Actions of the symmetric group on sets generated by Yamanouchi words

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Abstract

In this paper we consider words in a finite totally ordered alphabet which, restricted to a two consecutive letters subalphabet, are either Yamanouchi or dual Yamanouchi. We introduce coordinates or indexing sets of words and we show that there is a monoid isomorphism between words and classes of sequences of finite sets in $\mathbb{N}$. Considering words in a two consecutive letters subalphabet, we define maps acting on pairs of indexing sets which, by fixing a longest self-dual Yamanouchi subword, transform a Yamanouchi into a dual Yamanouchi word, and reciprocally. The pairs of indexing sets of Yamanouchi and dual Yamanouchi words are, respectively, comparable under an ordering and its dual in the power-set of $\{1, \ldots, n\}$. This family of transformations is induced by the witnesses of the comparable pairs. When minimal and maximal witnesses are considered, we recover those operators which satisfy the conditions of the symmetric group, defined by A. Lascoux and M. P. Schutzenberger in [10], [12]. Starting with given indexing sets of a Yamanouchi word, in a three-letters alphabet, we generate, under the action of these transformations, a set of indexing sets which gives rise to an action of the symmetric group $\mathcal{S}_3$. This group action of $\mathcal{S}_3$ is equivalent to an explicit decomposition of the given indexing sets of a Yamanouchi word in a three-letters alphabet. For transformations induced by minimal and maximal witnesses, we use this decomposition to define, recursively, an action of the symmetric group $\mathcal{S}_t$, $t \geq 3$, on a set generated by indexing sets of all Yamanouchi words in a t-letters alphabet. This group action coincides with the one described by A. Lascoux and M. P. Schutzenberger in [10] and [12], when restricted to the words under consideration.

The action of the symmetric group $\mathcal{S}_3$, on words or Young tableaux, has a natural matrix translation afforded by the obvious permutation action on a sequence of matrices over a local principal ideal domain with maximal ideal (p). Moreover, such a permutation action gives rise, directly, to the mentioned decomposition of the indexing sets of a Yamanouchi word in a three-letters alphabet. This is the content of a subsequent paper.

[1]Work supported by CMUC/FCT.
[2]Work supported by CMUC/FCT.

1 Introduction

Let M be the set of all re-arrangements of a sequence of t fixed integers in $\{1, ..., n\}$. We consider those Young tableaux $\mathcal{T}$ of weight $(m_1, ..., m_t)$ in M arising from a sequence of products of matrices over a local principal ideal domain with maximal ideal (p),

$$(\Delta_a, \Delta_a U(pI_{m_1} \oplus I_{n-m_1}), \Delta_a U \prod_{k=1}^{2}(pI_{m_k} \oplus I_{n-m_k}), ..., \Delta_a U \prod_{k=1}^{t}(pI_{m_k} \oplus I_{n-m_k})),$$

where Δ_a is an $n \times n$ diagonal matrix with invariant partition a, and U is an $n \times n$ unimodular matrix. When $(m_1, ..., m_t)$ is by decreasing order, $\mathcal{T}$ is a Littlewood-Richardson tableau [1],[2], [7]. Now, for each partition a and $n \times n$ unimodular matrix U, let $T_{(a,M)}(U)$ be the set of all sequences of matrices as above, with $(m_1, ..., m_t)$ running over M. The symmetric group $\mathcal{S}_t$ acts on M and on $T_{(a,M)}(U)$ in the obvious way. The action of the symmetric group on those sequences of matrices is equivalent to an action of the symmetric group on the set of Young tableaux realized by those sequences of matrices in $T_{(a,M)}(U)$. When $t = 3$, the action of $\mathcal{S}_3$, on $T_{(a,M)}(U)$, is described by an explicit decomposition of the indexing sets of the Littlewood-Richardson tableau in this set which generates the remaining indexing sets of the Young tableaux in $T_{(a,M)}(U)$. The indexing sets of those Young tableaux, generated in $T_{(a,M)}(U)$, are such that the corresponding words, restricted to a two consecutive letters subalphabet, are either Yamanouchi or dual Yamanouchi. This is analyzed in another paper.

A. Lascoux and M. P. Schutzenberger, [10], [12], have described an action of the symmetric group on Young tableaux by defining operators σ_k, $k = 1, \dots, t-1$, acting on words in a tottaly ordered alphabet $\{1, \dots, t\}$. Generically, the action of the operator σ_k, on a word w, may be described as follows: first extract from w a subword w' that contains the letters k and $k+1$ only. Second, in the word w' remove a longest self-dual Yamanouchi subword, in the two-letters subalphabet $\{k, k+1\}$. As a result, we obtain a subword of the type $k^r\,(k+1)^s$. Then, we replace it with the word $k^s\,(k+1)^r$ and, after this, we recover all the removed letters of w, including the ones different from k and $k+1$. In [10] and [12], the determination of a longest self-dual Yamanouchi subword, in a two-letters subalphabet $\{k, k+1\}$, is done by bracketing, consecutively, in w' every factor $k+1\,k$. The letters which are bracketed constitute a desired self-dual Yamanouchi word. The proof that the operators σ_k satisfy the relations of the symmetric group is done by using properties of the plactic monoid.

In this paper, the motivation for our analysis is the decomposition of the indexing sets of a Littlewood-Richardson tableau achieved in the matrix problem, described above, in the cases $t = 2$ [3], (already used in [5]), and $t = 3$. Introducing the notion of indexing sets of a word, and establishing a monoid isomorphism between words and classes of sequences of finite sets in $\mathbb{N}$, we rediscover this decomposition in theorem 4.4. We define a lattice on the power-set of $\{1, \dots, n\}$, and consider its dual induced by the reverse order in

$\{1,\ldots,n\}$. In this lattice and its dual, comparable pairs (A,B) are, respectively, indexing sets of Yamanouchi words and their duals, in a two-letters subalphabet. We give operations on comparable pairs of sets, afforded by their witnesses, such that the corresponding Yamanouchi word is mapped into a dual word, and reciprocally. Translating to words these operations, we describe procedures on indexing sets which are equivalent to put brackets on the letters of a word, in a two-letters subalphabet.Under the action of these operations, and starting with indexing sets of a Yamanouchi word, in a three-letters alphabet, we generate a set of indexing sets which (see definition 4.2 and theorem 4.2)is equivalent to a decomposition of the given indexing sets. This defines an action of the symmetric group $\mathcal{S}_3$. The witnesses of a comparable pair (A,B) are ordered by their images. The minimal and maximal witnesses of each comparable pair and its dual, respectively, afford operators Θ^* on the pairs of indexing sets of all Yamanouchi words w and their duals, in a two-letters alphabet. It turns out that $\Theta^*(w)=\sigma_1(w)$. Considering words in a t-letters alphabet which, restricted to a two consecutive letters subalphabet, are either Yamanouchi or dual Yamanouchi, we put $\Theta^*_k=\Theta^*_{|\{k,k+1\}}$, for $k=1,\ldots,t-1$. It is proven that Θ^*_k, $k=1,\ldots,t-1$, satisfy the conditions of the symmetric group.

The paper is organized as follows. In the section 2, the power-set of $\{1,\ldots,n\}$ is equipped with the structure of lattice, and is defined its dual induced by the reverse order in $\{1,\ldots,n\}$. We introduce the notion of witness of a comparable pair (A,B), and give procedures to calculate minimal and maximal witnesses of such pairs. A series of results are proved in order to support the last section.

In section 3, we define indexing sets of words and Young tableaux, and transfer to words the operations on indexing sets, defined in section 2. In particular, we recover the operators σ_k described in the platic monoid.

In section 4, we define actions of the symmetric group $\mathcal{S}_3$ on sets generated by indexing sets of Yamanouchi words, in a tree-letters alpahabet. The generation of these sets is equivalent to a decomposition of the given indexing sets of a Yamanouchi word.When the operators Θ^*_k, $k=1,2$, are used to decompose the given indexing sets, we obtain the action of symmetric group $\mathcal{S}_3$, described by A. Lascoux and M. P. Schutzenberger in [10], [12]. It is shown, recursively, that the operators Θ^*_k, $k=1,\ldots,t-1$, may be used to decompose indexing sets of a Yamanouchi word in a t-letters alpahbet, and that they satisfy the conditions of the symmetric group $\mathcal{S}_t$.

2 The lattice of the power-set of $[n]$ and its dual induced by the reverse order

2.1 The lattice $\mathcal{P}[n]$

Let $\mathbb{N}$ be the set of non-negative integers with the usual order " $\geq$ ". Given $n\in\mathbb{N}$, $[n]$ denotes the set $\{1,\ldots,n\}$, and $2^{[n]}$ the power-set of $[n]$.

Definition 2.1 *Let $A, B \subseteq [n]$. We write $A \geq B$ if there exists an injection $i : B \to A$ such that $b \leq i(b)$, for all $b \in B$. We call such an injection a witness for $A \geq B$.*

The relation $\geq$ defined by $A \geq B$ is a partial order on $2^{[n]}$, and we denote it by $\mathcal{P}[n]$, that is, given $(A, B) \in 2^{[n]} \times 2^{[n]}$, $(A, B) \in \mathcal{P}[n]$ iff $A \geq B$. This relation can be characterized in a number of ways as seen in the following proposition.

Given a finite set A, let $|A|$ denote the cardinality of A.

Proposition 2.1 *Given $A, B \subseteq [n]$, the following statements are equivalent:*

(a) There exists an injection $i : B \to A$ such that $b \leq i(b)$, for all $b \in B$.

(b)If $a = (a_1, a_2, ..., a_{|A|}, 0, ...)$ and $b = (b_1, ..., b_{|B|}, 0, ...)$ are the decreasing rearrangement of the elements of A and B as embedded into $\mathbb{N}^{\mathbb{N}}$, then $a \geq b$ in the componentwise order.

(c)For any $k \in \mathbb{N}$, it holds $|\{a \in A : a \geq k\}| \geq |\{b \in B : b \geq k\}|$.

(d) There exists an injection $i : B \to A$ as in (a), and satisfying additionally $A \cap B \subseteq i(B)$. In particular, $i_{|A\cap B} = id_{|A\cap B}$. (id stands for the identity map.)

Proof: It is a routine exercise to show that $(a) \Rightarrow (c) \Rightarrow (b) \Rightarrow (a)$.

$(a) \Rightarrow (d)$ Suppose condition (a) holds. Since we have already proved the equivalence of conditions (a) and (c), we have $|\{a \in A : a \geq k\}| \geq |\{b \in B : b \geq k\}|$, for all $k \in \mathbb{N}$. Then, we must have $|\{a \in A \setminus B : a \geq k\}| \geq |\{b \in B \setminus A : b \geq k\}|$, $k \in \mathbb{N}$. Thus, there exists an injection $j : B \setminus A \to A \setminus B$ such that $j(b) \geq b$, $b \in B \setminus A$. Define $\bar{j} : B \to A$ by $\bar{j}(b) = b$ if $b \in A \cap B$, and $\bar{j}(b) = j(b)$ if $b \notin A \cap B$. Clearly, $\bar{j}$ is a witness for $A \geq B$ and $\bar{j}_{|A\cap B} = id_{|A\cap B}$.

$(d) \Rightarrow (a)$ Trivial. □

Note that (b) characterizes $\mathcal{P}[n]$ as the component ordering. For simplicity, we shall write $A = \{a_1 > \ldots > a_{|A|}\}$ to mean $A = \{a_1, \ldots, a_{|A|}\}$ with $a_1 > \ldots > a_{|A|}$.

Proposition 2.2 *$\mathcal{P}[n]$ is a lattice in which the family of all subsets of a given cardinality forms a sublattice. The family of all finite subsets of $\mathbb{N}$, $\bigcup_{n\in\mathbb{N}} \mathcal{P}[n]$, is a lattice.*

Proof: It is a direct consequence of proposition 2.1, (b), and the observation that the family of all integral sequences with finite support defines a lattice with respect to the componentwise order: if $a = (a_i)_{i=1}^{\infty}$ and $b = (b_i)_{i=1}^{\infty}$ are two such sequences, we define $a \wedge b = (min\{a_i, b_i\})$ and $a \vee b = (max\{a_i, b_i\})$. □

Next proposition stresses the property that $\mathcal{P}[n]$ is an extension of any lattice of the family of all subsets of a given cardinality.

Proposition 2.3 *Given $A, B \subseteq [n]$, the following statements are equivalent:*

(a) $A \geq B$.

(b)There exits $X \subseteq A$ such that $|X| = |B|$ and $X \geq B$.

(c)There exits $X \subseteq A$ such that $|X| = |B|$, $A \cap B \subseteq X$ and $X \geq B$.

(d) $A \setminus Z \geq B \setminus Z$, with $Z \subseteq A \cap B$.

Proof: It is a direct consequence of proposition 2.1, (a) and (d) . □

Proposition 2.4 *$\mathcal{P}[n]$ has the following properties*

(a) If $B \subseteq A$ then $A \geq B$.

(b) Let $A \geq B$ and $C \geq D$ such that $A \cap C = \emptyset$ and $B \cap D = \emptyset$. Then $A \cup C \geq B \cup D$.

(c) If $A \geq B$ and $B = B_1 \cup B_2$, with $|A| = |B|$ and $B_1 \cap B_2 = \emptyset$, there exist $A_1, A_2 \subseteq A$ such that $A_1 \cup A_2 = A$, $A_1 \cap A_2 = A_1 \cap B_2 = A_2 \cap B_1 = \emptyset$ and $A_i \geq B_i$, $i = 1, 2$.

Proof: (a) and (b) are a direct consequence of proposition 2.3.

(c) Let $i : B \to A$ be a witness of $A \geq B$ such that $i_{|A\cap B} = id_{|A\cap B}$. Define $A_i := i(B_i)$, $i = 1, 2$. □

Let $A \geq B$ and i, j witnesses of $A \geq B$. We write $i \geq j$ if $i(B) \geq j(B)$. Given $A \geq B$, the relation $\geq$ defined by $i \geq j$ is a partial order on the set of witnesses of $A \geq B$.

Considering proposition 2.3 and since $\mathcal{P}[n]$ is a lattice, given $A \geq B$ we may define the least upper bound of B in 2^A.

Definition 2.2 *Let $A \geq B$. We write*

$$min_B A = min\{X \subseteq A : |X| = |B| \text{ and } X \geq B\}$$

as the least upper bound of B contained in A.

Note that $min_B A = min\{i(B) : i \text{ is a witness of } A \geq B\}$; $Y \in \mathcal{P}(A)$, $Y \geq B$ only if $Y \geq min_B A$; and $A \cap B \subseteq min_B A$. In particular, if $|A| = |B|$, $min_B A = A$, and if $B \subseteq A$, $min_B A = B$.

Definition 2.3 *Let i be a witness of $A \geq B$. We say that i is a minimal witness of $A \geq B$ if $i(B) = min_B A$.*

In the sublattice of all subsets of $[n]$ of a given cardinality every witness is minimal.

Given $A \geq B$, $min_B A$ may be computed in several ways, and, therefore, this leads to different minimal witnesses. The next algorithm exhibits a minimal witness of $A \geq B$.

Theorem 2.5 *Let $A \geq B$ with $B = \{b_1 > ... > b_m\}$. Let*

$$y_m = min\{a \in A : a \geq b_m\}, \text{ and } \tag{1}$$

$$y_{m-i} = min\{a \in A \setminus \{y_m, ..., y_{m-i+1}\} : a \geq b_{m-i}\}, \text{ for } i = 1, ..., m-1. \tag{2}$$

Then, $min_B A = \{y_1 > ... > y_m\}$.

Proof: By induction on m. If $m = 1$, $min_B A = \{y_1\}$. Let $m > 1$ and $(z_1^j, ..., z_m^j, 0, ...)$, for $j = 1, ..., k$, the decreasing rearrangement of all subsets of A of cardinal m such that $(z_1^j, ..., z_m^j, 0, ...) \geq (b_1, ..., b_m, 0, ...)$. Let $(x_1, ..., x_m, 0, \ldots) = \wedge_{j=1}^k (z_1^j, ..., z_m^j, 0, ...)$ be the decreasing rearrangement of $min_B A$. Clearly $\{y_1, ..., y_m\} \subseteq A$ and $(y_1, ..., y_m, 0, ...) \geq (b_1, ..., b_m, 0, ...)$. Therefore, $y_i \geq x_i$, $i = 1, ..., m$. On the other hand, by (1), $y_m = min_{\{b_m\}} A \leq x_m$. Hence $y_m = x_m = min_{\{b_m\}} A$. Now, notice that if $(q_1, ..., q_{m-1}, 0, ...)$ is the decreasing rearrangement of a subset of cardinal $m-1$ of $A \setminus \{y_m\}$ such that $(q_1, ..., q_{m-1}, 0, ...) \geq (b_1, ..., b_{m-1}, 0, ...)$, then the array $(q_1, ..., q_{m-1}, y_m, 0, ...)$ is the decreasing rearrangement of a subset of A of cardinal m such that $(q_1, ..., q_{m-1}, y_m, 0, ...) \geq (b_1, ..., b_{m-1}, b_m, 0, ...)$. Therefore, $(z_1^j, ..., z_{m-1}^j, 0, ...)$, $j = 1, ..., k$, are the decreasing rearrangement of all subsets of cardinal $m-1$ of $A \setminus \{y_m\}$ such that $(z_1^j, ..., z_{m-1}^j, 0, ...) \geq (b_1, ..., b_{m-1}, 0, ...)$. Hence, $(x_1, ..., x_{m-1}, \ldots) = \wedge_{j=1}^k (z_1^j, ..., z_{m-1}^j, 0, ...)$ is the decreasing rearrangement of $min_{(B\setminus\{b_m\})}(A \setminus \{y_m\})$, and

$$min_B A = min_{\{b_m\}} A \cup min_{(B\setminus\{b_m\})}(A \setminus \{y_m\}). \tag{3}$$

By induction and from (2), we have $min_{(B\setminus\{b_m\})}(A \setminus \{y_m\}) = \{y_1, ..., y_{m-1}\}$, and consequently, by (3), $min_B A = \{y_1, ..., y_m\}$. □

Corollary 2.6 *Let $A' \geq B$ with $|A'| = |B|$, $A' = \{a_1 > ... > a_m\}$ and $B = \{b_1 > ... > b_m\}$. Let $A, F \subseteq \{1, \ldots, n\}$ with $A' \subseteq A$. Then*

(I) the following conditions are equivalent

(a) $A' = min_B(A' \cup F)$.

(b) $F \subseteq \{a \in \{1, \ldots, n\} : \exists i \in \{0, 1, ..., m\}, b_i > a > a_{i+1}\}$.

(We convention $b_0 := +\infty$ and $a_{m+1} := -\infty$.)

(II) if $A' = min_B A$, it holds

(a) $min_B A = min_{\{b_m,\ldots,b_{i+1}\}} A \cup min_{\{b_i,\ldots,b_1\}}(A \setminus \{a_m, ..., a_{i+1}\})$, for $1 \leq i \leq m$.

(b) $min_B A = min_{\{b_m,\ldots,b_{i+1}\}} A \cup min_{\{b_i,\ldots,b_1\}} A$, if there exists $i \in \{1, ..., m-1\}$ such that $b_i > a_{i+1}$.

Proof: Straightforward from (1) and (2). □

Corollary 2.7 *Let A, B and $C \subseteq [n]$ such that $A \geq B$. Let $G, A' \subseteq A$ such that $A' \cap G = \emptyset$ and $A' \geq B$ with $|A'| = |B|$, $B = \{b_1 > ... > b_m\}$ and $A' = \{a_1 > ... > a_m\}$. Then*

(a) if $g \in G$ only if $b_i > g > a_{i+1}$, for some $i \in \{0, 1, ..., m\}$, it holds

$$min_B A = min_B(A \setminus G).$$

(b) $min_B(A \cup C) = min_B[(min_B A) \cup C]$.

Proof: (a) Let $min_B A = Y$ with $Y = \{y_1 > ... > y_m\}$. Then, $a_i \geq y_i \geq b_i$ for each $i \in \{1, ..., m\}$, and $g \in G$ only if $b_i > g > a_{i+1} \geq y_{i+1}$. By corollary 2.6, $min_B A = min_B Y = min_B(Y \cup (A \setminus G))$.

(b) Consequence of (a). □

The significance of the next two theorems will be clear in the last section.

Theorem 2.8 *Let $A, B, C \subseteq [n]$ such that $A \geq B$ and $A \cap C = \emptyset$. Let $A' = min_B A$. Then, $min_B(A \cup C) = X \cup Y$ with $X \subseteq A'$, $Y \subseteq C$, and $A' = X \cup Z$ such that $Z \geq Y$.*

Proof: By induction on $|C|$. Let $|C| = 1$ and $C = \{c\}$. Let $B = \{b_1 > ... > b_m\}$ and $A' = \{a_1 > ... > a_m\}$. By previous corollary, if $b_k > c > a_{k+1}$, for some $k \in \{0, 1, ..., m\}$, then $Y = \emptyset = Z$. Otherwise, $a_k > c > a_{k+1}$ and $a_k > c \geq b_k$, for some $k \in \{1, \ldots, m\}$. Since $A' \cap \{c\} = \{a_1 > \ldots > a_k > c > a_{k+1} > \ldots > a_m\}$ and using corollary 2.6, (II), we may write

$$\begin{aligned}
&min_B(A \cup \{c\}) = min_B(A' \cup \{c\}) \\
&= \{a_m, \ldots, a_{k+1}\} \cup min_{\{b_k\}}[(A \setminus \{a_m, \ldots, a_{k+1}\}) \cup \{c\}] \\
&\cup min_{\{b_{k-1},...,b_1\}}[A \setminus \{a_m, \ldots, a_{k+1}\}], \text{ and} \\
&min_B A = \{a_m, \ldots, a_{k+1}\} \cup min_{\{b_k\}}[A \setminus \{a_m, \ldots, a_{k+1}\}] \\
&\cup min_{\{b_{k-1},...,b_1\}}[A \setminus \{a_m, \ldots, a_{k+1}, a_k\}],
\end{aligned}$$

where $min_{\{b_{k-1},...,b_1\}}(A \setminus \{a_m, \ldots, a_{k+1}\}) = \{a_k, \ldots, a_{f+1}, a_{f-1}, \ldots, a_1\}$, with $1 \leq f \leq k$.

Define $X := A' \setminus \{a_f\}$, with $1 \leq f \leq k$, $Y := C$, $Z := \{a_f\}$. Thus, $A' = X \cup Z$ and $min_B(A \cup C) = X \cup Y$ with $Z > Y$.

Let $|C| > 1$ and $C = C' \cup \{c\}$. By induction $min_B(A \cup C') = X' \cup Y'$ with $X' \subseteq A'$, $Y' \subseteq C'$, and $A' = X' \cup Z'$ such that $Z' \geq Y'$. Now, either $min_B(A \cup C) = X' \cup Y'$ with $X' \subseteq A'$, $Y' \subseteq C$, and $A' = X' \cup Z'$ such that $Z' \geq Y'$, or $min_B(A \cup C) = [(X' \cup Y') \setminus \{x\}] \cup \{c\}$, with $x \in X' \cup Y'$ and $x > c$. In the last case, either we get $min_B(A \cup C) = X' \cup Y$ with $Y = (Y' \setminus \{x\}) \cup \{c\}$, and $A' = X' \cup Z$ with $Z = Z'$, or we get $min_B(A \cup C) = (X' \setminus \{x\}) \cup Y$ with $Y = Y' \cup \{c\}$, and $A' = (X' \setminus \{x\}) \cup Z$ with $Z = Z' \cup \{x\}$. □

Lemma 2.9 *Let $A, B \subseteq [n]$ where $B = \{b_1 > \ldots > b_m\}$ and $min_B A = \{a_1 > \ldots > a_m\}$. Let $Y = \{y_1 > \ldots > y_s\}$ such that $\exists i_k \in \{1, ..., m\}, b_{i_k} > y_k > a_{i_k+1}$, for $k = 1, \ldots, s$. Let $X = B \setminus \{b_{i_1}, \ldots, b_{i_s}\}$. Then, $min_B A = min_{(X \cup Y)}\{a_1, \ldots, a_m\}$.*

Proof: Straightforward from theorem 2.5 and corollary 2.6, (II), (b). □

Theorem 2.10 *Let G, F_2, F_3 be subsets of $[n]$ such that $G \geq F_2 \geq F_3$. Let $D \subseteq [n]$ such that $D \cap G = D \cap F_2 = \emptyset$, and $min_{F_2}(G \cup D) = G$. Let $F' = min_{F_3} F_2$. Then, if $min_{F_3}(F_2 \cup D) = F$, we have $min_F G = min_{F'} G$.*

Proof: Let $F_2 = \{a_1 > \ldots > a_t\}$, $F' = \{a_{s_1} > \ldots > a_{s_m}\}$ where $\{s_1 > \ldots > s_m\} \subseteq \{1, \ldots, t\}$, and $F_3 = \{b_1 > \ldots > b_m\}$. Using corollary 2.7 and theorem 2.8, we may write

$$min_{F_3}(F_2 \cup D) = min_{F_3}(F' \cup D) = X \cup Y,$$

with $X \subseteq F'$ and $Y \subseteq D$. Now, corollary 2.6 implies $y \in Y$ only if

$$\exists i \in \{1, \ldots, m\} : \quad a_{s_i} > y > a_{s_{i+1}}, \quad a_{s_i} > y \geq b_i, \quad \text{and} \\ a_{s_i} > \alpha \geq b_i \Rightarrow \alpha \notin F_2. \tag{4}$$

On the other hand, since $min_{F_2}(G \cup D) = G$, putting $G = \{g_1 > \ldots > g_t\}$, we have, from corollary 2.6,

$$y \in Y \Rightarrow \exists j \in \{1, \ldots, t\} : a_j > y > g_{j+1}.$$

Hence $j \in \{s_1, \ldots, s_m\}$. Otherwise, $a_{s_i} > a_j > y \geq b_i$, with $a_j \in F_2$. A contradiction with (4). Hence, from previous lemma, $min_{X \cup Y} G = min_F G$. □

Next we show that the result of the algorithm given in theorem 2.5 does not depend on the order in which the elements of B are considered. In particular, next algorithm leads to minimal witnesses of $A \geq B$ with different properties which shall be important in the sequel.

Theorem 2.11 *Let $A \geq B$ and $B = \{b_1 > ... > b_m\}$. Let $\sigma \in \mathcal{S}_m$ and*

$$\begin{aligned} z_m &= min\{a \in A : a \geq b_{\sigma(m)}\}, \\ z_{m-i} &= min\{a \in A \setminus \{z_m, ..., z_{m-i+1}\} : a \geq b_{\sigma(m-i)}\},\ i = 1, ..., m-1. \end{aligned}$$

Then, $min_B A = \{z_1, ..., z_m\}$.

Proof: By induction on m. If $m = 1$, it is trivial, $min_{\{b_1\}} A = \{z_1\}$. Let $m > 1$ and $min_B A = \{y_1 > ... > y_m\}$.

Let $\sigma(m) = j$, for some $j \in \{1, ..., m\}$, and $min_{\{b_j\}} A = \{z_m\}$. Then, by theorem 2.5, since $\{y_j\} = min_{\{b_j\}}(A \setminus \{y_m, ..., y_{j+1}\})$, either $z_m = y_j$ or $y_j > z_m > b_j$. In the last case, by corollary 2.6,(I), $z_m = y_k$ with $k > j$ and $y_m, ..., y_{k+1} < b_j$ and $y_{k-1}, ..., y_j, ..., y_1 > b_j$. From (1) and (2), we conclude that $A \setminus \{y_k\} \geq B \setminus \{b_j\}$ and $min_{\{b_m, ..., b_k, ..., b_{j+1}\}}(A \setminus \{y_k\}) = \{y_m, ..., y_{k+1}, y_{k-1}, ..., y_j\}$, and by corollary 2.6, (II),

$$min_{(B \setminus \{b_j\})}(A \setminus \{y_k\}) = min_{\{b_m, ..., b_{j+1}\}}(A \setminus \{y_k\}) \cup min_{\{b_{j-1}, ..., b_1\}}[A \setminus \{y_j, ..., y_m\}]$$

and, therefore, $min_{(B \setminus \{b_{\sigma(m)}\})}(A \setminus \{y_k\}) = \{y_m, ..., y_1\} \setminus \{y_k\}$. That is, since $\{z_m\} = min_{\{b_{\sigma(m)}\}} A$,

$$min_B A = min_{\{b_{\sigma(m)}\}} A \cup min_{(B \setminus \{b_{\sigma(m)}\})}(A \setminus \{y_k\}).$$

By induction, $min_{(B \setminus \{b_{\sigma(m)}\})}(A \setminus \{y_k\}) = \{z_{m-1}, ..., z_1\}$. So, the claim follows. □

Remark 1 *For each $\sigma \in \mathcal{S}_m$, this theorem defines the minimal witness $z_\sigma : B \longrightarrow A$ such that $z_\sigma(b_{\sigma(m-i)}) = z_{m-i}$, for $i = 0, 1, \ldots, m-1$. When $\sigma = id$ we have the minimal witness given by theorem 2.5. In particular, if $\sigma \in \mathcal{S}_m$ is such that $A \cap B = \{b_{\sigma(m)}, \ldots, b_{\sigma(t)}\}$, then $min_B A = (A \cap B) \cup min_{(B \setminus A)}(A \setminus B)$.*

Corollary 2.12 *Let $A \geq B$ and $C \subseteq B$. Let $min_C A = A'$. Then,*

$$min_B A = min_C A \cup min_{(B \setminus C)}(A \setminus A').$$

In particular, $A \setminus A' \geq B \setminus C$ and $A \cap C \subseteq A'$.

Proof: If $C = \emptyset$, $min_C A = \emptyset$ and it is done. Otherwise, let $B = \{b_1 > ... > b_m\}$, $m > 1$, and $C = \{b_{j_1} > ... > b_{j_r}\}$, $r \geq 1$. Let $\sigma \in \mathcal{S}_m$ such that $\sigma(m - i + 1) = j_i$, for $i = 1, ..., r$. The result now follows from theorem 2.11. $\square$

Note that $min_{(B \setminus C)}(A \setminus A') = min_B A \setminus min_C A$, with $A' = min_B A$.

As a consequence of choosing a particular $\sigma \in \mathcal{S}_m$ in the previous theorem and corollary, we have the following algorithms which will have a significant translation to words in the next section:

Algorithm I: Given $A \geq B$ and $B = \{b_1 > ... > b_m\}$, let $\{k, j, \ldots, l\} := \{i \in \{1, \ldots, m\} : \exists x \in A, b_{i-1} > x \geq b_i\}$ and $V_0 := \{b_k, b_j, \ldots, b_l\} \subseteq B$. For $\alpha \in \{k, j, \ldots, l\}$, define

$$z_\alpha := min\{x \in A : b_{\alpha-1} > x \geq b_\alpha\}, \tag{5}$$

and put $Z_0 := \{z_k, z_j, \ldots, z_l\} \subseteq A$. Let $A_1 := A \setminus Z_0$ and $B_1 := B \setminus V_0$. Then repeat the procedure with $A_1 \geq B_1$ defining $V_1 := \{b_l, b_p, \ldots, b_q\} \subseteq B_1$, $z_\alpha := min\{x \in A_1 : b_{\alpha-1} > x \geq b_\alpha\}$, for $\alpha \in \{l, p, \ldots, q\}$, and $Z_1 := \{z_l, z_p, \ldots, z_q\} \subseteq A_1$. There remains $A_2 := A_1 \setminus Z_1$ and $B_2 := B_1 \setminus V_1$. Continue this procedure with $A_2 \geq B_2$ until it stops, giving sets $A_k := A_{k-1} \setminus Z_{k-1}$ and $B_k := B_{k-1} \setminus V_{k-1} = \emptyset$. Then, $A = Z_0 \cup \ldots \cup Z_{k-1} \cup A_k$, $B = V_0 \cup \ldots \cup V_{k-1}$, and

$$\begin{aligned} min_B A &= min_{V_0} A \cup min_{V_1} A_1 \cup \ldots \cup min_{V_{k-1}} A_{k-1} \\ &= Z_0 \cup Z_1 \cup \ldots \cup Z_{k-1}. \end{aligned}$$

(We convention $b_0 := +\infty$.)

This defines the minimal witness $z^I : B \longrightarrow A$ such that $z^I = z_0^I \cup \ldots \cup z_{k-1}^I$ with $z_i^I : V_i \longrightarrow A_i$, $i = 0, 1, \ldots k-1$, given according to (5).

Algorithm II: Given $A \geq B$ and $A = \{a_1 > ... > a_n\}$, let $\{f, g, \ldots, h\} := \{i \in \{1, \ldots, n\} : \exists y \in B, a_i \geq y > a_{i+1}\}$ and $Q_0 := \{a_f, a_g, \ldots, a_h\} \subseteq A$. For $\alpha \in \{f, g, \ldots, h\}$, define

$$u_\alpha := max\{y \in B : a_\alpha \geq y > a_{\alpha+1}\}, \tag{6}$$

and put $U_0 := \{u_f, u_g, \ldots, u_h\} \subseteq B$. Let $B_1 := B \setminus U_0$ and $A_1 := A \setminus Q_0$. Then repeat the procedure with $A_1 \geq B_1$ defining $Q_1 := \{a_l, a_p, \ldots, a_q\} \subseteq A_1$, $u_\alpha := max\{y \in B_1 : b_{\alpha-1} \geq y > b_\alpha\}$, for $\alpha \in \{l, p, \ldots, q\}$, and $U_1 := \{u_l, u_p, \ldots, u_q\} \subseteq B_1$. There remains $B_2 := B_1 \setminus U_1$ and $A_2 := A_1 \setminus Q_1$. Continue this procedure with $A_2 \geq B_2$ until it stops, giving sets $A_k := A_{k-1} \setminus Q_{k-1}$ and $B_k := B_{k-1} \setminus U_{k-1} = \emptyset$. Then, $A = Q_0 \cup \ldots \cup Q_{k-1} \cup A_k$, $B = U_0 \cup \ldots \cup U_{k-1}$, and

$$min_B A = Q_0 \cup Q_1 \cup \ldots \cup Q_{k-1}.$$

(We convention $a_{n+1} := -\infty$.)

This defines the minimal witness $z^{II} : B \longrightarrow A$ such that $z^{II} = z_0^{II} \cup \ldots \cup z_{k-1}^{II}$ with $z_i^{II} : U_i \longrightarrow A_i$, $i = 0, 1, \ldots k-1$, given according to (6).

The next result will be used in the last section.

Lemma 2.13 *Let F, C and D subsets of $[n]$ such that $C \geq D$. Assume $C = \{g_1 > \ldots > g_r\}$ and $D = \{s_1 > \ldots > s_r\}$. Let $n \geq t \geq x \geq 1$ such that $x, t \notin C \cup D$, $min_{\{x\}}(\{t\} \cup F) = \{t\}$ and $s_u > x > g_{u+1}$ for some $u \in \{0, 1, \ldots, r\}$. Then,*

$$min_{(D \cup \{x\})}(F \cup C \cup \{t\}) = \{t\} \cup min_D(F \cup C).$$

Proof: Suppose that $g_{u-l} > t > g_{u-l+1}$, for some $l \in \{0, 1, \ldots, u\}$. Then, by corollary corollary 2.6, (II),

$$\begin{aligned} & min_{(D \cup \{x\})}(F \cup C \cup \{t\}) \\ & = min_{\{s_r, \ldots, s_{u+1}\}}(F \cup C \cup \{t\}) \cup min_{(\{s_u, \ldots, s_1\} \cup \{x\})}(F \cup C \cup \{t\}) \\ & = min_{\{s_r, \ldots, s_{u+1}\}}(F \cup C) \cup min_{(\{s_u, \ldots, s_1\} \cup \{x\})}(F \cup C \cup \{t\}). \end{aligned} \tag{7}$$

Now, by corollary 2.6, (II), since

$$min_{\{s_u, \ldots, s_{u-l+1}\}}(F \cup C \cup \{t\}) = \{g_u, \ldots, g_{u-l+1}\} = min_{\{s_u, \ldots, s_{u-l+1}\}}(F \cup C),$$

it holds,

$$\begin{aligned} & min_{(\{s_u, \ldots, s_1\} \cup \{x\})}(F \cup C \cup \{t\}) \\ & = min_{\{s_u, \ldots, s_{u-l+1}\}}(F \cup C) \cup min_{(\{x\} \cup \{s_{u-l}, \ldots, s_1\})}[F \cup (C \setminus \{g_u, \ldots, g_{u-l+1}\}) \cup \{t\}]. \end{aligned} \tag{8}$$

Again, by corollary 3, since $min_{\{x\}}[F \cup (C \setminus \{g_u, \ldots, g_{u-l+1}\}) \cup \{t\}] = \{t\}$, we have

$$\begin{aligned} & min_{(\{x\} \cup \{s_{u-l}, \ldots, s_1\})}[F \cup (C \setminus \{g_u, \ldots, g_{u-l+1}\}) \cup \{t\}] \\ & = \{t\} \cup min_{\{s_{u-l}, \ldots, s_1\}}[F \cup (C \setminus \{g_u, \ldots, g_{u-l+1}\})]. \end{aligned} \tag{9}$$

Therefore, by (7), (8), and (9),

$$\begin{aligned} & min_{(D \cup \{x\})}(C \cup \{t\}) \\ & = min_{\{s_r, \ldots, s_{u+1}\}}(F \cup C) \cup min_{\{s_u, \ldots, s_{u-l+1}\}}(F \cup C) \cup \{t\} \\ & \cup min_{\{s_{u-l}, \ldots, s_1\}}[F \cup (C \setminus \{g_u, \ldots, g_{u-l+1}\})] \\ & = \{t\} \cup min_D(F \cup C). \end{aligned}$$

Notice that, $min_D(F \cup C) = min_{\{s_r,\ldots,s_{u+1}\}}(F \cup C) \cup min_{\{s_{u-l},\ldots,s_1\}}(F \cup C)$, and

$$\begin{aligned} min_{\{s_u,\ldots,s_1\}}(F \cup C) &= min_{\{s_u,\ldots,s_{u-l+1}\}}(F \cup C) \\ &\cup min_{\{s_{u-l},\ldots,s_1\}}[F \cup (C \backslash \{g_u,\ldots,g_{u-l+1}\})]. \end{aligned} \tag{10}$$

□

Theorem 2.14 *Let A, B, C, D, $F \subseteq [n]$ with $|A| = |B|$, $|C| = |D|$, $A \cap C = B \cap D = \emptyset$. Assume $C = \{g_1 > \ldots > g_r\}$ and $D = \{s_1 > \ldots > s_r\}$ and suppose $A \geq B$ and $C \geq D$ are such that $min_B(A \cup F) = A$ and $x \in B$ only if $s_i > x > g_{i+1}$, for some $i \in \{0, 1, \ldots, r\}$. Then,*

$$min_{(B \cup D)}(A \cup C \cup F) = A \cup min_D(C \cup F).$$

Proof: Let $A = \{t_1 > \ldots > t_k\}$ and $B = \{x_1 > \ldots > x_k\}$. The proof will be handled by induction on $k = |A|$. The previous lemma proves the case $k = 1$.

Let $k > 1$, and suppose that $s_u > x_k > g_{u+1}$ for some $u \in \{0, 1, \ldots, r\}$, and $g_{u-l} > t_k > g_{u-l+1}$, with $l \in \{1, \ldots, u\}$.

By corollary 2.6, (II), we may write

$$min_{(B \cup D)}(A \cup C \cup F) = min_{\{s_r,\ldots,s_{u+1}\}}(A \cup C \cup F) \cup min_{(B \cup \{s_u,\ldots,s_1\})}(A \cup C \cup F). \tag{11}$$

But

$$\begin{aligned} &min_{(B \cup \{s_u,\ldots,s_1\})}(A \cup C \cup F) = min_{(\{s_u,\ldots,s_{u-l+1}\} \cup \{x_k\})}(A \cup C \cup F) \\ &\cup min_{[(B \backslash \{x_k\}) \cup \{s_{u-l},\ldots,s_1\}]}[(A \backslash \{t_k\}) \cup (C \backslash \{g_u,\ldots,g_{u-l+1}\}) \cup F]. \end{aligned} \tag{12}$$

Since, $min_{\{s_r,\ldots,s_{u+1}\}}(A \cup C \cup F) = min_{\{s_r,\ldots,s_{u+1}\}}((A \backslash \{t_k\}) \cup C \cup F)$, it follows, from (11) and (12),

$$\begin{aligned} &min_{B \cup D}(A \cup C \cup F) \\ &= min_{((D \backslash \{s_u,\ldots,s_{u-l+1}\}) \cup (B \backslash \{x_k\}))}[(A \backslash \{t_k\}) \cup (C \backslash \{g_u,\ldots,g_{u-l+1}\}) \cup F] \\ &\cup min_{(\{s_u,\ldots,s_{u-l+1}\} \cup \{x_k\})}[(A \cup C \cup F]. \end{aligned} \tag{13}$$

On the other hand, attending to $k = 1$, we have

$$\begin{aligned} &min_{(\{s_u,\ldots,s_{u-l+1}\} \cup \{x_k\})}[(A \cup C \cup F] \\ &= min_{\{s_u,\ldots,s_{u-l+1}\}}(A \cup C \cup F) \cup min_{\{x_k\}}[A \cup (C \backslash \{g_u,\ldots,g_{u-l+1}\}) \cup F] \\ &= min_{\{s_u,\ldots,s_{u-l+1}\}}[(A \backslash \{t_k\}) \cup C \cup F] \cup \{t_k\}. \end{aligned} \tag{14}$$

Hence, from (13) and (14),

$$min_{(B\cup D)}(A\cup C\cup F)=min_{\{s_u,\ldots,s_{u-l+1}\}}[(A\backslash\{t_k\})\cup C\cup F]$$
$$\cup min_{[(D\backslash\{s_u,\ldots,s_{u-l+1}\})\cup(B\backslash\{x_k\})]}[(A\backslash\{t_k\})\cup(C\backslash\{g_u,\ldots,g_{u-l+1}\})\cup F]\cup\{t_k\}$$
$$=min_{(D\cup(B\backslash\{x_k\}))}[(A\backslash\{t_k\})\cup C\cup F]\cup\{t_k\}.$$

By induction,

$$min_{(B\cup D)}(A\cup C\cup F)=min_{[(D\cup(B\backslash\{x_k\})]}[(A\backslash\{t_k\})\cup C\cup F]\cup\{t_k\}$$
$$=(A\backslash\{t_k\})\cup min_D(C\cup F)\cup\{t_k\}=A\cup min_D(C\cup F).$$

□

2.2 The dual lattice of $\mathcal{P}[n]$ induced by the reverse order

Considering proposition 2.3, we start with the following

Definition 2.4 *Let $A,B\subseteq[n]$. We write $A\geq_{op}B$ if $A\geq X$, for some $X\subseteq B$ with $|X|=|A|$.*

The relation $\geq_{op}$ is a partial order on $2^{[n]}$, and we denote it by $\mathcal{P}^{op}[n]$. Clearly, $\mathcal{P}^{op}[n]$ is also a lattice in which the family of all subsets of a given cardinality is a sublattice. Furthermore, $\bigcup_{n\in\mathbb{N}}\mathcal{P}^{op}[n]$, the family of all finite subsets of $\mathbb{N}$ is a lattice with the relation defined by $A\geq_{op}B$.

Note that if $A\geq_{op}B$ then $|A|\leq|B|$. On the other hand, if $|A|=|B|$, $A\geq B$ iff $A\geq_{op}B$. This means that the sublattice of the family of all subsets of a given cardinality of $\mathcal{P}[n]$ and $\mathcal{P}^{op}[n]$ respectively, are isomorphic under the identity map.

The relation $\geq_{op}$ has also many characterizations: $A\geq_{op}B$ iff there exists an injection $j:A\longrightarrow B$ with $a\geq j(a)$, for all $a\in A$. Such an injection j is called a witness of $A\geq_{op}B$.

Let $A\geq_{op}B$ and i,j witnesses of $A\geq_{op}B$. We write $i\geq j$ if $i(A)\geq j(A)$. Given $A\geq_{op}B$, the relation $\geq$ defined by $i\geq j$ is a partial order on the set of all witnesses of $A\geq_{op}B$.

Since $\mathcal{P}^{op}[n]$ is a lattice, given $A\geq_{op}B$ we may define the greatest lower bound of A in 2^B. We write

$$max_AB:=max\{X\subseteq B:|X|=|A|\text{ and }A\geq X\}$$

as the greatest lower bound of A contained in B. Note that $max_AB=max\{i(A):i\text{ is a witness of }A\geq_{op}B\}$ and $A\cap B\subseteq max_AB$. In particular, if $|A|=|B|$, $max_AB=B$.

Let i be a witness of $A\geq_{op}B$. We say that i is a *maximal witness* of $A\geq_{op}B$ if $i(A)=max_AB$. In the sublattice of all subsets of $[n]$ of a given cardinality every witness is both minimal and maximal.

Let us denote by $\mathbb{N}_{op}$ as the set $\mathbb{N}$ with the reverse order, $\geq_{op}$, that is, $x \geq_{op} y \Longleftrightarrow x \leq y$. Given $n \in \mathbb{N}$, if op denotes the reverse permutation of $[n]$, $op(k) = n - k + 1$, then one may also look at the sublattice $[n]_{op}$, as the set $[n]$ with the order induced by the anti-automorphism op in the poset $([n], \geq)$, that is, $x \geq_{op} y \Longleftrightarrow op(x) \leq op(y)$. The map op is an involution in $2^{[n]}$ and it is an anti-automorphism on the sublattice of $\mathcal{P}[n]$ defined by the family of all subsets of a given cardinality, that is, $A \geq B$ iff $op(B) \geq op(A)$.

Now, we will look at the relation $\geq_{op}$ in $2^{[n]}$ either as a partial order in $2^{[n]}$, with respect to the reverse order in $[n]$, or the order induced in $2^{[n]}$ by the involution op.

Proposition 2.15 *Given $A, B \subseteq [n]$, the following assertions are equivalent*

(a) $A \geq_{op} B$.

(b) If $a = (\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_{|A|}, n+1, ...)$ and $b = (\tilde{b}_1, ..., \tilde{b}_{|B|}, n+1, ...)$ are the decreasing rearrangement of the elements of A and B, with respect to the reverse order, as embedded into $\mathbb{N}_{op}^{\mathbb{N}}$, then $b \geq a$ in the component reverse ordering order, that is, $\tilde{b}_i \geq_{op} \tilde{a}_i$ in $\mathbb{N}_{op}$, for all i.

(c) $op(B) \geq op(A)$.

Proof: $(a) \Longrightarrow (b)$ If $A \geq X$ with $X \subseteq B$, then putting $A = \{a_1 > \ldots > a_{|A|}\}$ and $B = \{b_1 > \ldots > b_{|A|} > \ldots > b_{|B|}\}$, we get $a_{|A|} \geq b_{|B|}, \ldots, a_1 \geq b_{|B|-|A|+1}$. That is, $\tilde{b}_i \geq_{op} \tilde{a}_i$ with $\tilde{a}_i = a_{|A|-i+1}$ and $\tilde{b}_i = b_{|B|-i+1}$.

$(b) \Longrightarrow (c)$ If (b) then there is an injection $j : A \longrightarrow B$ such that $j(a) \geq_{op} a$, for all $a \in A$, that is , $op(j(a)) \geq op(a)$, for all $a \in A$. This means that $op(B) \geq op(A)$.

$(a) \Longleftrightarrow (c)$ $A \geq X$, with $X \subseteq B$ and $|X| = |A|$ iff $op(X) \geq op(A)$, with $op(X) \subseteq op(B)$. $\square$

Since $A \geq B$ iff $op(B) \geq_{op} op(A)$, the operation op is an anti-automorphism between the posets $\mathcal{P}[n]$ and $\mathcal{P}^{op}[n]$. We have $op(A \wedge B) = op(A) \vee op(B)$ and $op(A \vee B) = op(A) \wedge op(B)$. Thus, $\mathcal{P}^{op}[n]$ is the dual lattice of $\mathcal{P}[n]$ induced by the reverse order in $[n]$.

Let $A \geq_{op} B$. Note that $\tilde{i}$ is a witness of $op(B) \geq op(A)$ iff there exists an injection $j : A \longrightarrow B$ with $a \geq j(a)$ and such that $op.\tilde{i}.op = j$. That is, j is a witness of $A \geq_{op} B$ iff $op.j.op$ is a witness $op(B) \geq op(A)$. Hence, $op(Y) = min_{(op(A))} op(B)$ iff $Y = max_A B$. In other words, j is a maximal witness of $A \geq_{op} B$ iff $opjop$ is a minimal witness of $op(B) \geq op(A)$.

Let $A \geq_{op} B$ and $A = \{a_1 > \ldots > a_m\}$. By the principle of duality [8], we have the following dual statements.

The dual of theorem 2.5: $max_A B = \{x_1, \ldots, x_m\}$ where

$$\begin{aligned} x_1 &= max\{b \in B : a_1 \geq b\}, \text{ and} \\ x_i &= max\{b \in B \setminus \{x_1, ..., x_{i-1}\} : a_i \geq b\}, \text{ for } i = 1, ..., m-1. \end{aligned} \tag{15}$$

The dual of Algorithm I : let $\{k, j, \ldots, l\} := \{i \in \{1, \ldots, m\} : \exists y \in B, a_i \geq y > a_{i+1}\}$ and $U_0 := \{a_k, a_j, \ldots, a_l\} \subseteq A$. For $\alpha \in \{k, j, \ldots, l\}$, define

$$z_\alpha := max\{y \in B : a_{\alpha-1} \geq y > a_\alpha\},$$

and put $Z_0 := \{z_k, z_j, \ldots, z_l\} \subseteq B$. Let $A_1 := A \setminus U_0$ and $B_1 := B \setminus Z_0$. Then, repeat the procedure with $A_1 \geq_{op} B_1$ until it stops, giving sets $A_k := A_{k-1} \setminus V_{k-1} = \emptyset$, and $B_k := B_{k-1} \setminus Z_{k-1}$, and we get

$$\begin{aligned} max_A B &= max_{U_0} B \cup max_{U_1} B_1 \cup \ldots \cup max_{U_{k-1}} B_{k-1} \\ &= Z_0 \cup Z_1 \cup \ldots \cup Z_{k-1}. \end{aligned} \tag{16}$$

The dual of Algorithm II is: let $B = \{b_1 > \ldots > b_n\}$ and $\{k, j, \ldots, l\} := \{i \in \{1, \ldots, n\} : \exists x \in A, b_{i-1} > x \geq b_i\}$ and $V_0 := \{b_k, b_j, \ldots, b_l\} \subseteq B$. For $\alpha \in \{k, j, \ldots, l\}$, define

$$q_\alpha := min\{x \in A : b_{\alpha-1} > x \geq b_\alpha\},$$

and put $Q_0 := \{q_k, q_j, \ldots, q_l\} \subseteq A$. Let $A_1 := A \setminus Q_0$ and $B_1 := B \setminus V_0$. Then, repeat the procedure with $A_1 \geq_{op} B_1$ until it stops, giving sets $A_k := A_{k-1} \setminus Q_{k-1} = \emptyset$, and $B_k := B_{k-1} \setminus V_{k-1}$, and we get

$$\begin{aligned} max_A B &= max_{Q_0} B \cup max_{Q_1} B_1 \cup \ldots \cup max_{Q_{k-1}} B_{k-1} \\ &= V_0 \cup V_1 \cup \ldots \cup V_{k-1}. \end{aligned} \tag{17}$$

These procedures describe maximal witnesses of $A \geq_{op} B$.

If $A \geq B$ is a chain in $\mathcal{P}[n]$ then $op(B) \geq_{op} op(A)$ is a chain in $\mathcal{P}^{op}[n]$. Clearly, $A \geq B \longrightarrow op(B) \geq_{op} op(A)$ defines a bijection between chains of length 2 in the lattice $\mathcal{P}[n]$, and chains in the dual lattice $\mathcal{P}^{op}[n]$. On the other hand, $A \geq B$ iff $X \geq B$ for some $X \subseteq A$ with $|X| = |B|$, and $F \geq_{op} G$ iff $F \geq Y$, for some $Y \subseteq G$ with $|Y| = |F|$. Our aim now is to set a bijection between chains in $\mathcal{P}[n]$ and chains in $\mathcal{P}^{op}[n]$ avoiding the lattice anti-automorphism op but instead stressing the ideas of the last characterization.

According to proposition 2.3, for each $F_1 \geq F_2$, we may choose $G_1, G_2 \subseteq [n]$, such that

$$|G_1| = |F_2|, \; G_1 \subseteq F_1, \; G_1 \geq F_2, \; F_1 \cap F_2 \subseteq G_1, \text{ and } G_2 = F_2 \cup (F_1 \setminus G_1). \tag{18}$$

Clearly $G_1 \geq_{op} G_2$, and if $|F_1| = |F_2|$, $G_i = F_i$, $i = 1, 2$.

That is, given $F_1 \geq F_2$, there is a witness s with $F_1 \cap F_2 \subseteq s(F_2)$, such that (F_1, F_2) and $(s(F_2), F_2 \cup (F_1 \setminus s(F_2))$ satisfy (18). Clearly, an injection $g : s(F_2) \longrightarrow F_2 \cup (F_1 \setminus s(F_2))$ such that $gs = id$ is a witness of $s(F_2) \geq_{op} F_2 \cup (F_1 \setminus s(F_2))$. If s_* is a minimal witness and $gs_* = id$ then g is a maximal witness.

On the other hand, given $G_1 \geq_{op} G_2$, there exist $F_1 \geq F_2$ and a witness s with $F_1 \cap F_2 \subseteq s(F_2)$, such that $(G_1, G_2) = (s(F_2), (F_1 \setminus s(F_2)) \cup F_2)$. For, if j is a witness of $G_1 \geq_{op} G_2$ with $G_1 \cap G_2 \subseteq j(G_1)$, then an injection $g : j(G_1) \longrightarrow (G_2 \setminus j(G_1)) \cup G_1$ such that $jg = id$ will do the claim. If j^* is a maximal witness and $j^* g = id$ then g is a minimal witness.

Let $\mathbf{H} := \{(F_1, F_2) : F_1 \geq F_2\}$ be the set of comparable elements in $\mathcal{P}[n]$, and let $\mathbf{H}^{op} := \{(G_1, G_2) : G_1 \geq_{op} G_2\}$ be the set of comparable elements in $\mathcal{P}^{op}[n]$.

For each $F_1 \geq F_2$ in $\mathbf{H}$, let s be a witness of $F_1 \geq F_2$ such that $F_1 \cap F_2 \subseteq s(F_2)$, and define the map

$$\mathbf{s} : \mathbf{H} \longrightarrow \mathbf{H}^{op}, \tag{19}$$

where $\mathbf{s}(F_1, F_2) = (s(F_2), F_2 \cup (F_1 \setminus s(F_2))$.

In particular, for each $F_1 \geq F_2$, we may choose a minimal witness s_*, that is, $s(F_1) = min_{F_2} F_1$. According to (19), any collection of minimal witnesses of $\mathbf{H}$ induces a same bijection $\mathbf{s}_* : \mathbf{H} \longrightarrow \mathbf{H}^{op}$. Any collection of maximal witnesses of $\mathbf{H}^{op}$ induces a same bijection $\mathbf{s}^* : \mathbf{H}^{op} \longrightarrow \mathbf{H}$ such that ${\mathbf{s}_*}^{-1} = \mathbf{s}^*$. The map $\mathbf{s}$ in (19) is a bijection iff $\mathbf{s} = \mathbf{s}_*$.

Let $\Theta^* := \mathbf{s}_* \cup {\mathbf{s}_*}^{-1}$, that is, ${\Theta^*}_{|\mathbf{H}} = \mathbf{s}_*$ and ${\Theta^*}_{|\mathbf{H}^{op}} = \mathbf{s}_*^{-1}$. Therefore, $\Theta^* : \mathbf{H} \cup \mathbf{H}^{op} \longrightarrow \mathbf{H} \cup \mathbf{H}^{op}$ is an involution which fixes the elements of $\mathbf{H} \cap \mathbf{H}^{op}$, that is, the pairs (F_1, F_2) such that $|F_1| = |F_2|$.

3 Words, indexing sets and Young tableaux

3.1 Words and indexing sets

Let $t \in \mathbb{N}$ and $M([t])$ be the free monoid of all words in the totally ordered alphabet $[t]$. Let $[\mathbb{N}]_t$ be the set of all t-sequences of finite subsets of $\mathbb{N}$. The elements of $[\mathbb{N}]_t$ may be represented by words in a grid as with matrices: the first coordinate, the row index, increases as one goes downwards, and the second coordinate, the column index, increases as one goes from left to right. Each sequence $(J_1, \dots, J_t)$ of finite subsets of $\mathbb{N}$ gives rise to a word $w(J_1, \dots, J_t)$ in $M([t])$ called the *word generated* by $(J_1, \dots, J_t)$, obtained reading the grid from top to down, along each row, from right to left, by assigning a label i to each dot in column i, for $i = 1, \dots, t$. The empty word Λ is generated by $(\emptyset, \dots, \emptyset)$.

For instance, let $J_1 = \{2, 3, 6, 9\}$, $J_2 = \{4, 5\}$, and $J_3 = \{1, 7, 8\}$, then $w(J_1,\ J_2, J_3) = 311221331$. Also, if $F_1 = \{1, 2, 4, 6\}$, $F_2 = \{3, 4\}$, and $F_3 = \{1, 5, 6\}$, we have $w(F_1, F_2, F_3) = 311221331$.

We identify the elements of $[\mathbb{N}]_t$ which generate the same word,

$$(J_1, \dots, J_t) \sim (J'_1, \dots, J'_t) \ \text{ if } \ w(J_1, \dots, J_t) = w(J'_1, \dots, J'_t).$$

The relation "$\sim$" is an equivalence relation in $[\mathbb{N}]_t$, and we write $M([\mathbb{N}]_t) := [\mathbb{N}]_t / \sim$.

Given $m \geq 0$ and a finite set $F = \{x_1, \dots, x_n\} \subseteq \mathbb{N}$, we write $m + F := \{m + x_1, \dots, m + x_n\}$. Clearly, $F \sim m + F$, for any $m \geq 0$. (We have $m + \emptyset = \emptyset$.)

$M([\mathbb{N}]_t)$ is a monoid with the operation $\cup$ defined as

$$[(J_1, \dots, J_t)] \cup [(F_1, \dots, F_t)] = [(J_1 \cup (m + F_1), \dots, J_t \cup (m + F_t))],$$

such that $J_i \subseteq [m]$, for $i = 1, \dots, t$.

As usual we denote the length of a word w by $|w|$. Let w be a word in $M([t])$, we write $|w|_k$, $k \in [t]$, to mean the multiplicity of the letter k in the word w.

Proposition 3.1 *$M([\mathbb{N}]_t)$ and $M([t])$ are isomorphic monoids.*

Proof: The map $\phi : M([\mathbb{N}]_t) \longrightarrow M([t])$ defined by $\phi([(J_1, \dots, J_t)]) = w(J_1, \dots, J_t)$ is a monoid isomorphism. Let w in $M([t])$ and define $J_k = \{i \in \{1, \dots, |w|\} : i-th$ letter of w is $k\}$, for $k = 1, \dots, t$. Then, $w(J_1, \dots, J_t) = w$. That is, every word w in $M([t])$ arises from $M([\mathbb{N}]_t)$, and ϕ is a bijection. Clearly, $\phi([(J_1, \dots, J_t)] \cup [(F_1, \dots, F_t)]) = w(J_1, \dots, J_t)w(F_1, \dots, F_t)$. □

For instance, the word $w = 311221331$ is generated by $J_1 = \{2, 3, 6, 9\}$, $J_2 = \{4, 5\}$, and $J_3 = \{1, 7, 8\}$.

Given $w \in M([t])$ and $(J_1, \dots, J_t) \in [\mathbb{N}]_t$ such that $w = w(J_1, \dots, J_t)$, we call $(J_1, \dots, J_t)$ the *indexing sets* of w, and $[(J_1, \dots, J_t)]$ the class of indexing sets of w.

A word w in $M([t])$ is a Yamanouchi word if any right factor v of w satisfies $|v|_1 \geq |v|_2 \geq \dots \geq |v|_t$. Recalling proposition 2.1, this is equivalent to say that if $(J_1, \dots, J_t)$ are indexing sets of w, then every pair (J_i, J_{i+1}), $i = 1, \dots, t-1$, satisfy condition (c) of that proposition. Henceforth, $w(J_1, \dots, J_t)$ is a Yamanouchi word in the alphabet $[t]$ iff $J_1 \geq \dots \geq J_t$.

The dual word of $w = x_1 \dots x_k \in M([t])$ is $w_{op} := op(x_k) \dots op(x_1)$ a word in the dual alphabet $op([t])$,with $op(i) = t - i + 1$. Clearly, $w(J_1, \dots, J_t) = w$ iff $w(op(J_t), \dots, op(J_1)) = w_{op}$. Hence, $w(J_1, \dots, J_t)$ is a *dual Yamanouchi* word iff $J_1 \geq_{op} \dots \geq_{op} J_t$. The map $w \longrightarrow w_{op}$ determines an anti-isomorphism in $M([t])$: $(w_1w_2)_{op} = (w_2)_{op}(w_1)_{op}$. Now we search for a bijection between Yamanouchi words and their dual avoiding this anti-automorphism.

Given $w \in M([t])$, w' is a subword of w iff $(J_1, \dots, J_t)$ are indexing sets of w then $w' = w(F_1, \dots, F_t)$, for some $F_i \subseteq J_i$, $i = 1, \dots, t$.

Definition 3.1 *Let w be a word in the two letters alphabet $\{i, i+1\}$. A subword w' of w is called a basis of w if (A, B) are given indexing sets of w, there exist $X \subseteq A$ and $Y \subseteq B$ such that*

(i) $|X| = |Y|$, $X \geq Y$ and $w(X, Y) = w'$,

(ii) $w(A \setminus X, B \setminus Y) = i^r(i+1)^s$, where $r = |A| - |X|$ and $s = B - |Y|$.

A basis is a self-dual Yamanouchi word of longest length. When w is a Yamanouchi (dual Yamanouchi) word any basis is of type $w(X, B)$ $(w(A, Y))$, $(r = 0\ )$ $s = 0$. We identify a basis of a word, in a two letters alphabet, with its class of indexing sets. So, we say that (X, Y) is a basis of (A, B). In particular, if the word is either Yamanouchi or dual Yamanouchi every basis may be determined by a witness, and every witness determines a basis of a word.

As a basis is a self-dual Yamanouchi word, the calculation of a basis may be done either using the procedures to determine minimal witnesses for Yamanouchi words, for instance theorem 2.5, or for dual Yamanouchi words, as the dual of theorem 2.5, or those which are common to both, like algorithms I and II.

For example, let $w = 112121122122112$ with indexing sets $A = \{1, 2, 4, 6, 7, 10, 13, 14\}$ and $B = \{3, 5, 8, 9, 11, 12, 15\}$:

(a) (Applying the dual of theorem 2.5.) Extract from the word a subword w' containing letters i and $i+1$ only. Remove the right most subword $i+1\,i$ of w': put a bracket in the right most letter i, and then a bracket in the right most letter $i+1$, to the left of the just brackted i. The remaining letters, the ones which are not brackted, constitute a subword v_1 of w'. Then remove the right most subword $i+1\,i$ of v_1. There remains a subword v_2. Continue this procedure until it stops, giving a word v_q of type $v_q = i^r\,(i+1)^s$, with $|w|_i - q = r$ and $|w|_{i+1} - q = s$. The basis, in the subalphabet $\{i, i+1\}$, is the subword constituted by the bracketed letters. In our example, we have

$$11(21(21)1)2(21)(2(21)1)2,$$

$X = \{6, 7, 10, 13, 14\} \geq Y = \{3, 5, 9, 11, 12\}$,

$$w(X, Y) = 2211212211, \quad \text{and} \quad w(A \setminus X, B \setminus Y) = 11122.$$

Note that $max_B X = Y$.

(b) (Applying theorem 2.5.) Do the procedure above with the leftmost subword $i+1\,i$ of w: put a bracket in the left most letter $i+1$, and then a bracket in the left most letter i, to the right of the just brackted $i+1$. When the procedure stops we get a subword u_q of type $u_q = i^r\,(i+1)^s$, with $|w|_i - q = r$ and $|w|_{i+1} - q = s$. The basis, in the subalphabet $\{i, i+1\}$, is the subword constituted by the bracketed letters. In our example, we have

$$11(21)(21)1(2(21)(221)1)2,$$

$X' = \{4, 6, 10, 13, 14\} \geq Y' = \{3, 5, 8, 9, 11\}$,

$$w(X', Y') = 2121221211, \quad \text{and} \quad w(A \setminus X', B \setminus Y') = 11122.$$

Note that $min_{Y'} A = X'$.

(c)[10], [12] (Applying either algorithms I or II.) Extract from the word a subword w' containing letters i and $i+1$ only. Bracket every factor $i+1\,i$ of w. The letters which are not bracketed constitute a subword w_1 of w. Then bracket every factor $i+1\,i$ of w_1. There remains a subword w_2. Continue this procedure until it stops, giving a word w_k of type $w_k = i^r\,(i+1)^s$. The basis, in the subalphabet $\{i, i+1\}$, is the subword constituted by the bracketed letters. In our example, we have

$$11(21)(21)12(21)(2(21)1)2,$$

$X'' = \{4, 6, 10, 13, 14\} \geq Y'' = \{3, 5, 9, 11, 12\}$,

$$w(X'', Y'') = 2121212211, \quad \text{and} \quad w(A \setminus X'', B \setminus Y'') = 11122.$$

Note that $min_{Y''}A = X''$, $max_B X'' = Y''$, and $X'' = X'$ and $Y'' = Y$.

If w is a Yamanouchi (dual Yamanouchi), $X = X' = X'' \subseteq A$ and $Y = Y' = Y'' = B$ (and $X = X' = X'' = A$ and $Y = Y' = Y'' \subseteq B$) and both procedures (b) and (c) ((a) and (c)) coincide as was shown in theorem 2.11.

The procedure (c) is the transformation given by M. P. Schutzenberger and A. Lascoux in [10] and [12], and its translation to indexing sets, in the case of Yamanouchi and dual Yamanouchi words in a two-letters alphabet, is given either by algorithm I or II, as well as their dual algorithms, respectively. Moreover, given a Yamanouchi (dual Yamanouchi) word $w = w(A, B)$, the procedure (c) on w is equivalent to putting brackets in the subword $w(min_B A, B)$ ($w(A, max_B A)$). There remains a word i^r ($(i+1)^s$), with $r = |A| - |B|$ ($s = |B| - |A|$). That is, according to theorem 2.11, bracketing every factor $i+1\,i$ on a Yamanouchi (dual Yamanouchi) word, as in (c), is an instance of a minimal (maximal) witness of $A \geq B$ ($A \geq_{op} B$).

The next proposition gives a representative of each class $[(A, B)]$.

Proposition 3.2 *Given A and B two finite subsets of $\mathbb{N}$, let $Y \subseteq B$ and $X \subseteq A$ such that $w(X, Y)$ is the basis of w obtained according to the procedure (c). Then, $w(A, B) = w((A \setminus X) \cup Y, B) = w(A, (B \setminus Y) \cup X)$.*

Proof: Let w be the word generated by (A, B), then w is a word in a two-letters alphabet [2]. Bracket every factor 21 of w. The subword constituted by the bracketed factors is generated by (X, Y). In this case, $min_Y A = X$ and $max_B X = Y$. The remaining word is generated by $(A \setminus X, B \setminus Y)$ and the word generated by $((A \setminus X) \cup Y, B)$ is w. □

Let $\mathbf{s}_* : \mathbf{H} \longrightarrow \mathbf{H}^{op}$ be the bijection induced by any collection of minimal witnesses of $\mathbf{H}$, defined by $\mathbf{s}_*(A, B) = (min_B A, B \cup (A \setminus min_B A))$; and $\mathbf{s}_*^{-1} : \mathbf{H}^{op} \longrightarrow \mathbf{H}$ be the bijection induced by any collection of maximal witnesses of $\mathbf{H}^{op}$, $\mathbf{s}_*^{-1}(C, D) = (C \cup (D \setminus max_C D), max_C D)$. Recall that $min_B A$ and $max_C D$ may be achieved by any minimal witness and maximal witness respectively. For instance, as mentioned above, the procedure (c), when restricted to Yamanouchi words and their dual, is the translation of algorithms I, II to words:

$$\begin{aligned} w_1 &= w(V_0, z^I(V_0)) = w(U_0, z^{II}(U_0)) \\ &\vdots \\ w_k &= w(V_{k-1}, z^I(V_{k-1})) = w(U_{k-1}, z^{II}(U_{k-1})). \end{aligned}$$

For $k = 1, \ldots, t-1$, let σ_k be the involutions, defined in [10] and [12], acting on $\mathbb{Z}([t])$, the free algebra on $[t]$. Then,

$$w(\mathbf{s}_*(A, B)) = \sigma_1(w(A, B)), \quad w(\mathbf{s}_*^{-1}(C, D)) = \sigma_1(w(C, D)).$$

In the last section, we have introduced the involution $\Theta^* : \mathbf{H} \cup \mathbf{H}^{op} \longrightarrow \mathbf{H} \cup \mathbf{H}^{op}$. Now we extend it to $(J_1, \ldots, J_t) \in [\mathbb{N}]_t$ such that either $J_i \geq J_{i+1}$ or $J_i \geq_{op} J_{i+1}$, for $i = 1, \ldots, t-1$, as follows

$$\Theta_k^* = \Theta^*_{|(J_k, J_{k+1})} \ \ k = 1, \ldots, t-1.$$

The translation of the involutions Θ_k^*, $k = 1, \ldots, t$, on indexing sets to words, are the involutions σ_k, $k = 1, \ldots, t-1$, restricted to words which with respect to the subalphabet $\{k, k+1\}$ are either Yamanouchi words or dual Yamanouchi words.

Let $\Theta^*(J_k, J_{k+1}) = (J_k^*, J_{k+1}^*)$, then

$$\sigma_k(w(J_1, \ldots, J_t)) = w(J_1, \ldots, J_k^*, J_{k+1}^*, \ldots, J_t).$$

For instance, let $w = 122111211$ be the two-letter word. Using the involution h_1,

$$w = 1(2(21)1)1(21)1 \longrightarrow \sigma_1(w) = 222112212.$$

On the other hand, let $J_1 = \{1, 4, 5, 6, 8, 9\}$ and $J_2 = \{2, 3, 7\}$ be indexing sets of the word w. We have $min_{J_2} J_1 = \{4, 5, 8\}$,

$$\Theta^*(J_1, J_2) = (\{4, 5, 8\}, \{1, 2, 3, 6, 7, 9\})$$

and the word generated by $\Theta^*(J_1, J_2)$ is $222112212 = \sigma_1(w)$.

3.2 Young tableaux

A partition is a sequence of non negative integers, $a = (a_1, a_2, ...)$, all but a finite number of which are non zero, such that $a_1 \geq a_2 \geq ...$ The number $|a| := \sum_i a_i$ is called the *weight* of a; the maximum value of i for which $a_i > 0$ is called the *length* of a and is denoted by $l(a)$. If $l(a) = 0$ we have the null partition $a = (0, 0, ...)$. If $l(a) = k$, we shall often write $a = (a_1, ..., a_k)$.

Sometimes it is convenient to use the notation

$$a = (a_1^{m_1}, a_2^{m_2}, ..., a_k^{m_k}),$$

where $a_1 > a_2 > ... > a_k$ and $a_i^{m_i}$, with $m_i \geq 0$, means that a_i appears m_i times as a part of a.

We say that a is an *elementary partition* if there is an $m \in \{1, ..., n\}$ such that $a = (1^m)$.

Suppose $a = (a_1, ..., a_k)$ is a partition of length k with $|a| = n$. The *Young diagram* of a is an array of n boxes having k left-justified rows with row i containing a_i boxes for $1 \leq i \leq k$. We shall identify a partition with its Young diagram.

Given two partitions a and c, we write $a \subseteq c$ to mean $a_i \leq c_i$, for all i. Graphically, this means that the Young diagram of a is contained in the Young diagram of c.

Let a and c be partitions such that $a \subseteq c$. We define

$$c/a := \{(i,j) \in c \ : \ (i,j) \notin a\},$$

called a *skew-diagram*. We write $|c/a| := |c| - |a|$.

A skew-diagram is called a vertical [horizontal] *m-strip*, where $m > 0$, if it has m boxes and at most one box in each row [column].

Let a and c be partitions such that $a \subseteq c$ and $(m_1, ..., m_t)$ a sequence of positive integers. A *Young tableau* $\mathcal{T}$ of type $(a, (m_1, ..., m_t), c)$ is a sequence of partitions

$$\mathcal{T} = (a^0, a^1, ..., a^t)$$

such that $a = a^0 \subseteq a^1 \subseteq ... \subseteq a^t = c$ and, for each $k = 1, ..., t$, the skew-diagram a^k/a^{k-1} is a vertical strip labelled by k with $m_k = |a^k/a^{k-1}|$.

The *indexing sets* $J_1, ..., J_t$ of $\mathcal{T}$ are the subsets of $\{1, ..., n\}$ given by

$$J_k = \{i \ : a_i^k - a_i^{k-1} \neq 0\}, \ \ 1 \leq k \leq t.$$

That is, J_k is defined by the row indices of the boxes of c/a labelled by k, $1 \leq k \leq t$. Notice that $(|J_1|, ..., |J_t|) = (m_1, ..., m_t)$.

The skew-diagram c/a is called the *shape* of the tableau $\mathcal{T}$ and $(m_1, ..., m_t)$ the *weight* of $\mathcal{T}$.

For example,

$$\begin{array}{ccccc} x & x & x & 1 & 4 \\ x & x & 1 & 3 & 4 \\ x & 1 & & & \\ 2 & & & & \end{array}$$

is a Young tableau of type $((3,2,1),(3,1,1,2),(5,5,2,1))$, with indexing sets $J_1 = \{1,2,3\}$, $J_2 = \{4\}$, $J_3 = \{2\}$ and $J_4 = \{1,2\}$.

We will introduce now the notion of Littlewood-Richardson sequence, following the terminology in [1, 2, 3].

Definition 3.2 *[1] Let $\mathcal{T}$ be a Young tableau of type $(a, (m_1, ..., m_t), c)$. We say that $\mathcal{T}$ is a Littlewood-Richardson (LR for short) sequence if its indexing sets satisfy $J_1 \geq J_2 \geq ... \geq J_t$.*

Proposition 2.1 (c) shows that this definition is equivalent to the one given in [9].

If $a = (a_1, \ldots, a_n)$ and $c = (c_1, \ldots, c_n)$ with $a \subseteq c$ (the diagram of a is contained in the diagram of c), we put $\overline{a} := (c_1 - a_n, \ldots, c_1 - a_2, c_1 - a_1)$ and $\overline{c} = (c_1 - c_n, c_1 - c_{n-1}, \ldots, c_1 - c_2, 0)$. Note that $\overline{a}$ and $\overline{c}$ are respectively the complements of a and c in the $n \times c_1$ rectangle.

Given a tableau $\mathcal{T}$ of type $(a, (m_1, \ldots, m_t), c)$ and indexing sets $J_1, \ldots, J_t$, its complement, denoted by $\overline{\mathcal{T}}$, is the tableau of type $(\overline{c}, (m_t, \ldots, m_1), \overline{a})$ with indexing sets $op(J_t), \ldots, op(J_1)$. When $\mathcal{T}$ is an LR tableau, we have $op(J_t) \geq_{op} \cdots \geq_{op} op(J_1)$.

Geometrically, $\overline{\mathcal{T}}$, the *complement tableau* of $\mathcal{T}$, is obtained by reflecting $\mathcal{T}$ once about the horizontal axis, then once about the vertical axis and then substituting labels $i \in \{1,\ldots,t\}$ by $op(i) = t - i + 1$, with $op \in \mathcal{S}_t$, *i.e.*, J_i by $op(J_{t-i+1})$, with $op \in \mathcal{S}_n$, for $i = 1,\ldots,t$. As usual if $\mathcal{T}$ is of type $(a,(m_1,\ldots,m_t),c)$ we say that c/a is the skew-shape of $\mathcal{T}$. We define the complement of the skew-shape of $\mathcal{T}$, as being the skew-shape of $\overline{\mathcal{T}}$, $\overline{a}/\overline{c}$.

Definition 3.3 *The word of a tableau $\mathcal{T}$, denoted by $\omega(\mathcal{T})$ on the alphabet $[t]$, is the word generated by the indexing sets of $\mathcal{T}$.*

This definition agrees with the one given in [6]: the word of a tableau may be obtained by listing the labels from right to left in each row, starting in the top and moving to bottom. We may also characterize LR tableaux and complement tableaux, using the language of words. Clearly, $\mathcal{T}$ is an LR tableau iff its word is a Yamanouchi word. Let $\mathcal{T}$ be a tableau of type $(a,(m_1,\ldots,m_t),c)$ with word $x_1\ldots x_r$, and $\mathcal{H}$ a tableau of type $(\overline{c},(m_t,\ldots,m_1),\overline{a})$. $\mathcal{H}$ is the complement of $\mathcal{T}$ iff $\omega(\mathcal{H}) = op(x_r)\ldots op(x_1)$ the dual word of $\mathcal{T}$.

In previous example, $\omega(\mathcal{T}) = 4143112$ and $\omega(\overline{\mathcal{T}}) = 3442141$.

There is a bijection between LR tableaux and complement LR tableaux given by the bijection between chains and dual chains, that is,

$$J_1 \geq \ldots \geq J_t \longrightarrow op(J_t) \geq \ldots \geq op(J_1).$$

Definition 3.4 *We say that a tableau $\mathcal{T}$ of type $(a,(m_1,\ldots,m_t),c)$ is an op-LR (reversing LR tableau) if its indexing sets are such that $J_1 \geq_{op} \ldots \geq_{op} J_t$.*

Therefore, $\mathcal{T}$ is an *op*-LR tableau iff its word is the dual of a Yamanouchi word.

Hence, $\mathcal{T}$ is an *op*-LR tableau iff its complement is an LR tableau. This means that our definition of reversing *LR* tableau is based on the notion of complementation. Our aim now is twice-fold: avoid duality and to interpret an *op*-LR as a consequence of the action of the symmetric group $\mathcal{S}_t$ on a family of indexing sets in order to introduce σ-LR tableaux where σ is any permutation; and exhibit a bijection between LR tableaux and σ-LR tableaux of a given type.

In [5] we have interpreted the LR reverse filling as a permutation of the LR rule and we have exhibited a bijection between *op*-LR tableaux and LR tableaux. In this way we have also established a bijection between LR tableaux and their conjugate.

4 Actions of the symmetric group

Let $t \geq 2$, and consider the transpositions of consecutive positive integers $s_i = (i\; i+1)$, $1 \leq i \leq t-1$. Denote the identity by s_0. The symmetric group $\mathcal{S}_t$, $t \geq 1$, is generated by these involutions s_i, $i = 1,\ldots,t-1$, written $\mathcal{S}_t =< s_1,\ldots,s_{t-1} >$, which satisfy the Moore-Coxeter relations: $s_i^2 = s_0$, $s_is_j = s_js_i$, if $|i-j| \neq 1$, and $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$.

The elements of $\mathcal{S}_t$, $t \geq 1$, may be written as words on the alphabet $\{s_1, ..., s_{t-1}\}$, defined recursively as follows:

$$\begin{aligned} \mathcal{S}_1 &= \{s_0\}, \\ \mathcal{S}_t &= \left\{ \begin{array}{cc} \omega & \\ s_{t-1}\omega & \\ s_{t-2}s_{t-1}\omega & , \omega \in \mathcal{S}_{t-1} \\ \vdots & \\ s_1 s_2 ... s_{t-1}\omega & \end{array} \right\}, \quad \text{if } t \geq 2. \end{aligned} \tag{20}$$

We call these words, *canonical words* of $\mathcal{S}_t$.

Definition 4.1 *Let $F_1 \geq F_2$ and $\mathbb{F} = \{(F_1^\sigma, F_2^\sigma) : \sigma \in < s_1 >\} \subseteq \mathbf{H} \cup \mathbf{H}^{op}$. We say that $\mathbb{F}$ is generated by (F_1, F_2), if $(F_1^{s_0}, F_2^{s_0}) = (F_1, F_2)$, and there exists a map $\mathbf{s} : \mathbf{H} \to \mathbf{H}^{op}$ such that $\mathbf{s}(F_1, F_2) = (F_1^{s_1}, F_2^{s_1})$.*

Given $\mathbb{F} = \{(F_1^\sigma, F_2^\sigma) : \sigma \in < s_1 >\} \subseteq \mathbf{H} \cup \mathbf{H}^{op}$ generated by (F_1, F_2), we may define the involutions on $\mathbb{F}$, $\psi_{s_i}(F_1^\sigma, F_2^\sigma) = (F_1^{s_i\sigma}, F_2^{s_i\sigma})$, $i = 0, 1$, with $\sigma \in \{s_0, s_1\}$. Thus, the map $\psi : \mathcal{S}_2 \to \mathcal{S}_F$ defined by $\psi(s_0) = \psi_{s_0}$ and $\psi(s_1) = \psi_{s_1}$, is an isomorphism if $|F_1| > |F_2|$, and an homomorphism if $|F_1| = |F_2|$. Therefore, ψ defines an action of the symmetric group $\mathcal{S}_2$ on $\mathbb{F}$.

As $\Theta^* : \mathbf{H} \cup \mathbf{H}^{op} \to \mathbf{H} \cup \mathbf{H}^d$ is an involution, if we put $\psi(s_1) = \Theta^*$ and $\psi(s_0) = id$, ψ defines an action of the symmetric group $\mathcal{S}_2$ on $\mathbf{H} \cup \mathbf{H}^{op}$.

Recall that $s_0, s_1, s_2, s_2s_1, s_1s_2, s_1s_2s_1$ are the canonical words of $\mathcal{S}_3$.

Definition 4.2 *Given $F_1 \geq F_2 \geq F_3$ and $\mathbb{F} = \{(F_1^\sigma, F_2^\sigma, F_3^\sigma) \in \mathcal{P}(n)^3 : \sigma \in < s_1, s_2 >\}$, with $(F_1^{s_0}, F_2^{s_0}, F_3^{s_0}) = (F_1, F_2, F_3)$, we say that $\mathbb{F}$ is generated by (F_1, F_2, F_3) if*

(I) *(a)* $F_3^{s_1} = F_3$ *and* $\{(F_1^\sigma, F_2^\sigma) : \sigma \in < s_1 >\}$ *is generated by* (F_1, F_2).

(b) $F_1^{s_2} = F_1$ *and* $\{(F_2^\sigma, F_3^\sigma) : \sigma \in < s_2 >\}$ *is generated by* (F_2, F_3).

(II) *(a)* $F_1^{s_2s_1} = F_1^{s_1}$ *and* $\{(F_2^{\sigma s_1}, F_3^{\sigma s_1}) : \sigma \in < s_2 >\}$ *is generated by* $(F_2^{s_1}, F_3^{s_1})$ *and* $\mathbf{s}(F_2^{s_1}, F_3^{s_1}) = (F_2^{s_2s_1}, F_3^{s_2s_1})$ *such that* $F_2^{s_2s_1} = X \cup Y$ *with* $X \subseteq F_2^{s_2}$ *and* $Y \subseteq F_1 \setminus F_1^{s_1}$ *only if* $F_2^{s_2} = X \cup Z$ *with* $Z \geq Y$.

(b) $F_3^{s_1s_2} = F_3^{s_2}$ *and* $\{(F_1^{\sigma s_2}, F_2^{\sigma s_2}) : \sigma \in < s_1 >\}$ *is generated by* $(F_1^{s_2}, F_2^{s_2})$ *and* $\mathbf{s}(F_1^{s_2}, F_2^{s_2}) = (F_1^{s_1s_2}, F_2^{s_1s_2})$ *such that* $F_1^{s_1s_2} \subseteq F_1^{s_1}$ *and* $F_1^{s_1} \setminus F_1^{s_1s_2} \geq F_2 \setminus F_2^{s_2}$.

(III) $F_3^{s_1s_2s_1} = F_3^{s_2s_1}$ *and* $\{(F_1^{\sigma s_2s_1}, F_2^{\sigma s_2s_1}) : \sigma \in < s_1 >\}$ *is generated by* $(F_1^{s_2s_1}, F_2^{s_2s_1})$ *and* $\mathbf{s}(F_1^{s_2s_1}, F_2^{s_2s_1}) = (F_1^{s_1s_2s_1}, F_2^{s_1s_2s_1})$ *such that* $F_1^{s_1s_2s_1} = F_1^{s_1s_2}$.

Proposition 4.1 *Let $F_1 \geq F_2 \geq F_3$.*

(a) *There exists always a set* $\mathbb{F}$ *generated by* (F_1, F_2, F_3).

(b) *If* $\mathbb{F}$ *is generated by* (F_1, F_2, F_3), *then* $F_2^{s_1 s_2} \geq F_2^{s_1}$ *and* $F_2^{s_2} \geq F_2^{s_2 s_1}$.

(c) *If* $\mathbb{F}$ *is generated by* (F_1, F_2, F_3), *then, for each* $\sigma \in < s_1, s_2 >$,

  (i) *either* $F_i^\sigma \geq F_{i+1}^\sigma$ *or* $F_i^\sigma \geq_{op} F_{i+1}^\sigma$, $1 \leq i \leq 2$;

  (ii) $F_3 \subseteq F_3^\sigma$;

  (iii) *there exist* $G_i^\sigma \subseteq F_i^\sigma$ *for* $i = 1, 2, 3$, *with* $|G_i^\sigma| = |F_3|$ *such that* $G_3^\sigma = F_3$ *and* $G_1^\sigma \geq G_2^\sigma \geq F_3$.

Proof: Since $F_1 \geq F_2$ and $F_2 \geq F_3$, let $\{(F_1^\sigma, F_2^\sigma) : \sigma \in < s_1 >\}$ be generated by (F_1, F_2), and let $\{(F_2^\sigma, F_3^\sigma) : \sigma \in < s_2 >\}$ be generated by (F_2, F_3). Clearly $F_2^{s_1} \geq F_3^{s_1}$ and $F_1^{s_2} \geq F_2^{s_2}$.

We may choose a witness s of $F_2^{s_1} \geq F_3^{s_1}$ with $F_2^{s_1} \cap F_3^{s_1} \subseteq s(F_3^{s_1})$, such that $\mathbf{s}(F_2^{s_1}, F_3^{s_1}) = (F_2^{s_2 s_1}, F_3^{s_2 s_1})$ satisfies

$$F_2^{s_2 s_1} = X \cup Y, \text{ with } X \subseteq F_2^{s_2},\ Y \subseteq F_1 \setminus F_1^{s_1}, \text{ only if}$$
$$F_2^{s_2} = X \cup Z, \text{ with } Z \geq Y. \tag{21}$$

For $F_2^{s_1} = F_2 \cup (F_1 \setminus F_1^{s_1})$, $F_2^{s_2} \subseteq F_2$, $F_2 \cap F_3 \subseteq F_2^{s_2}$, $F_2^{s_2} \geq F_3$, and $|F_2^{s_2 s_1}| = |F_2^{s_2}| = |F_3|$. Theorem 2.8, asserts that (21) is feasible. Note that from (21), we have

$$F_2^{s_2} \geq F_2^{s_2 s_1}. \tag{22}$$

We may choose a witness s of $F_1^{s_2} \geq F_2^{s_2}$ with $F_1^{s_2} \cap F_2^{s_2} \subseteq s(F_2^{s_2})$, such that $\mathbf{s}(F_1^{s_2}, F_2^{s_2}) = (F_1^{s_1 s_2}, F_2^{s_1 s_2})$ satisfies

$$F_1^{s_1 s_2} \subseteq F_1^{s_1},$$
$$F_1^{s_1} \setminus F_1^{s_1 s_2} \geq F_2 \setminus F_2^{s_2}. \tag{23}$$

For, $F_1 \cap F_2 \subseteq F_1^{s_1}$, $F_1^{s_1} \geq F_2$, and $F_2 = F_2^{s_2} \cup (F_3^{s_2} \setminus F_3)$. Again, proposition (2.4), (c), asserts that (23) is feasible.

Notice that $F_2^{s_1 s_2} \geq F_3^{s_1 s_2} = F_3^{s_2}$. Note also that $F_2^{s_1 s_2} = F_2^{s_2} \cup (F_1 \setminus F_1^{s_1 s_2}) = F_2^{s_2} \cup (F_1 \setminus F_1^{s_1}) \cup (F_1^{s_1} \setminus F_1^{s_1 s_2})$ and $F_2^{s_1} = F_2 \cup (F_1 \setminus F_1^{s_1}) = F_2^{s_2} \cup (F_2 \setminus F_2^{s_2}) \cup (F_1 \setminus F_1^{s_1})$. Henceforth, by (23), we find

$$F_2^{s_1 s_2} \geq F_2^{s_1}. \tag{24}$$

Finally, we may choose a witness s of $F_1^{s_2 s_1} \geq F_2^{s_2 s_1}$ with $F_1^{s_2 s_1} \cap F_2^{s_2 s_1} \subseteq s(F_2^{s_2 s_1})$ such that $\mathbf{s}(F_1^{s_2 s_1}, F_2^{s_2 s_1}) = (F_1^{s_1 s_2 s_1}, F_2^{s_1 s_2 s_1})$, satisfies $F_1^{s_1 s_2 s_1} = F_1^{s_1 s_2}$. ¿From (23) and recalling that $F_1^{s_2 s_1} = F_1^{s_1}$, it is enough to show that $(F_1^{s_2 s_1}, F_2^{s_2 s_1})$ and $(F_1^{s_1 s_2}, F_2^{s_1 s_2 s_1})$ satisfy (18), that is,

$$F_1^{s_1 s_2} \subseteq F_1^{s_1} = F_1^{s_2 s_1},$$

and using (21) and (22),

$$F_1^{s_2s_1} \cap F_2^{s_2s_1} \subseteq F_1^{s_2} \cap F_2^{s_2} \subseteq F_1^{s_1s_2}, \quad F_1^{s_1s_2} \geq F_2^{s_2} \geq F_2^{s_2s_1}.$$

Conditions of definition 4.2 are, therefore, fulfilled. Hence, $\mathbb{F}$ is generated by (F_1, F_2, F_3).

(b) was obtained in (22) and (24).

(c) (i) and (ii) are consequence of (18).

(iii) Note that $F_1^{s_1s_2} \geq F_2^{s_2} \geq F_3$ and since $F_2^{s_2} \geq F_2^{s_2s_1}$, it follows $F_1^{s_1s_2} \geq F_2^{s_2s_1} \geq F_3$, with $|F_1^{s_1s_2}| = |F_2^{s_2}| = |F_2^{s_2s_1}| = |F_3|$. The sets in the first sequence are subsets of $F_1^\sigma, F_2^\sigma, F_3^\sigma$, respectively, for $\sigma \in \{s_0, s_1, s_2, s_1s_2\}$, and the sets in the second sequence are subsets of $F_1^\sigma, F_2^\sigma, F_3^\sigma$, for $\sigma \in \{s_2s_1, s_1s_2s_1\}$. □

Theorem 4.2 *Let* $\mathbb{F} = \{(F_1^\sigma, F_2^\sigma, F_3^\sigma) : \sigma \in < s_1, s_2 >\}$. $\mathbb{F}$ *is be generated by* (F_1, F_2, F_3) *iff*

(I) *(a)* $F_3^{s_1} = F_3$ *and* $\{(F_1^\sigma, F_2^\sigma) : \sigma \in < s_1 >\}$ *is generated by* (F_1, F_2).

(b) $F_1^{s_2} = F_1$ *and* $\{(F_2^\sigma, F_3^\sigma) : \sigma \in < s_2 >\}$ *is generated by* (F_2, F_3).

(II) *(a)* $F_1^{s_2s_1} = F_1^{s_1}$ *and* $\{(F_2^{\sigma s_1}, F_3^{\sigma s_1}) : \sigma \in < s_2 >\}$ *is generated by* $(F_2^{s_1}, F_3^{s_1})$.

(b) $F_3^{s_1s_2} = F_3^{s_2}$ *and* $\mathbb{F}_{s_1s_2} := \{(F_1^{\sigma s_2}, F_2^{\sigma s_2}) : \sigma \in < s_1 >\}$ *is generated by* $(F_1^{s_2}, F_2^{s_2})$.

(III) *(a)* $F_3^{s_1s_2s_1} = F_3^{s_2s_1}$ *and* $\{(F_1^{\sigma s_2s_1}, F_2^{\sigma s_2s_1}) : \sigma \in < s_1 >\}$ *is generated by* $(F_1^{s_2s_1}, F_2^{s_2s_1})$ *and* $\mathbf{s}(F_1^{s_2s_1}, F_2^{s_2s_1}) = (F_1^{s_1s_2s_1}, F_2^{s_1s_2s_1})$ *such that* $F_1^{s_1s_2s_1} = F_1^{s_1s_2}$.

(b) $\{(F_2^{s_1s_2}, F_3^{s_1s_2}); (F_2^{s_1s_2s_1}, F_3^{s_1s_2s_1})\}$ *is generated by* $(F_2^{s_1s_2}, F_3^{s_1s_2})$.

(IV) $F_2^{s_1s_2} \geq F_2^{s_1}$ *and* $F_2^{s_2} \geq F_2^{s_2s_1}$.

Proof: The *only if* part. Suppose that $\mathbb{F}$ satisfy the conditions of definition 4.2. We have only to prove (III), (b). Note that $F_2^{s_1s_2} \geq F_3^{s_1s_2}$. We shall show that we may choose a witness s of $F_2^{s_1s_2} \geq F_3^{s_1s_2}$ with $F_2^{s_1s_2} \cap F_3^{s_1s_2} \subseteq s(F_2^{s_1s_2})$ such that

$$\mathbf{s}(F_2^{s_1s_2}, F_3^{s_1s_2}) = (F_2^{s_1s_2s_1}, F_3^{s_1s_2s_1}).$$

In fact, recalling definition 4.2, we have $F_1^{s_1s_2s_1} = F_1^{s_1s_2}$, $F_2^{s_1s_2} = F_2^{s_2} \cup (F_1 \setminus F_1^{s_1s_2})$ and $F_2^{s_1s_2s_1} = F_2^{s_2s_1} \cup (F_1^{s_2s_1} \setminus F_1^{s_1s_2})$, and using (21),(23), we get

$$F_2^{s_1s_2s_1} \subseteq F_2^{s_1s_2}.$$

On the other hand, from $F_1^{s_1} \setminus F_1^{s_1s_2} \geq F_2 \setminus F_2^{s_2} = F_3^{s_2} \setminus F_3$, and since $F_2^{s_2s_1} \geq F_3$, we have

$$F_2^{s_1s_2s_1} = F_2^{s_2s_1} \cup (F_1^{s_2s_1} \setminus F_1^{s_1s_2}) \geq F_3 \cup (F_3^{s_2} \setminus F_3) = F_3^{s_1s_2}.$$

Now, using (21), notice that $(F_1 \setminus F_1^{s_1}) \cap F_3^{s_2} \subseteq Y$, and write $F_2^{s_1s_2s_1} = X \cup Y \cup (F_1^{s_1} \setminus F_1^{s_1s_2})$, with $X \subseteq F_2^{s_2}$ and $Y \subseteq F_1 \setminus F_1^{s_1}$. Then, recalling that $F_3^{s_1s_2} = F_3^{s_2}$, we get

$$\begin{aligned} F_2^{s_1s_2} \cap F_3^{s_1s_2} &= (F_2^{s_2} \cap F_3^{s_2}) \cup ((F_1 \setminus F_1^{s_1s_2}) \cap F_3^{s_2}) \\ &= (F_2^{s_2} \cap F_3) \cup \Big(((F_1 \setminus F_1^{s_1}) \cup (F_1^{s_1} \setminus F_1^{s_1s_2})) \cap F_3^{s_2}\Big) \\ &\subseteq X \cup \Big((F_1 \setminus F_1^{s_1}) \cap F_3^{s_2}\Big) \cup \Big((F_1^{s_1} \setminus F_1^{s_1s_2}) \cap F_3^{s_2}\Big) \\ &\subseteq X \cup Y \cup ((F_1^{s_1} \setminus F_1^{s_1s_2}) = F_2^{s_1s_2s_1}. \end{aligned}$$

Finally, recalling that $F_2 \cap (F_1 \setminus F_1^{s_1}) = \emptyset$, we have

$$\begin{aligned} F_3^{s_1s_2s_1} = F_3^{s_2s_1} &= F_3 \cup (F_2^{s_1} \setminus F_2^{s_2s_1}) \\ &= F_3 \cup \Big((F_2 \cup (F_1 \setminus F_1^{s_1})) \setminus F_2^{s_2s_1}\Big) \\ &= F_3 \cup (F_2 \setminus F_2^{s_2s_1}) \cup \Big((F_1 \setminus F_1^{s_1}) \setminus F_2^{s_2s_1}\Big), \end{aligned}$$

On the other hand, since $F_2^{s_2} \cap (F_1 \setminus F_1^{s_1s_2}) = \emptyset$ and $(F_1^{s_1} \setminus F_1^{s_1s_2}) \cap F_2^{s_2} = \emptyset$, $F_1^{s_2s_1} = F_1^{s_1}$, and $F_1^{s_1s_2} \subseteq F_1^{s_1}$, we get

$$\begin{aligned} F_3^{s_1s_2s_1} &= F_3^{s_1s_2} \cup (F_2^{s_1s_2} \setminus F_2^{s_1s_2s_1}) \\ &= F_3 \cup (F_2 \setminus F_2^{s_2}) \cup \Big((F_2^{s_2} \cup (F_1 \setminus F_1^{s_1s_2})) \setminus (F_2^{s_2s_1} \cup (F_1^{s_2s_1} \setminus F_1^{s_1s_2}))\Big) \\ &= F_3 \cup (F_2 \setminus F_2^{s_2}) \cup \Big(F_2^{s_2} \setminus (F_2^{s_2s_1} \cup (F_1^{s_2s_1} \setminus F_1^{s_1s_2}))\Big) \cup \\ &\cup \Big((F_1 \setminus F_1^{s_1s_2}) \setminus (F_2^{s_2s_1} \cup (F_1^{s_2s_1} \setminus F_1^{s_1s_2}))\Big) \\ &= F_3 \cup (F_2 \setminus F_2^{s_2}) \cup (F_2^{s_2} \setminus F_2^{s_2s_1}) \cup \Big((F_1 \setminus F_1^{s_1s_2}) \setminus (F_2^{s_2s_1} \cup (F_1^{s_1} \setminus F_1^{s_1s_2}))\Big) \\ &= F_3 \cup (F_2 \setminus F_2^{s_2s_1}) \cup \Big[\Big((F_1 \setminus F_1^{s_1s_2}) \setminus (F_1^{s_1} \setminus F_1^{s_1s_2})\Big) \setminus F_2^{s_2s_1}\Big] \\ &= F_3 \cup (F_2 \setminus F_2^{s_2s_1}) \cup ((F_1 \setminus F_1^{s_1}) \setminus F_2^{s_2s_1}). \end{aligned}$$

Henceforth, the map $\mathbf{s}$ is well defined.

The *if part* . Suppose that $\mathbb{F} = \{(F_1^\sigma, F_2^\sigma, F_3^\sigma) : \sigma \in < s_1, s_2 >\}$ satisfy $(I), (II)$, (III) and (IV) as above. To show that $\mathbb{F}$ is generated by (F_1, F_2, F_3) it remains to prove (II) of definition 4.2.

Since $\{(F_2^{\sigma s_1}, F_3^{\sigma s_1}) : \sigma \in < s_2 >\}$ is generated by $(F_2^{s_1}, F_3^{s_1})$, we have $F_2^{s_2s_1} \subseteq F_2^{s_1} = F_2 \cup (F_1 \setminus F_1^{s_1})$ and, therefore,

$$F_2^{s_2s_1} = X \cup Y, \text{ where } X \subseteq F_2 \text{ and } Y \subseteq F_1 \setminus F_1^{s_1}.$$

¿From (III),(a) and (b) above, we get $F_1^{s_1s_2} = F_1^{s_1s_2s_1} \subseteq F_1^{s_1}$.

We may write

$$F_2^{s_1s_2s_1} = F_2^{s_2s_1} \cup (F_1^{s_1} \setminus F_1^{s_1s_2}) = X \cup Y \cup (F_1^{s_1} \setminus F_1^{s_1s_2}), \tag{25}$$

and

$$F_2^{s_1s_2} = F_2^{s_2} \cup (F_1 \setminus F_1^{s_1s_2}) = F_2^{s_2} \cup (F_1 \setminus F_1^{s_1}) \cup (F_1^{s_1} \setminus F_1^{s_1s_2}). \tag{26}$$

On the other hand, from (III), (b), we have the inclusion $F_2^{s_1s_2s_1} \subseteq F_2^{s_1s_2}$. As $X \subseteq F_2^{s_2s_1}$ and $F_2^{s_2s_1}$, we get from (25), (26), $X \subseteq F_2^{s_2}$. Let $Z := F_2^{s_2} \setminus X$. Since $F_2^{s_2} \geq F_2^{s_2s_1}$, we must have $Z \geq Y$.

Finally, from proposition 2.3, using $F_2^{s_1s_2} \geq F_2^{s_1}$, (26), and $F_2^{s_1} = F_2 \cup (F_1 \setminus F_1^{s_1})$, we get $F_1^{s_1} \setminus F_1^{s_1s_2} \geq F_2 \setminus F_2^{s_2}$.

Therefore, the set $\mathbb{F}$ is generated by the sequence (F_1, F_2, F_3). □

Remark 2 *If in definition 4.2 we replace condition $(II)(a)$ by*

$$F_2^{s_2s_1} = X \cup Y \text{ with } X \subseteq F_2^{s_2} \text{ and } Y \subseteq F_1 \setminus F_1^{s_1} \text{ only if } F_1^{s_1s_2} \geq X \cup Y$$

then proposition 4.1 follows except condition (a), as well as theorem 4.2 except condition (IV).

Corollary 4.3 *Let $F_1 \geq F_2 \geq F_3$ and $\mathbb{F} = \{(F_1^\sigma, F_2^\sigma, F_3^\sigma) : \sigma \in < s_1, s_2 >\}$ generated by (F_1, F_2, F_3). Let $\psi_{s_i} : \mathbb{F} \to \mathbb{F}$ defined by $\psi_{s_i}(F_1^\sigma, F_2^\sigma, F_3^\sigma) = (F_1^{s_i\sigma}, F_2^{s_i\sigma}, F_3^{s_i\sigma})$, $i = 0, 1, 2$, $\sigma \in < s_1, s_2 >$. Then, $\psi_{s_i}^2 = id$, $i = 1, 2$, and $\psi_{s_1}\psi_{s_2}\psi_{s_1} = \psi_{s_2}\psi_{s_1}\psi_{s_2}$. That is, the symmetric group $\mathcal{S}_3$ acts on the set $\mathbb{F}$.*

The generation of the set $\mathbb{F}$ by $F_1 \geq F_2 \geq F_3$ is equivalent to a decomposition of $F_1 \geq F_2 \geq F_3$.

Theorem 4.4 *Let $F_1 \geq F_2 \geq F_3$. The following assertions are equivalent:*

(a) $\mathbb{F} = \{(F_1^\sigma, F_2^\sigma, F_3^\sigma) : \sigma \in < s_1, s_2 >\}$ is generated by (F_1, F_2, F_3).

(b) The sequence $F_1 \geq F_2 \geq F_3$ has a decomposition $F_1 = \cup_{j=1}^5 A_1^j$, $F_2 = \cup_{j=3}^5 A_2^j$, $F_3 = A_3^5 \cup A_3^2$,

$$F_1, F_2, F_3 = \begin{array}{ccc} A_1^1 & & \\ A_1^2 & & A_3^2 \\ A_1^3 & A_2^3 & \\ A_1^4 & A_2^4 & \\ A_1^5 & A_2^5 & A_3^5 \end{array} \tag{27}$$

satisfying:

1. $A_1^4 \geq A_2^4 > A_1^2 \geq A_3^2$, *with* $|A_1^4| = |A_2^4| = |A_1^2| = |A_3^2|$,
   $A_1^5 \geq A_2^5 \geq A_3^5$, *with* $|A_1^5| = |A_2^5| = |A_3^5|$,
   $A_1^3 \geq A_2^3$, *with* $|A_1^3| = |A_2^3|$.
2. $A_1^i \cap A_1^j = \emptyset$, *if* $i \neq j$,
   $A_2^i \cap A_2^j = \emptyset$, *if* $i \neq j$,
   $A_3^2 \cap A_3^5 = \emptyset$.
3. $F_1 \cap A_2^5 \subseteq A_1^5$,
   $(F_1 \setminus A_1^5) \cap A_2^4 \subseteq A_1^4$,
   $[F_1 \setminus (A_1^5 \cup A_1^4)] \cap A_2^3 \subseteq A_1^3$,
   $[F_2 \cup (A_1^2 \cup A_1^1)] \cap A_3^2 \subseteq A_1^2$, *and*
   $[F_2 \cup (A_1^2 \cup A_1^1)] \cap A_3^5 \subseteq A_2^5$.

such that the sets $F_1^\sigma, F_2^\sigma, F_3^\sigma$, *with* $\sigma \in \{s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1\}$, *are obtained from* F_1, F_2, F_3 *as follows:*

$$F_1^{s_1}, F_2^{s_1}, F_3^{s_1} = \begin{array}{ccc} & A_1^1 & \\ & A_1^2 & A_3^2 \\ A_1^3 & A_2^3 & \\ A_1^4 & A_2^4 & \\ A_1^5 & A_2^5 & A_3^5 \end{array}, \quad F_1^{s_2s_1}, F_2^{s_2s_1}, F_3^{s_2s_1} = \begin{array}{ccc} & & A_1^1 \\ & A_1^2 & A_3^2 \\ A_1^3 & & A_2^3 \\ A_1^4 & & A_2^4 \\ A_1^5 & A_2^5 & A_3^5 \end{array}, \tag{28}$$

$$F_1^{s_1s_2s_1}, F_2^{s_1s_2s_1}, F_3^{s_1s_2s_1} = \begin{array}{ccc} & & A_1^1 \\ & A_1^2 & A_3^2 \\ & A_1^3 & A_2^3 \\ A_1^4 & & A_2^4 \\ A_1^5 & A_2^5 & A_3^5 \end{array}, \tag{29}$$

$$F_1^{s_2}, F_2^{s_2}, F_3^{s_2} = \begin{array}{ccc} A_1^1 & & \\ A_1^2 & & A_3^2 \\ A_1^3 & & A_2^3 \\ A_1^4 & A_2^4 & \\ A_1^5 & A_2^5 & A_3^5 \end{array}, \quad F_1^{s_1s_2}, F_2^{s_1s_2}, F_3^{s_1s_2} = \begin{array}{ccc} & A_1^1 & \\ & A_1^2 & A_3^2 \\ & A_1^3 & A_2^3 \\ A_1^4 & A_2^4 & \\ A_1^5 & A_2^5 & A_3^5 \end{array}. \tag{30}$$

Proof: $(a) \Rightarrow (b)$ Let $\mathbb{F} = \{(F_1^\sigma, F_2^\sigma, F_3^\sigma) : \sigma \in < s_1, s_2 >$ be generated by the sequence (F_1, F_2, F_3), and consider the involutions ψ_{s_1}, ψ_{s_2} defined in corollary 4.3.

Since $\psi_{s_1}(F_1, F_2, F_3) = (F_1^{s_1}, F_2^{s_1}, F_3^{s_1})$ and $\psi_{s_2}(F_1, F_2, F_3) = (F_1^{s_2}, F_2^{s_2}, F_3^{s_2})$ we find that $F_1^{s_1} \subseteq F_1$, $F_1 \cap F_2 \subseteq F_1^{s_1}$, $F_2^{s_1} = F_2 \cup (F_1 \setminus F_1^{s_1})$, and $F_2^{s_2} \subseteq F_2$, $F_2 \cap F_3 \subseteq F_2^{s_2}$, $F_3^{s_2} = F_3 \cup (F_2 \setminus F_2^{s_2})$.

Consider now the sequence $(F_1^{s_2 s_1}, F_2^{s_2 s_1}, F_3^{s_2 s_1}) = \psi_{s_2}(F_1^{s_1}, F_2^{s_1}, F_3^{s_1})$. We have $F_2^{s_2 s_1} \subseteq F_2^{s_1} = F_2 \cup (F_1 \setminus F_1^{s_1})$. Note that F_2 and $F_1 \setminus F_1^{s_1}$ are disjoint, since $F_1 \cap F_2 \subseteq F_1^{s_1}$. So we may write

$$F_2^{s_2 s_1} = A_2^5 \cup A_1^2, \tag{31}$$

where $A_2^5 \subseteq F_2$ and $A_1^2 \subseteq F_1 \setminus F_1^{s_1}$. Let $A_1^1 := F_1 \setminus (F_1^{s_1} \cup A_1^2)$.

¿From (31), since $F_2^{s_2 s_1} \geq F_3$ and $|F_2^{s_2 s_1}| = |F_3|$, by we may factorize F_3 as

$$F_3 = A_3^5 \cup A_3^2,$$

where $A_2^5 \geq A_3^5$, $A_1^2 \geq A_3^2$, $|A_2^5| = |A_3^5|$, $|A_1^2| = |A_3^2|$, and $A_2^5 \cap A_2^3 = A_1^2 \cap A_3^5 = \emptyset$. Recalling that $F_2^{s_1} \cap F_3^{s_1} \subseteq F_2^{s_2 s_1} = A_3^5 \cup A_1^2$, we find that $F_2^{s_1} \cap A_3^5 \subseteq A_2^5$ and $F_2^{s_1} \cap A_3^2 \subseteq A_1^2$.

Now, $(F_1^{s_1 s_2 s_1}, F_2^{s_1 s_2 s_1}, F_3^{s_1 s_2 s_1}) = \psi_{s_1}(F_1^{s_2 s_1}, F_2^{s_2 s_1}, F_3^{s_2 s_1})$. Since $F_1^{s_1 s_2 s_1} \geq F_2^{s_2 s_1} = A_2^5 \cup A_1^2$, we may define $A_1^5 := min_{A_2^5} F_1^{s_1 s_2 s_1}$ and $A_1^4 := F_1^{s_1 s_2 s_1} \setminus A_1^5$. Since $F_1^{s_1 s_2 s_1} \subseteq F_1^{s_1}$, define $A_1^3 := F_1^{s_1} \setminus F_1^{s_1 s_2 s_1}$. Then we obtain

$$F_1^{s_1 s_2 s_1} = A_1^5 \cup A_1^4, \quad \text{and } F_1^{s_1} = A_1^5 \cup A_1^4 \cup A_1^3, \tag{32}$$

where $A_1^4 \geq A_1^2$, $A_1^5 \geq A_2^5$, $|A_1^4| = |A_1^2|$, $|A_1^5| = |A_2^5|$, and $F_1^{s_1 s_2 s_1} \cap A_2^5 \subseteq A_1^5$. Note that $F_1 \cap A_2^5 \subseteq A_1^5$, since $F_1 \cap F_2 \subseteq F_1^{s_1}$ and $F_1^{s_1} \cap F_2^{s_2 s_1} \subseteq F_1^{s_1 s_2 s_1}$. Since $F_2^{s_1 s_2 s_1} = F_2^{s_2 s_1} \cup (F_1^{s_1} \setminus F_1^{s_1 s_2 s_1})$, we have

$$F_2^{s_1 s_2 s_1} = A_2^5 \cup A_1^2 \cup A_1^3. \tag{33}$$

Consider the sequence $(F_1^{s_1 s_2}, F_2^{s_1 s_2}, F_3^{s_1 s_2}) = \psi_{s_1}(F_1^{s_2}, F_2^{s_2}, F_3^{s_2})$. We have $F_1^{s_1 s_2} \subseteq F_1$, $F_1^{s_1 s_2} \geq F_2^{s_2}$, and $F_2^{s_1 s_2} = F_2^{s_2} \cup (F_1 \setminus F_1^{s_1 s_2})$. Finally, consider

$$\psi_{s_2}(F_1^{s_1 s_2}, F_2^{s_1 s_2}, F_3^{s_1 s_2}) = (F_1^{s_1 s_2 s_1}, F_2^{s_1 s_2 s_1}, F_3^{s_1 s_2 s_1})$$

Thus, we have

$$F_1^{s_1 s_2} = F_1^{s_1 s_2 s_1} = A_1^5 \cup A_1^4.$$

¿From (33) and the inclusion $A_2^5 \cup A_1^2 \cup A_1^3 = F_2^{s_2 s_1 s_2} \subseteq F_2^{s_1 s_2} = F_2^{s_2} \cup A_1^3 \cup A_1^2 \cup A_1^1$, it follows that $A_2^5 \subseteq F_2^{s_2} \cup A_1^1$. But since the sets A_2^5 and A_1^1 are disjoints, we have $A_2^5 \subseteq F_2^{s_2}$. Let $A_2^4 := F_2^{s_2} \setminus A_2^5$ and $A_2^3 := F_2 \setminus F_2^{s_2}$. Thus,

$$F_2^{s_2} = A_2^5 \cup A_2^4.$$

Since $|F_2^{s_2}| = |F_1^{s_1 s_2}|$ we have $|A_1^4| = |A_2^4|$ and $|A_1^3| = |A_2^3|$. Moreover, from the inequality $F_1^{s_1 s_2} \geq F_2^{s_2}$ and the definition of A_1^5, we obtain $A_1^4 \geq A_2^4$.

¿From the inequalities $F_2^{s_2} \geq F_2^{s_2 s_1}$ and $F_2^{s_1 s_2} \geq F_2^{s_1}$, we get $A_2^4 \geq A_1^2$ and $A_1^3 \geq A_2^3$.

$(b) \Rightarrow (a)$ Let $\mathbb{F} := \{(F_1^\sigma, F_2^\sigma, F_3^\sigma) : \sigma \in < s_1, s_2 >\}$ be the set defined in (27), (28), (29) and (30). Clearly, $\mathbb{F}$ is generated by (F_1, F_2, F_3). □

We show now that the transformations Θ_k^*, $k=1,2$, may be used to obtain a decomposition of a sequence $F_1 \geq F_2 \geq F_3$ according to the rules of the previous theorem.

Proposition 4.5 *Given $F_1 \geq F_2 \geq F_3$, there exists a set $\mathbb{F}^*$ generated by (F_1, F_2, F_3) such that, for all $\sigma \in \mathcal{S}_3$, and $j=1,2$, $\Theta_j^*(F_j^\sigma, F_{j+1}^\sigma) = (F_j^{s_j\sigma}, F_{j+1}^{s_j\sigma})$ and $F_p^{s_j\sigma} = F_p^\sigma$, $p \neq j, j+1$. $\mathbb{F}^*$ is uniquely determined by $F_1 \geq F_2 \geq F_3$.*

We say that $\mathbb{F}^*$ is the set $*$-generated by (F_1, F_2, F_3).

Proof: We have to show that conditions (I), (II), (III) and (IV) of theorem 4.2 are feasible with $\mathbf{s}_*$. Let $(F_1^{s_1}, F_2^{s_1}, F_3^{s_1})$, where $F_3^{s_1} = F_3$ and $\mathbf{s}_*(F_1, F_2) = (F_1^{s_1}, F_2^{s_1})$, and $(F_1^{s_2}, F_2^{s_2}, F_3^{s_2})$, where $F_1^{s_2} = F_1$ and $\mathbf{s}_*(F_1, F_2) = (F_1^{s_2}, F_2^{s_2})$.

Define now $(F_1^{s_2s_1}, F_2^{s_2s_1}, F_3^{s_2s_1})$, where $F_1^{s_2s_1} = F_1^{s_1}$ and $(F_2^{s_2s_1}, F_3^{s_2s_1}) = \mathbf{s}_*(F_2^{s_1}, F_3^{s_1})$. Since $F_2^{s_1} = F_2 \cup (F_1 \setminus F_1^{s_1})$ and $F_2^{s_2} = min_{F_3} F_2$, by theorem 2.8, we have

$$F_2^{s_2s_1} = min_{F_3}[F_2 \cup (F_1 \setminus F_1^{s_1})] = min_{F_3}[F_2^{s_2} \cup (F_1 \setminus F_1^{s_1})] = X \cup Y, \tag{34}$$

and $F_2^{s_2} = X \cup Z$, where $X \subseteq F_2^{s_2}$, $Y \subseteq F_1 \setminus F_1^{s_1}$ and $Z \geq Y$.

Recalling that $F_1^{s_2} = F_1 \geq F_2^{s_2}$, define $(F_1^{s_1s_2}, F_2^{s_1s_2}, F_3^{s_1s_2})$, such that $F_3^{s_1s_2} = F_3^{s_2}$ and $(F_1^{s_1s_2}, F_2^{s_1s_2}) = \mathbf{s}_*(F_1^{s_2}, F_2^{s_2})$. By corollary 2.12, we have

$$F_1^{s_1} = min_{F_2^{s_2}} F_1 \cup min_{(F_2 \setminus F_2^{s_2})}(F_1 \setminus F_1^{s_1s_2}), \tag{35}$$

Thus, since $F_1^{s_1s_2} = min_{F_2^{s_2}} F_1$, we must have $F_1^{s_1s_2} \subseteq F_1^{s_1}$. By corollary 2.6, $F_1^{s_1s_2} = min_{F_2^{s_2}} F_1^{s_1}$, and from (35), $F_1^{s_1} \setminus F_1^{s_1s_2} = min_{(F_2 \setminus F_2^{s_2})}(F_1 \setminus F_1^{s_1s_2})$. Hence $F_1^{s_1} \setminus F_1^{s_1s_2} \geq F_2 \setminus F_2^{s_2}$.

Finally, define $(F_1^{s_1s_2s_1}, F_2^{s_1s_2s_1}, F_3^{s_1s_2s_1})$, where $F_3^{s_1s_2s_1} = F_3^{s_2s_1}$ and $(F_1^{s_1s_2s_1}, F_2^{s_1s_2s_1}) = s_*(F_1^{s_2s_1}, F_2^{s_2s_1})$. Notice that $F_1^{s_1s_2} = min_{F_2^{s_2}} F_1 = min_{X \cup Z} F_1^{s_1}$, and $F_1^{s_1s_2s_1} = min_{F_2^{s_2s_1}} F_1^{s_1}$ $= min_{X \cup Y} F_1^{s_1}$, where $Z \subseteq F_2^{s_2}$, $Y \subseteq F_1 \setminus F_1^{s_1}$ and $Z \geq Y$.

Using theorem 2.10, we have $F_1^{s_1s_2s_1} = min_{F_2^{s_2s_1}} F_1^{s_1} = F_1^{s_1s_2}$.

Note that $\mathbf{s}_*(F_2^{s_1s_2}, F_3^{s_1s_2}) = (F_2^{s_2s_1s_2}, F_3^{s_2s_1s_2})$. In fact, $min_{F_3^{s_1s_2}} F_2^{s_1s_2} = min_{F_3^{s_2}} F_2^{s_1s_2}$ and

$$min_{F_3^{s_2}} F_2^{s_1s_2} = min_{[F_3 \cup (F_2 \setminus F_2^{s_2})]}[F_2^{s_2} \cup (F_1^{s_1} \setminus F_1^{s_1s_2}) \cup (F_1 \setminus F_1^{s_1})], \tag{36}$$

where $min_{(F_2 \setminus F_2^{s_2})}[(F_1^{s_1} \setminus F_1^{s_1s_2}) \cup (F_1 \setminus F_1^{s_1})] = F_1^{s_1} \setminus F_1^{s_1s_2}$.

Then, by theorem 2.14,

$$(36) = (F_1^{s_1} \setminus F_1^{s_1s_2}) \cup min_{F_3}[F_2^{s_2} \cup (F_1 \setminus F_1^{s_1})] = F_2^{s_1s_2s_1}.$$

By theorem 4.2 we conclude the proof.

Therefore, the set $\mathbb{F}^* := \{(F_1^\sigma, F_2^\sigma, F_3^\sigma) : \sigma \in < s_1, s_2 >\}$, is generated by (F_1, F_2, F_3), and satisfy, for all $\sigma \in \mathcal{S}_3$ and $j=1,2$, $\Theta_j^*(F_j^\sigma, F_{j+1}^\sigma) = (F_j^{s_j\sigma}, F_{j+1}^{s_j\sigma})$ and $F_p^{s_j\sigma} = F_p^\sigma$, $p \neq j, j+1$. $\square$

Corollary 4.6 *Let $\mathbb{F}^*$ be the set $*$-generated by $(F_1,\ F_2,\ F_3)$. Then, for $j = 1,\ 2$, $\Theta_j^*(F_i^\sigma)_{i=1}^3 = (F_i^{s_j\sigma})_{i=1}^3$, $\sigma \in < s_1, s_2 >$, satisfy:*

(a) $(\Theta_j^*)^2 = id$, $j = 1, 2$.

(b) $\Theta_1^*\Theta_2^*\Theta_1^* = \Theta_2^*\Theta_1^*\Theta_2^*$.

That is, $\mathcal{S}_3$ acts on $\mathbb{F}^*$.

Proof: Use previous proposition. □

Let us consider the canonical words of $\mathcal{S}_{k+1}$, $k \geq 1$, (20), written as : $\theta_r\omega$, with $\theta_r := s_r\theta_{r+1}$, $\theta_{k+1} := id$, and $\omega \in \mathcal{S}_k$, for $r = 1, \dots, k$.

Definition 4.3 *Let $k \geq 1$ and $F_1 \geq \dots \geq F_k \geq F_{k+1}$. We define, recursively, the set $\mathbb{F}^* = \{(F_1^\sigma, .., F_{k+1}^\sigma) : \sigma \in< s_1, .., s_k >\}$ $*$-generated by $(F_1, .., F_k, F_{k+1})$ as follows:*

- *If $k = 1$, $\mathbb{F}^* := \{(F_1, F_2), \mathbf{s}_*(F_1, F_2)\}$.*
- *If $k > 1$, $\mathbb{F}^* := \{(F_i^\omega)_{i=1}^{k+1} : \omega \in \mathcal{S}_k\} \cup \{(F_i^{\theta_r\omega})_{i=1}^{k+1} : 1 \leq r \leq k,\ \omega \in \mathcal{S}_k\}$, where*

  (i) $\{(F_i^\omega)_{i=1}^k : \omega \in \mathcal{S}_k\}$ is $*$-generated by $(F_1, \dots, F_k)$, and $F_{k+1}^\omega = F_{k+1}$, for all $\omega \in \mathcal{S}_k$;

  (ii) for each $r = 1, ..., k$, and $\omega \in \mathcal{S}_k$,

$$\mathbf{s}_*(F_r^{\theta_{r+1}\omega}, F_{r+1}^{\theta_{r+1}\omega}) = (F_r^{s_r\theta_{r+1}\omega}, F_{r+1}^{s_r\theta_{r+1}\omega}), \quad \text{and} \quad F_p^{s_r\theta_{r+1}\omega} = F_p^{\theta_{r+1}\omega}, \quad p \neq r, r+1. \tag{37}$$

Proposition 4.7 *Let $k \geq 1$ and $F_1 \geq ... \geq F_k \geq F_{k+1}$. The following statements hold*

(I) There exists always $\mathbb{F}^*$ $*$-generated by $(F_i)_{i=1}^{k+1}$.

(II) If $\mathbb{F}^*$ is $*$-generated by $(F_i)_{i=1}^{k+1}$, then, for each $r \in \{1, \dots, k+1\}$, and $\omega \in \mathcal{S}_k$, there exist $G_i^\omega \subseteq F_i^\omega$, with $|G_i^\omega| = |F_{k+1}|$, for $1 \leq i < r$, such that

$$G_1^\omega \geq ... \geq G_{r-1}^\omega \geq F_r^{\theta_r\omega} \geq \dots \geq F_k^{\theta_k\omega} \geq F_{k+1},$$

where $F_i^{\theta_i\omega} \subseteq F_i^\omega$, $|F_i^{\theta_i\omega}| = |F_{k+1}|$, for $i = r, \dots, k$, and $F_{k+1} \subseteq F_{k+1}^{\theta_k\omega}$.

Proof: By double induction on k and r. When $k = 1$, this is definition 4.1 with s_*.When $k = 2$, this is definition 4.2 with s_*, and this set was constructed in proposition 4.5. We have to show that (37) is feasible, for $k \geq 3$. By induction, there exists a set $\widetilde{\mathbb{F}} = \{(F_i^\omega)_{i=1}^k : \omega \in \mathcal{S}_k\}$ $*$-generated by $(F_i)_{i=1}^k$, satisfying (II) above. Fix ω arbitrarily in $\mathcal{S}_k$. Then, there exist, for $i = 1, ..., k$, $G_i^\omega \subseteq F_i^\omega$, with $G_k^\omega = F_k$ and $|G_i^\omega| = |F_k|$, such that

$$G_1^\omega \geq ... \geq G_k^\omega = F_k \geq F_{k+1}.$$

Since $G_k^\omega = F_k \subseteq F_k^\omega$, it follows that $F_k^\omega \geq F_{k+1}$. So, putting $F_{k+1}^\omega = F_{k+1}$, we may define $\mathbf{s}_*(F_k^\omega, F_{k+1}^\omega) = (F_k^{s_k\omega}, F_{k+1}^{s_k\omega})$ and $F_p^{s_k\omega} = F_p^\omega$, $p \neq k, k+1$. We have $F_k^{s_k\omega} = min_{F_{k+1}} F_k^\omega$ and $F_{k+1}^{s_k\omega} = F_{k+1} \cup (F_k^\omega \setminus F_k^{s_k\omega})$. Thus, $G_k^\omega \geq F_k^{s_k\omega}$ and $G_1^\omega \geq ... \geq G_{k-1}^\omega \geq F_k^{s_k\omega} \geq F_{k+1}$, with $F_{k+1} \subseteq F_{k+1}^{s_k\omega}$, $F_k^{s_k\omega} \subseteq F_k^\omega$, and $|F_k^{s_k\omega}| = |F_{k+1}|$. Clearly we may consider $|G_i^\omega| = |F_{k+1}|$, for $i = 1, \ldots, k-1$.

Suppose we have already defined $(F_i^{s_{r+1}\ldots s_k\omega})_{i=1}^{k+1}$, for $1 < r+1 \leq k$, such that

$$G_1^\omega \geq ... \geq G_r^\omega \geq F_{r+1}^{s_{r+1}\ldots s_k\omega} \geq ... \geq F_k^{s_k\omega} \geq F_{k+1},$$

with $F_i^{s_i\ldots s_k\omega} \subseteq F_i^{s_{i+1}\ldots s_k\omega}$, $|F_i^{s_i\ldots s_k\omega}| = |F_{k+1}|$, $i = r+1, ..., k$, $G_i^\omega \subseteq F_i^\omega = F_i^{s_{r+1}\ldots s_k\omega}$, $|G_i^\omega| = |F_{k+1}|$, $i = 1, \ldots, r$.

So we must have $F_r^\omega = F_r^{s_{r+1}\ldots s_k\omega} \geq F_{r+1}^{s_{r+1}\ldots s_k\omega}$, and we may define

$$\mathbf{s}_*(F_r^{s_{r+1}\ldots s_k\omega}, F_{r+1}^{s_{r+1}\ldots s_k\omega}) = (F_r^{s_r s_{r+1}\ldots s_k\omega}, F_{r+1}^{s_r s_{r+1}\ldots s_k\omega})$$

and $F_p^{s_r s_{r+1}\ldots s_k\omega} = F_p^{s_{r+1}\ldots s_k\omega}$, $p \neq r, r+1$.

As $F_r^{s_r\ldots s_k\omega} = min_{F_{r+1}^{s_{r+1}\ldots s_k\omega}} F_r^{s_{r+1}\ldots s_k\omega}$ and $F_{r+1}^{s_r\ldots s_k\omega} = F_{r+1}^{s_{r+1}\ldots s_k\omega} \cup (F_r^{s_{r+1}\ldots s_k\omega} \setminus F_r^{s_r\ldots s_k\omega})$, then, $G_{r-1}^\omega \geq G_r^\omega \geq F_r^{s_r\ldots s_k\omega}$, and

$$G_1^\omega \geq ... \geq G_{r-1}^\omega \geq F_r^{s_r\ldots s_k\omega} \geq ... \geq F_k^{s_k\omega} \geq F_{k+1},$$

with $F_r^{s_r\ldots s_k\omega} \subseteq F_r^{s_{r+1}\ldots s_k\omega} = F_r^\omega$ and $|F_{r-1}^{s_{r-1}\ldots s_k\omega}| = |F_{k+1}|$. By induction, the result follows. □

Next we show that the operators Θ_k^*, $k = 1, \ldots, t-1$, may be used to decompose a sequence $F_1 \geq \ldots \geq F_t$, $t \geq 3$, such that we have an action of $\mathcal{S}_t$.

Theorem 4.8 *Let* $F_1 \geq ... \geq F_k \geq F_{k+1}$, $k \geq 2$, *and* $\mathbb{F}^* = \{(F_i^\sigma)_{i=1}^{k+1} : \sigma \in \mathcal{S}_{k+1}\}$ *$*$-generated by* $(F_i)_{i=1}^k$. *For each* $t, t+1, t+2 \in \{1, \ldots, k+1\}$ *and* $\sigma \in \mathcal{S}_{k+1}$, *let* $G = \{(F_t^{\alpha\sigma}, F_{t+1}^{\alpha\sigma}, F_{t+2}^{\alpha\sigma}) : \alpha \in < s_t, s_{t+1} >\} \subseteq \mathbb{F}^*$. *Then,*

$$\Theta_j^*(F_j^{\alpha\sigma}, F_{j+1}^{\alpha\sigma}) = (F_j^{s_j\alpha\sigma}, F_{j+1}^{s_j\alpha\sigma}), \ \ \textit{for all} \ \ \alpha \in < s_t, s_{t+1} >, \ \ j = t, t+1,$$

and $F_p^{s_j\alpha\sigma} = F_p^{\alpha\sigma}$, *for* $p \neq j, j+1$.

Proof: By induction on k. When $k = 2$, we have $\{\alpha\sigma : \alpha \in < s_1, s_2 >\} =< s_1, s_2 >$, for all $\sigma \in \mathcal{S}_2$, and this is proposition 4.5.

Let $k \geq 3$ and $\widetilde{\mathbb{F}} = \{(F_i^\omega)_{i=1}^k : \omega \in \mathcal{S}_k\}$ $*$-generated by $(F_1, \ldots, F_k)$. By induction hypothesis, for each $\omega \in \mathcal{S}_k$, and $i, i+1 \in \{1, \ldots, k-1\}$,

$$\Theta_i^*(F_i^\omega, F_{i+1}^\omega) = (F_i^{s_i\omega}, F_{i+1}^{s_i\omega}), \tag{38}$$

and $F_p^{s_i\omega} = F_p^\omega$, $p \neq i, i+1$.

Fix $\omega \in \mathcal{S}_k$ arbitrarily. If $t+1=k$, since $F_k^\omega \geq F_{k+1}$ and

$$\Theta^*_{k-1}(F^\omega_{k-1}, F^\omega_k) = (F^{s_{k-1}\omega}_{k-1}, F^{s_{k-1}\omega}_k),$$

it follows , by (38),

$$F^\omega_{k-1} \geq F^\omega_k \geq F_{k+1}, \quad \text{or} \tag{39}$$
$$F^{s_{k-1}\omega}_{k-1} \geq F^{s_{k-1}\omega}_k \geq F_{k+1}. \tag{40}$$

Notice that $\{\alpha s_{k-1} : \alpha \in < s_{k-1}, s_k >\} =< s_{k-1}, s_k >$. Then, $G = \{(F^{\alpha\omega}_{k-1}, F^{\alpha\omega}_k, F^{\alpha\omega}_{k+1}) : \theta \in < s_{k-1}, s_k >\}$ is $*$-generated by (39) or by (40). Again, by proposition 4.5 the claim is true.

Now, let $\sigma = \theta_r\omega$, where $\theta_r = s_r\theta_{r+1}$, $1 \leq r \leq k$, with $\theta_{k+1} = id$, and suppose the claim is true for all θ_i, with $r < i \leq k$.

If $t+2 < r$, by induction hypothesis, the claim is true.

If $t > r$, then $s_t s_r \dots s_k\omega = s_r \dots s_k\omega'$, with $\omega' = s_{t-1}\omega \in \mathcal{S}_k$. So it is enough to analyze the cases $t = r$, $t+1 = r$ and $t+2 = r$.

Case $t+2 = r$. We have $F^\sigma_t = F^\omega_t$, $F^\sigma_{t+1} = F^\omega_{t+1}$. Then by previous theorem, (II), $F^\omega_{r-1} \geq F^\sigma_r$, and, by induction, $\Theta^*_j(F^\sigma_j, F^\sigma_{j+1}) = \Theta^*_t(F^\omega_t, F^\omega_{t+1}) = (F^{s_t\omega}_t, F^{s_t\omega}_{t+1})$. So,

$$F^{s_{r-2}\omega}_{r-2} \geq F^{s_{r-2}\omega}_{r-1} \geq F^\sigma_r \quad \text{or} \tag{41}$$
$$F^\omega_{r-2} \geq F^\omega_{r-1} \geq F^\sigma_r. \tag{42}$$

Then $G = \{(F^{\alpha\sigma}_{r-2}, F^{\alpha\sigma}_{r-1}, F^{\alpha\sigma}_r) : \alpha \in < s_{r-2}, s_{r-1} >\}$ is $*$-generated by (41) or by (42). Again proposition 4.5 shows the claim.

Case $t+1 = r$. We have by previous theorem, (II), $F^\sigma_{r-1} = F^\omega_{r-1} \geq F^\sigma_r \geq_{op} F^\sigma_{r+1}$ and $\Theta^*_r(F^\sigma_r, F^\sigma_{r+1}) = (F^{s_r\sigma}_r, F^{s_r\sigma}_{r+1})$. So, $F^{s_r\sigma}_r = F^\omega_r$ and $F^{s_r\sigma}_{r+1} = F^{s_{r+1}\dots s_k\sigma}_{r+1}$. By induction, $\Theta^*_{r-1}(F^\omega_{r-1}, F^\omega_r) = (F^{s_{r-1}\omega}_{r-1}, F^{s_{r-1}\omega}_r)$. Hence,

$$F^\omega_{r-1} \geq F^\omega_r \geq F^{s_{r+1}\dots s_k\omega}_{r+1}, \text{ or} \tag{43}$$
$$F^{s_{r-1}\omega}_{r-1} \geq F^{s_{r-1}\omega}_r \geq F^{s_{r+1}\dots s_k\omega}_{r+1}. \tag{44}$$

Then G is $*$-generated by (43) or by (44). Again apply proposition 4.5

Case $t = r$. By previous theorem, $F^\sigma_r \geq_{op} F^\sigma_{r+1}$ and

$$\Theta^*_r(F^\sigma_r, F^\sigma_{r+1}) = (F^{s_r\sigma}_r, F^{s_r\sigma}_{r+1}).$$

Then, using previous theorem, (II), $F^\omega_r \geq F^{s_{r+1}\dots s_k\omega}_{r+1} \geq_{op} F^\sigma_{r+2} = F^{s_{r+1}\dots s_k\omega}_{r+2}$, and

$$\Theta^*_{r+1}(F^{s_{r+1}\dots s_k\omega}_{r+1}, F^{s_{r+1}\dots s_k\omega}_{r+2}) = (F^{s_{r+2}\dots s_k\omega}_{r+1}, F^{s_{r+2}\dots s_k\omega}_{r+2}) = (F^\omega_{r+1}, F^{s_{r+2}\dots s_k\omega}_{r+2}).$$

By induction $\Theta^*_r(F^\omega_r, F^\omega_{r+1}) = (F^{s_r\omega}_r, F^{s_r\omega}_{r+1})$. Hence,

$$F^\omega_r \geq F^\omega_{r+1} \geq F^{s_{r+2}\dots s_k\omega}_{r+2} \quad \text{or} \tag{45}$$
$$F^{s_r\omega}_r \geq F^{s_r\omega}_{r+1} \geq F^{s_{r+2}\dots s_k\omega}_{r+2}. \tag{46}$$

Therefore, G is $*$-generated by (45) or by (46). Again proposition 4.5 shows the claim. □

Corollary 4.9 *Let $k \geq 2$, $F_1 \geq ... \geq F_k$, and $\mathbb{F}^* = \{(F_i^\sigma)_{i=1}^k : \sigma \in \mathcal{S}_k\}$ $*$-generated by $(F_i)_{i=1}^k$. Then, for $j = 1, ..., k-1$, $\Theta_j^*((F_i^\sigma)_{i=1}^k) = (F_i^{s_j\sigma})_{i=1}^k$, $\sigma \in \mathcal{S}_k$, satisfy:*

(a) $(\Theta_j^*)^2 = id$, $1 \leq j \leq k-1$.

(b) $\Theta_j^*\Theta_r^* = \Theta_r^*\Theta_j^*$, $|j-r| > 1$.

(c) $\Theta_j^*\Theta_{j+1}^*\Theta_j^* = \Theta_{j+1}^*\Theta_j^*\Theta_{j+1}^*$, $1 \leq j \leq k-2$.
That is, $\mathcal{S}_k$ acts on $\mathbb{F}^*$.

Proof: Use previous theorem. □

In view of the results above, we may give the following definitions.

Definition 4.4 *Given $J_1, ..., J_t \subseteq \{1, ..., n\}$ and $\sigma \in \mathcal{S}_t$, we say that $(J_1, ..., J_t)$ is a σ-LR sequence of sets if there exists $F_1 \geq ... \geq F_t$ such that $(J_1, ..., J_t) = (F_1^\sigma, ..., F_t^\sigma) \in \mathbb{F}^*$, with $\mathbb{F}^*$ $*$-generated by $(F_1, ..., F_t)$.*

Proposition 4.10 *Given $J_1, ..., J_t \subseteq \{1, ..., n\}$, the sequence $(J_1, ..., J_t)$ is a σ-LR sequence of sets if and only if $\Theta_{i_1}^*\Theta_{i_2}^*...\Theta_{i_r}^*(J_1, ..., J_t)$ is a LR sequence of sets, where $s_{i_1}s_{i_2}...s_{i_r}$ is a word of σ in the alphabet $\{s_1, ..., s_{t-1}\}$.*

Proof: Use previous corollary. □

Now we shall characterize σ-LR sequences of lenght 3, with $\sigma \in \mathcal{S}_3$, and we show that the elements generated in a set $\mathbb{F}$, by an LR sequence, may also be generated by the transformations Θ_k^*, $k = 1, 2$, by some other LR sequence.

Theorem 4.11 *Let $J_1, J_2, J_3 \subseteq \{1, ..., n\}$. Then,*

(a) (J_1, J_2, J_3) *is a s_1-LR sequence iff* $\exists A \subseteq J_2$ *with* $|A| = |J_1|$ *such that* $J_1 \geq A \geq J_3$.

(b) (J_1, J_2, J_3) *is a s_2-LR sequence iff*

(i) $\exists A_1^1 \subseteq J_1$, $A_3^1 \subseteq J_3$ *with* $|A_1^1| = |A_3^1| = |J_2|$ *such that* $A_1^1 \geq J_2 \geq A_3^1$,

(ii) $\exists A_1^2 \subseteq J_1 \setminus A_1^1$ *with* $|A_1^2| = |J_3 \setminus A_3^1|$ *such that* $A_1^2 \geq J_3 \setminus A_3^1$.

(c) (J_1, J_2, J_3) *is a s_1s_2-LR sequence iff*

(i) $\exists A_2^1 \subseteq J_2$, $A_3^1 \subseteq J_3$ *with* $|J_1| = |A_2^1| = |A_3^1|$ *such that* $J_1 \geq A_2^1 \geq A_3^1$,

(ii) $\exists A_2^2 \subseteq J_2 \setminus A_2^1$ *with* $|A_2^2| = |J_3 \setminus A_3^1|$ *such that* $A_2^2 \geq J_3 \setminus A_3^1$.

(d) (J_1, J_2, J_3) *is a s_2s_1-LR sequence iff*

(i) $\exists A_1^1 \subseteq J_1$, $A_3^1 \subseteq J_3$ *with* $|A_1^1| = |J_2| = |A_3^1|$ *such that* $A_1^1 \geq J_2 \geq A_3^1$,

(ii) $\exists A_3^2 \subseteq J_3 \setminus A_3^1$ *with* $|A_3^2| = |J_1 \setminus A_1^1|$ *such that* $J_1 \setminus A_1^1 \geq A_3^2$.

(e) (J_1, J_2, J_3) *is a* $s_1s_2s_1$*-LR sequence iff*

(i) $\exists A_2^1 \subseteq J_2$, $A_3^1 \subseteq J_3$ *with* $|J_1| = |A_2^1| = |A_3^1|$ *such that* $J_1 \geq A_2^1 \geq A_3^1$,

(ii) $\exists A_3^2 \subseteq J_3 \setminus A_3^1$ *with* $|A_3^2| = |J_2 \setminus A_2^1|$ *such that* $J_2 \setminus A_2^1 \geq A_3^2$.

Proof: The *only if* part. If (J_1, J_2, J_3) is a σ-LR sequence, then there exists $F_1 \geq F_2 \geq F_3$ such that $(J_1, J_2, J_3) = (F_1^\sigma, F_2^\sigma, F_3^\sigma) \in \mathbb{F}^*$, the set $*$-generated by (F_1, F_2, F_3). Using theorem 4.4, the conclusion follows easily.

The *if part.* (a) and (b) are obvious.

(c) It is enough to prove that $\Theta_1^*(J_1, J_2, J_3)$ is a s_2-LR sequence.

Define $\widetilde{A} := max\{X \subseteq J_2 : |X| = |J_1|, J_1 \geq X\}$ and let $F_2 := \widetilde{A}$ and $F_1 := J_1 \cup (J_2 \setminus F_2)$. Clearly, $\Theta_1^*(J_1, J_2, J_3) = (F_1, F_2, F_3)$, with $F_3 = J_3$.

Since $J_1 \geq \widetilde{A} \geq A_2^1$, we may write $\widetilde{A} = X_1 \cup Y_1 \cup Z$, $A_2^1 = X_1 \cup X_2 \cup X_3$, and $A_2^2 = Y_1 \cup Y_2$, union of pairwise disjoint subsets, with $Z \subseteq J_2 \setminus (A_2^1 \cup A_2^2)$, $|Y_1| = |X_2|$, and $|Z| = |X_3|$ such that $Y_1 \geq X_2$, $Z \geq X_3$. Let $A_3^2 := J_3 \setminus A_3^1$. Since $A_2^1 \geq A_3^1$ and $A_2^2 \geq A_3^2$, we may also write $A_3^1 = X_1' \cup X_2' \cup X_3'$ and $A_3^2 = Y_1' \cup Y_2'$, union of pairwise disjoint subsets, where $|X_i'| = |X_i|$, $X_i \geq X_i'$ $i = 1, 2, 3$, and $|Y_i'| = |Y_i|$, $Y_i \geq Y_i'$ $i = 1, 2$.

We have: (i) $J_1 \subseteq F_1$ and $\widetilde{A}_3 := X_1' \cup X_3' \cup Y_1' \subset F_3$ with $|J_1| = |F_2| = |\widetilde{A}_3|$ such that $J_1 \geq F_2 \geq \widetilde{A}_3$; and (ii) $X_2 \cup Y_2 \subseteq F_1 \setminus J_1$ with $|X_2 \cup Y_2| = |F_3 \setminus \widetilde{A}_3|$ such that $X_2 \cup Y_2 \geq F_3 \setminus \widetilde{A}_3$. Hence, by (b), $\Theta_1^*(J_1, J_2, J_3)$ is a s_2-LR sequence.

(d) It is enough to prove that $\Theta_2^*(J_1, J_2, J_3)$ is a s_1-LR sequence. Let $F_1 := J_1$, $F_3 := max\{X \subseteq J_3 : |X| = |J_2|, J_2 \geq X\}$ and $F_2 := J_2 \cup (J_3 \setminus F_3)$. Clearly, $\Theta_2^*(J_1, J_2, J_3) = (F_1, F_2, F_3)$. Since $A_1^1 \geq J_2 \geq A_3^1$ and $J_1 \setminus A_1^1 \geq A_3^2$, we may write $F_3 = X_1' \cup Y_1' \cup Z$, $J_2 = X_1 \cup X_2 \cup X_3$, $A_3^1 = X_1' \cup X_2' \cup X_3'$, $J_1 \setminus A_1^1 = Y_1 \cup Y_2$, and $A_3^2 = Y_1' \cup Y_2'$, union of pairwise disjoint subsets, with $|X_i| = |X_i'|$, $X_i \geq X_i'$, $i = 1, 2, 3$, $|Y_i| = |Y_i'|$, $Y_i \geq Y_i'$, $i = 1, 2$, and $|Z| = |X_3|$, such that $X_3 \geq Z \geq X_3'$ and $X_2 \geq Y_1' \geq X_2'$.

Therefore, $F_1 = A_1^1 \cup Y_1 \cup Y_2 \geq J_2 \cup X_2' \cup Y_2'$, with $J_2 \cup X_2' \cup Y_2' \subseteq F_2$ and $|J_2 \cup X_2' \cup Y_2'| = |F_1|$. Hence, by (a), $\Theta_2^*(J_1, J_2, J_3)$ is a s_1-LR sequence.

(e) It is enough to prove that $\Theta_1^*(J_1, J_2, J_3)$ is a s_2s_1-LR sequence. If $F_3 := J_3$, $F_2 := max\{X \subseteq J_2 : |X| = |J_1|, J_1 \geq X\}$ and $F_1 := J_1 \cup (J_2 \setminus F_2)$, then clearly, $\Theta_1^*(J_1, J_2, J_3) = (F_1, F_2, F_3)$. We may write $J_1 = X_1 \cup X_2$, $A_2^1 = Y_1 \cup Y_2$, $J_2 \setminus A_2^1 = Y_3 \cup Y_4$, $A_3^1 = Z_1 \cup Z_2$, $A_3^2 = Z_3 \cup Z_4$, and $F_2 = Y_1 \cup Y_3$, union of pairwise disjoint subsets, with $|X_i| = |Y_i| = |Z_i|$, $X_i \geq Y_i \geq Z_i$, $i = 1, 2$, and $|Y_i| = |Z_i|$, $Y_i \geq Z_i$, $i = 3, 4$, such that $|Y_3| = |Y_2|$ and $X_2 \geq Y_3 \geq Y_2$.

Therefore, (i) $X_1 \cup X_2 \subseteq F_1$ and $Z_2 \cup Z_3 \subseteq F_3$ with $|X_1 \cup X_3| = |F_2| = |Z_1 \cup Z_3|$, such that $X_1 \cup X_2 \geq F_2 \geq Z_1 \cup Z_3$; and (ii) $Z_2 \cup Z_4 \subseteq F_3 \setminus (Z_1 \cup Z_3)$, with $|Z_2 \cup Z_4| = |F_1 \setminus (X_1 \cup X_2)|$, such that $F_1 \setminus (X_1 \cup X_2) \geq Z_2 \cup Z_3$. Hence, by (d), $\Theta_1^*(J_1, J_2, J_3)$ is a s_2s_1-LR sequence. $\square$

Definition 4.5 *Given $\sigma \in \mathcal{S}_t$, we say that $\mathcal{T}$ is a σ-LR tableau of type $(a, \sigma m, c)$, with m a partition, if its indexing sets are a σ-LR sequence.*

Let a, m, c be partitions of length $\leq n$ with $a, m \subseteq c$. Given $\sigma \in \mathcal{S}_t$, we denote by $LR_\sigma(a, \sigma m, c)$ the set constituted by the indexing sets of all σ-LR tableau of type $(a, \sigma m, c)$. If $\{(A_i)_{i=1}^t, (B_i)_{i=1}^t, ..., (F_i)_{i=1}^t\}$ are the indexing sets of all LR tableaux of type (a, m, c), let $\mathbb{F}_A^*, \mathbb{F}_B^*, ..., \mathbb{F}_F^*$ be, respectively, the sets $*$-generated by those indexing sets. Then,

$$LR_\sigma(a, \sigma m, c) = \{(A_i^\sigma)_{i=1}^t, (B_i^\sigma)_{i=1}^t, ..., (F_i^\sigma)_{i=1}^t\}.$$

Hence,

$$\bigcup_{\sigma \in \mathcal{S}_t} LR_\sigma(a, \sigma m, c) = \mathbb{F}_A^* \cup \mathbb{F}_B^* \cup ... \cup \mathbb{F}_F^*.$$

If $J_1, ..., J_t$ are the indexing sets of a σ-LR tableau of type $(a, \sigma m, c)$, then $\Theta_i^*(J_1, ..., J_t)$ are the indexing sets of a $s_i\sigma$-LR tableau of type $(a, s_i\sigma m, c)$, for all $i = 1, \ldots, t$. Hence, given a tableau $\mathcal{T}$ of type $(a, \sigma m, c)$ with indexing sets $J_1, ..., J_t$, $\mathcal{T}$ is a σ-LR tableau if and only if $\Theta_{i_1}^* \Theta_{i_2}^* ... \Theta_{i_r}^*(J_1, ..., J_t)$ are the indexing sets of an LR tableau of type (a, m, c), where $s_{i_1} s_{i_2} ... s_{i_r}$ is a word of σ in the alphabet $\{s_1, ..., s_{t-1}\}$.

Given $\varepsilon, \sigma \in \mathcal{S}_t$ and $s_{i_1} s_{i_2} ... s_{i_r}$ a word of $\varepsilon\sigma^{-1}$ in the alphabet $\{s_1, ..., s_{t-1}\}$, $\Theta_{i_1}^* \Theta_{i_2}^* ... \Theta_{i_r}^*$ defines a bijection between $LR_\sigma(a, \sigma m, c)$ and $LR_\varepsilon(a, \varepsilon m, c)$.

Theorem 4.12 *The involutions Θ_i^*, $i = 1, ..., t-1$, define an action of the symmetric group $\mathcal{S}_t$ on $\cup_{\sigma \in \mathcal{S}_t} LR_\sigma(a, \sigma m, c)$.*

Acknowledgement: We are very grateful to Ana Paula Escada and Alexander Kovačec for useful comments. In particular, we are much obliged to Alexander Kovačec for the concept of "witness" which led us to a better understanding of the operations under consideration in this paper.

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