A $q$–SAMPLING THEOREM RELATED TO THE $q$–HANKEL TRANSFORM

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Abstract: A $q$–version of the sampling theorem is derived using the $q$–Hankel transform introduced by Koornwinder and Swarttouw. The sampling points are the zeros of the third Jackson $q$–Bessel function.

Keywords: Sampling theorem, reproducing kernel, $q$–Bessel functions, $q$–Hankel transform.

AMS Subject Classification (2000): Primary 33D15, 33D05; Secondary 94A20.

1. Introduction

The classical sampling theorem asserts that every function $f$ in the Paley-Wiener space defined by

$$PW = \left\{ f \in L^2(\mathbb{R}) : f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{ixt} u(t) dt, u \in L^2(0,1) \right\}$$

can be represented by the interpolation series

$$f(x) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi(x-n)}{\pi(x-n)}$$

Hardy’s proof of this fact [4] used properties from the kernel of the Fourier transform. Relying on properties of the Hankel’s transform kernel, Higgins [5] used the theory of reproducing kernels to obtain a sampling theorem where the sampling points are the zeros of the Bessel function. In this note, a $q$–Bessel analogue of the sampling theorem is derived by considering the kernel of the $q$–Hankel transform, $H_q^\nu$, introduced by Koornwinder and Swarttouw [8]

$$(H_q^\nu f)(x) = \int_0^\infty (xt)^{\frac{3}{2}} J_\nu^{(3)}(xt;q^2) f(t) dt$$

Partial financial assistance by Fundação para a Ciência e Tecnologia and Centro de Matemática da Universidade de Coimbra.
where $J_{\nu}^{(3)}$ denotes the third Jackson $q$–Bessel function defined by the power series
\[
J_{\nu}^{(3)}(x; q) = \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} x^\nu \sum_{n=0}^\infty (-1)^n \frac{q^{n(n+1)/2}}{(q^{\nu+1}; q)_n (q; q)_n} x^{2n}
\] (1.1)
with $0 < q < 1$, $(a; q)_n = (1 - a)(1 - aq) \ldots (1 - aq^{n-1})$ and $(a; q)_\infty = \lim_{n \to \infty} (a; q)_n$. We are using the definition of the $q$–integral. The $q$–integral in the interval $(0, 1)$ is defined as
\[
\int_0^1 f(t) \, dq_t = (1 - q) \sum_{n=0}^\infty f(q^n) q^n
\] (1.2)
and in the interval $(0, \infty)$ as
\[
\int_0^\infty f(t) \, dq_t = (1 - q) \sum_{n=-\infty}^\infty f(q^n) q^n
\] (1.3)

The sampling points will turn out to be $q j_{nw}(q^2)$, where $j_{nw}(q^2)$ is the $n^{th}$ zero of $J_{\nu}^{(3)}(x; q^2)$. In [2] it was proved that $j_{nw}(q^2) = q^{-n+\epsilon_n}, 0 < \epsilon_n < 1$. This shows how big is the spacing between the sampling points.

2. Preliminaries on reproducing kernels

Let $H$ be a class of complex valued functions, defined in a set $X \subset \mathbb{C}$, such that $X$ is a Hilbert space with the norm of $L^2(X, \mu)$. $g(s, x)$ is a reproducing kernel to $H$ if

i) $g(t, x) \in H$ for every $x \in X$;

ii) $f(x) = \langle f(t), g(t, x) \rangle$ for every $f \in H$, $x \in X$.

The next result lists the properties of Hilbert spaces with reproducing kernel that will be used in the remainder. Properties (a), (c) and (d) are proved in [5]. Property (b) is a well known property of the reproducing kernels, of primary importance, because it relates two different kinds of convergence. A proof of (b) can be found in [10], together with an introduction to the general theory.

**Proposition 1.** In the Hilbert space $L^2([a, b], \mu)$, an operator is defined by
\[
Ku = \langle K(x, t), u(t) \rangle_{L^2([a, b], \mu)}
\]
The following properties hold:
(a) If $K^{-1}$ is bounded, the range of $K$, denoted by $N$, is a Hilbert space with reproducing kernel.

(b) If the sequence $\{f_n\}$ converges strongly to $f$ in the norm of $H$, with reproducing kernel $g$, then $\{f_n\}$ converges pointwise in $X$ to $f$. The convergence is uniform in every set of $X$ where $g(s, x)$ is bounded.

(c) If $K$ is an isometry, then $g(s, x) = \langle K(s, t), K(x, t) \rangle_{L^2(a, b)}$.

(d) Let $\{f_n\}$ be a complete orthonormal sequence in $H$ and $(x_n)$ such that $f_n(x_m) = \delta_{nm}$. Then

$$f_n(t) = \frac{g(t, x_n)}{g(x_n, x_n)}$$

We will suppose $N \subset L^2(X, \mu)$, and this implies $K$ bounded. $K^{-1}$ is a transformation of $N$ over $L^2([a, b], \mu)$, also bounded.

3. A $q$-sampling theorem

We introduce a $q$-Bessel version of the Paley-Wiener space, and call it $PW^q$:

$$PW^q = \left\{ f \in L^2_q(0, \infty) : f(x) = \int_0^1 (tx)^{\frac{1}{2}} J_{\nu}^{(3)}(xt; q^2) u(t) d_q t, u \in L^2_q(0, 1) \right\}$$

(3.1)

The notation $L^2_q(0, 1)$ stands for the Hilbert space associated to the measure of the $q$-integral in $(0, 1)$. In [8] it was proved the inversion formula

$$f(t) = \int_0^{\infty} (xt)^{\frac{1}{2}} (H_q^v f)(x) J_{\nu}^{(3)}(xt; q^2) d_q x = (H_q^v (H_q^v f))(t)$$

(3.2)

Let $f \in L^2_q(0, \infty)$ such that $(H_q^v f)(q^{-n}) = 0$, $n = 1, 2, \ldots$. Then $f \in PW^q$.

To see this use the formula (3.2) and compare (1.2) and (1.3) to write $f$ as an element of $PW^q$.

Now, in the language of the preceding section, consider $X = (0, \infty)$, $(a, b) = (0, 1)$ and the kernel $K(x, t) = (xt)^{\frac{1}{2}} J_{\nu}^{(3)}(xt; q^2)$. The corresponding operator $K$ is

$$\langle Ku, x \rangle = \langle K(x, t), u(t) \rangle_{L^2(0, 1)} = \int_0^1 (xt)^{\frac{1}{2}} J_{\nu}^{(3)}(xt; q^2) u(t) d_q t$$

By (3.2), $H_q^v$ is a self-inverse operator and consequently, an isometry. Thus, $K$ is also an isometry. The range of $K$, $N$, is the set of functions $f \in L^2_q(0, \infty)$
such that \( f = Ku \) for some \( u \in L^2_q(0,1) \). By (3.1), \( N = PW^\nu_q \). In the next Lemma, the reproducing kernel of the space \( PW^\nu_q \) is evaluated.

Lemma 1. The set \( PW^\nu_q \) is a Hilbert space with reproducing kernel given by

\[
g(s, x) = (1 - q)^{q^\nu} \frac{J_{\nu+1}^3(sq^{-1}; q^2) \left[ \left( \frac{J_{\nu+1}^3(s; q^2) J_{\nu+1}^3(xq^{-1}; q^2)}{x^2 - s^2} \right) \right]}{x^2 - s^2} \tag{3.3}
\]

Proof: By Proposition 1 (a), \( PW^\nu_q \) is a space with reproducing kernel \( g(s, x) \). From Proposition 1 (c), since \( K \) is an isometry,

\[
g(s, x) = \langle K(s, t), K(x, t) \rangle_{L^2_q(0,1)} = \int_0^1 t (xs)^{\frac{\nu}{2}} J_{\nu+1}^3(zt; q^2) J_{\nu+1}^3(st; q^2) \, dt
\]

In [7], the following formula was proved

\[
a^2 - b^2 \int_0^\infty t (xs)^{\frac{\nu}{2}} J_{\nu}^3(aqt; q^2) J_{\nu}^3(bqt; q^2) \, dt = (1 - q)^{q^\nu - 1} \left[ a J_{\nu+1}^3(aq; q^2) J_{\nu}^3(bq; q^2) - b J_{\nu+1}^3(bq; q^2) J_{\nu}^3(aq; q^2) \right] \tag{3.4}
\]

Setting \( z = 1, a = xq^{-1} \) and \( b = sq^{-1} \) in (3.4), (3.3) follows.

The \( q \)-sampling theorem can now be stated and proved.

Theorem 1. If \( f \in PW^\nu_q \) then \( f \) has the unique representation

\[
f(x) = \sum_{n=1}^{\infty} f \left( qj_{\nu n}(q^2) \right) \frac{2 \left( xqj_{\nu n}(q^2) \right)^{\frac{\nu}{2}} J_{\nu}^3(x; q^2)}{dx \left[ J_{\nu}^3(x; q^2) \right]_{x=qj_{\nu n}(q^2)} (a^2 - q^2j_{\nu n}^2(q^2))} \tag{3.5}
\]

where \( \{ j_{\nu n}(q^2) \} \) denotes the sequence of positive zeros of \( J_{\nu}^3(x; q^2) \). The series converges uniformly in compact subsets of \((0, \infty)\).

Proof: Consider the sequence \( \{ f_n(x) \} \) defined by

\[
f_n(x) = \left( xqj_{\nu n}(q^2) \right)^{\frac{\nu}{2}} J_{\nu}^3(qxj_{\nu n}(q^2); q^2)
\]

It was proved in [1] that \( \{ f_n(x) \} \) is a complete orthogonal sequence in \( L^2_q(0,1) \). Taking into account that \( K \) is an isometry, the sequence \( (Kf_n)(x) \) is also orthogonal and complete in \( PW^\nu_q \). Now set

\[
F_n(x) = \frac{(Kf_n)(x)}{(Kf_n)(qj_{\nu n}(q^2))}
\]
The orthogonality of \( \{ f_n (x) \} \) implies
\[
F_n (q j_{m} (q^2)) = \delta_{nm} \tag{3.6}
\]

Proposition 1 (d) allows to write
\[
F_n (x) = \frac{g(x, q j_{m} (q^2))}{g(q j_{m} (q^2), q j_{m} (q^2))}
\]
Substituting in (3.3) yields
\[
F_n (x) = \frac{2 (x q j_{m} (q^2))^{\frac{1}{2}} J_{\nu}^{(3)} (x; q^2)}{d \left[ J_{\nu}^{(3)} (x; q^2) \right]_{x=q j_{m} (q^2)} \left( x^2 - q^2 j_{m}^2 (q^2) \right)}
\]

\( F_n (x) \) is an orthonormal complete sequence in \( N \). Thus, every \( f \in PW_q \) has a unique series expansion in the form
\[
f(x) = \sum_{n=1}^{\infty} a_n F_n (x) \tag{3.7}
\]
where \( a_n \) are the Fourier coefficients of \( f \) in \( \{ F_n (x) \} \). The series in (3.7) is convergent in the norm of \( L^2_q (0, 1) \) and also in the norm of \( PW_q \). The real-valued function \( g(x, x) \) is continuous, thus bounded in every compact subset of \( (0, \infty) \). It follows from Proposition 1 (b) that (3.7) converges uniformly in compact subsets of \( (0, \infty) \). Finally, setting \( x = q j_{m} (q^2) \) in (3.7), (3.6) implies \( f (q j_{m} (q^2)) = a_m \) and thus, (3.7) can be written in the form (3.5).

4. Application

The following formula is a consequence of the product representation for the classical Bessel function
\[
\frac{d}{dx} J_{\nu} (x) = 2x \sum_{n=1}^{\infty} \frac{1}{j_{nm}^2 - x^2} + \nu \frac{x}{x} \tag{4.1}
\]
Using the recurrence \( x \frac{d}{dx} J_{\nu} (x) - \nu J_{\nu} (x) = -x J_{\nu+1} (x) \), (4.1) becomes
\[
\frac{J_{\nu+1} (x)}{J_{\nu} (x)} = -2x \sum_{n=1}^{\infty} \frac{1}{j_{nm}^2 - x^2} \tag{4.2}
\]
where \( j_{nm} \) stands for the zeros of \( J_{\nu} (x) \). In the case of the \( q \)-analogues of the Bessel function, this analysis cannot be done, for there are no formulas.
to establish a simple relation between a $q$-Bessel function and its derivative. While the $q-$analogue of (4.1) is very simple to derive from the Hadamard factorization theorem or using residues, the $q$-analogue of (4.2) is harder to obtain. In [6], Ismail studied the second Jackson $q$-Bessel function, $J^{(2)}_\nu(x; q)$, and found such $q-$analogue using the orthogonality measure of the modified $q$-Lommel polynomials associated to $J^{(2)}_\nu(x; q)$. Kvitsinsky [9] found a recurrence relation for the coefficients $h_n$ in the identity

$$
\frac{J^{(3)}_{\nu+1}(x; q)}{J^{(3)}_\nu(x; q)} = \sum_{n=1}^{\infty} h_n x^{2n-1}
$$

(4.3)

In this section an explicit formula for the coefficients $h_n$ will be obtained as a special case of the expansion of a particular function as a sampling series. Preliminary to this expansion, a $q-$integral formula connecting two $q$-Bessel functions of different order is established.

**Lemma 2.** For $y > 0$, $\nu > -\frac{1}{2}$ and $x \in \mathbb{R}$, the following relation holds

$$\frac{(q; q)_\infty}{(q^n; q)_\infty} x^{-y} J^{(3)}_{\nu+y}(x; q) = \int_0^1 t^{x-1} \frac{(tq; q)_\infty}{(tq^n; q)_\infty} J^{(3)}_\nu(x t^{2}; q) d_q t
$$

(4.4)

**Proof:** The $q-$analogues of the gamma and the beta function will be critical in the proof. According to [3, 1.10], the $q-$gamma function, $\Gamma_q(x)$, is defined by

$$\Gamma_q(x) = \frac{(q; q)_\infty}{(q^x; q)_\infty} (1 - q)^{1-x}
$$

(4.5)

and the $q-$beta function, $\beta_q(x, y)$ by

$$\beta_q(x, y) = \frac{\Gamma_q(x) \Gamma_q(y)}{\Gamma_q(x+y)}
$$

(4.6)

The $q-$beta function has the $q$-integral representation

$$\beta_q(x, y) = \int_0^1 t^{x-1} \frac{(tq; q)_\infty}{(tq^n; q)_\infty} d_q t, \text{ Re}(x) > 0, y \neq 0, -1, -2, ...
$$

(4.7)

Using the series representation (1.1) and the $q-$integral representation (4.7) it is easy to see that, if $\nu > -\frac{1}{2}$ and $y > 0$,

$$\int_0^1 t^{x} \frac{(tq; q)_\infty}{(tq^n; q)_\infty} J^{(3)}_\nu(x t^{2}; q) d_q t = x^\nu \sum_{k=0}^{\infty} (-1)^k \frac{q^{\nu(k+1)}}{(q; q)_k (q^{n+1}; q)_k} x^{2k} \beta_q(k+\nu+1, y)
$$

(4.8)
Now use (4.5) and (4.6) to express $q(k + \nu + 1, y)$ as a quotient of infinite products. Then, some algebraic manipulations using the formula $(a; q)_\infty = (a; aq^\gamma; q)_\infty$ allow us to see that the right hand member of (4.8) is equal to the left hand member of identity (4.4).

Before moving to the next Theorem, it is convenient to point out that, from the definition (1.2), one can verify the relation:

$$\int_0^1 f(t^2) dq t = (1 + q) \int_0^1 tf(t) dq t$$

(4.9)

**Theorem 2.** If $u > \nu > -\frac{1}{2}$, the following identity holds

$$x^{\nu-u} \frac{J_u^{(3)}(x; q^2)}{J_0^{(3)}(x; q^2)} = -2 \sum_{n=1}^\infty \frac{\left( q j_{2n}(q^2) \right)^{u-n+1} J_u^{(3)}(q j_{2n}(q^2); q^2)}{(q^{2u-2v}; q^2)_\infty} (q^{2u}; q^2) \frac{J_{u+1}^{(3)}(qt^2; q^2)}{(q^{2u-2v}; q^2)_\infty}$$

(4.10)

**Proof:** Setting $y = u - \nu$ in (4.4) and replacing $q$ by $q^2$, the result is, if $u > \nu$,

$$\frac{(q^2; q^2)_\infty}{(q^{2u-2v}; q^2)_\infty} x^{\nu-u} J_u^{(3)}(x; q^2) = \int_0^1 t^2 \frac{(tq^2; q^2)_\infty}{(tq^{2u-2v}; q^2)_\infty} J_{u+1}^{(3)}(xt^2; q^2) dq t$$

Taking (4.9) into account, this can be rewritten as

$$\frac{(q^2; q^2)_\infty}{(q^{2u-2v}; q^2)_\infty} x^{\nu-u} J_u^{(3)}(x; q^2) = (1 + q) \int_0^1 t^{u+1} \frac{(tq^2; q^2)_\infty}{(tq^{2u-2v}; q^2)_\infty} J_{u+1}^{(3)}(xt; q^2) dq t$$

(4.11)

Considering

$$u(t) = t^{u+1} \frac{(1 + q) (q^{2u-2v}; q^2)_\infty}{(q^{2u}; q^2)_\infty} \frac{(tq^2; q^2)_\infty}{(tq^{2u-2v}; q^2)_\infty}$$

relation (4.11) yields

$$x^{\nu-u+\frac{1}{2}} J_u^{(3)}(x; q^2) = \int_0^1 (tx)^{\frac{1}{2}} J_{u+1}^{(3)}(xt; q^2) u(t) dq t$$

Thus,

$$f(x) = x^{\nu-u+\frac{1}{2}} J_u^{(3)}(x; q^2) \in PW_{\nu}$$

And it is possible to apply Theorem 1 to $f$. The result of this application is (4.10).
Taking $u = \nu + 1$ in (4.10) and replacing $q^2$ by $q$, the result is the analogue of (4.2) previously mentioned:

$$
\frac{J_{\nu+1}^{(3)}(x; q)}{J_{\nu}^{(3)}(x; q)} = -2x \sum_{n=1}^{\infty} \frac{d}{dx} \left[ J_{\nu}^{(3)}(x; q) \right]_{x=q^2 j_{\nu}(q)} \frac{1}{q^{2n} j_{\nu}(q) - x^2} \quad (4.12)
$$

Expanding $1/ \left( j_{\nu}^2(q) - x^2 \right)$ in power series of $x$ and substituting in (4.12), the coefficients $h_n$ in (4.3) can be seen to be

$$
h_n = \sum_{k=1}^{\infty} \frac{d}{dx} \left[ J_{\nu}^{(3)}(x; q) \right]_{x=q^2 j_{\nu}(q)} \left( \frac{1}{q^{2n} j_{\nu}(q)} \right)^{2n} \quad (\text{4.12})
$$

References