The Ramanujan Journal (2024) 63:253–258
https://doi.org/10.1007/s11139-023-00759-5

On the positivity of a certain function related with the Digamma function

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Received: 21 October 2022 / Accepted: 31 May 2023 / Published online: 2 August 2023

Abstract
In this note, we prove that

$$\left(\frac{x^n}{1-e^{-x}}\right)^{(n)} > 0$$

for all $x \in (\log 2, \infty)$ and $n \in \mathbb{N}$. This result improves a theorem of Al-Musallam and Bustoz (Ramanujan J 11:399–402, 2006).

Keywords Completely monotonic function · Digamma function · Bernoulli numbers

Mathematics Subject Classification 33B15 · 11B68

1 Introduction

A function $f : (a, b) \subset \mathbb{R} \longrightarrow \mathbb{R}$ is completely monotonic if it is infinitely differentiable and

$$(-1)^n f^{(n)}(x) \geq 0$$

for all $x \in (a, b)$ and $n \in \mathbb{N}$. A function $f(-x)$ is called absolutely monotonic on $(-b, -a)$ if and only if $f(x)$ is completely monotonic on (a, b). Absolutely monotonic functions were introduced by Bernstein. Bernstein himself, and later Widder independently, discovered that a necessary and sufficient condition for f to be completely monotonic on $(0, \infty)$ is that

This work was supported by the Centre for Mathematics of the University of Coimbra-UIDB/00324/2020, funded by the Portuguese Government through FCT/ MCTES.

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$$f(x) = \mathcal{L}(\mu)(x) = \int e^{-xt}\, \mathrm{d}\mu(t),$$

where μ is a positive measure on $[0, \infty)$ and the integral converges for all positive x. (These and other classical results on absolutely/completely monotonic functions can be found in [8, Chapter IV] and [3].) As it was remarked in [2], by Bernstein's theorem, it is easy to see that the absolute value of the derivatives of digamma function (the polygamma functions), $\psi^{(n)} = (\Gamma'/\Gamma)^{(n)}$, are completely monotonic functions on $(0, \infty)$. Indeed,

$$(-1)^{n+1}\psi^{(n)}(x) = \mathcal{L}\left(\frac{t^n}{1-e^{-t}}\right)(x)$$

for all $x \in (0, \infty)$ and $n \in \mathbb{N} \setminus \{0\}$. (The digamma function ψ and its absolute value are neither completely monotonic on $x \in (0, \infty)$.) In [4], Clark and Ismail introduced the functions

$$F_m(x) = x^m \psi(x), \quad G_m(x) = -x^m \psi(x).$$

These authors proved that $F_m^{(m+1)}$ is completely monotonic on $(0, \infty)$ for $m \in \mathbb{N} \setminus \{0\}$ [4, Theorem 1.2] and that $G_m^{(m)}$ is completely monotonic on $(0, \infty)$ for $m = 1, 2, \ldots, 16$ [4, Theorem 1.3]. Afterwards, they wrote *"We believe Theorem* 1.3 [$G_m^{(m)}$*is completely monotonic on* $(0, \infty)$]*is true for all m* [...]". However, Alzer, Berg, and Koumandos [2, Theorem 1.1] proved that there exists an integer m_0 such that for all $m \geq m_0$, the function $G_m^{(m)}$ is not completely monotonic. From this and the relation [4, (2.4)]

$$G_m^{(m)}(x) = \mathcal{L}\left(t^m \left(\frac{t^m}{1-e^{-t}}\right)^{(m)}\right)(x),$$

it follows that the following conjecture of Clark and Ismail [4, Conjecture 1.4] is false:

Conjecture *The inequality*

$$\left(\frac{x^n}{1-e^{-x}}\right)^{(n)} > 0 \tag{1}$$

holds for all $x \in (0, \infty)$ *and* $n \in \mathbb{N}$.

Since (1) holds for $n = 1, 2, \ldots, 16$, Clark and Ismail proved that $G_n^{(n)}$ is (strictly) completely monotonic on $(0, \infty)$ for these values of n. Regardless of the fact that the conjecture is not true, the inequality (1) is of interest in its own right. It remains an open problem to determine the smallest positive number a (respectively, positive integer n_0) such that (1) remains positive for all $x \in (a, \infty)$ and $n \in \mathbb{N}$ (respectively, $x \in (0, \infty)$ and $n > n_0$ with $n \in \mathbb{N}$). This problem was placed in [2, Section 4]. In [1, Theorem 2.1],[1] Al-Musallam and Bustoz proved that (1) holds for all $x \in (2\log 2, \infty)$ and $n \in \mathbb{N}$. This was also proved independently in [2, p. 112] using the same idea: an inequality proved by Szegő [8, Theorem 17a, p. 168]. Our main theorem, which improves the result in [1], reads as follows:

Theorem *The inequality* (1) *holds for all* $x \in (\log 2, \infty)$ *and* $n \in \mathbb{N}$.

As in [1, Theorem 3.1], the next result follows from the theorem above. The details are left to the reader.

Corollary *The inequality*

$$\left(\frac{x^{n+\alpha}}{1-e^{-x}}\right)^{(n)} > 0$$

holds for all $\alpha \in (0, \infty)$*,* $x \in (\log 2, \infty)$*, and* $n \in \mathbb{N}$.

Let us give now an application of our main result.

Example It is easy to obtain from [5, (1), p. 11], for $n \in \mathbb{N}\backslash\{0\}$, the power series

$$\left(\frac{x^n}{1-e^{-x}}\right)^{(n)} = \frac{n!}{2} + \sum_{j=2}^{\infty} \frac{(j+n-1)!}{(j-1)!} \frac{B_j}{j!} x^{j-1}$$

valid in the disk $|x| < 2\pi$ which extends to the nearest singularities $x = \pm 2\pi i$ of $x/(e^x - 1)$. The coefficients B_j are the Bernoulli numbers. The odd Bernoulli numbers are all zero after the first, but it is a highly complex task to determine the even Bernoulli numbers. Let us imagine that we are questioned about the sign of the following sum:

$$\begin{aligned} S_n &= \frac{(n+0)!}{0!\,1!} B_0 + \frac{(n+1)!}{1!\,2!} B_2 + \frac{(n+3)!}{3!\,4!} B_4 + \frac{(n+5)!}{4!\,5!} B_6 + \cdots \\ &= n! + \sum_{j=1}^{\infty} \frac{(2j+n-1)!}{(2j-1)!} \frac{B_{2j}}{(2j)!}. \end{aligned}$$

$(Recall\ that\, B_0 = 1.)$ Note that

$$\left.\left(\frac{x^n}{1-e^{-x}}\right)^{(n)}\right|_{x=1} = S_n - \frac{n!}{2}.$$

Since $\log 2 \approx 0.693147 < 1 < 2\pi$, our main result gives

$$S_n > \frac{n!}{2},$$

[1] This paper was submitted on May 21, 2003 and accepted for publication on October 24, 2003. J. Bustoz passed away on August 13, 2003.

$n \in \mathbb{N} \setminus \{0\}$. It is worth pointing out that from the results obtained in [1, 2], it is not possible to conclude this because $2 \log 2 \approx 1.38629 > 1$. Now it only remains to check that S_n converges, which follows from

$$\lim_{j \to \infty} \sqrt[2j]{\frac{(2j+n-1)!}{(2j-1)!} \frac{|B_{2j}|}{(2j)!}} = \frac{1}{2\pi} < 1.$$

In [2], the relation of the function given in (1) with a function of Hardy and Littlewood was extensively explored.

2 Proof of the theorem

Set

$$f_n(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left(\frac{x^n}{1 - e^{-x}} \right).$$

If $c > 0$ is arbitrary and fixed, the series

$$\frac{1}{1 - e^{-x}} = \sum_{j=0}^{\infty} e^{-jx}$$

converges uniformly on $[c, \infty)$. We then write f_n in the form

$$f_n(x) = \sum_{j=0}^{\infty} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left(e^{-jx} x^n \right).$$

Recall that [6, (5), p. 188] $n! L_n(x) = e^x (\mathrm{d}^n/\mathrm{d}x^n) \left(e^{-x} x^n \right)$, L_n being the Laguerre polynomial of degree n, and so

$$n! e^{-jx} L_n(jx) = \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left(e^{-jx} x^n \right).$$

Hence,

$$f_n(x) = n! \sum_{j=0}^{\infty} L_n(jx) e^{-jx}$$

on $[c, \infty)$. There is a well-known connection between the Laguerre and Hermite polynomials due to Feldheim [6, (33), p. 195]:

$$\int_0^{\infty} e^{-t^2} H_n^2(t) \cos(2^{1/2} y\, t)\, \mathrm{d}t = \sqrt{\pi} 2^{n-1} n! L_n(y^2),$$

H_n being the Hermite polynomial of degree n. Write

$$y^2 = j\,x.$$

From the above expressions, we have

$$f_n(x) = \frac{1}{\sqrt{\pi}\,2^{n-1}} \sum_{j=0}^{\infty} \int_0^{\infty} g_j(t)\,\mathrm{d}t,$$

where

$$g_j(t) = e^{-t^2} e^{-jx} H_n^2(t) \cos(\sqrt{2j\,x}\,t)$$

for all $x \in [c, \infty)$. Recall that it it is not true that uniform convergence is sufficient to allow interchange of the sum and integral when the integral is over an infinite interval. However, we claim that the function g_j is integrable and

$$\sum_{j=0}^{\infty} \int_0^{\infty} |g_j(t)|\,\mathrm{d}t < 0,$$

for all $x \in [c, \infty)$. (These conditions allow the interchanging of the above sum and integral, see, for instance, [7, Corollary 17.4.7].) Indeed, since

$$|g_j(t)| < e^{-t^2} e^{-j\,x} H_n^2(t),$$

we see at once that

$$\begin{aligned}\int_0^{\infty} |g_j(t)|\,\mathrm{d}t &< e^{-j\,x} \int_0^{\infty} e^{-t^2} H_n^2(t)\,\mathrm{d}t \\ &< e^{-j\,x} \int_{-\infty}^{\infty} e^{-t^2} H_n^2(t)\,\mathrm{d}t \le \sqrt{\pi} 2^n n!\,e^{-j\,c} < \infty,\end{aligned}$$

and the integrability of g_j is guaranteed. Moreover,

$$\begin{aligned}\sum_{j=0}^{\infty} \int_0^{\infty} |g_j(t)|\,\mathrm{d}t &< \sqrt{\pi}\,2^n\,n! \sum_{j=0}^{\infty} e^{-j\,x} \\ &\le \sqrt{\pi}\,2^n\,n!\,\frac{e^c}{e^c - 1} < \infty.\end{aligned}$$

Consequently, we can interchange the sum and integral to obtain

$$\sqrt{\pi} 2^{n-1} f_n(x) = \int_0^{\infty} e^{-t^2} H_n^2(t) \sum_{j=0}^{\infty} e^{-jx} \cos(\sqrt{2j\,x}\,t)\,\mathrm{d}t.$$

Finally, note that

$$\sum_{j=0}^{\infty} e^{-jx} \cos(\sqrt{2j\,x}\,t) > 1 - \sum_{j=1}^{\infty} e^{-jx} = 1 - \frac{1}{e^x - 1} = g(x).$$

Thus $g(x) \geq 0$ if and only if $x > \log 2$. This completes the proof.

Author contributions KC wrote the manuscript text and reviewed the manuscript.

Funding Open access funding provided by FCT|FCCN (b-on).

Declarations

Competing interests The author declare no competing interests.

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