COMPLEMENTARITY AND GENETIC ALGORITHMS FOR AN OPTIMIZATION SHELL PROBLEM

Isabel N. Figueiredo, Joaquim J. Júdice

Departamento de Matemática
Universidade de Coimbra, Coimbra, Portugal
e–mail: isabelf@mat.uc.pt, judice@mat.uc.pt

Pedro N. Oliveira

Departamento de Produção e Sistemas
Universidade do Minho, Braga, Portugal
e–mail: pno@ci.uminho.pt

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Abstract. *The application of complementarity and genetic algorithms to an optimization thin laminated shallow shell problem is discussed. The discrete form of the problem leads to a Mathematical Program with Equilibrium Constraints (MPEC) [1], whose constraint set consists of a variational inequality and a set of equality constraints. Furthermore the variables are discrete. Special instances of the general problem are considered and indicate that the choice of the algorithm depends on the problem to be linear or nonlinear.*

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1 Introduction

Let S be a thin elastic laminated shallow shell, made of $2n$ laminas which are symmetrical disposed, both from a material and a geometric properties standpoint, with respect to the middle surface of the shell. Each lamina i, for $i = 1, ..., n$, is supposed to be made of a material m_i, having a monoclinic behaviour through the thickness of the laminate. The thickness t_i of lamina i is defined by $t_i = h_i - h_{i-1}$, where h_i is the distance, measured along the direction of the unit normal vector to the middle surface, to the upper face of lamina i. In addition, the shell is subject to a vertical load, clamped on the boundary, and the vertical displacement of the middle surface is constrained by an obstacle. Moreover, upper bounds on the global cost, weight and thickness of the laminated shell are also imposed.

Given discrete sets of materials $M = \{m_j : j = 1, 2, 3, ..., q \ (q > n)\}$, of thickness $T = \{t_k : k = 1, 2, 3, ..., p \ (p > n)\}$ and of functions, $\Phi = \{\vec{\phi}_l : l = 1, 2, 3, ..., r \ (r > 2)\}$ defining the middle surface of the shell, the objective is to select the materials and thickness, m_i and t_i of each lamina i, and a function $\vec{\phi}$, in order to minimize the strain energy of the two-dimensional laminated shallow shell model, for the linear and nonlinear cases.

The variational formulation of this problem takes the following form

$$\begin{cases} \min\limits_{s \in C} F(s, \vec{u}^s) \\ \text{subject to:} \begin{cases} \vec{u}^s \in V \\ \Pi^s(\vec{u}^s) = \min\limits_{\vec{v} \in V} \Pi^s(\vec{v}) \end{cases} \end{cases} \tag{1}$$

The discrete optimization variable s is defined by $s = (s_M, s_T, s_\Phi)$ where s_M is a vector of materials with components in M, s_T is a vector of thickness with components in T and s_Φ is a vector with only one component, belonging to the set Φ. The set V contains the admissible displacements of the middle surface; Π^s and F are, respectively, the total potential energy and the strain energy of the laminated shell. The set C is a subset of $M \times T$, that imposes constraints as the global cost, weight and thickness of the laminated shell.

The main goal of this paper is to describe and to investigate the properties of the bilevel problem (1) and its numerical solution. The inner optimization problem in (1) can be reformulated as a variational inequality, that is more useful for its numerical solution by the finite element method. Some hybrid algorithms are discussed that combine complementarity path-following techniques [2], [3] for solving the variational inequality, with genetic algorithms [4] for the minimization of the functional. Two special instances of problem (1) are also discussed, namely the obstacle problem for a nonlinear elastic beam [5] and the compliance minimization of a composite laminated plate [6].

The rest of the paper is organized as follows: in section 2 and 3 notations and hypotheses on the geometry, the material properties of the shell, the expression of the strain and curvature tensors and the exact definitions of V, Π^s and F are introduced. Equivalent formulations and properties of problem (1) and its discrete formulation are discussed in the next two sections. The complementarity and genetic algorithms for the discrete problem (1) are briefly described in section 6. Finally some conclusions are stated in the last part of the paper.

2 Notations and Hypotheses

In the next section the definitions of V, Π^s and F are presented for the linear and nonlinear shallow shell models. For this purpose, one must introduce some notations and hypotheses.

As far as the notations are concerned, greek indexes or exponents α, β, μ,... belong to the set $\{1,2\}$ and the latin indexes or exponents i, j, k,... belong to the set $\{1,2,3\}$. The summation convention with respect to repeated indexes and exponents is used; the euclidean scalar and vector product of two vectors $\vec{u}$ and $\vec{v}$ in R^3 are denoted by $\vec{u}\cdot\vec{v}$ and $\vec{u}\times\vec{v}$ respectively, and $|.|$ will denote the euclidean norm in R^3.

The hypotheses of the models, which are concerned with the geometry and material properties of the shell, the strain and curvature tensors of the middle surface, are discussed next.

2.1 Geometry of the shell

The middle surface $\overline{\Omega} \subset R^3$ of the shell is the image of an open, connected, bounded subset $\omega \subset R^2$, by a sufficiently smooth mapping $\vec{\phi}$.

The covariant and the contravariant basis, $(\vec{a}_\alpha)$ and $(\vec{a}^\beta)$, of the tangent plane of the middle surface are defined by $\vec{a}_\alpha = \vec{\phi}_{,\alpha}$ and $\vec{a}^\beta \cdot \vec{a}_\alpha = \delta_\alpha^\beta$, where δ_α^β is the Kronecker's symbol, that is, $\delta_\alpha^\beta = 1$, if $\alpha = \beta$ and $\delta_\alpha^\beta = 0$, if $\alpha \neq \beta$ and $._{,\alpha}$ means the usual derivation with respect to the component ξ^α of the variable $\xi = (\xi^1,\xi^2)$ in ω.

The unit normal vector is $\vec{a}_3 = \vec{a}^3 = \frac{\vec{a}_1\times\vec{a}_2}{|\vec{a}_1\times\vec{a}_2|}$ and ξ^3 denotes the variable along the vertical axis with the direction of $\vec{a}_3$.

A shell S with middle surface $\vec{\phi}(\overline{\omega})$ and constant thickness t is the set of points P in R^3 defined by

$$S = \{\vec{OP}: \quad \vec{OP} = \vec{\phi}(\xi^1,\xi^2) + \xi^3\vec{a}_3(\xi^1,\xi^2), \quad -\frac{t}{2} \leq \xi^3 \leq \frac{t}{2}\} \tag{2}$$

where O is the origin of the reference system, $\vec{\phi}(\xi^1,\xi^2)$ represents the projection of $\vec{OP}$ in the middle surface and $|\xi^3|$ is the distance from P to its projection, measured along the direction of the unit normal vector $\vec{a}^3$.

In particular a shallow shell is a shell which has a weak curvature, that is, a shell such that $b_{\alpha\beta}$ and $b_{\alpha\beta|\lambda}$ are very small when compared to the unity.

The Christoffel symbols $\Gamma^{\alpha}_{\beta\gamma}$ and the covariant components $a_{\alpha\beta}$ and $b_{\alpha\beta}$ of the first and second fundamental forms of the middle surface are given by

$$\Gamma^{\alpha}_{\beta\gamma} = \vec{a}^{\alpha} \cdot \vec{a}_{\gamma,\beta} = \vec{a}^{\alpha} \cdot \vec{a}_{\beta,\gamma} = \Gamma^{\alpha}_{\gamma\beta}, \quad a_{\alpha\beta} = \vec{a}_{\alpha} \cdot \vec{a}_{\beta}, \quad b_{\alpha\beta} = -\vec{a}_{\alpha} \cdot \vec{a}_{3,\beta}. \tag{3}$$

Furthermore $a = det(a_{\alpha\beta}) = a_{11}a_{22} - a_{12}^2 \neq 0$.

The covariant derivatives of a vector field $\vec{v}$ defined on the middle surface are denoted by a vertical bar $|$, that is,

$$v^{\alpha}_{|\mu} = v^{\alpha}_{,\mu} + \Gamma^{\alpha}_{\lambda\mu} v^{\lambda}, \quad v_{\alpha|\mu} = v_{\alpha,\mu} - \Gamma^{\lambda}_{\alpha\mu} v_{\lambda}, \quad v_{3|\alpha} = v_{3,\alpha}, \quad v_{3|\alpha\beta} = v_{3,\alpha\beta} - \Gamma^{\lambda}_{\alpha\beta} v_{3,\lambda} \tag{4}$$

where v^{α} and v_{α} are the contravariant and the covariant components of the vector field $\vec{v}$, respectively.

2.2 Material properties of the shell

Each lamina i is supposed to be made of an anisotropic and nonhomogeneous material, with elastic symmetry with respect to the surface ξ^3 = constant, that is, a monoclinic material whose elastic coefficients C_i^{jklm}, for each lamina i, satisfy ([7], [8])

$$\begin{cases} C_i^{jklm} = C_i^{kjlm} = C_i^{jkml} = C_i^{mljk}, \\ C_i^{\alpha\beta\lambda 3} = C_i^{\alpha 333} = 0, \\ \exists c > 0 : \quad C_i^{jklm} \tau_{jk} \tau_{lm} \geq \sum_{j,k=1}^{3} |\tau_{jk}|^2, \quad \forall (\tau_{jk}) \text{ symmetric tensor.} \end{cases} \tag{5}$$

2.3 Strain and curvature tensors of the middle surface

Two thin elastic shallow shell models are adopted. The expressions of the covariant components $\gamma_{\alpha\beta}(.)$ and $\rho_{\alpha\beta}(.)$ of the strain tensor and the change of curvature tensor of the middle surface are given by

$$\gamma_{\alpha\beta}(\vec{v}) = \frac{1}{2}(v_{\alpha|\beta} + v_{\beta|\alpha}) - b_{\alpha\beta} v_3, \qquad \rho_{\alpha\beta}(\vec{v}) = v_{3|\alpha\beta}, \tag{6}$$

for the linear case [9], and by

$$\gamma_{\alpha\beta}(\vec{v}) = \frac{1}{2}(v_{\alpha|\beta} + v_{\beta|\alpha}) - b_{\alpha\beta} v_3 + \frac{1}{2} v_{3,\alpha} v_{3,\beta}, \qquad \rho_{\alpha\beta}(\vec{v}) = v_{3|\alpha\beta}. \tag{7}$$

for the nonlinear case [10], [11].

3 Definition of V, Π^s and F

The set of admissible displacements $\vec{u}$ of the middle surface of the shell is defined by

$$V = \{\vec{u} \equiv (u_1, u_2, u_3) \equiv (\underline{u}, u_3) \in [H_0^1(\omega)]^2 \times K\} \tag{8}$$

with

$$K = \{z \in H_0^2(\omega) : \quad z(\xi_1, \xi_2) \geq \psi(\xi_1, \xi_2), \quad a.e.\ \omega\}, \tag{9}$$

where ψ is the function representing the obstacle and $H_0^1(\omega)$, $H_0^2(\omega)$ are Sobolev spaces defined by

$$H_0^1(\omega) = \{v \in H^1(\omega) : \ v_{|\partial\omega} = 0\}, \qquad H_0^2(\omega) = \{v \in H^2(\omega) : \ v_{|\partial\omega} = \frac{\partial v}{\partial n} = 0\} \tag{10}$$

with $\partial\omega$ the boundary of ω and $\frac{\partial}{\partial n}$ the normal derivative.

The functional $\Pi^s(\vec{v})$ is the total potential energy of the shell given by

$$\Pi^s(\vec{v}) = \frac{1}{2} b^s(\vec{v}, \vec{v}) - L(\vec{v}). \tag{11}$$

The form $L(.)$ is a linear scalar form in V, related to the vertical force, acting on the shell

$$L(\vec{v}) = \int_\omega f^3\, v_3 \sqrt{a}\, d\xi^1\, d\xi^2, \tag{12}$$

with $f^3 \in L^2(\omega)$ the given intensity of vertical force. The form $\frac{1}{2} b^s(\vec{v}, \vec{v})$ is the strain energy of the laminated shallow shell and its expression is

$$\begin{cases} b^s(\vec{v}, \vec{v}) = 2 \sum_{i=1}^{n} \int_\omega \Big[\Big(\int_{h_{i-1}}^{h_i} A_i^{\alpha\beta\lambda\mu}\, d\xi^3 \Big)\, \gamma_{\alpha\beta}(\vec{v})\, \gamma_{\lambda\mu}(\vec{v}) + \\ \qquad \Big(\int_{h_{i-1}}^{h_i} (\xi^3)^2 A_i^{\alpha\beta\lambda\mu}\, d\xi^3 \Big)\, \rho_{\alpha\beta}(\vec{v})\, \rho_{\lambda\mu}(\vec{v}) \Big] \sqrt{a}\, d\xi^1 d\xi^2. \end{cases} \tag{13}$$

where $\gamma_{\alpha\beta}(.)$ and $\rho_{\alpha\beta}(.)$ are, respectively, the covariant components of the strain tensor and the change of curvature tensor of the middle surface, of the linear or nonlinear model (6) or (7), and $A_i^{\alpha\beta\lambda\mu}$ are the reduced elasticity coefficients, of lamina i, defined by

$$A_i^{\alpha\beta\lambda\mu} = C_i^{\alpha\beta\lambda\mu} - \frac{C_i^{\alpha\beta 33}\, C_i^{33\lambda\mu}}{C_i^{3333}}. \tag{14}$$

A justifcation of formula (13) is given in [12], by the asymptotic development technique, with the half-thickness of the laminate as a small parameter.

Another justification of (13) can be obtained directly from the formula of the strain energy of the three dimensional shell model, that is, from the calculus of the integral

$$\int_S \sigma^{kj}\, \varepsilon_{kj} \tag{15}$$

with $\sigma^{kj} = C^{kjlm}\, \varepsilon_{lm}$ the components of the three dimensional stress tensor, ε_{lm} the components of the three dimensional strain tensor and C^{kjlm} the elastic coefficients of the laminate, and assuming that $\varepsilon_{\alpha\beta} = \gamma_{\alpha\beta} + \xi^3 \rho_{\alpha\beta}$ ($\gamma_{\alpha\beta}$, $\rho_{\alpha\beta}$ given by (6) or (7)), $\varepsilon_{\alpha 3} = 0$,

$\sigma^{33} = 0$ and $C^{kjlm} = C_i^{kjlm}$ in lamina i. Moreover, cross products of the type $\gamma_{\alpha\beta}(\vec{v})\, \rho_{\alpha\beta}(\vec{v})$ do not appear in (13), because the laminas are symmetrical with respect to the middle surface of the shell.

Finally the objective functional in (1) is defined by

$$F(s, \vec{u}^s) = \frac{1}{2} b^s(\vec{u}^s, \vec{u}^s). \tag{16}$$

4 Variational inequality formulation of the inner problem

In this section the inner mathematical problem

$$\begin{cases} \vec{u}^s \in V \\ \Pi^s(\vec{u^s}) = \min\limits_{\vec{v} \in V} \Pi^s(\vec{v}) \end{cases} \tag{17}$$

that is the constrained set of problem (1), is briefly studied. It is shown that problem (17) can be reformulated as a variational inequality that is more useful for the numerical procedure to be discussed. The functions involved in this variational inequality depend on the problem to be linear or nonlinear.

For a fixed s, the solution of (17) is the triple composed by the covariant components (u_1^s, u_2^s, u_3^s) of the displacement $\sum_{i=1}^3 u_i^s\, \vec{a}^i$ of the points of the middle surface $\vec{\phi}(\overline{\omega})$ of the shell, when it is subject to the action of a vertical force, and the normal displacement $u_3^s\, \vec{a}^3$ is constrained by the obstacle ψ.

As the function Π^s is Gâteaux differentiable in $[H_0^1(\omega)]^2 \times H_0^2(\omega)$, problem (17) is equivalent to the following variational inequality [13]

$$\begin{cases} \vec{u}^s \in V \\ < D\Pi^s(\vec{u^s}), \vec{v} - \vec{u^s} > \geq 0, \quad \forall \vec{v} \in V \end{cases} \tag{18}$$

where $< D\Pi^s(\vec{u^s}), \vec{v} >$ is the Gâteaux derivative of Π^s at $\vec{u^s}$ in the direction of $\vec{v}$.

As $V = [H_0^1(\omega)]^2 \times K$, choosing in (18), $\vec{v} = (0, v_3)$ and subsequently $\vec{v} = (\underline{v} + \underline{u}, 0)$ and $\vec{v} = (-\underline{v} + \underline{u}, 0)$ it is easy to show that the variational inequality (18) is equivalent to a system composed of another variational inequality and an equation. The expressions of these systems for the linear and nonlinear cases are stated below.

- Linear case

$$\begin{cases} \text{Find } \vec{u^s} \equiv (u_1, u_2, u_3) \equiv (\underline{u}, u_3) \in V, \text{ such that} \\ A^s(u_3, v_3 - u_3) + a^s(\underline{u}, v_3 - u_3) - L(v_3 - u_3) \geq 0, \quad \forall v_3 \in K \\ B^s(\underline{u}, \underline{v}) + c^s(u_3, \underline{v}) = 0, \quad \forall \underline{v} \in [H_0^1(\omega)]^2 \end{cases} \tag{19}$$

- Nonlinear case

$$\left\{\begin{array}{l} \text{Find } \vec{u}^s \equiv (u_1, u_2, u_3) \equiv (\underline{u}, u_3) \in V, \text{ such that} \\ A^s(u_3, v_3 - u_3) + a^s(\underline{u}, u_3; v_3 - u_3) - L(v_3 - u_3) \geq 0, \quad \forall v_3 \in K \\ B^s(\underline{u}, \underline{v}) + d^s(u_3, \underline{v}) = 0, \quad \forall \underline{v} \in [H_0^1(\omega)]^2 \end{array}\right. \tag{20}$$

The definitions of the forms in these problems are presented next:

$$\left\{\begin{array}{l} A^s(u_3, v_3) = 2\sum_{i=1}^{n} \int_\omega \Big[\Big(\int_{h_{i-1}}^{h_i} A_i^{\alpha\beta\lambda\mu}\, d\xi^3\Big)\, b_{\alpha\beta} u_3\, b_{\lambda\mu} v_3 + \\ \qquad \Big(\int_{h_{i-1}}^{h_i} (\xi^3)^2 A_i^{\alpha\beta\lambda\mu}\, d\xi^3\Big)\, u_{3|\alpha\beta}\, v_{3|\lambda\mu}\Big] \sqrt{a}\, d\xi^1 d\xi^2 \end{array}\right. \tag{21}$$

$$B^s(\underline{u}, \underline{v}) = 2\sum_{i=1}^{n} \int_\omega \Big[\Big(\int_{h_{i-1}}^{h_i} A_i^{\alpha\beta\lambda\mu}\, d\xi^3\Big) \frac{1}{4}(u_{\alpha|\beta} + u_{\beta|\alpha})\,(v_{\lambda|\mu} + v_{\mu|\lambda})\Big] \sqrt{a}\, d\xi^1 d\xi^2 \tag{22}$$

$$c^s(u_3, \underline{v}) = -2\sum_{i=1}^{n} \int_\omega \Big[\Big(\int_{h_{i-1}}^{h_i} A_i^{\alpha\beta\lambda\mu}\, d\xi^3\Big)\, b_{\alpha\beta} u_3\, \frac{1}{2}\,(v_{\lambda|\mu} + v_{\mu|\lambda})\Big] \sqrt{a}\, d\xi^1 d\xi^2 \tag{23}$$

$$d^s(u_3, \underline{v}) = 2\sum_{i=1}^{n} \int_\omega \Big[\Big(\int_{h_{i-1}}^{h_i} A_i^{\alpha\beta\lambda\mu}\, d\xi^3\Big)\, (-b_{\alpha\beta} u_3 + \frac{1}{2} u_{3,\alpha} u_{3,\beta}) \frac{1}{2}(v_{\lambda|\mu} + v_{\mu|\lambda})\Big] \sqrt{a}\, d\xi^1 d\xi^2 \tag{24}$$

$$\left\{\begin{array}{l} a^s(\underline{u}, u_3; v_3) = 2\sum_{i=1}^{n} \int_\omega \Big[\Big(\int_{h_{i-1}}^{h_i} A_i^{\alpha\beta\lambda\mu}\, d\xi^3\Big)\Big(- b_{\alpha\beta} u_3\, u_{3,\mu}\, v_{3,\lambda} + \\ \qquad \frac{1}{2}[u_{\alpha|\beta} + u_{\beta|\alpha} + u_{3,\alpha} u_{3,\beta}][u_{3,\mu} v_{3,\lambda} - b_{\lambda\mu} v_3]\Big)\Big] \sqrt{a}\, d\xi^1 d\xi^2 \end{array}\right. \tag{25}$$

$$a^s(\underline{u}, v_3) = c^s(v_3, \underline{u}) \tag{26}$$

$$L(v_3) = \int_\omega f^3\, v_3 \sqrt{a}\, d\xi^1 d\xi^2 \tag{27}$$

It is also worthwhile to mention that for the linear model

$$< D\Pi^s(\vec{u^s}), \vec{v} >= b^s(\vec{u^s}, \vec{v}) - L(\vec{v}) \tag{28}$$

so the variational inequality (18) or the system (19) are equivalent to

$$\left\{\begin{array}{l} \vec{u}^s \in V \\ b^s(\vec{u^s}, \vec{v} - \vec{u^s}) - L(\vec{v} - \vec{u^s}) \geq 0, \quad \forall \vec{v} \in V. \end{array}\right. \tag{29}$$

As mentioned before, the analysis of the properties of the operators and forms defining the systems (19) and (20) are important, because they determine the choice of the numerical procedure to solve (17). For the case where the laminate has only one ply and the material is homogeneous and isotropic, the following results hold:

- For the linear problem (19), or equivalently (29), the bilinear form $b^s(.,.)$ is elliptic [9], under the hypothesis that $|b_\alpha^\beta| \leq \epsilon$ and $|b_{\alpha|\lambda}^\beta| \leq \epsilon$, for $\epsilon > 0$ a real number small enough, where b_α^β are the mixed components of the second form of the surface; this implies the existence and uniqueness of solution by the Lions-Stampacchia theorem [14].

- For the nonlinear case, the problem (20) can be transformed into a variational inequality whose operator is nonlinear, pseudo-monotone and coercive [11], so it has at least a solution [14].

For the laminated linear or nonlinear shell problems (19) and (20), with more than one material, the same properties and results hold, using arguments similar to [9] and [11], because the reduced elastic coefficients $A_i^{\alpha\beta\gamma\mu}$ are smooth enough and satisfy the following symmetric and ellipticity conditions [15]

$$\begin{cases} A_i^{\alpha\beta\gamma\mu} = A_i^{\alpha\beta\mu\gamma} = A_i^{\mu\gamma\alpha\beta} = A_i^{\gamma\mu\alpha\beta} \\ \exists c > 0 : \quad A_i^{\alpha\beta\gamma\mu}\tau_{\alpha\beta}\tau_{\gamma\mu} \geq \sum_{\alpha,\beta=1}^{2} |\tau_{\alpha\beta}|^2, \quad \forall(\tau_{\alpha\beta}) \text{ symmetric tensor.} \end{cases} \tag{30}$$

5 Discrete Formulation

As is usual in the solution of these type of variational models, the finite element method is used to get a discrete problem that approximates the original continuous problem (1); see for instance [9], for the details of the use of this method in shell models. Due to the constraints involved in the continuous problem (1), the discrete problem takes the form of a Mathematical Program with Equilibrium Constraints (MPEC) [1]. In this section the definiton of the resulting MPEC is introduced.

Consider a finite element mesh of the domain ω, with m global degrees of freedom. Let L_1, L, H and I be four subsets of the index set $\{1, 2, 3, .., m\}$ such that:

- L_1 represents the indexes of the degrees of freedom of the vertical displacement u_3 and its derivatives, at the nodes in the interior of the mesh;

- L is a subset of L_1, corresponding to the degrees of freedom of the vertical displacement u_3;

- I contains the indexes of the degrees of freedom of the displacement $\vec{u}$ at the boundary nodes;

- H is the complementary in $\{1, 2, 3, .., m\}$ of the sets L_1 and I, that is $H = \{1, 2, 3, .., m\} \setminus (I \cup L_1)$.

Let v be a vector in R^m, and denote by v_L, v_I, v_{L_1} or v_H the subvectors of v, whose components have indexes, that vary in L, I, L_1 or H respectively.

Moreover, let $\psi_L = (\psi_i)_{i\in L}$ be the vector whose components are the values of the obstacle ψ at the nodes belonging to the set L, and denote by K the set that approximates the original set (9), that is

$$K = \{z \in R^m : \quad z_L \geq \psi_L \Leftrightarrow z_i \geq \psi_i, \ i \in L\}. \tag{31}$$

Then, the discrete problem corresponding to (19) or (20) takes the following form

$$\begin{cases} \text{Find } u \in R^m, \text{ such that} \\ u_I = 0, \quad u_L \geq \psi_L \\ (z_{L_1} - u_{L_1})^T G^s_{L_1}(u) \geq 0 \\ z \in R^m, \quad z_L \geq \psi_L \\ G^s_H(u) = 0, \end{cases} \tag{32}$$

where u is the finite element approximation of $\vec{u}^s$, depending on s, and $G^s_{L_1}(u) = (G^s_i(u))_{i\in L_1}$, $G^s_H(u) = (G^s_j(u))_{j\in H}$ are the functions obtained from the finite element discretization of the variational inequality and the equation of systems (19) or (20). These functions are linear or nonlinear, depending on the continuous problem to be linear or nonlinear.

If $F(s,u)$ is the finite element approximation of (16), then the discrete formulation of (1) is the following MPEC

$$\begin{cases} \min\limits_{s\in C} F(s,u) \\ \text{subject to:} \begin{cases} \text{Find } u \in R^m, \text{ such that} \\ u_I = 0, \quad u_L \geq \psi_L \\ (z_{L_1} - u_{L_1})^T G^s_{L_1}(u) \geq 0, \quad z_L \geq \psi_L \\ G^s_H(u) = 0 \end{cases} \end{cases} \tag{33}$$

where s is the discrete optimization variable, defined in (1).

6 Numerical Procedure

In this section, the complementarity and genetic algorithms, that can be applied for the solution of problem (33), are discussed.

6.1 Complementarity algorithms

These algorithms are designed to process (32), that is, the inner problem of (33). Let $n = m - |I|$, where $|I|$ denotes the number of elements of the set I and let $J = \{1, 2, 3, .., m\} \setminus (I \cup L)$. It is well known that (32) is equivalent to the following mixed complementarity problem

$$\begin{cases} \text{Find } u = (u_J, u_L) \in R^n, \text{ such that} \\ G^s_J(u) = 0, \quad u_J \quad \text{free} \\ 0 \leq (u_L - \psi_L) \perp G^s_L(u) \geq 0, \end{cases} \tag{34}$$

where the symbol $\perp$ means orthogonality for the usual scalar product in R^n and the vector $u_I = 0$ has been eliminated from further consideration.

Next, an interior-point method, for the solution of this complementarity problem, is discussed. To this end, it is convenient to formulate (34) in the following equivalent form:

$$\text{Find } (u, w) \in R^n \times R^n, \text{ such that}$$

$$G^s(u) - w = 0 \tag{35}$$

$$(U_L - \Psi_L)^T W_L \, e_L = 0 \tag{36}$$

$$w_J = 0 \tag{37}$$

$$u_L \geq \psi_L, \quad w_L \geq 0 \tag{38}$$

where U_L, Ψ_L, W_L are diagonal matrices with diagonal elements equal to u_L, ψ_L and w_L repectively, e_L is a vector of ones with $|L|$ components, the exponent T means transposition and $G^s(u) = (G^s_J(u), G^s_L(u))$.

The interior-point method is an iterative technique based on Newton method for the solution of system (35)-(38), such that each iterate must satisfy the condition (38) strictly. The steps of the algorithm are as follows:

Step 1 - Let (u^0, w^0) be such that

$$u^0_L > \psi_L, \quad w^0_L > 0, \quad w^0_J = 0. \tag{39}$$

Step 2 - For $k = 0, 1, 2, 3, ..$ determine Δu^k in $R^{|J|+|L|}$ and Δw^k in $R^{|L|}$

$$\begin{bmatrix} & \nabla G^s(u^k) & \vdots & -I_L \\ & & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ W^k_L & 0 & \vdots & U^k_L - \Psi_L \end{bmatrix} \begin{bmatrix} \Delta u^k \\ \\ \\ \Delta w^k \end{bmatrix} = \begin{bmatrix} w^k - G^s(u^k) \\ \\ \\ \mu_k e_L - (U^k_L - \Psi_L) W^k_L e_L \end{bmatrix} \tag{40}$$

where

$$\mu_k = \frac{\delta}{|L|} (u^k_L - \psi_L)^T w^k_L = \frac{\delta}{|L|} \sum_{i \in L} (u^k_i - \psi_i) w^k_i \tag{41}$$

with $0 < \delta < 1$ a fixed parameter and I_L the unit matrix of order $|L|$.

Step 3 - Set

$$u^{k+1} = u^k + \alpha_k \, \Delta u^k, \quad w^{k+1}_L = w^k_L + \alpha_k \, \Delta w^k, \quad w^{k+1}_J = 0, \tag{42}$$

where α_k is a stepsize defined by

$$\alpha_k = \nu_k \min \left\{ \begin{array}{ll} \min \left\{ \frac{u^k_i - \psi_i}{-(\Delta u^k)_i} : \quad (\Delta u^k)_i < 0, \ i \in L \right\}, \\ \\ \min \left\{ \frac{w^k_i}{-(\Delta w^k)_i} : \quad (\Delta w^k)_i < 0, \ i \in L \right\} \end{array} \right\} \tag{43}$$

for some $0 < \nu_k < 1$.

Step 4 - Stop if

$$\|w^{k+1} - G^s(u^{k+1})\|_{R^n} < \varepsilon_1, \qquad \sum_{i\in L}(u_i^{k+1} - \psi_i)w_i^{k+1} < \varepsilon_2 \tag{44}$$

for some positive tolerances ε_1 and ε_2.

In order to apply this method, it is necessary to impose that the matrix

$$\begin{bmatrix} & \nabla G^s(u^k) & \vdots & -I_L \\ & & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ W_L^k & 0 & \vdots & U_L^k - \Psi_L \end{bmatrix} \tag{45}$$

is nonsingular. Furthermore global convergence is assured only for special choices of δ and ν_k [3].

Moreover, in step 2 of this algorithm one computes the Newton direction of system (35)-(37) with equation (36) replaced by

$$(U_L - \Psi_L)^T W_L\, e_L = \mu_k e_L. \tag{46}$$

This change forces the products $(u_i^k - \psi_i)w_i^k$ to be strictly positive and to decrease to zero at the same rate. This has proven to work quite well in practice.

This method is well suited to solve the variational inequality of the linear model. In fact, in this case the function $G^s(u)$ is of the form

$$G^s(u) = M^s u + q = \begin{bmatrix} M_{JJ}^s & M_{JL}^s \\ M_{LJ}^s & M_{LL}^s \end{bmatrix} \begin{bmatrix} u_J \\ u_L \end{bmatrix} + \begin{bmatrix} q_J \\ q_L \end{bmatrix} \tag{47}$$

where q is a constant non zero vector, related to the force f^3 and M^s is a symmetric positive definite matrix, because $b^s(\vec{u}, \vec{v})$ is a symmetric and elliptic bilinear form. Thus the matrix (45) in step 2 is always nonsingular in each iteration.

In particular, this method was successfully applied to a nonlinear obstacle beam problem [5], whose nonlinear discrete operator G^s was of the monotone type, that is

$$(v - u)^T(G^s(v) - G^s(u)) \geq 0, \quad \forall u, v \in R^n. \tag{48}$$

Nevertheless the same method has not performed well for a nonlinear plate problem, obtained from the nonlinear shallow shell problem for the particular case $b_{\alpha\beta} = 0$, that is, the unit normal vector is constant and the curvature is zero. In fact, in this case the discrete operator is not of the monotone type, and this is a drawback for the application of the interior-point method. These objections seem to indicate that this last procedure is not the best choice for processing the nonlinear inner problem.

The path-following algorithm PATH [2] has been successfully applied for the solution of another type of nonlinear elasticity problem, an eigenvalue problem [16], for which the operator does not satisfy the monotonicity property (48). This algorithm is essentially a Newton-type method based on the reformulation of the problem as a system of nonlinear and nonsmooth equations. The reformulation of (34) uses the definition of the normal map, that is, (34) is equivalent to the following nonsmooth system of equations

$$G^s_B(x) = 0, \quad \text{with} \quad G^s_B(x) = G^s(\Pi_B(x)) + x - \Pi_B(x) \tag{49}$$

where G^s_B is the so-called normal map, $B = \{x \in R^n : \; x_L \geq \psi_L\}$ and $\Pi_B(x)$ is the projection of x onto the set B. For a detailed description of this algorithm, see [2], where a global convergence result is also proven. This algorithm should process efficiently the nonlinear shallow shell complementarity inner problem (34).

6.2 Genetic Algorithms

It follows from its definition that the discrete problem (33) is a combinatorial problem well suited for the solution via genetic algorithms, combined with the complementarity algorithms. Genetic algorithms are search and optimization algorithms that model the process of natural evolution. Their main disadvantage is that they require a great number of evaluations, although they do not request any derivative information.

A brief description of an implementation of a genetic algorithm adapted to problem (33) requires the following steps:

Step 1 A coding technique, that assigns to each variable s a binary string, referred to as a chromosome.

To exemplify this coding technique, consider for instance, that the laminated shell has 2×3 laminas, and there are 7 admissible materials $M = \{1, 2, 3, ..., 7\}$, 15 admissible thickness $T = \{1, 2, 3, ..., 15\}$ and 3 admissible functions, defining the middle surface of the shell, $\Phi = \{1, 2, 3\}$. A possible distribution of materials and thickness and a possible choice for the function $\vec{\phi}$, indicated by the vector s in (1), is

$$s = (s_M, s_T, s_\Phi) = (\underbrace{4, 1, 7}_{materials}, \underbrace{3, 11, 15}_{thickness}, \underbrace{2}_{function}) \tag{50}$$

Note that component i $(i = 1, 2, 3)$ of subvectors s_M and s_T coincide with the number of the lamina, that is, laminas 1, 2, 3 correspond to the materials 4, 1, 7 and the thickness 3, 11, 15, respectively.

By expressing these numbers (4,1,..) in the binary system, the following binary string represents the vector s

$$\underbrace{100\,001\,111}_{materials}\ \underbrace{0011\,1011\,1111}_{thickness}\ \underbrace{10}_{function} \tag{51}$$

This is called a chromosome. Thus, with this coding, each chromosome has a total of 23 bits, being 3 bits for each material, 4 bits for each thickness and 2 bits for the function.

Step 2 An initialization procedure, that is, a random set of initial points s (generated from the admissible cartesian set $M \times T \times \Phi$), which is the initial population of chromosomes.

This population of chromosomes is the set where the search of the optimum of problem (33) will be performed, using the so-called genetic operators mentioned in step 4.

Step 3 An evaluation objective function, which is, in this case, the discretized strain energy $F(s,u)$ of the shell plus a penalized function, corresponding to the constraints defined in the set C of (1).

To evaluate the objective function, for each chromosome s, it is necessary first to combine the finite element code with the complementarity algorithm, in order to obtain the solution u of problem (34). The computation of $F(s,u)$ is done using these two quantities s and u.

Step 4 Genetic operators act on the chromosomes and generate successively new populations of chromosomes, from the original one, based on probabilistic rules. The most usual operators are crossover, mutation and reproduction [4], that are briefly explained below.

1. The crossover operator starts by randomly selecting two chromosomes s_1 and s_2, see (50)-(51), from the population; next, the bits between two randomly selected positions, along their common length, are swapped, and define two new chromosomes s_3 and s_4 in the search set. For example, if the bits between positions 6 and 18 in s_1 and s_2 are swapped, the new chromosomes s_3 and s_4 are defined by

$$\begin{array}{rcllcl}
s_1 & = & 100\,001 \; \overbrace{111\,0011\,1011}^{positions\ 7-17\ of\ s_1} \; 1111\,10 & = & (\overbrace{4,1,7}^{materials}, \overbrace{3,11,15}^{thickness}, \overbrace{2}^{function}) \\
s_2 & = & 001\,011 \; \underbrace{101\,0001\,0011}_{positions\ 7-17\ of\ s_2} \; 1110\,01 & = & (\underbrace{1,3,5}_{materials}, \underbrace{1,3,14}_{thickness}, \underbrace{1}_{function}) \\
s_3 & = & 100\,001 \; \overbrace{101\,0001\,0011}^{positions\ 7-17\ of\ s_2} \; 1111\,10 & = & (\overbrace{4,1,5}^{materials}, \overbrace{1,3,15}^{thickness}, \overbrace{2}^{function}) \\
s_4 & = & 001\,011 \; \underbrace{111\,0011\,1011}_{positions\ 7-17\ of\ s_1} \; 1110\,01 & = & (\underbrace{1,3,7}_{materials}, \underbrace{3,11,14}_{thickness}, \underbrace{1}_{function})
\end{array} \tag{52}$$

which means that the material of lamina 3 and the thickness of laminas 1 and 2 also change.

2. The mutation operator randomly selects a position in the chromosome s_1 and changes the corresponding bit with a given probability, thus defining a new chromosome s_5. For example, if the position 11 in s_1 is selected, the bit 0 changes to 1, and the thickness of lamina 1 changes from 3 to 7. The new chromosome s_5 is

$$\begin{array}{lclcl} s_1 & = & 100\,001\,111\,0\underline{0}11\,1011\,1111\,10 & = & (\overbrace{4,1,7}^{materials},\overbrace{3,11,15}^{thickness},\overbrace{2}^{function}) \\ s_5 & = & 100\,001\,111\,0\underline{1}11\,1011\,1111\,10 & = & (\underbrace{4,1,7}_{materials},\underbrace{7,11,15}_{thickness},\underbrace{2}_{function}) \end{array} \quad (53)$$

3. The reproduction operator defines the process by which the new generation is created from the previous one. The chromosomes in one generation are transferred into the next generation, with a probability according to the value of their objective function; thus, a higher proportion of the chromosomes with the best objective function values will be present in the next generation.

Step 5 A stopping criterium, that can be, for instance, a maximum number of generations of chromosomes.

The steps 1-5 present a summary of a genetic algorithm for the discrete optimization problem (1). For the details of implementation of this type of genetic algorithms see [4].

It is worthwhile to mention that a special case of problem (1) has been solved using genetic algorithms [6]. It is the compliance minimization of a linear, composite, laminated plate. The discrete optimization variables are the materials and the angle of orientation of the fibers, in each ply of the plate. The thickness of each ply is constant and a constraint on the global cost of the materials is imposed.

Unlike problem (1), the vertical displacement of the plate is free; it is not subject to any obstacle, so in this case, the index set L in problem (34) is empty and there is no need to apply a complementarity algorithm. Problem (34) reduces to a system of linear equations, which can be solved by a standard method. For this plate problem the genetic algorithm successfully identified, in each ply, the materials and the angles of orientation, corresponding to the minimum compliance of the plate.

7 Conclusions

In this paper a linear and a nonlinear optimization laminated shallow shell models, involving a variational inequality and a discrete feasible set, are described and analysed, and some numerical algorithms are proposed for their solutions. The success in the solution of some special instances of these models indicate that a combination of complementarity and genetic algorithms may be efficient for the solution of the concrete shallow shell models presented in this paper.

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