We recall from [5] that a commutative square in **Set**

$$\begin{tikzcd} W \arrow[r, "l"] \arrow[d, "g"'] & Z \arrow[d, "h"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

is said to be a *BC-square* if the following diagram commutes in **Rel**

$$\begin{tikzcd} W \arrow[r, "l_\circ", "\shortmid" marking] & Z \\ X \arrow[u, "g^\circ", "\shortmid" marking] \arrow[r, "f_\circ"', "\shortmid" marking] & Y, \arrow[u, "h^\circ"', "\shortmid" marking] \end{tikzcd}$$

where, given a map $t\colon A \to B$, $t_\circ\colon A \rightarrow\!\!\!\!\!\!\mapsto B$ denotes the relation defined by t and $t^\circ\colon B \rightarrow\!\!\!\!\!\!\mapsto A$ its opposite. Since $t_\circ \dashv t^\circ$ in **Rel**, this is in fact a kind of Beck-Chevalley condition. A **Set**-endofunctor T is said to satisfy BC if it preserves BC-squares, while a natural transformation $\alpha\colon T \to T'$ between two **Set**-endofunctors satisfies BC if, for each map $f\colon X \to Y$, its naturality square

$$\begin{tikzcd} TX \arrow[r, "\alpha_X"] \arrow[d, "Tf"'] & T'X \arrow[d, "T'f"] \\ TY \arrow[r, "\alpha_Y"'] & T'Y \end{tikzcd}$$

is a BC-square.

In our situation, for endofunctors and natural transformations in V-**Cat**, the role of **Rel** is played by V-**Dist**.

DEFINITION 3.1. A commutative square in V-**Cat**

$$\begin{tikzcd} (W,d) \arrow[r, "l"] \arrow[d, "g"'] & (Z,c) \arrow[d, "h"] \\ (X,a) \arrow[r, "f"'] & (Y,b) \end{tikzcd} \tag{3.i}$$

is said to be a *BC*-square* if the following diagram commutes in V-**Dist**

$$\begin{tikzcd} (W,d) \arrow[r, "l_*", "\circ" marking] & (Z,c) \\ (X,a) \arrow[u, "g^*", "\circ" marking] \arrow[r, "f_*"', "\circ" marking] & (Y,b) \arrow[u, "h^*"', "\circ" marking] \end{tikzcd}$$

(or, equivalently, $h^* \cdot f_* \leq l_* \cdot g^*$).

Lemma 3.2. *For a V-functor $f\colon (X,a) \to (Y,b)$, to be fully faithful is equivalent to*

$$\begin{tikzcd}
(X,a) \arrow[r, "1"] \arrow[d, "1"'] & (X,a) \arrow[d, "f"] \\
(X,a) \arrow[r, "f"'] & (Y,b)
\end{tikzcd}$$

being a BC-square (exactly in parallel with the characterisation of monomorphisms via BC-squares).*

Remark 3.3. We point out that, contrarily to the case of BC-squares, in BC*-squares the horizontal and the vertical arrows play different roles; that is, the fact that diagram (3.i) is a BC*-square is not equivalent to

$$\begin{tikzcd}
(W,d) \arrow[r, "g"] \arrow[d, "l"'] & (X,a) \arrow[d, "f"] \\
(Z,c) \arrow[r, "h"'] & (Y,b)
\end{tikzcd}$$

being a BC*-square; it is indeed equivalent to its *dual*

$$\begin{tikzcd}
(W,d^\circ) \arrow[r, "g"] \arrow[d, "l"'] & (X,a^\circ) \arrow[d, "f"] \\
(Z,c^\circ) \arrow[r, "h"'] & (Y,b^\circ)
\end{tikzcd}$$

being a BC*-square.

Definitions 3.4. (1) A *functor* $T\colon V\text{-}\mathbf{Cat} \to V\text{-}\mathbf{Cat}$ *satisfies BC** if it preserves BC*-squares.

(2) Given two endofunctors T, T' on V-**Cat**, a *natural transformation* $\alpha\colon T \to T'$ *satisfies BC** if the naturality diagram

$$\begin{tikzcd}
TX \arrow[r, "\alpha_X"] \arrow[d, "Tf"'] & T'X \arrow[d, "T'f"] \\
TY \arrow[r, "\alpha_Y"'] & T'Y
\end{tikzcd}$$

is a BC*-square for every morphism f in V-**Cat**.

(3) *A 2-monad* $\mathbb{T} = (T, \mu, \eta)$ on V-**Cat** is said to satisfy *fully BC** if T, μ, and η satisfy BC*.

REMARK 3.5. In the case of **Set** and **Rel**, since the condition of being a BC-square is equivalent, under the Axiom of Choice (AC), to being a weak pullback, a **Set**-monad $\mathbb{T}$ *satisfies fully BC* if, and only if, it is *weakly cartesian* (again, under (AC)). This, together with the fact that there are relevant **Set**-monads – like for instance the ultrafilter monad – whose functor and multiplication satisfy BC but the unit does not, led the authors of [5] to name such monads as *BC-monads*. This is the reason why we use *fully BC** instead of BC* to identify these 2-monads.

As a side remark we recall that, still in the **Set**-context, a partial BC-condition was studied by Manes in [19]: for a **Set**-monad $\mathbb{T} = (T, \mu, \eta)$ to be *taut* requires that T, μ, η satisfy BC for commutative squares where f is monic.

Our first use of BC* is the following characterisation of lax idempotency for a 2-monad $\mathbb{T}$ on V-**Cat**.

PROPOSITION 3.6. *Let $\mathbb{T} = (T, \mu, \eta)$ be a 2-monad on V-**Cat**.*

(1) *The following assertions are equivalent:*

  (i) *$\mathbb{T}$ is lax idempotent.*

  (ii) *For each V-category X, the diagram*

(3.ii)
$$\begin{tikzcd} TX \arrow[r, "T\eta_X"] \arrow[d, "\eta_{TX}"'] & TTX \arrow[d, "\mu_X"] \\ TTX \arrow[r, "\mu_X"'] & TX \end{tikzcd}$$

is a BC-square.*

(2) *If $\mathbb{T}$ is lax idempotent, then μ satisfies BC*.*

Proof. (1) (i) $\Rightarrow$ (ii): The monad $\mathbb{T}$ is lax idempotent if, and only if, for every V-category X, $T\eta_X \dashv \mu_X$, or, equivalently, $\mu_X \dashv \eta_{TX}$. These two conditions are equivalent to $(T\eta_X)_* = (\mu_X)^*$ and $(\mu_X)_* = (\eta_{TX})^*$. Hence $(\mu_X)^*(\mu_X)_* = (T\eta_X)_*(\eta_{TX})^*$ as claimed.

(ii) $\Rightarrow$ (i): From $(\mu_X)^*(\mu_X)_* = (T\eta_X)_*(\eta_{TX})^*$ it follows that

$$(\mu_X)_* = (\mu_X)_*(\mu_X)^*(\mu_X)_* = (\mu_X \cdot T\eta_X)_*(\eta_{TX})^* = (\eta_{TX})^*,$$

that is, $\mu_X \dashv \eta_{TX}$.

(2) BC* for μ follows directly from lax idempotency of $\mathbb{T}$, since

$$\begin{tikzcd} TTX \arrow[r, "(\mu_X)_*", "\circ"{marking}] & TX \\ TTY \arrow[u, "(TTf)^*", "\circ"{marking}] \arrow[r, "(\mu_Y)_*"', "\circ"{marking}] & TY \arrow[u, "(Tf)^*"', "\circ"{marking}] \end{tikzcd} \quad = \quad \begin{tikzcd} TTX \arrow[r, "(\eta_{TX})^*", "\circ"{marking}] & TX \\ TTY \arrow[u, "(TTf)^*", "\circ"{marking}] \arrow[r, "(\eta_{TY})^*"', "\circ"{marking}] & TY \arrow[u, "(Tf)^*"', "\circ"{marking}] \end{tikzcd}$$

and the latter diagram commutes trivially by naturality of η. $\square$

Thanks to Remark 3.3 we get also a characterisation of oplax idempotent 2-monad:

Lemma 3.7. *$\mathbb{T}$ is oplax idempotent if, and only if, the diagram*

$$\begin{tikzcd} TX \arrow[r, "\eta_{TX}"] \arrow[d, "T\eta_X"'] & TTX \arrow[d, "\mu_X"] \\ TTX \arrow[r, "\mu_X"'] & TX \end{tikzcd}$$

is a BC-square.*

Theorem 3.8. *The presheaf monad $\mathbb{P}=(P,\mathfrak{m},\mathfrak{y})$ satisfies fully BC*.*

Proof. (1) *P satisfies BC**: Given a BC*-square

$$\begin{tikzcd} (W,d) \arrow[r, "l"] \arrow[d, "g"'] & (Z,c) \arrow[d, "h"] \\ (X,a) \arrow[r, "f"'] & (Y,b) \end{tikzcd}$$

in V-**Cat**, we want to show that

$$\begin{tikzcd} PW \arrow[r, "(Pl)_*", "\circ" marking] & PZ \\ PX \arrow[u, "(Pg)^*", "\circ" marking] \arrow[r, "(Pf)_*"', "\circ" marking] \arrow[ur, phantom, "\geq"] & PY. \arrow[u, "(Ph)^*"', "\circ" marking] \end{tikzcd} \tag{3.iii}$$

For each $\varphi\in PX$ and $\psi\in PZ$, we have

$$\begin{aligned}
(Ph)^*(Pf)_*(\varphi,\psi) &= (Ph)^\circ\cdot\widetilde{b}\cdot Pf(\varphi,\psi)\\
&=\widetilde{b}(Pf(\varphi),Ph(\psi))\\
&=\bigwedge_{y\in Y}\hom(\varphi\cdot f^*(y),\psi\cdot h^*(y))\\
&\leq\bigwedge_{x\in X}\hom(\varphi\cdot f^*\cdot f_*(x),\psi\cdot h^*\cdot f_*(x))\\
&\leq\bigwedge_{x\in X}\hom(\varphi(x),\psi\cdot l_*\cdot g^*(x)) \qquad (\varphi\leq\varphi\cdot f^*\cdot f_*,\ (3.\mathrm{iii})\text{ is BC*})\\
&=\widetilde{a}(\varphi,\psi\cdot l_*\cdot g^*)\\
&\leq\widetilde{a}(\varphi,\psi\cdot l_*\cdot g^*)\otimes\widetilde{c}(\psi\cdot l_*\cdot l^*,\psi) \qquad (\text{because } \psi\cdot l_*\cdot l^*\leq\psi)
\end{aligned}$$

$$\begin{aligned}&= \widetilde{a}(\varphi, Pg(\psi\cdot l_*)\otimes \widetilde{c}(Pl(\psi\cdot l_*),\psi)\\ &\le \bigvee_{\gamma\in PW} \widetilde{a}(\varphi, Pg(\gamma))\otimes \widetilde{c}(Pl(\gamma),\psi)\\ &= (Pl)_*(Pg)^*(\varphi,\psi).\end{aligned}$$

(2) μ *satisfies BC**: For each V-functor $f\colon X\to Y$, from the naturality of y it follows that the following diagram

$$\begin{tikzcd} PPX \arrow[r, "(\mathsf{y}_{PX})^*", "\circ" marking] & PX \\ PPY \arrow[u, "(PPf)^*", "\circ" marking] \arrow[r, "(\mathsf{y}_{PY})^*"', "\circ" marking] & PY \arrow[u, "(Pf)^*"', "\circ" marking] \end{tikzcd}$$

commutes. Lax idempotency of $\mathbb{P}$ means in particular that $\mathsf{m}_X \dashv \mathsf{y}_{PX}$, or, equivalently, $(\mathsf{m}_X)_* = (\mathsf{y}_{PX})^*$, and therefore the commutativity of this diagram shows BC* for m.

(3) y *satisfies BC**: Once again, for each V-functor $f\colon (X,a)\to(Y,b)$, we want to show that the diagram

$$\begin{tikzcd} X \arrow[r, "(\mathsf{y}_X)_*", "\circ" marking] & PX \\ Y \arrow[u, "f^*", "\circ" marking] \arrow[r, "(\mathsf{y}_Y)_*"', "\circ" marking] & PY \arrow[u, "(Pf)^*"', "\circ" marking] \end{tikzcd}$$

commutes. Let $y\in Y$ and $\varphi\colon X \mathrel{\circ\!\!\to} E$ belong to PX. Then

$$\begin{aligned}((Pf)^*(\mathsf{y}_Y)_*)(y,\varphi) &= ((Pf)^\circ\cdot\widetilde{b}\cdot \mathsf{y}_Y)(y,\varphi) = \widetilde{b}(\mathsf{y}_Y(y), Pf(\varphi)) = Pf(\varphi)(y)\\ &= \bigvee_{x\in X} b(y,f(x))\otimes\varphi(x) = \bigvee_{x\in X} b(y,f(x))\otimes \widetilde{a}(\mathsf{y}_X(x),\varphi)\\ &= (\widetilde{a}\cdot \mathsf{y}_X\cdot f^\circ\cdot b)(y,\varphi) = (\mathsf{y}_X)_*\cdot f^*(y,\varphi),\end{aligned}$$

as claimed. □

COROLLARY 3.9. *Let $\mathbb{T}=(T,\mu,\eta)$ be a 2-monad on V-**Cat**, and $\sigma\colon\mathbb{T}\to\mathbb{P}$ be a monad morphism, pointwise fully faithful. Then $\mathbb{T}$ is lax idempotent.*

Proof. We know that $\mathbb{P}$ is lax idempotent, and so, for every V-category X, $(\mathsf{m}_X)_* = (\mathsf{y}_{PX})^*$. Consider diagram (2.i). The commutativity of the diagram on the right gives that $(\mu_X)_* = (\sigma_X)^*(\sigma_X)_*(\mu_X)_* = (\sigma_X)^*(\mathsf{m}_X)_*(P\sigma_X)_*(\sigma_{TX})_*$; using the equality above, and preservation of fully faithful V-functors by $\mathbb{P}$ – which follows from BC* – we obtain:

$$\begin{aligned}(\mu_X)_* &= (\sigma_X)^*(\mathsf{y}_{PX})^*(P\sigma_X)_*(\sigma_{TX})_* = (\sigma_X)^*(\eta_{PX})^*(\sigma_{PX})^*(P\sigma_X)_*(\sigma_{TX})_* =\\ &= (\eta_{TX})^*\cdot(\sigma_{TX})^*(P\sigma_X)^*(P\sigma_X)_*(\sigma_{TX})_* = (\eta_{TX})^*.\end{aligned}$$

□

4. Presheaf submonads and Beck-Chevalley conditions. In this section, for a general 2-monad $\mathbb{T}=(T,\mu,\eta)$ on V-**Cat**, we relate its BC* properties with the existence of a (sub)monad morphism $\mathbb{T}\to\mathbb{P}$. We remark that a necessary condition for $\mathbb{T}$ to be a submonad of $\mathbb{P}$ is that TX is separated for every V-category X, since PX is separated and separated V-categories are stable under monomorphisms.

We start by stating a consequence of [6, Lemma 2.7]:

LEMMA 4.1. *If $\mathbb{T}$ is a lax idempotent monad on V-**Cat** with TX separated for every V-category X, then there is at most one monad morphism $\mathbb{T}\to\mathbb{P}$.*

LEMMA 4.2. *Let $\mathbb{T}=(T,\mu,\eta)$ a 2-monad on V-**Cat**. If η satisfies BC*, then:*

(1) *The morphisms*

$$TX\xrightarrow{y_{TX}}PTX\xrightarrow{Q\eta_X}PX\ ,$$

*for $X\in V$-**Cat**, define a natural transformation $T\to P$.*

(2) *Moreover, $\alpha=Q\eta\cdot y_T\colon\mathbb{T}\to\mathbb{P}$ is a monad morphism.*

Proof. (1) For each V-functor $f\colon X\to Y$, consider the following diagram

$$\begin{tikzcd} TX \arrow[r, "y_{TX}"] \arrow[d, "Tf"'] \arrow[dr, phantom, "\boxed{1}"] & PTX \arrow[r, "Q\eta_X"] \arrow[d, "PTf"] \arrow[dr, phantom, "\boxed{2}"] & PX \arrow[d, "Pf"] \\ TY \arrow[r, "y_{TY}"'] & PTY \arrow[r, "Q\eta_Y"'] & PY. \end{tikzcd}$$

Then $\boxed{1}$ is always commutative, since y is a natural transformation, and BC* for η implies that $\boxed{2}$ is commutative.

(2) It remains to show that α *is a monad morphism*: for each V-category (X,a), we have $(\,TX\xrightarrow{\alpha_X}PX\,)=(\,TX\xrightarrow{y_{TX}}PTX\xrightarrow{Q\eta_X}PX\,)$; that is, denoting by $\widehat{a}$ the V-category structure on TX, $\alpha_X(\mathfrak{x})=(\,X\xrightarrow{\eta_X}TX\xrightarrow{\widehat{a}}\!\!\!\!\!\!\!\!\!\!|\;\;\;\;\;TX\xrightarrow{\mathfrak{x}^\circ}\!\!\!\!\!\!\!\!\!\!|\;\;\;\;\;E\,)=\widehat{a}(\eta_X(\ ),\mathfrak{x})$ for each $\mathfrak{x}\in TX$. Hence, for each V-category (X,a) and $x\in X$,

$$(\alpha_X\cdot\eta_X)(x)=\widehat{a}(\eta_X(\ ),\eta_X(x))=a(-,x)=x^*=y_X(x),$$

and so $\alpha\cdot\eta=y$. To check that, for every V-category (X,a), the following diagram commutes

$$\begin{tikzcd} TTX \arrow[r, "\alpha_{TX}"] \arrow[d, "\mu"'] & PTX \arrow[r, "P\alpha_X"] & PPX \arrow[d, "m_X"] \\ TX \arrow[rr, "\alpha_X"'] & & PX, \end{tikzcd}$$

let $\mathfrak{X} \in TTX$. We have

$$\begin{aligned}
&m_X \cdot P\alpha_X \cdot \alpha_{TX}(\mathfrak{X}) \\
&\quad = (\, X \xrightarrow{y_X} PX \overset{\widetilde{a}}{\longrightarrow\!\!\!\!\!\!\!\!\!\;+\;\,} PX \overset{\alpha_X^\circ}{\longrightarrow\!\!\!\!\!\!\!\!\!\;+\;\,} TX \xrightarrow{\eta_{TX}} TTX \overset{\widehat{\widehat{a}}}{\longrightarrow\!\!\!\!\!\!\!\!\!\;+\;\,} TTX \overset{\mathfrak{X}^\circ}{\longrightarrow\!\!\!\!\!\!\!\!\!\;+\;\,} E \,) \\
&\quad = (\, X \xrightarrow{\eta_X} TX \overset{\widehat{a}}{\longrightarrow\!\!\!\!\!\!\!\!\!\;+\;\,} TX \xrightarrow{\eta_{TX}} TTX \overset{\widehat{\widehat{a}}}{\longrightarrow\!\!\!\!\!\!\!\!\!\;+\;\,} TTX \overset{\mathfrak{X}^\circ}{\longrightarrow\!\!\!\!\!\!\!\!\!\;+\;\,} E \,),
\end{aligned}$$

since $\alpha_X^\circ \cdot \widetilde{a} \cdot y_X(x,\mathfrak{x}) = \widetilde{a}(y_X(x), \alpha_X(\mathfrak{x})) = \alpha_X(\mathfrak{x})(x) = \widehat{a} \cdot \eta_X(x,\mathfrak{x})$, and

$$\alpha_X \cdot \mu_X(\mathfrak{x}) = (\, X \xrightarrow{\eta_X} TX \overset{\widehat{a}}{\longrightarrow\!\!\!\!\!\!\!\!\!\;+\;\,} TX \overset{\mu_X^\circ}{\longrightarrow\!\!\!\!\!\!\!\!\!\;+\;\,} TTX \overset{\mathfrak{X}^\circ}{\longrightarrow\!\!\!\!\!\!\!\!\!\;+\;\,} E \,).$$

Hence the commutativity of the diagram follows from the equality $\widehat{\widehat{a}} \cdot \eta_{TX} \cdot \widehat{a} \cdot \eta_X = \mu_X^\circ \cdot \widehat{a} \cdot \eta_X$ we show next. Indeed,

$$\begin{aligned}
\widehat{\widehat{a}} \cdot \eta_{TX} \cdot \widehat{a} \cdot \eta_X = (\eta_{TX})_*(\eta_X)_* = (\eta_{TX} \cdot \eta_X)_* = (T\eta_X \cdot \eta_X)_* \\
= (T\eta_X)_*(\eta_X)_* = \mu_X^*(\eta_X)_* = \mu_X^\circ \cdot \widehat{a} \cdot \eta_X. \ \square
\end{aligned}$$

THEOREM 4.3. *For a 2-monad $\mathbb{T} = (T, \mu, \eta)$ on V-**Cat** with TX separated for every V-category X, the following assertions are equivalent:*

(i) *$\mathbb{T}$ is a submonad of $\mathbb{P}$.*

(ii) *$\mathbb{T}$ is lax idempotent and satisfies BC*, and both natural transformations $\eta\colon \mathrm{Id} \to T$ and $Q\eta \cdot y_T\colon T \to P$ are fully faithful.*

(iii) *$\mathbb{T}$ is lax idempotent, μ and η satisfy BC*, and both natural transformations $\eta\colon \mathrm{Id} \to T$ and $Q\eta \cdot y_T\colon T \to P$ are fully faithful.*

(iv) *$\mathbb{T}$ is lax idempotent, η satisfies BC*, and both natural transformations $\eta\colon \mathrm{Id} \to T$ and $Q\eta \cdot y_T\colon T \to P$ are fully faithful.*

Proof. (i) $\Rightarrow$ (ii): By (i) there exists a monad morphism $\sigma\colon \mathbb{T} \to \mathbb{P}$ with σ_X an embedding for every V-category X. By Corollary 3.9, with $\mathbb{P}$, also $\mathbb{T}$ is lax idempotent. Moreover, from $\sigma_X \cdot \eta_X = y_X$, with y_X, also η_X is fully faithful. (In fact this is valid for any monad with a monad morphism into $\mathbb{P}$.)

To show that $\mathbb{T}$ satisfies BC* we use the characterisation of Theorem 2.4; that is, we know that there is an admissible class Φ of distributors so that $\mathbb{T} = \Phi$. Then BC* for T follows directly from the fact that Φf is a (co)restriction of Pf, for every V-functor f.

BC* for η follows from BC* for y and full faithfulness of σ since, for any commutative diagram in V-**Cat**

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{1}"] & \cdot \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\boxed{2}"] & \cdot \arrow[d] \\
\cdot \arrow[r] & \cdot \arrow[r, "g"'] & \cdot
\end{tikzcd}$$

with $\boxed{1}\boxed{2}$ satisfying BC*, and f and g fully faithful, also $\boxed{1}$ satisfies BC*.

Thanks to Proposition 3.6, BC* for μ follows directly from lax idempotency of $\mathbb{T}$.

Now, using the previous lemmas, by uniqueness of the monad morphism we may conclude that $\sigma = Q\eta \cdot y_T$, and so $Q\eta \cdot y_T$ is fully faithful.

The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are obvious.

(iv) $\Rightarrow$ (i): By Lemma 4.2, under these conditions $Q\eta \cdot y_T$ is the desired monad morphism, which is an embedding by assumption, since it is fully faithful and TX is separated for every X. $\square$

The proof of the theorem allows us to conclude immediately the following result.

Corollary 4.4. *Given a 2-monad $\mathbb{T} = (T, \mu, \eta)$ on V-**Cat** such that η satisfies BC*, there is a monad morphism $\mathbb{T} \to \mathbb{P}$ if, and only if, η is pointwise fully faithful.*

5. On algebras for submonads of $\mathbb{P}$: a survey. In the remainder of this paper we will study, given a submonad $\mathbb{T}$ of $\mathbb{P}$, the category $(V\text{-}\mathbf{Cat})^{\mathbb{T}}$ of (Eilenberg-Moore) $\mathbb{T}$-algebras. Here we collect some known results which will be useful in the following sections. We will denote by $\Phi(\mathbb{T})$ the admissible class of distributors that induces the monad $\mathbb{T}$ (defined in (2.ii)).

The following result, which is valid for any lax-idempotent monad $\mathbb{T}$, asserts that, for any V-category, to be a $\mathbb{T}$-algebra is a property (see, for instance, [9] and [6]).

Theorem 5.1. *Let $\mathbb{T}$ be lax idempotent monad on V-**Cat**.*

1. *For a V-category X, the following assertions are equivalent:*
   (i) *$\alpha\colon TX \to X$ is a $\mathbb{T}$-algebra structure on X;*
   (ii) *there is a V-functor $\alpha\colon TX \to X$ such that $\alpha \dashv \eta_X$ with $\alpha \cdot \eta_X = 1_X$;*
   (iii) *there is a V-functor $\alpha\colon TX \to X$ such that $\alpha \cdot \eta_X = 1_X$;*
   (iv) *$\alpha\colon TX \to X$ is a split epimorphism in V-**Cat**.*
2. *If (X, α) and (Y, β) are $\mathbb{T}$-algebra structures, then every V-functor $f\colon X \to Y$ satisfies $\beta \cdot Tf \le f \cdot \alpha$.*

Next we formulate characterisations of $\mathbb{T}$-algebras that can be found in [11, 2], using *injectivity* with respect to certain *embeddings*, and using the existence of certain *weighted colimits*, notions that we recall very briefly in the sequel.

Definition 5.2. ([8]) A V-functor $f\colon X \to Y$ is a *T-embedding* if Tf is a left adjoint right inverse; that is, there exists a V-functor $Tf_\sharp$ such that $Tf \dashv Tf_\sharp$ and $Tf_\sharp \cdot Tf = 1_{TX}$.

For each submonad $\mathbb{T}$ of $\mathbb{P}$, the class $\Phi(\mathbb{T})$ allows us to identify easily the T-embeddings.

Proposition 5.3. *For a V-functor $h\colon X \to Y$, the following assertions are equivalent:*

(i) *h is a T-embedding.*

(ii) *h is fully faithful and h_* belongs to $\Phi(\mathbb{T})$.*

In particular, P-embeddings are exactly the fully faithful V-functors.

Proof. (ii) $\Rightarrow$ (i): Let h be fully faithful with $h_* \in \Phi(\mathbb{T})$. As in the case of the presheaf monad, $\Phi h : \Phi X \to \Phi Y$ has always a right adjoint whenever $h_* \in \Phi(\mathbb{T})$, $\Phi^{\dashv} h := (-) \cdot h_*\colon \Phi Y \to \Phi X$; that is, for each distributor $\psi : Y \mathrel{-\!\!\circ\!\!\rightarrow} E$ in ΦY, $\Phi^{\dashv} h(\psi) = \psi \cdot h_*$, which is well defined because by hypothesis $h_* \in \Phi(\mathbb{T})$. If h is fully faithful, that is, if $h^* \cdot h_* = (1_X)^*$, then $(\Phi^{\dashv} h \cdot \Phi h)(\varphi) = \varphi \cdot h^* \cdot h_* = \varphi$.

(i) $\Rightarrow$ (ii): If $\Phi^{\dashv} h$ is well-defined, then $y^* \cdot h_*$ belongs to $\Phi(\mathbb{T})$ for every $y \in Y$, hence $h_* \in \Phi(\mathbb{T})$, by 2.2(3), and so $h_* \in \Phi(\mathbb{T})$. Moreover, if $\Phi^{\dashv} h \cdot \Phi h = 1_{\Phi X}$, then in particular $x^* \cdot h^* \cdot h_* = x^*$, for every $x \in X$, which is easily seen to be equivalent to $h^* \cdot h_* = (1_X)^*$. $\square$

In V-**Dist**, given a V-distributor $\varphi\colon (X,a) \mathrel{-\!\!\circ\!\!\rightarrow} (Y,b)$, the functor $(\ )\cdot\varphi$ preserves suprema, and therefore it has a right adjoint $[\varphi,-]$ (since the hom-sets in V-**Dist** are complete ordered sets):

$$\begin{tikzcd} \mathbf{Dist}(X,Z) \arrow[r, bend left, "{[\varphi,-]}"] \arrow[r, phantom, "\top"] & \mathbf{Dist}(Y,Z). \arrow[l, bend left, "(\ )\cdot\varphi"] \end{tikzcd}$$

For each distributor $\psi\colon X \mathrel{-\!\!\circ\!\!\rightarrow} Z$,

$$\begin{tikzcd} X \arrow[r, "\psi", "\circ"{marking}] \arrow[d, "\varphi"', "\circ"{marking}] \arrow[dr, phantom, "\leq"] & Z \\ Y \arrow[ur, "{[\varphi,\psi]}"', "\circ"{marking}] & \end{tikzcd}$$

$[\varphi,\psi]\colon Y \mathrel{-\!\!\circ\!\!\rightarrow} Z$ is defined by

$$[\varphi,\psi](y,z) = \bigwedge_{x\in X} \hom(\varphi(x,y),\psi(x,z)).$$

Definitions 5.4. (1) Given a V-functor $f\colon X \to Z$ and a distributor (here called *weight*) $\varphi\colon X \mathrel{-\!\!\circ\!\!\rightarrow} Y$, a *$\varphi$-weighted colimit* of f (or simply a *φ-colimit* of f), whenever it exists, is a V-functor $g\colon Y \to Z$ such that $g_* = [\varphi, f_*]$. One says then that g *represents* $[\varphi, f_*]$.

(2) A V-category Z is called *φ-cocomplete* if it has a colimit for each weighted diagram with weight $\varphi\colon (X,a) \mathrel{-\!\!\circ\!\!\rightarrow} (Y,b)$; i.e. for each V-functor $f\colon X \to Z$, the φ-colimit of f exists.

(3) Given a class Φ of V-distributors, a V-category Z is called Φ-*cocomplete* if it is φ-cocomplete for every $\varphi \in \Phi$. When $\Phi = V\text{-}\mathbf{Dist}$, then Z is said to be *cocomplete*.

The proof of the following result can be found in [11, 2].

Theorem 5.5. *Given a submonad $\mathbb{T}$ of $\mathbb{P}$, for a V-category X the following assertions are equivalent:*

(i) *X is a $\mathbb{T}$-algebra.*

(ii) *X is injective with respect to T-embeddings.*

(iii) *X is $\Phi(\mathbb{T})$-cocomplete.*

$\Phi(\mathbb{T})$-cocompleteness of a V-category X is guaranteed by the existence of some special weighted colimits, as we explain next. (Here we present very briefly the properties needed. For more information on this topic see [20].)

Lemma 5.6. *For a distributor $\varphi\colon X \to Y$ and a V-functor $f\colon X \to Z$, the following assertions are equivalent:*

(i) *There exists the φ-colimit of f.*

(ii) *There exists the $(\varphi \cdot f^*)$-colimit of 1_Z.*

(iii) *For each $y \in Y$, there exists the $(y^* \cdot \varphi)$-colimit of f.*

Proof. (i) $\Leftrightarrow$ (ii): It is straightforward to check that

$$[\varphi, f_*] = [\varphi \cdot f^*, (1_Z)_*].$$

(i) $\Leftrightarrow$ (iii): Since $[\varphi, f_*]$ is defined pointwise, it is easily checked that, if g represents $[\varphi, f_*]$, then, for each $y \in Y$, the V-functor $E \xrightarrow{\;y\;} Y \xrightarrow{\;g\;} Z$ represents $[y^* \cdot \varphi, f_*]$.

Conversely, if, for each $y\colon E \to Y$, $g_y\colon E \to Z$ represents $[y^* \cdot \varphi, f_*]$, then the map $g\colon Y \to Z$ defined by $g(y) = g_y(*)$ is such that $g_* = [\varphi, f_*]$; hence, as stated in Remark 1.3, g is automatically a V-functor. $\square$

Corollary 5.7. *Given a submonad $\mathbb{T}$ of $\mathbb{P}$, a V-category X is a $\mathbb{T}$-algebra if, and only if, $[\varphi, (1_X)_*]$ has a colimit for every $\varphi \in TX$.*

Remark 5.8. Given $\varphi\colon X \mathrel{-\!\!\circ\!\!\rightarrow} E$ in TX, in the diagram

$$\begin{tikzcd} X \arrow[r, "a", "\circ" marking] \arrow[d, "\varphi"', "\circ" marking] & X \\ Y \arrow[ur, "{[\varphi,a]}"', "\circ" marking] \arrow[ur, phantom, "\leq", shift left=3ex] & \end{tikzcd}$$

$$[\varphi, a](*, x) = \bigwedge_{x' \in X} \hom(\varphi(x', *), a(x', x)) = TX(\varphi, x^*).$$

Therefore, if $\alpha\colon TX \to X$ is a $\mathbb{T}$-algebra structure, then

$$[\varphi, a](*, x) = TX(\varphi, x^*) = X(\alpha(\varphi), x),$$

that is, $[\varphi, a] = \alpha(\varphi)_*$; this means that α assigns to each distributor $\varphi\colon X \,{\circ\!\!\to}\, E$ the representative of $[\varphi, (1_X)_*]$.

Hence, we may describe the category of $\mathbb{T}$-algebras as follows.

THEOREM 5.9. (1) *A map $\alpha\colon TX \to X$ is a $\mathbb{T}$-algebra structure if, and only if, for each distributor $\varphi\colon X \,{\circ\!\!\to}\, E$ in TX, $\alpha(\varphi)_* = [\varphi, (1_X)_*]$.*

(2) *If X and Y are $\mathbb{T}$-algebras, then a V-functor $f\colon X \to Y$ is a $\mathbb{T}$-homomorphism if, and only if, f preserves φ-weighted colimits for any $\varphi \in TX$, i.e., if $x \in X$ represents $[\varphi, (1_X)_*]$, then $f(x)$ represents $[\varphi \cdot f^*, (1_Y)_*]$.*

6. On algebras for submonads of $\mathbb{P}$: the special case of the formal ball monad. From now on we will study more in detail $(V\text{-}\mathbf{Cat})^{\mathbb{T}}$ for special submonads $\mathbb{T}$ of $\mathbb{P}$. In our first example, the formal ball monad $\mathbb{B}$, we will need to consider the (co)restriction of $\mathbb{B}$ and $\mathbb{P}$ to $V\text{-}\mathbf{Cat}_{\mathrm{sep}}$. We point out that the characterisations of $\mathbb{T}$-algebras of Theorem 5.5 remain valid for these (co)restrictions.

The space of formal balls is an important tool in the study of (quasi-)metric spaces. Given a metric space (X, d) its *space of formal balls* is simply the collection of all pairs (x, r), where $x \in X$ and $r \in [0, \infty[$. This space can itself be equipped with a (quasi-)metric. Moreover this construction can naturally be made into a monad on the category of (quasi-)metric spaces (cf. [10, 17] and references there).

This monad can readily be generalised to V-categories, using a V-categorical structure in place of the (quasi-)metric. We will start by considering an extended version of the formal ball monad, the *extended formal ball monad* $\mathbb{B}_\bullet$, which we define below.

DEFINITIONS 6.1. The *extended formal ball monad* $\mathbb{B}_\bullet = (B_\bullet, \eta, \mu)$ is given by the following:

- a functor $B_\bullet\colon V\text{-}\mathbf{Cat} \to V\text{-}\mathbf{Cat}$ which maps each V-category X to $B_\bullet X$ with underlying set $X \times V$ and

$$B_\bullet X((x, r), (y, s)) = \hom(r, X(x, y) \otimes s)$$

  and every V-functor $f\colon X \to Y$ to the V-functor $B_\bullet f\colon B_\bullet X \to B_\bullet Y$ with $B_\bullet f(x, r) = (f(x), r)$;
- natural transformations $\eta\colon 1 \to B_\bullet$ and $\mu\colon B_\bullet B_\bullet \to B_\bullet$ with $\eta_X(x) = (x, k)$ and $\mu_X((x, r), s) = (x, r \otimes s)$, for every V-category X, $x \in X$, $r, s \in V$.

The *formal ball monad* $\mathbb{B}$ is the submonad of $\mathbb{B}_\bullet$ obtained when we only consider balls with radius different from $\bot$.

REMARK 6.2. Note that $\mathbb{B}_\bullet X$ is not separated if X has more than one element (for any $x, y \in X$, $(x, \perp) \simeq (y, \perp)$), while, as shown in 6.13, for X separated, separation of $\mathbb{B}X$ depends on an extra property of the quantale V.

Using Corollaries 4.4 and 3.9, it is easy to check that

PROPOSITION 6.3. *There is a pointwise fully faithful monad morphism $\sigma\colon \mathbb{B}_\bullet \to \mathbb{P}$. In particular, both $\mathbb{B}_\bullet$ and $\mathbb{B}$ are lax-idempotent.*

Proof. First of all let us check that η satisfies BC*, i.e., for any V-functor $f\colon X \to Y$,

$$\begin{tikzcd} X \arrow[r, "(\eta_X)_*", "\circ" marking] & B_\bullet X \\ Y \arrow[u, "f^*", "\circ" marking] \arrow[r, "(\eta_Y)_*"', "\circ" marking] \arrow[ur, phantom, "\geq"] & B_\bullet Y \arrow[u, "(B_\bullet f)^*"', "\circ" marking] \end{tikzcd}$$

For $y \in Y$, $(x, r) \in B_\bullet X$,

$$\begin{aligned} ((B_\bullet f)^*(\eta_Y)_*)(y, (x, r)) &= B_\bullet Y((y, k), (f(x), r)) = Y(y, f(x)) \otimes r \\ &\leq \bigvee_{z \in X} Y(y, f(z)) \otimes X(z, x) \otimes r \\ &= \bigvee_{z \in X} Y(y, f(z)) \otimes B_\bullet X((z, k), (x, r)) \\ &= ((\eta_X)_* f^*)(y, (x, r)). \end{aligned}$$

Then, by Corollary 4.4, for each V-category X, σ_X is defined as in the proof of Theorem 4.3, i.e. for each $(x, r) \in B_\bullet X$, $\sigma_X(x, r) = B_\bullet X((-, k), (x, r))\colon X \to V$; more precisely, for each $y \in X$, $\sigma_X(x, r)(y) = X(y, x) \otimes r$.

Moreover, σ_X is fully faithful: for each $(x, r), (y, s) \in B_\bullet X$,

$$\begin{aligned} B_\bullet X((x, r), (y, s)) &= \hom(r, X(x, y) \otimes s) \geq \hom(X(x, x) \otimes r, X(x, y) \otimes s) \\ &\geq \bigwedge_{z \in X} \hom(X(z, x) \otimes r, X(z, y) \otimes s) = PX(\sigma(x, r), \sigma(y, s)). \end{aligned}$$ □

It is clear that $\sigma\colon \mathbb{B}_\bullet \to \mathbb{P}$ is not pointwise monic; indeed, if $r = \perp$, then $\sigma_X(x, \perp)\colon X \to\!\!\!\!\circ\, E$ is the distributor that is constantly $\perp$, for any $x \in X$. Still it is interesting to identify the $\mathbb{B}_\bullet$-algebras via the existence of special weighted colimits.

PROPOSITION 6.4. *For a V-category X, the following conditions are equivalent:*

(i) *X has a $\mathbb{B}_\bullet$-algebra structure $\alpha\colon B_\bullet X \to X$.*

(ii) $(\forall x \in X)\ (\forall r \in V)\ (\exists x \oplus r \in X)\ (\forall y \in X)\ \ X(x \oplus r, y) = \hom(r, X(x, y))$.

(iii) *For all $(x,r) \in B_\bullet X$, every diagram of the sort*

$$\begin{tikzcd} X \arrow[r, "(1_X)_*", "\circ" marking] \arrow[d, "\sigma_X(x,r)"', "\circ" marking] \arrow[dr, phantom, "\leq"] & X \\ E \arrow[ur, "{[\sigma_X(x,r),(1_X)_*]}"', "\circ" marking] & \end{tikzcd}$$

has a (weighted) colimit.

Proof. (i) $\Rightarrow$ (ii): The adjunction $\alpha \dashv \eta_X$ gives, via Remark 1.3,

$$X(\alpha(x,r),y) = B_\bullet X((x,r),(y,k)) = \hom(r, X(x,y)).$$

For $x \oplus r := \alpha(x,r)$, condition (ii) follows.

(ii) $\Rightarrow$ (iii): The calculus of the distributor $[\sigma_X(x,r),(1_X)_*]$ shows that it is represented by $x \oplus r$:

$$[\sigma_X(x,r),(1_X)_*](*,y) = \hom(r, X(x,y)).$$

(iii) $\Rightarrow$ (i): For each $(x,r) \in B_\bullet X$, let $x \oplus r$ represent $[\sigma_X(x,r),(1_X)_*]$. In case $r = k$, we choose $x \oplus k = x$ to represent the corresponding distributor (any $x' \simeq x$ would fit here but x is the right choice for our purpose). Then $\alpha\colon B_\bullet X \to X$ defined by $\alpha(x,r) = x \oplus r$ is, by construction, left adjoint to η_X, and $\alpha \cdot \eta_X = 1_X$. $\square$

The V-categories X satisfying (iii), and therefore satisfying the above (equivalent) conditions, are called *tensored*. This notion was originally introduced in the article [1] by Borceux and Kelly for general V-categories (for our special V-categories we suggest to consult [20]).

Note that, thanks to condition (ii), we get the following characterisation of tensored categories.

Corollary 6.5. *A V-category X is tensored if, and only if, for every $x \in X$,*

$$\begin{tikzcd} X \arrow[r, bend left, "X(x,-)"] \arrow[r, phantom, "\top"] & V \arrow[l, bend left, "x\oplus -"] \end{tikzcd}$$

*is an adjunction in V-**Cat**.*

We now shift our attention to the formal ball monad $\mathbb{B}$. The characterisation of $\mathbb{B}_\bullet$-algebras given by the Proposition 6.4 may be adapted to obtain a characterisation of $\mathbb{B}$-algebras. Indeed, the only difference is that a $\mathbb{B}$-algebra structure $BX \to X$ does not include the existence of $x \oplus \bot$ for $x \in X$, which, when it exists, is the top element with respect to the order in X. Moreover, the characterisation of $\mathbb{B}$-algebras given in [10, Proposition 3.4] can readily be generalised to V-**Cat** as follows.

PROPOSITION 6.6. *For a V-functor $\alpha\colon BX \to X$ the following conditions are equivalent.*

(i) *α is a $\mathbb{B}$-algebra structure.*

(ii) *For every $x \in X$, $r, s \in V \setminus \{\bot\}$, $\alpha(x, k) = x$ and $\alpha(x, r \otimes s) = \alpha(\alpha(x, r), s)$.*

(iii) *For every $x \in X$, $r \in V \setminus \{\bot\}$, $\alpha(x, k) = x$ and $X(x, \alpha(x, r)) \geq r$.*

(iv) *For every $x \in X$, $\alpha(x, k) = x$.*

Proof. By definition of $\mathbb{B}$-algebra, (i) $\Leftrightarrow$ (ii), while (i) $\Leftrightarrow$ (iv) follows from Theorem 5.1, since $\mathbb{B}$ is lax-idempotent. (iii) $\Rightarrow$ (iv) is obvious, and so it remains to prove that, if α is a $\mathbb{B}$-algebra structure, then $X(x, \alpha(x, r)) \geq r$, for $r \neq \bot$. But

$$X(x, \alpha(x, r)) \geq r \;\Leftrightarrow\; k \leq \hom(r, X(x, \alpha(x, r)) = X(\alpha(x, r), \alpha(x, r)),$$

because $\alpha(x, -) \dashv X(x, -)$ by Corollary 6.5. $\square$

Since we know that, if X has a $\mathbb{B}$-algebra structure α, then $\alpha(x, r) = x \oplus r$, we may state the conditions above as follows.

COROLLARY 6.7. *If $BX \xrightarrow{-\oplus-} X$ is a $\mathbb{B}$-algebra structure, then, for $x \in X$, $r, s \in V \setminus \{\bot\}$:*

(1) *$x \oplus k = x$;*

(2) *$x \oplus (r \otimes s) = (x \oplus r) \oplus s$;*

(3) *$X(x, x \oplus r) \geq r$.*

LEMMA 6.8. *Let X and Y be V-categories equipped with $\mathbb{B}$-algebra structures $BX \xrightarrow{-\oplus-} X$ and $BY \xrightarrow{-\oplus-} Y$. Then a map $f : X \to Y$ is a V-functor if and only if*

$$f \text{ is monotone and } f(x) \oplus r \leq f(x \oplus r),$$

for all $(x, r) \in BX$.

Proof. Assume that f is a V-functor. Then it is, in particular, monotone, and, from Theorem 5.1 we know that $f(x) \oplus r \leq f(x \oplus r)$.

Conversely, assume that f is monotone and that $f(x) \oplus r \leq f(x \oplus r)$, for all $(x, r) \in BX$. Let $x, x' \in X$. Then $x \oplus X(x, x') \leq x'$ since $(x \oplus -) \dashv X(x, -)$ by Corollary 6.5, and then

$$\begin{aligned} f(x) \oplus X(x, x') &\leq f(x \oplus X(x, x')) && \text{(by hypothesis)} \\ &\leq f(x') && \text{(by monotonicity of } f). \end{aligned}$$

Now, using the adjunction $f(x) \oplus - \dashv Y(f(x), -))$, we conclude that

$$X(x, x') \leq Y(f(x), f(x')).$$

$\square$

The following results are now immediate:

COROLLARY 6.9. (1) *Let* $(X, \oplus), (Y, \oplus)$ *be* $\mathbb{B}$*-algebras. Then a map* $f : X \to Y$ *is a* $\mathbb{B}$*-algebra morphism if and only if, for all* $(x, r) \in BX$,

$$f \text{ is monotone and } f(x \oplus r) = f(x) \oplus r.$$

(2) *Let* $(X, \oplus), (Y, \oplus)$ *be* $\mathbb{B}$*-algebras. Then a* V*-functor* $f : X \to Y$ *is a* $\mathbb{B}$*-algebra morphism if and only if, for all* $(x, r) \in BX$,

$$f(x \oplus r) \leq f(x) \oplus r.$$

EXAMPLE 6.10. If $X \subseteq [0, \infty]$, with the V-category structure inherited from hom, then

(1) X is a $\mathbb{B}_\bullet$-algebra if, and only if, $X = [a, b]$ for some $a, b \in [0, \infty]$.

(2) X is a $\mathbb{B}$-algebra if, and only if, $X =]a, b]$ or $X = [a, b]$ for some $a, b \in [0, \infty]$.

Let X be a $\mathbb{B}_\bullet$-algebra. From Proposition 6.4 one has

$$(\forall x \in X)\ (\forall r \in [0, \infty])\ (\exists x \oplus r \in X)\ (\forall y \in X)\ \ y \ominus (x \oplus r) = (y \ominus x) \ominus r = y \ominus (x + r).$$

This implies that, if $y \in X$, then $y > x \otimes r \iff y > x + r$. Therefore, if $x + r \in X$, then $x \oplus r = x + r$, and, moreover, X is an interval: given $x, y, z \in [0, \infty]$ with $x < y < z$ and $x, z \in X$, then, with $r = y - x \in [0, \infty]$, $x + r = y$ must belong to X:

$$z \ominus (x \oplus r) = z - (x + r) = z - y > 0 \Rightarrow z \ominus (x \oplus r) = z - (x \oplus r) = z - y \iff y = x \oplus r \in X.$$

In addition, X must have bottom element (that is a maximum with respect to the classical order of the real half-line): for any $x \in X$ and $b = \sup X$, $x \oplus (b - x) = \sup\{z \in X\, ;\, z \leq b\} = b \in X$. For $r = \infty$ and any $x \in X$, $x \oplus \infty$ must be the top element of X, so $X = [a, b]$ for $a, b \in [0, \infty]$.

Conversely, if $X =]a, b]$, for $x \in X$ and $r \in [0, \infty[$, define $x \oplus r = x + r$ if $x + r \in X$ and $x \oplus r = b$ elsewhere. It is easy to check that condition (ii) of Proposition 6.4 is satisfied for $r \neq \infty$.

Analogously, if $X = [a, b]$, for $x \in X$ and $r \in [0, \infty]$, we define $x \oplus r$ as before in case $r \neq \infty$ and $x \oplus \infty = a$.

As we will see, (co)restricting $\mathbb{B}$ to V-$\mathbf{Cat}_{\mathrm{sep}}$ will allows us to obtain some interesting results. Unfortunately X being separated does not entail BX being so. Because of this we will need to restrict our attention to the *cancellative* quantales which we define and characterize next.

DEFINITION 6.11. A quantale V is said to be *cancellative* if

$$\forall r, s \in V, r \neq \perp :\ r = s \otimes r \Rightarrow s = k. \qquad \text{(6.i)}$$

Remark 6.12. We point out that this notion of cancellative quantale does not coincide with the notion of cancellable ccd quantale introduced in [4, before Proposition 1.4]. On the one hand cancellative quantales are quite special, since, for instance, when V is a locale, and so with $\otimes = \wedge$ is a quantale, V is not cancellative since condition (6.i) would mean, for $r \neq \bot$, $r = s \wedge r \;\Rightarrow\; s = \top$. On the other hand, $[0,1]_\odot$, that is $[0,1]$ with the usual order and having as tensor product the Łukasiewicz sum, is cancellative but not cancellable. In addition we remark that every *value quantale* [17] is cancellative.

Proposition 6.13. *Let V be an integral quantale. The following assertions are equivalent:*

(i) *BV is separated.*

(ii) *V is cancellative.*

(iii) *If X is separated then BX is separated.*

Proof. (i) $\Rightarrow$ (ii): Let $r, s \in V$, $r \neq \bot$ and $r = s \otimes r$. Note that

$$BV((k,r),(s,r)) = \hom(r, \hom(k,s) \otimes r) = \hom(r, s \otimes r) = \hom(r,r) = k$$

and

$$\begin{aligned} BV((s,r),(k,r)) = \hom(r, \hom(s,k) \otimes r) &= \hom(r, \hom(s,k) \otimes s \otimes r) \\ &= \hom(s \otimes r, s \otimes r) = k. \end{aligned}$$

Therefore, since BV is separated, $(s,r) = (k,r)$ and it follows that $s = k$.

(ii) $\Rightarrow$ (iii): If $(x,r) \simeq (y,s)$ in BX, then

$$BX((x,r),(y,s)) = k \Leftrightarrow r \leq X(x,y) \otimes s, \text{ and}$$

$$BX((y,s),(x,r)) = k \Leftrightarrow s \leq X(y,x) \otimes r.$$

Therefore $r \leq s$ and $s \leq r$, that is $r = s$. Moreover, since $r \leq X(x,y) \otimes r \leq r$ we have that $X(x,y) = k$. Analogously, $X(y,x) = k$ and we conclude that $x = y$.

(iii) $\Rightarrow$ (i): Since V is separated it follows immediately from (iii) that BV is separated. $\square$

We can now show that $\mathbb{B}$ is a submonad of $\mathbb{P}$ in the adequate setting. *From now on we will be working with a cancellative and integral quantale V, and $\mathbb{B}$ will be the (co)restriction of the formal ball monad to V-$\mathbf{Cat}_{\mathrm{sep}}$.*

Proposition 6.14. *Let V be a cancellative and integral quantale. Then $\mathbb{B}$ is a submonad of $\mathbb{P}$ in V-$\mathbf{Cat}_{\mathrm{sep}}$.*

Proof. Thanks to Proposition 6.3, all that remains is to show that σ_X is injective on objects, for any V-category X. Let $\sigma(x,r)=\sigma(y,s)$, or, equivalently, $X(-,x)\otimes r=X(-,y)\otimes s$. Then, in particular,

$$r=X(x,x)\otimes r=X(x,y)\otimes s\leq s=X(y,y)\otimes s=X(y,x)\otimes r\leq r.$$

Therefore $r=s$ and $X(y,x)=X(x,y)=k$. We conclude that $(x,r)=(y,s)$. $\square$

Thanks to Theorem 5.5 $\mathbb{B}$-algebras are characterized via an injectivity property with respect to special embeddings. We end this section studying in more detail these embeddings. Since we are working in $V\text{-}\mathbf{Cat}_{\mathrm{sep}}$, a B-embedding $h\colon X\to Y$, being fully faithful, is injective on objects. Therefore, for simplicity, we may think of it as an inclusion. With $Bh_\sharp\colon BY\to BX$ the right adjoint and left inverse of $Bh\colon BX\to BY$, we denote $Bh_\sharp(y,r)$ by (y_r,r_y).

LEMMA 6.15. *Let $h\colon X\to Y$ be a B-embedding. Then:*

(1) $(\forall y\in Y)\ (\forall x\in X)\ (\forall r\in V)\ BY((x,r),(y,r))=BY((x,r),(y_r,r_y))$;

(2) $(\forall\, y\in Y)\colon k_y=Y(y_k,y)$;

(3) $(\forall\, y\in Y)\ (\forall x\in X)\colon\ \ Y(x,y)=Y(x,y_k)\otimes Y(y_k,y)$.

Proof. (1) From $Bh_\sharp\cdot Bh=1_{BX}$ and $Bh\cdot Bh_\sharp\leq 1_{BY}$ one gets, for any $(y,r)\in BY$, $(y,r)\leq(y_r,r_y)$, i.e. $BY((y,r),(y_r,r_y))=\hom(r_y,Y(y_r,y)\otimes r)=k$. Therefore, for all $x\in X$, $y\in Y$, $r\in V$,

$$\begin{aligned}BY((x,r),(y,r))&\leq BX((x,r),(y_r,r_y))=BY((x,r),(y_r,r_y))\\&=BY((x,r),(y_r,r_y))\otimes BY((y_r,r_y),(y,r))\leq BY((x,r),(y,r)),\end{aligned}$$

that is

$$BY((x,r),(y,r))=BY((x,r),(y_r,r_y)).$$

(2) Let $y\in Y$. Then

$$Y(y_k,y)=BY((y_k,k),(y,k))=BY((y_k,k),(y_k,k_y))=k_y.$$

(3) Let $y\in Y$ and $x\in X$. Then

$$\begin{aligned}Y(x,y)=BY((x,k),(y,k))&=BY((x,k),(y_k,k_y))\\&=Y(x,y_k)\otimes k_y=Y(x,y_k)\otimes Y(y_k,y).\end{aligned}$$ $\square$

PROPOSITION 6.16. *Let X and Y be V-categories. A V-functor $h\colon X\to Y$ is a B-embedding if and only if h is fully faithful and*

(6.ii) $$(\forall y\in Y)\ (\exists! z\in X)\ (\forall x\in X)\ \ Y(x,y)=Y(x,z)\otimes Y(z,y).$$

Proof. If h is a B-embedding, then it is fully faithful by Proposition 5.3 and, for each $y \in Y$, $z = y_k \in X$ fulfils the required condition. To show that such z is unique, assume that $z, z' \in X$ verify the equality of condition (6.ii). Then

$$Y(z,y) = Y(z,z') \otimes Y(z',y) \leq Y(z',y) = Y(z',z) \otimes Y(z,y) \leq Y(z,y),$$

and therefore, because V is cancellative, $Y(z',z) = k$; analogously one proves that $Y(z,z') = k$, and so $z = z'$ because Y is separated.

To prove the converse, for each $y \in Y$ we denote by $\overline{y}$ the only $z \in X$ satisfying (6.ii), and define

$$Bh_\sharp(y,r) = (\overline{y}, Y(\overline{y},y) \otimes r).$$

When $x \in X$, it is immediate that $\overline{x} = x$, and so $Bh_\sharp \cdot Bh = 1_{BX}$. Using Remark 1.3, to prove that $Bh_\sharp$ is a V-functor and $Bh \dashv Bh_\sharp$ it is enough to show that

$$BX((x,r), Bh_\sharp(y,s)) = BY(Bh(x,r),(y,s)),$$

for every $x \in X$, $y \in Y$, $r, s \in V$. By definition of $Bh_\sharp$ this means

$$BX((x,r),(\overline{y}, Y(\overline{y},y) \otimes s)) = BY((x,r),(y,s)),$$

that is,

$$\hom(r, Y(x,\overline{y}) \otimes Y(\overline{y},y) \otimes s) = \hom(r, Y(x,y) \otimes s),$$

which follows directly from (6.ii). □

COROLLARY 6.17. *In* **Met**, *if* $X \subseteq [0,\infty]$, *then its inclusion* $h\colon X \to [0,\infty]$ *is a* B*-embedding if, and only if,* X *is a closed interval.*

Proof. If $X = [x_0, x_1]$, with $x_0, x_1 \in [0,\infty]$, $x_0 \leq x_1$, then it is easy to check that, defining $\overline{y} = x_0$ if $y \leq x_0$, $\overline{y} = y$ if $y \in X$, and $\overline{y} = x_1$ if $y \geq x_1$, for every $y \in [0,\infty]$, condition (6.ii) is fulfilled.

We divide the proof of the converse in two cases:

(1) If X is not an interval, i.e. if there exists $x, x' \in X$, $y \in [0,\infty] \setminus X$ with $x < y < x'$, then either $\overline{y} < y$, and then

$$0 = y \ominus x' \neq (y \ominus x') + (y \ominus \overline{y}) = y - \overline{y},$$

or $\overline{y} > y$, and then

$$y - x = y \ominus x \neq (\overline{y} \ominus x) + (y \ominus \overline{y}) = \overline{y} - x.$$

(2) If $X = [x_0, x_1[$ and $y > x_1$, then there exists $x \in X$ with $\overline{y} < x < y$, and so

$$y - x = y \ominus x \neq (\overline{y} \ominus x) + (y \ominus \overline{y}) = y - \overline{y}.$$

An analogous argument works for $X =]x_0, x_1]$. □

7. On algebras for submonads of $\mathbb{P}$ and their morphisms. In the following $\mathbb{T}=(T,\mu,\eta)$ is a submonad of the presheaf monad $\mathbb{P}=(P,m,y)$ in V-$\mathbf{Cat}_{\mathrm{sep}}$ For simplicity we will assume that the injective and fully faithful components of the monad morphism $\sigma : T \to P$ are inclusions. Theorem 5.1 gives immediately that:

PROPOSITION 7.1. *Let (X,a) be a V-category and $\alpha : TX \to X$ be a V-functor. The following are equivalent:*

(1) *(X,α) is a $\mathbb{T}$-algebra.*

(2) *$\forall\, x\in X : \alpha(x^*)=x$.*

We would like to identify the $\mathbb{T}$-algebras directly, as we did for $\mathbb{B}_\bullet$ or $\mathbb{B}$ in Proposition 6.4. First of all, we point out that a $\mathbb{T}$-algebra structure $\alpha\colon TX\to X$ must satisfy, for every $\varphi\in TX$ and $x\in X$,

$$X(\alpha(\varphi),x)=TX(\varphi,x^*),$$

and so, in particular,

$$\alpha(\varphi)\le x \;\Leftrightarrow\; \varphi\le x^*;$$

hence α must assign to each $\varphi\in TX$ an $x_\varphi\in X$ so that

$$x_\varphi=\min\{x\in X\,;\,\varphi\le x^*\}.$$

Moreover, for such map $\alpha\colon TX\to X$, α is a V-functor if, and only if,

$$\begin{aligned}
&(\forall\varphi,\rho\in TX)\;\; TX(\varphi,\rho)\le X(x_\varphi,x_\rho)=TX(X(-,x_\varphi),X(-,x_\rho))\\
\Leftrightarrow\;&(\forall\varphi,\rho\in TX)\;\; TX(\varphi,\rho)\le\bigwedge_{x\in X}\hom(X(x,x_\varphi),X(x,x_\rho))\\
\Leftrightarrow\;&(\forall x\in X)\;(\forall\varphi,\rho\in TX)\;\; X(x,x_\varphi)\otimes TX(\varphi,\rho)\le X(x,x_\rho).
\end{aligned}$$

PROPOSITION 7.2. *A V-category X is a $\mathbb{T}$-algebra if, and only if:*

(1) *for all $\varphi\in TX$ there exists $\min\{x\in X\,;\,\varphi\le x^*\}$;*

(2) *for all $\varphi,\rho\in TX$ and for all $x\in X$, $X(x,x_\varphi)\otimes TX(\varphi,\rho)\le X(x,x_\rho)$.*

We remark that condition (2) can be equivalently stated as:

(2') for each $\rho\in TX$, the distributor $\rho_1=\displaystyle\bigvee_{\varphi\in TX}X(-,x_\varphi)\otimes TX(\varphi,\rho)$ satisfies $x_{\rho_1}=x_\rho$,

which is the condition corresponding to condition (2) of Corollary 6.7.

Finally, as for the formal ball monad, Theorem 5.1 gives the following characterisation of $\mathbb{T}$-algebra morphisms.

COROLLARY 7.3. *Let $(X,\alpha),(Y,\beta)$ be $\mathbb{T}$-algebras. Then a V-functor $f : X \to Y$ is a $\mathbb{T}$-algebra morphism if and only if*

$$(\forall \varphi \in TX)\ \ \beta(\varphi \cdot f^*) \geq f(\alpha(\varphi)).$$

EXAMPLE 7.4. *The Lawvere monad.* Among the examples presented in [2] there is a special submonad of $\mathbb{P}$ which is inspired by the crucial remark of Lawvere in [18] that Cauchy completeness for metric spaces is a kind of cocompleteness for V-categories. Indeed, the submonad $\mathbb{L}$ of $\mathbb{P}$ induced by

$$\Phi = \{\varphi\colon X {\longrightarrow\!\!\!\!\!\!\circ\,\,} Y\,;\, \varphi \text{ is a right adjoint } V\text{-distributor}\}$$

has as $\mathbb{L}$-algebras the *Lawvere complete V-categories.* These were studied also in [3], and in [14] under the name L-complete V-categories. When $V = [0,\infty]_+$, using the usual order in $[0,\infty]$, for distributors $\varphi\colon X {\longrightarrow\!\!\!\!\!\!\circ\,\,} E$, $\psi\colon E {\longrightarrow\!\!\!\!\!\!\circ\,\,} X$ to be adjoint

$$\begin{tikzcd} X \arrow[r, bend left, "\varphi"] \arrow[r, phantom, "\top"] & E \arrow[l, bend left, "\psi"] \end{tikzcd}$$

means that

$$\begin{aligned}(\forall x,x' \in X)\ \ X(x,x') &\leq \varphi(x) + \psi(x'),\\ 0 &\geq \inf_{x\in X}(\psi(x)+\varphi(x)).\end{aligned}$$

This means in particular that

$$(\forall n \in \mathbb{N})\ (\exists x_n \in X)\ \ \psi(x_n) + \varphi(x_n) \leq \frac{1}{n},$$

and, moreover,

$$X(x_n,x_m) \leq \varphi(x_n) + \psi(x_m) \leq \frac{1}{n} + \frac{1}{m}.$$

This defines a *Cauchy sequence* $(x_n)_n$, so that

$$(\forall \varepsilon > 0)\ (\exists p \in \mathbb{N})\ (\forall n,m \in \mathbb{N})\ n \geq p\ \wedge\ m \geq p\ \Rightarrow\ \ X(x_n,x_m) + X(x_m,x_n) < \varepsilon.$$

Hence, any such pair induces a (equivalence class of) Cauchy sequence(s) $(x_n)_n$, and a representative for

$$\begin{tikzcd} X \arrow[r, "(1_X)_*"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\leq"] & X \\ E \arrow[ur, "{[\varphi,(1_X)_*]}"'] & \end{tikzcd}$$

is nothing but a limit point for $(x_n)_n$. Conversely, it is easily checked that every Cauchy sequence $(x_n)_n$ in X gives rise to a pair of adjoint distributors

$$\varphi = \lim_n X(-,x_n) \text{ and } \psi = \lim_n X(x_n,-).$$

We point out that the $\mathbb{L}$-embeddings, i.e. the fully faithful and fully dense V-functors $f\colon X\to Y$ do not coincide with the $\mathbb{L}$-dense ones (so that f_* is a right adjoint). For instance, assuming for simplicity that V is integral, a V-functor $y\colon E\to X$ ($y\in X$) is fully dense if and only if $y\simeq x$ for all $x\in X$, while it is an $\mathbb{L}$-embedding if and only if $y\leq x$ for all $x\in X$. Indeed, $y\colon E\to X$ is $\mathbb{L}$-dense if, and only if,

– there is a distributor $\varphi\colon X{\longrightarrow\!\!\!\!\!\!\circ\;\;} E$, i.e.

$$(\forall x,x'\in X)\;\; X(x,x')\otimes\varphi(x')\leq\varphi(x), \tag{7.i}$$

such that

– $k\geq\varphi\cdot y_*$, which is trivially true, and $a\leq y_*\cdot\varphi$, i.e.

$$(\forall x,x'\in X)\;\; X(x,x')\leq\varphi(x)\otimes X(y,x'). \tag{7.ii}$$

Since (7.i) follows from (7.ii),

$$y \text{ is } \mathbb{L}\text{-dense} \;\;\Leftrightarrow\;\; (\forall x,x'\in X)\;\; X(x,x')\leq\varphi(x)\otimes X(y,x').$$

In particular, when $x=x'$, this gives $k\leq\varphi(x)\otimes X(y,x)$, and so we can conclude that, for all $x\in X$, $y\leq x$ and $\varphi(x)=k$. The converse is also true; that is

$$y \text{ is } \mathbb{L}\text{-dense} \;\;\Leftrightarrow\;\; (\forall x\in X)\;\; y\leq x.$$

Still, it was shown in [14] that injectivity with respect to fully dense and fully faithful V-functors (called L-dense in [14]) characterizes also the $\mathbb{L}$-algebras.

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