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CONVERSE TO THE PARTER-WIENER THEOREM: THE CASE OF NON-TREES

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ABSTRACT: Through a sucession of results, it is known that if the graph of an Hermitian matrix A is a tree and if for some index j, $\lambda \in \sigma(A) \cap \sigma(A(j))$, then there is an index i such that the multiplicity of λ in $\sigma(A(i))$ is one *more* than that in A. We exhibit a converse to this result by showing that it is generally true only for trees. In particular, it is shown that the minimum rank of a positive semidefinite matrix with a given graph G is $\leq n-2$ when G is not a tree. This raises the question of how the minimum rank of a positive semidefinite matrix depends upon the graph in general.

In a series of papers over 40 years, [PAR], [WIE], [JLS], a remarkable fact about multiple eigenvalues, in an Hermitian matrix A whose graph is a tree, has emerged. If $\mathrm{m}_A(\lambda)$, the multiplicity of λ as an eigenvalue of A, is greater than one, then there is an index i such that in $A(i)$, the $(n-1)$-by-$(n-1)$ principal submatrix of A with row and column i deleted, $\mathrm{m}_{A(i)}(\lambda) = \mathrm{m}_A(\lambda) + 1$: the multiplicity of λ necessarily goes up in passing to a smaller submatrix! The same conclusion holds even if $\mathrm{m}_A(\lambda) = 1$, as long as $\mathrm{m}_{A(j)}(\lambda) \geq 1$ for some index j. Much more information is available about such indices i (see [JLS]), but our primary purpose here is to prove a converse to this remarkable fact: *it is generally true only for trees.* In the process, facts of possible independent interest are proven, raising further questions.

Throughout, let G denote an (undirected) graph on n vertices. G need not be connected; but, generally, our claims are easily verified in the non-connected case, so that we concentrate on the connected case. We denote by $\mathcal{S}(G)$ the set of all real symmetric matrices (equivalently, complex Hermitian matrices in case G is a tree). The following (and more) is proven in [JLS], and it has substantial antecedents in [PAR], [WIE] .

Theorem 1. *Let G be a tree. If $A \in \mathcal{S}(G)$ and there is an index j such that $\lambda \in \sigma(A) \cap \sigma(A(j))$ then there is an index i such that* $\mathrm{m}_{A(i)}(\lambda) = \mathrm{m}_A(\lambda) + 1$.

It should be noted that there may be several such indices i, and it may be that j is not among them.

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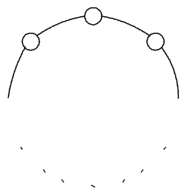
Our primary goal is to prove a rather strong converse to Theorem 1.

We begin with an illustrative example. A simple connected non-tree is the cycle, C, on n vertices



in which we may number the vertices consecutively around the cycle. If one of the vertices (of degree 2), say n, is deleted, a path, T, remains. Now, suppose that, for this path, a positive semidefinite (PSD) matrix $B \in \mathcal{S}(T)$ with non-positive off-diagonal entries and row sums zero is constructed. This is an example of a (singular) M-matrix. Then construct a matrix $A \in \mathcal{S}(C)$ with $A(n) = B$; if the last row and column of A are chosen so that the sum of the off-diagonal entries is zero (note that there are two nonzero off-diagonal entries because of the graph) and the diagonal entry is sufficiently large and positive, the result will be a PSD matrix such that $\mathrm{m}_A(0) = \mathrm{m}_{A(n)}(0) = 1$, but $\mathrm{m}_{A(j)}(0) = 0$, $1 \leq i < n$. According to Theorem 1, this can nor happen for a tree. For example, when n=4, we may have

$$A = \begin{bmatrix} 1 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 1 & 1 \\ -1 & 0 & 1 & 10 \end{bmatrix}$$

and the claim above may be checked directly. The vector $(1, 1, 1, 0)^T$ spans the null space of A.

For a general (connected) non-tree we need not have a vertex that is both degree at least two and such that its removal leaves a connected graph. The above strategy may be generalized for connected non-trees; the only difference will be that $\mathrm{m}_{A(i)}(0) = 1$ for some additional indices i.

Our main result is the following

Theorem 2. *Suppose that G is a graph on n vertices that is not a tree. Then:*
(1) There is a matrix $A \in \mathcal{S}(G)$ with an eigenvalue λ such that there is an index j so that $\mathrm{m}_A(\lambda) = \mathrm{m}_{A(j)}(\lambda) = 1$ and $\mathrm{m}_{A(i)}(\lambda) \leq 1$ for every i,

$i = 1, \ldots, n$.
(2) There is a matrix $B \in \mathcal{S}(G)$ with an eigenvalue λ such that $\mathrm{m}_B(\lambda) \geq 2$ and $\mathrm{m}_{B(i)}(\lambda) = \mathrm{m}_B(\lambda) - 1$ for every i, $i = 1, \ldots, n$.

Several "converses" to theorem 1 might be imagined, but theorem 2 is stronger than what might be asked. For any non-tree , it guarantees examples of matrices A, with that graph, and in which the multiplicity of some (multiple) eigenvalue of A is *less* in any principal submatrix of size one smaller, and, even when the multiplicity is one in both A and a submatrix, the multiplicity does not go up. Our proof rests upon 3 lemmas that include constructions that may be carried out only for non-trees. Since M-matrices are frequently used, see [HJ2, Ch. 2] as a general reference for this topic.

Lemma 3. *Suppose that A is an n-by-n Hermitian matrix with eigenvalues $\lambda_1 < \lambda_2 < \ldots < \lambda_k$ with respective multiplicities $m_1, m_2, \ldots, m_k$, $\sum_{i=1}^{k} m_i = n$. Then, for any i, $1 \leq i \leq n$, $\mathrm{m}_{A(i)}(\lambda_1) \leq m_1$ and $\mathrm{m}_{A(i)}(\lambda_k) \leq m_k$. Moreover, if $G(A)$ is a tree, $m_1 = m_k = 1$, and, for each i, $1 \leq i \leq n$, $\mathrm{m}_{A(i)}(\lambda_1) = \mathrm{m}_{A(i)}(\lambda_k) = 0$.*

Proof: For each claim, the cases of 1 and k are equivalent via replacing A by $-A$. The first claim follows from the interlacing inequalities for Hermitian matrices (*e g* [HJ1, Ch. 4]). The only possibility that need be precluded is $\mathrm{m}_{A(i)}(\lambda_1) = m_1 + 1$. But, by the interlacing inequalities, the $(m_1 + 1)$st smallest eigenvalue of $A(i)$ is at least $\lambda_2 > \lambda_1$ (*e g* between λ_2 and λ_3), so that $\mathrm{m}_{A(i)}(\lambda_1) = m_1 + 1$ is not possible. The "moreover" claim may be either proven from theorem 1, together with the first claim of this lemma, or independently using the Perron-Frobenius theory of irreducible nonnegative matrices (*e g* [HJ1, Ch. 8]). Via diagonal similarity and translation by a scalar matrix, A may be made entry-wise nonnegative without altering the hypothesis, when the graph is a tree. But, then, because of the irreducibility, the largest eigenvalue has multiplicity one and for any proper principal submatrix the largest eigenvalue strictly decreases. The first part of the moreover claim is known and has several proofs (see *e g* [JLS]). ■

We note that none of the claims of lemma 3 is generally true for intermediate eigenvalues (λ_i, $1 < i < k$).

The fact that generalizes the example of the cycle given above is the following.

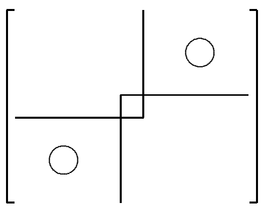
Lemma 4. *Let G be a graph on n vertices that is not a tree. Then, there is a matrix $A \in \mathcal{S}(G)$, an eigenvalue λ of A, and an index j, $1 < j < n$ such that* $\mathrm{m}_A(\lambda) = \mathrm{m}_{A(j)}(\lambda) = 1$ *and* $\mathrm{m}_{A(i)}(\lambda) \leq 1$ *for all i, $1 \leq i \leq n$.*

Proof: First, let G be connected and not a tree. We construct a PSD matrix of rank n-1, such that at least one $(n-1)$-by-$(n-1)$ principal submatrix is rank deficient by one and none is rank deficient by two. Thus, in this case, the eigenvalue $\lambda = 0$ will satisfy the claims of the lemma.

Since G is connected and not a tree, it contains a cycle C of at least three vertices. We consider two possibilities, at least one of which must occur: (1) there is a vertex v of C that is not a cut-vertex of G (of course, $\deg_G v \geq 2$); and (2) there is a vertex u of C that is a cut-vertex of G.

In case (1), construct matrix $A_1 \in \mathcal{S}(G - v)$ with positive diagonal entries, nonpositive off-diagonal elements and zero row sums. Then A_1 is a singular M-matrix and, as $G - v$ is connected, A_1 is PSD of rank defiency one ([HJ2, Ch. 2]). Now embed A_1 in $A \in \mathcal{S}(G)$ by choosing the sum of the additional off-diagonal entries (in the new row and column) to be zero. Since $\deg_G v \geq 2$, this is possible. And choose the new diagonal entry to be sufficiently large and positive, so that A is PSD of rank deficiency one. This is straightforward as each proper principal submatrix of A_1 is positive definite. Now, A has the desired properties: $\mathrm{m}_A(0) = 1$, $\mathrm{m}_{A(v)}(0) = 1$ and $\mathrm{m}_{A(i)}(0) \leq 1$ for all i, $1 \leq i \leq n$.

In case (2), call the graph induced by the vertices of the component of $G - u$ containing $C - u$ together with u, G_1 and let G_2 be the subgraph induced by all the other vertices together with u. By numbering the vertices of $C - u$ first then u and then the remaining vertices, any matrix in $\mathcal{S}(G)$ appears as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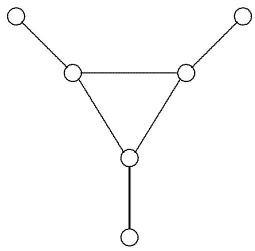
in which the upper left principal block corresponds to G_1, the lower right to G_2 and the lone overlapping entry to u. Now, as in case (1), construct a singular M-matrix A_1 in $\mathcal{S}(G_1 - u)$ and embed it in a PSD matrix A_2 of rank deficiency one in $\mathcal{S}(G_1)$. Then, choose a positive definite (PD) matrix A_3 in $\mathcal{S}(G_1)$ and superimpose it, as depicted (adding the entries from A_2 and A_3

in the position corresponding to u) to obtain the matrix $A \in \mathcal{S}(G)$. Now, A is PSD of rank deficiency one, as is any principal submatrix resulting from the deletion of a vertex of G_2. Again, as every proper principal submatrix of A_1 is PD (and A_3 is PD), deletion of no row and column leaves a matrix of rank deficiency more than one. As before, A meets the desired requirements.

If G is not connected, choose one of the components, and for it construct a singular M-matrix, as A_1 was constructed above. Choose a PD matrix for each other component to produce A. Then zero is an eigenvalue of multiplicity one of A and of each of principal matrix resulting from the deletion of a vertex not in the first component and the requirements of the lemma are met. ■

It is an interesting question in how few of the $A(i)$ must we have $\mathrm{m}_{A(i)} = 1$ (as opposed to zero). It may be only one if C includes all vertices of G (or in the case (1) of the proof of lemma 4). But it may be more, as in the graph



or if G is not connected.

We next give our key lemma that allows us to prove the second claim of theorem 2.

Lemma 5. *Let G be a graph on n vertices that is* not *a tree. Then, there is a PSD matrix $A \in \mathcal{S}(G)$ such that* $\operatorname{rank} A = k \leq n-2$ *and such that for any i, $1 \leq i \leq n$,* $\operatorname{rank} A(i) = k$.

Proof: First, note that for any G there is a PSD (and not PD) matrix $A \in \mathcal{S}(G)$: Choose $B \in \mathcal{S}(G)$ and let $A = B - \lambda_{\min}(B)$, in which $\lambda_{\min}(B)$ denotes the smallest eigenvalue of B.

We first suppose that G is connected (and not a tree). The case of any not connected G will be straightforward later.

Since G is connected and not a tree, there are vertices i, j such that (1) $\{i, j\}$ is an edge of G and (2) there is a path in G, not involving $\{i, j\}$, from i to j: $\{i, p_1\}$, $\{p_1, p_2\}$, $\ldots$, $\{p_k, j\}$. Without loss of generality, suppose

$i = 1$, $j = 2$. We will construct the desired matrix A as follows:

$$A = \begin{bmatrix} A_1 & B \\ B^T & A_2 \end{bmatrix}.$$

Let A_2 be a PD M-matrix, $A_2 \in \mathcal{S}(G')$, $G' = (G - \{1, 2\})$, the subgraph induced by vertices $\{3, 4, \ldots, n\}$. Such an A_2 may be easily found by choosing negative off-diagonal entries in the positions allowed by G' (0 off-diagonal elements otherwise) and then choosing positive diagonal entries, to achieve strict diagonal dominance. Let B be nonnegative with *positive* entries corresponding to the edges of G and 0's elsewhere. Finally, let $A_1 = BA_2^{-1}B^T$, a nonnegative PSD, 2-by-2 matrix. By Schur complements (see *e g* [CAR]) $\operatorname{rank} A = \operatorname{rank} A_2 + \operatorname{rank}(A_1 - BA_2^{-1}B^T) = \operatorname{rank} A_2 + \operatorname{rank} 0 = n-2+0 = n-2$, and A is PSD (in fact the interlacing inequalities applied to the eigenvalues of A and A_2 (see [HJ1, Ch.4]) shows that A cannot have negative eigenvalues) of rank $n-2$.

Now, it suffices to show that A_1 has its two off-diagonal entries positive, so that $A \in \mathcal{S}(G)$. But B has a positive entry in the 1, p_1 position and in the 2, p_k position. Moreover, because A_2 is an M-matrix, $A_2^{-1} \geq 0$, and, as there is a path in G' from p_1 to p_k, the p_1, p_k (and p_k, p_1) entry of A_2^{-1} is positive. By matrix multiplication, the 1, 2 entry of the symmetric matrix $BA_2^{-1}B^T$ is then positive .

Now, we turn to the second claim in the connected case: that $\operatorname{rank} A(i) = \operatorname{rank} A$ for $1 \leq i \leq n$ and the A just defined. If $i \in \{1, 2\}$, $A(i)$ contains A_2 as a principal submatrix and as $\operatorname{rank} A = \operatorname{rank} A_2$, $\operatorname{rank} A(i) = \operatorname{rank} A$ as claimed. On the other hand, if $i \in \{3, 4, \ldots, n\}$ $\operatorname{rank} A_2(i) = n-3$, and as $A_2(i)^{-1} \leq A_2^{-1}(i)$ (entry-wise, because A_2 is an M-matrix; see *e g* [JS, theorem 2.1]), we have $B(i)A_2(i)^{-1}B^T(i) \lneqq (BA_2^{-1}B^T)(i) = A_1$. Here for i, we retain the numbering in A and by $B(i)$ ($B^T(i)$) we mean B with only its i-th column deleted (B^T with only its i-th row deleted). Thus, the Schur complement $A_1 - B(i)A_2(i)^{-1}B^T(i) \neq 0$ and its rank must be 1. We conclude that $\operatorname{rank} A(i) = n-3+1 = n-2$, in this case, as well, and the proof is complete in the case of connected graphs.

Finally if G is not connected, A may be constructed for each connected component as follows: if the component is an isolated vertex the corresponding submatrix is zero. If the component is a tree, let the corresponding principal submatrix be any PSD matrix of rank one less than the number of vertices in the graph is that comprises that component. It follows from the lemma 3 that any principal submatrix of such a submatrix is then PD. If the component is neither a vertex nor a tree (a connected graph that is not a tree), construct the corresponding principal submatrix as in the earlier part of this proof. It is then easily checked that both parts of the conclusion of the lemma hold for such an A, completing the proof. ■

Remark. Following the same proof as lemma 5 , if G contains a k-clique C such that for any 2 vertices i, j in C there is also a path from i to j through C' (the complement of C in G), then there is a PSD $A \in \mathcal{S}(G)$ such that $\operatorname{rank} A \leq n-k$. Further, if there is a subgraph H of G induced by k vertices $i_1,\ i_2,\ \ldots,\ i_k$ such that for every pair of vertices in H, either they are connected by an edge of H and by a path through $G-H$ or they are connected neither by an edge of H nor a path via $G-H$, then there is a PSD matrix $A \in \mathcal{S}(G)$ such that $\operatorname{rank} A \leq n-k$. Note that the second case occurs, even in a connected G, if all paths between the two vertices use edges in H and not in H.
Problem. The lemma raises the very interesting question of, for a given graph G on n vertices, what is

$$\min_{A\in\mathcal{S}(G),\, A\,\mathrm{PSD}} \operatorname{rank} A = \min \mathrm{PSD}\,\mathrm{Rank}(G)\,.$$

If G is a tree or an isolated vertex the minimum is $n-1$. For all other graphs the mimimum is $\leq n-2$. It would be of interest to be able to describe the mimimum in terms of the graph.

<u>Proof of theorem 2:</u> To conclude, theorem 2 now follows easily from lemmas 4 and 5. Claim number (1) is the content of lemma 4 and number (2) the content of lemma 5. ■

From theorems 1 and 2 the following corollary follows easily.

Corollary 6. *For an undirectd graph G on n vertices the following are equivalent:*

(1) *G is a tree.*
(2) $\min_{A\in\mathcal{S}(G),\, A\, PSD} \operatorname{rank} A = n-1$
(3) *For any $A \in \mathcal{S}(G)$ and any $\lambda \in \sigma(A)$ such that $\mathrm{m}_A(\lambda) > 1$, there is an index i, $1 \leq i \leq n$, such that $\mathrm{m}_{A(i)}(\lambda) = \mathrm{m}_A(\lambda)+1$.* ■

References

[CAR] D. Carlson, What are Schur complements, anyway? *Linear Algebra Appl.* 74 (1986) 257-275.

[HJ1] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge Univ. Press, New York, 1985.

[HJ2] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge Univ. Press, New York, 1991.

[JL] C. R. Johnson and A. Leal Duarte, The maximum multiplicity of an eigenvalue in a matrix whose graph is a tree, *Linear and Multilinear Algebra* 46 (1999) 139-144.

[JLS] C. R. Johnson and A. Leal Duarte, Carlos M. Saiago, The Parter-Wiener Theorem: Refinement and generalization, *SIAM Journal of Matrix Analysis and its Applications*, 25 (2) 352-361 (2003).

[JS] C. R. Johnson and R. L. Smith, Almost principal minors of inverse M-matrices, *Linear Algebra Appl.* 337 (2001) 253-265.

[PAR] S. Parter, On the eigenvalues and eigenvectors of a class of matrices, *J. Soc. Indust. Appl. Math* 8 (1960) 376-388.

[WIE] G. Wiener, Spectral multiplicity and splitting results for a class of qualitative matrices, *Linear Algebra Appl.* 61 (1984) 15-29.

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