Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 04–07

ON THE DOUBLY SINGULAR EQUATION $\gamma(u)_t = \Delta_p u$

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ABSTRACT: We prove that local weak solutions of a nonlinear parabolic equation with a doubly singular character are locally continuous. One singularity occurs in the time derivative and is due to the presence of a maximal monotone graph; the other comes up in the principal part of the PDE, where the p-Laplace operator is considered. The paper extends to the singular case $1 < p < 2$, the results obtained previously by the second author for the degenerate case $p > 2$; it completes a regularity theory for a type of PDEs that model phase transitions for a material obeying a nonlinear law of diffusion.

KEYWORDS: doubly singular PDE, regularity theory, intrinsic scaling, phase transition.
AMS SUBJECT CLASSIFICATION (2000): 35B65, 35D10, 35K65.

1. Introduction

Questions of regularity concerning partial differential equations that are either degenerate or singular, or both, have been addressed quite intensively in the last two decades. The understanding by DiBenedetto (cf. [7]) of the crucial relation between the structure of these PDEs and the geometry in which they have to be analyzed set up a vast research programme that is still far from being completed.

One interesting family is that of equations of the form

$$\gamma(u)_t - \operatorname{div} \mathbf{a}(x,t,u,\nabla u) = \mathrm{b}(x,t,u,\nabla u) \qquad \text{in} \quad \mathcal{D}'(\Omega_T) \ ,$$

where γ is a maximal monotone graph and $\mathbf{a}$ and b are measurable functions satisfying the usual appropriate structure conditions. The equality is to be interpreted in the sense of the graphs, *i.e.*, for a choice $v \in \gamma(u)$. To fix ideas, let us restrict our attention to the case

$$\mathbf{a}(x,t,u,\nabla u) = |\nabla u|^{p-2}\nabla u \ ; \qquad \mathrm{b}(x,t,u,\nabla u) \equiv 0$$

that is, to the equation for the p-Laplace operator

$$v_t = \Delta_p u \ , \quad v \in \gamma(u) \ ; \qquad p > 1 \ . \tag{1}$$

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Research supported by CMUC/FCT and Project POCTI/34471/MAT/2000.

When γ has a single jump at the origin, this equation generalizes to a nonlinear setting the modelling of the classical Stefan problem that corresponds to the case $p = 2$ and describes a phase transition at constant temperature for a substance obeying Fourier's law. There is a vast literature concerning the Stefan problem: see [14] or [16] and the references therein.

The table below sums up what we already know, from the point of view of the local regularity properties of weak solutions of (1), in terms of the range of values p can assume (the case $p = 2$ obviously corresponds to the Laplacian as principal part in the equation) and the number of jumps of the maximal monotone graph γ; the original reference for each result is also indicated:

range for p	**# of jumps of** γ	**regularity**	**paper**
$p = 2$	one	continuity	[1] and [4]
$p = 2$	several	continuity	[10]
$p > 2$	none	Hölder continuity	[6]
$p > 2$	one	continuity	[18]
$1 < p < 2$	none	Hölder continuity	[2]

Our objective is to enlarge the table, treating the case $1 < p < 2$ in the presence of *one* jump and showing that local weak solutions are *locally continuous*.

To be precise, consider a bounded domain $\Omega \subset \mathbf{R}^N$, with a regular boundary $\partial\Omega$, fix a time interval $(0, T)$, for some $T > 0$, and denote with $\Omega_T = \Omega \times (0, T)$ the space-time domain. Consider the maximal monotone graph H associated with the Heaviside function

$$H(s) = \begin{cases} 0 & \text{if } \; s < 0 \\ [0,1] & \text{if } \; s = 0 \\ 1 & \text{if } \; s > 0 \end{cases}$$

and let $\gamma(s) = s + \lambda H(s)$, where λ is a positive constant (physically, the latent heat of the phase transition). The equation

$$v_t = \Delta_p u \ , \quad v \in \gamma(u) \ ; \qquad 1 < p < 2 \tag{2}$$

has a *double singularity*: since $p < 2$, its modulus of ellipticity $|\nabla u|^{p-2}$ blows up at points where $|\nabla u| = 0$, and, concerning the time derivative, we have what can be heuristically described as $\gamma'(0) = \infty$.

For strongly elliptic/parabolic equations the technique used to obtain results on the continuity of weak solutions (including quantitative information on their modulus of continuity) is based on energy (and logarithmic) estimates. These are essential to set forward an iterative argument consisting of showing that, for every point in the domain, we can find a sequence of nested and shrinking cylinders such that, as the cylinders shrink to the point, the essential oscillation of the solution in the cylinders converges to zero. This iterative argument was introduced for strongly elliptic equations by DeGiorgi in [3], revisited by Moser in [15], and adapted later by the Russian school to the parabolic case (cf. [12]). It is essential to the reasoning that the equation is strongly elliptic/parabolic so that the integral norms appearing in the estimates are homogeneous.

In the case of (2), that has a double singularity, this homogeneity is lost. On the one hand, the singularity in the principal part gives rise to a power $p < 2$ in the energy estimates. On the other hand, dealing with the maximal monotone graph involves a regularization procedure and the proof that the family of approximate solutions is equicontinuous. The consequence of estimating uniformly the regularization of the maximal monotone graph is the appearance of a third power (power 1) in the energy estimates. We are thus dealing with three powers (1, p and 2) and we have $1 < p < 2$.

The technique introduced by DiBenedetto to deal with singular (and degenerate) parabolic PDEs is called *intrinsic scaling*. The main idea consists in looking at the equation in its own geometry, *i.e.*, in a geometry dictated by its singular (or degenerate) structure. This amounts to rescale the standard parabolic cylinders by a factor depending on the oscillation of the solution. The procedure, which can be called accommodation of the singularity/degeneracy, allows for the recovering of the homogeneity in the energy estimates if written over these rescaled cylinders. It can be said heuristically that *the equation behaves in its own geometry like the heat equation.*

In the case of three different powers, the price to be paid for the recovering of the homogeneity in the energy estimates is a dependence on the oscillation in the various constants that are determined along the proof. Owing to this fact we will no longer be able to exhibit a modulus of continuity but only to define it implicitly independently of the regularization. This is enough to obtain a continuous solution for the original problem, via Ascoli's theorem, but the Hölder continuity, that holds in the case $\gamma(s) = s$, is lost.

We anticipate that the choice of the right intrinsic geometry is the main difficulty of the problem. To fully understand what is at stake, let us observe that a bridge between the singularity in time and a degeneracy in space can be made through rewriting equation (2) in terms of v only, taking into account that $u = \gamma^{-1}(v)$ and γ^{-1} is now a well defined function, such that $\gamma^{-1}(s) = 0$ for $0 \leq s \leq \lambda$. It is clear that the time singularity in the u-equation becomes a space degeneracy for the v-equation. Now, the existent literature shows that a singularity in the principal part of an equation requires a rescaling in the space variables while a degeneracy requires a rescaling in the time variable. In the case of equation (1) with $p > 2$, that was treated in [18], we were in the presence of two types of degeneracy and that explains why a rescaling in time was enough. Also, in the case $1 < p < 2$ but with no jumps (*i.e.*, for $\gamma(s) = s$), there is only a singularity, so a rescaling in space suffices (see [7, Ch. 4]). Here, we have the equivalent of both a singularity and a degeneracy in the principal part and so we need both rescalings; it is the first time, to the best of our knowledge, that an intrinsic scaling has to be performed simultaneously in the space and time dimensions, although in [10] a similar procedure has been adopted but keeping the natural space-time homogeneity.

The technical implementation of the proof consists in the study of an alternative. In the first part of the alternative, we deal with the singularity in time (degeneracy in space) so what dominates the geometry is the scaling in time; the type of partition of the cylinders that is considered is a reflection of this fact. In the second part of the alternative, everything takes place above the singularity in time, the singular character of the principal part thus being the dominant factor; it comes with no surprise that the type of partition considered there is a partition in space.

2. Regularization and local estimates

The proof that local weak solutions of (2) are locally continuous consists in defining an approximated problem and showing that the sequence of approximate solutions, which can be shown to be uniformly bounded, is also equicontinuous. The use of Ascoli's theorem completes the reasoning. We will not be concerned with problems of existence of the weak solution for boundary value problems associated with (2) or the convergence of the sequence of approximate solutions to the weak solution. This problem was treated in [17] (see also, the now classical, reference [13]).

Let $0 < \epsilon \ll 1$ and consider the function

$$\gamma_\epsilon(s) = s + \lambda H_\epsilon(s) \ ,$$

where H_ϵ is a C^∞ approximation of the Heaviside function, such that

$$H_\epsilon(s) = 0 \quad \text{if} \quad s \leq 0 \ ; \qquad H_\epsilon(s) = 1 \quad \text{if} \quad s \geq \epsilon \ ,$$

$H'_\epsilon(s) \geq 0$, $s \in \mathbf{R}$ and $H_\epsilon \longrightarrow H$ uniformly in compact subsets of $\mathbf{R} \setminus \{0\}$ as $\epsilon \to 0$. The function γ_ϵ is bilipschitzian and satisfies

$$1 \leq \gamma'_\epsilon(s) \leq 1 + \lambda L_\epsilon \ , \quad s \in \mathbf{R} \ , \tag{3}$$

where $L_\epsilon \equiv \mathcal{O}(\frac{1}{\epsilon})$ is the Lipschitz constant of H_ϵ. Its inverse $\beta_\epsilon = \gamma_\epsilon^{-1}$ satisfies

$$0 < \frac{1}{1 + \lambda L_\epsilon} \leq \beta'_\epsilon(s) \leq 1 \ , \quad s \in \mathbf{R} \ . \tag{4}$$

The approximated problem consists in finding a function

$$\theta_\epsilon \in H^1\left(0, T; L^2(\Omega)\right) \cap L^\infty\left(0, T; W_0^{1,p}(\Omega)\right) \cap L^\infty(\Omega_T)$$

such that, for a.e. $t \in (0, T)$,

$$\int_{\Omega\times\{t\}} \partial_t(\gamma_\epsilon(\theta_\epsilon))\, \varphi + \int_{\Omega\times\{t\}} |\nabla\theta_\epsilon|^{p-2}\nabla\theta_\epsilon \cdot \nabla\varphi = 0 \ , \quad \forall \varphi \in W_0^{1,p}(\Omega) \ . \tag{5}$$

We assume that the approximate solutions satisfy the uniform bound

$$\|\theta_\epsilon\|_{L^\infty(\Omega_T)} \leq M \ .$$

It is easily seen that (5) corresponds, in the sense of distributions, to an equation of the type

$$u_t - \operatorname{div} \mathbf{a}(x, t, u, \nabla u) = 0 \ ,$$

with $u = \gamma_\epsilon(\theta_\epsilon)$ and $\mathbf{a}(x, t, u, \nabla u) = |\nabla\beta_\epsilon(u)|^{p-2}\nabla\beta_\epsilon(u)$, that satisfies the structure conditions

$$\mathbf{a}(x, t, u, \nabla u) \cdot \nabla u \geq \left(\frac{1}{1 + \lambda L_\epsilon}\right)^{p-1} |\nabla u|^p$$

and

$$|\mathbf{a}(x, t, u, \nabla u)| \leq |\nabla u|^{p-1} \ .$$

As a consequence of this fact, we conclude (using the theory developed in [7]) that the approximate solutions are Hölder continuous. But the constant $C_0(\epsilon) := (\frac{1}{1+\lambda L_\epsilon})^{p-1}$ deteriorates when $\epsilon \to 0$ and, in this way, nothing is obtained in the limit. We will next obtain a series of estimates that are independent of the approximating parameter ϵ and will allow us to show that the approximate solutions are continuous independently of the approximation. The main result of this paper then follows from Ascoli's theorem.

Theorem 1. *The sequence $(\theta_\epsilon)_\epsilon$ is locally equicontinuous, i.e., for each θ_ϵ, there exists an interior modulus of continuity that is independent of ϵ. Therefore equation (2) has, at least, one locally continuous solution.*

Let us start with some notation. Given $x_0 \in \mathbf{R}^N$, define the N-dimensional cube

$$[x_0 + K_\rho] := \left\{ x \ : \ \max_{1\leq i\leq N} |x_i - x_{0i}| < \rho \right\} \ .$$

Given the pair $(x_0, t_0) \in \mathbf{R}^{N+1}$, define the cylinder

$$[(x_0, t_0) + Q(\tau, \rho)] := [x_0 + K_\rho] \times (t_0 - \tau, t_0) \ .$$

The energy and logarithmic estimates that we are about to present will be written for a cylinder with "vertex" at $(0, 0)$. It is easy to obtain analogous estimates for a cylinder with "vertex" at a generic point (x_0, t_0).

Consider a cylinder $Q(\tau, \rho) \subset \Omega_T$, and piecewise smooth cutoff functions ξ in $Q(\tau, \rho)$ such that $0 \leq \xi \leq 1$, $|\nabla \xi| < \infty$ and $\xi = 0$, $x \notin K_\rho$. Let $k < M$ and $\varphi = -(\theta_\epsilon - k)_- \zeta^p$ in (5). Integrating in time over $(-\tau, t)$ for $t \in (-\tau, 0)$, the first term gives

$$\begin{aligned}
-\int_{-\tau}^{t}\int_{K_\rho} \partial_t \left[\gamma_\epsilon(\theta_\epsilon)\right] \, ((\theta_\epsilon - k)_- \zeta^p) &= \int_{K_\rho}\int_{-\tau}^{t} \partial_t \left(\int_0^{(\theta_\epsilon - k)_-} \gamma_\epsilon'(k-s) s \, \mathrm{d}s \right) \zeta^p \\
= \int_{K_\rho\times\{t\}} \left(\int_0^{(\theta_\epsilon - k)_-} \gamma_\epsilon'(k-s) s \, \mathrm{d}s \right) \zeta^p &- \int_{K_\rho\times\{-\tau\}} \left(\int_0^{(\theta_\epsilon - k)_-} \gamma_\epsilon'(k-s) s \, \mathrm{d}s \right) \zeta^p \\
&- p \int_{-\tau}^{t}\int_{K_\rho} \left(\int_0^{(\theta_\epsilon - k)_-} \gamma_\epsilon'(k-s) s \, \mathrm{d}s \right) \zeta^{p-1} \partial_t \zeta \\
\geq \frac{1}{2} \int_{K_\rho\times\{t\}} (\theta_\epsilon - k)_-^2 \zeta^p &- 2(M+\lambda) \int_{K_\rho\times\{-\tau\}} (\theta_\epsilon - k)_- \zeta^p \\
&- 2p(M+\lambda) \int_{-\tau}^{t}\int_{K_\rho} (\theta_\epsilon - k)_- \zeta^{p-1} \partial_t \zeta \ , \qquad (6)
\end{aligned}$$

since we have, recalling (3),

$$\int_0^{(\theta_\epsilon-k)_-} \gamma'_\epsilon(k-s)s\,\mathrm{d}s \geq \int_0^{(\theta_\epsilon-k)_-} s\,\mathrm{d}s = \frac{1}{2}(\theta_\epsilon-k)_-^2$$

and

$$\int_0^{(\theta_\epsilon-k)_-} \gamma'_\epsilon(k-s)s\,\mathrm{d}s \leq (\theta_\epsilon-k)_- \int_0^{(\theta_\epsilon-k)_-} \gamma'_\epsilon(k-s)\,\mathrm{d}s$$

$$= (\theta_\epsilon-k)_-[\gamma_\epsilon(k)-\gamma_\epsilon(\theta_\epsilon)] \leq 2(M+\lambda)\,(\theta_\epsilon-k)_- \ .$$

Concerning the other term, we have

$$\int_{-\tau}^t \int_{K_\rho} |\nabla\theta_\epsilon|^{p-2}\nabla\theta_\epsilon\cdot\nabla[\,-(\theta_\epsilon-k)_-\zeta^p] = \int_{-\tau}^t \int_{K_\rho} |\nabla(\theta_\epsilon-k)_-\,\zeta|^p$$

$$-p\int_{-\tau}^t \int_{K_\rho} |\nabla\theta_\epsilon|^{p-2}\nabla\theta_\epsilon\cdot\nabla\zeta\,[\zeta^{p-1}(\theta_\epsilon-k)_-]$$

$$\geq \frac{1}{2}\int_{-\tau}^t \int_{K_\rho} |\nabla(\theta_\epsilon-k)_-\,\zeta|^p - C(p)\int_{-\tau}^t \int_{K_\rho} (\theta_\epsilon-k)_-^p\,|\nabla\zeta|^p\ , \tag{7}$$

using Young's inequality. Since $t\in(-\tau,0)$ is arbitrary, we can combine estimates (6) and (7) to obtain

Proposition 1. *Let θ_ϵ be a solution of* (5) *and $k<M$. Then there exists a constant C, that is independent of ϵ, such that for every cylinder $Q(\tau,\rho)\subset\Omega_T$,*

$$\sup_{-\tau<t<0}\int_{K_\rho\times\{t\}} (\theta_\epsilon-k)_-^2\xi^p + \int_{-\tau}^0\int_{K_\rho} |\nabla(\theta_\epsilon-k)_-\xi|^p \leq C\int_{-\tau}^0\int_{K_\rho}(\theta_\epsilon-k)_-^p|\nabla\xi|^p$$

$$+C\int_{K_\rho\times\{-\tau\}}(\theta_\epsilon-k)_-\xi^p + C\int_{-\tau}^0\int_{K_\rho}(\theta_\epsilon-k)_-\xi^{p-1}\xi_t\ . \tag{8}$$

Remark 1. *This estimate and the ones to follow in this section were first obtained in* [18]. *We reproduce them here for the sake of completeness. It is obvious that they are the same irrespective of the value of p being greater of less than two.*

The second and third terms in the right hand side of (8) involve a power 1 that is due to the fact that ${\gamma'}_\epsilon$ is not uniformly bounded above near the singularity. When $k>\epsilon$, we are above the singularity and the energy estimate reads

Proposition 2. *Let θ_ϵ be a solution of (5) and $k > \epsilon$. Then there exists a constant C, that is independent of ϵ, such that for every cylinder $Q(\tau,\rho) \subset \Omega_T$,*

$$\sup_{-\tau<t<0} \int_{K_\rho\times\{t\}} (\theta_\epsilon - k)_+^2 \xi^p + \int_{-\tau}^0 \int_{K_\rho} |\nabla(\theta_\epsilon - k)_+ \xi|^p \leq C \int_{-\tau}^0 \int_{K_\rho} (\theta_\epsilon - k)_+^p |\nabla \xi|^p$$

$$+C \int_{K_\rho\times\{-\tau\}} (\theta_\epsilon - k)_+^2 \xi^p + C \int_{-\tau}^0 \int_{K_\rho} (\theta_\epsilon - k)_+^2 \xi^{p-1} \xi_t \ . \tag{9}$$

Next, we obtain logarithmic estimates. Let a, b, c be real constants verifying $0 < c < a$. Define the nonnegative function

$$\begin{aligned} \psi^\pm_{\{a,b,c\}}(s) \ &\equiv \ \left(\ln\left\{\tfrac{a}{(a+c)-(s-b)_\pm}\right\}\right)_+ \\ &= \left\{ \begin{array}{lll} \ln\left\{\tfrac{a}{(a+c)\pm(b-s)}\right\} & \text{if} & b\pm c \lesseqgtr s \lesseqgtr b \pm (a+c) \\ \\ 0 & \text{if} & s \lesseqqgtr b \pm c \end{array} \right. \end{aligned}$$

whose first derivative is

$$\left(\psi^\pm_{\{a,b,c\}}\right)'(s) \ = \ \left\{ \begin{array}{lll} \frac{1}{(b-s)\pm(a+c)} & \text{if} & b\pm c \lesseqgtr s \lesseqgtr b \pm (a+c) \\ \\ 0 & \text{if} & s \lesseqgtr b \pm c \end{array} \right. \qquad \gtreqqless 0$$

and second derivative, off $s = b \pm c$, is

$$\left(\psi^\pm_{\{a,b,c\}}\right)'' = \left\{\left(\psi^\pm_{\{a,b,c\}}\right)'\right\}^2 \geq 0 \ .$$

Now, given a bounded function u in a cylinder $Q(\tau,\rho)$ and $k \in \mathbf{R}$, define the constant

$$H^\pm_{u,k} \equiv \operatorname*{ess\,sup}_{Q(\tau,\rho)} (u-k)_\pm \ .$$

The following function was introduced in [5] and since then has been used as a recurrent tool in the proof of results concerning the local behaviour of solutions of singular and degenerate PDE's:

$$\Psi^\pm\left(H^\pm_{u,k}, (u-k)_\pm, c\right) \equiv \psi^\pm_{\left\{H^\pm_{u,k},k,c\right\}}(u) \ , \quad 0 < c < H^\pm_{u,k} \ . \tag{10}$$

We will write it as $\psi^{\pm}(u)$, omitting the subscripts whose meaning will be clear from the context. This function is essential in the derivation of the following logarithmic estimates (see [18] for the details):

Proposition 3. *Let θ_ϵ be a solution of* (5)*, $k \in \mathbf{R}$ and $0 < c < H^-_{\theta_\epsilon,k}$. There exists a constant $C > 0$, that is independent of ϵ, such that for every cylinder $Q(\tau,\rho) \subset \Omega_T$,*

$$\sup_{-\tau<t<0} \int_{K_\rho\times\{t\}} \left[\psi^-(\theta_\epsilon)\right]^2 \xi^p \;\leq\; \int_{K_\rho\times\{-\tau\}} \left(\int_k^{\theta_\epsilon} 2\gamma_\epsilon'(s)\psi^-(s)(\psi^-)'(s)\,\mathrm{d}s\right)_+ \xi^p$$
$$+C\int_{-\tau}^0\int_{K_\rho} \psi^-(\theta_\epsilon)|(\psi^-)'(\theta_\epsilon)|^{2-p}|\nabla\xi|^p \;. \quad (11)$$

Remark 2. *In this estimate there is a dependence on ϵ through γ_ϵ'. We will see later how to get over this difficulty.*

Proposition 4. *Let θ_ϵ be a solution of* (5)*, $k > \epsilon$ and $0 < c < H^+_{\theta_\epsilon,k}$. There exists a constant $C > 0$, that is independent of ϵ, such that for every cylinder $Q(\tau,\rho) \subset \Omega_T$,*

$$\sup_{-\tau<t<0} \int_{K_\rho\times\{t\}} \left[\psi^+(\theta_\epsilon)\right]^2 \xi^p \leq \int_{K_\rho\times\{-\tau\}} \left[\psi^+(\theta_\epsilon)\right]^2 \xi^p$$
$$+C\int_{-\tau}^0\int_{K_\rho} \psi^+(\theta_\epsilon)\left[(\psi^+)'(\theta_\epsilon)\right]^{2-p}|\nabla\xi|^p \;. \quad (12)$$

3. The intrinsic geometry

The study of the interior regularity of the approximate solution, namely showing that it is continuous independently of ϵ, requires the choice of a geometry that somehow reflects the two singularities in the equation. This means that, instead of working with the standard parabolic cylinders, we have to consider cylinders whose dimensions are suitably stretched in a way that is directly related to the structure of the PDE. In order to simplify the notation, from now on we drop the subscript ϵ in θ_ϵ.

Given $R > 0$, sufficiently small such that

$$Q\left(R, R^{\frac{1}{2}}\right) \subset \Omega_T \;,$$

define

$$\mu^- := \operatorname*{ess\,inf}_{Q(R,R^{\frac12})} \theta \;\;;\quad \mu^+ := \operatorname*{ess\,sup}_{Q(R,R^{\frac12})} \theta \;\;;\quad \omega := \operatorname*{ess\,osc}_{Q(R,R^{\frac12})} \theta = \mu^+ - \mu^-$$

and then construct the cylinder

$$Q\left(a_0R^p, c_0R\right)\ , \qquad a_0=\left(\frac{\omega}{A}\right)^{(1-p)(2-p)}\quad ; \qquad c_0=\left(\frac{\omega}{B}\right)^{p-2}\ , \tag{13}$$

where $A=2^s$ and $B=2^{\bar{s}}$, for some $s, \bar{s}>1$ to be chosen. Observe that for $p=2$, $a_0=c_0=1$, and these are the standard parabolic cylinders, reflecting the natural homogeneity of the space and time variables.

We assume, without loss of generality, that $\omega \le 1$ and also that

$$\omega > \max\left\{AR^{\frac{1}{2-p}}\ ,\ BR^{\frac{1}{2(2-p)}}\right\}\ . \tag{14}$$

Then $Q(a_0R^p, c_0R) \subset Q(R, R^{\frac{1}{2}})$ and

$$\operatorname*{ess\,osc}_{Q(a_0R^p, c_0R)} \theta \le \omega\ . \tag{15}$$

Remark 3. *If (14) does not hold then the oscillation is comparable to the radius (in a way given by the reverse of the above inequality) and there is nothing to prove. Note also that (15) is not verified for any given cylinder, since its dimensions have to be defined in terms of the essential oscillation of the solution within it.*

We now consider the cube K_{c_0R} partitioned in disjoint subcubes, each one of them congruent to K_{d_*R}, with $d_*=(\frac{\omega}{2^{n_*+1}})^{p-2}$, for n_* to be determined:

$$[\bar{x}+K_{d_*R}]\ , \qquad \bar{x}\in K_{\mathcal{R}(\omega)}\ , \qquad \mathcal{R}(\omega):=c_0R-d_*R=L_1d_*R$$

$$L_1=\left(\frac{B}{2^{n_*+1}}\right)^{2-p}-1\ , \qquad B>2^{n_*+1}\ .$$

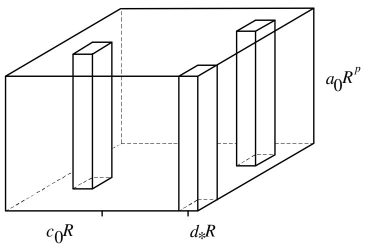

FIGURE 1. Partition in space of $Q(a_0R^p, c_0R)$.

Since we may arrange L_1 to be an integer, we can look at K_{c_0R} as the disjoint union, up to a set of measure zero, of L_1^N cubes of the above type. Then we may regard the cylinder $Q(a_0R^p, c_0R)$ as the disjoint union, up to a set of measure zero, of L_1^N subcylinders of the type $[(\bar{x}, 0) + Q(a_0R^p, d_*R)]$.

We next consider subcylinders of $[(\bar{x}, 0) + Q(a_0R^p, d_*R)]$ of the form

$$[(\bar{x}, \bar{t}) + Q(dR^p, d_*R)] \ , \qquad d = \left(\frac{\omega}{2}\right)^{(1-p)(2-p)} \ ,$$

which are contained in $[(\bar{x}, 0) + Q(a_0R^p, d_*R)]$ assuming that $A > 2$ and

$$\bar{t} \in \mathcal{I}(\omega) := (-a_0R^p + dR^p, 0) \ . \tag{16}$$

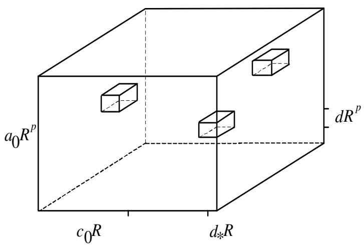

Figure 2. Partition in space and time of $Q(a_0R^p, c_0R)$.

The proof of the equicontinuity relies on the study of two complementary cases and the achievement of the same conclusion for both, namely the reduction of the oscillation. For a given constant $\nu_0 \in (0, 1)$, which will be determined later only in terms of the data and ω, we assume that either

the first alternative

there exists $\bar{t} \in \mathcal{I}(\omega)$ such that, for all $\bar{x} \in K_{\mathcal{R}(\omega)}$,

$$\frac{\left|(x,t) \in [(\bar{x}, \bar{t}) + Q(dR^p, d_*R)] \ : \ \theta(x,t) < \mu^- + \frac{\omega}{2}\right|}{|Q(dR^p, d_*R)|} \leq \nu_0 \tag{17}$$

or

the second alternative

for every $\bar{t} \in \mathcal{I}(\omega)$, there exists $\bar{x} \in K_{\mathcal{R}(\omega)}$ such that

$$\frac{\left|(x,t) \in [(\bar{x}, \bar{t}) + Q(dR^p, d_*R)] \ : \ \theta(x,t) > \mu^+ - \frac{\omega}{2}\right|}{|Q(dR^p, d_*R)|} < 1 - \nu_0 \ . \tag{18}$$

4. The first alternative

Assume that (17) holds in $[(\bar{x},\bar{t})+Q(dR^p,d_*R)]$, where $[\bar{x}+K_{d_*R}]$ is any subcube of the partition of K_{c_0R} and $\bar{t}$ is the same in all these cylinders. Since K_{c_0R} is the disjoint union, up to a set of measure zero, of subcubes of the form $[\bar{x}+K_{d_*R}]$, the first alternative implies that there exists a cylinder of the type $[(0,\bar{t})+Q(dR^p,c_0R)]$ in which

$$\frac{\left|(x,t)\in[(0,\bar{t})+Q(dR^p,c_0R)]\ :\ \theta(x,t)<\mu^-+\frac{\omega}{2}\right|}{|Q(dR^p,c_0R)|}\le\nu_0\ . \tag{19}$$

Lemma 1. *There exists a constant $\nu_0\in(0,1)$, depending only upon the data and ω, such that if (17) holds for some $\bar{t}\in\mathcal{I}(\omega)$ and all $\bar{x}\in K_{\mathcal{R}(\omega)}$,*

$$\theta(x,t)\ge\mu^-+\frac{\omega}{4}\ ,\qquad \text{a.e }(x,t)\in\left[(0,\bar{t})+Q\left(d\left(\frac{R}{2}\right)^p,c_0\frac{R}{2}\right)\right]\ . \tag{20}$$

Proof. According to what we have just remarked, (19) is in force. After translation, we may assume that $\bar{t}=0$. Define two decreasing sequences of numbers

$$R_n=\frac{R}{2}+\frac{R}{2^{n+1}}\ ,\qquad k_n=\mu^-+\frac{\omega}{4}+\frac{\omega}{2^{n+2}}\ ;\qquad n=0,1,2,\dots$$

and construct the family of nested and shrinking cylinders $Q_n\equiv Q(dR_n^p,c_0R_n)$. Write the energy estimate (8) for the functions $(\theta-k_n)_-$, over Q_n, and for smooth cutoff functions $0\le\xi_n\le1$, defined in Q_n, and such that

$$\begin{cases}\xi_n\equiv1\ \text{ in }\ Q_{n+1}\ ;\ \ \xi_n\equiv0\ \text{ on }\ \partial_pQ_n\\[2mm] |\nabla\xi_n|\le\frac{2^{n+2}}{c_0R}\ ;\qquad 0<\partial_t\xi_n\le\frac{2^{(n+2)p}}{dR^p}\end{cases}$$

Since $(\theta-k_n)_-\le\frac{\omega}{2}$, we get

$$\sup_{-dR_n^p<t<0}\int_{K_{c_0R_n}\times\{t\}}(\theta-k_n)_-^2\xi_n^p+\iint_{Q_n}|\nabla(\theta-k_n)_-\xi_n|^p$$

$$\le C\left(\frac{\omega}{2}\right)^p\frac{2^{(n+2)p}}{c_0^pR^p}\iint_{Q_n}\chi_{[\theta<k_n]}+C\left(\frac{\omega}{2}\right)\frac{2^{(n+2)p}}{dR^p}\iint_{Q_n}\chi_{[\theta<k_n]}$$

$$\le C\left(\frac{\omega}{2}\right)^p\frac{2^{(n+2)p}}{R^p}\left\{c_0^{-p}+d^{-1}\left(\frac{\omega}{2}\right)^{1-p}\right\}\iint_{Q_n}\chi_{[\theta<k_n]}\ .$$

Now observe that, since $1 < p < 2$, and $(\theta - k_n)_- \leq \frac{\omega}{2}$,

$$(\theta - k_n)_-^2 = (\theta - k_n)_-^{(1-p)(2-p)}(\theta - k_n)_-^{p(2-p)}(\theta - k_n)_-^p$$

$$\geq \left(\frac{\omega}{2}\right)^{(1-p)(2-p)}(\theta - k_n)_-^{p(2-p)}(\theta - k_n)_-^p = d(\theta - k_n)_-^{p(2-p)}(\theta - k_n)_-^p.$$

Introducing the level

$$k_{n+1} < \bar{k}_n \equiv \frac{k_n + k_{n+1}}{2} < k_n \ ,$$

we have

$$\int_{K_{c_0 R_n}} (\theta - k_n)_-^2 \xi_n^p \geq d \int_{K_{c_0 R_n}} (\theta - k_n)_-^{p(2-p)}(\theta - k_n)_-^p \xi_n^p$$

$$\geq d\left(k_n - \bar{k}_n\right)^{p(2-p)} \int_{K_{c_0 R_n}} (\theta - \bar{k}_n)_-^p \xi_n^p$$

$$= d\left(\frac{\omega}{2}\right)^{p(2-p)}\left(\frac{1}{2^{n+3}}\right)^{p(2-p)} \int_{K_{c_0 R_n}} (\theta - \bar{k}_n)_-^p \xi_n^p \ .$$

Consequently,

$$\sup_{-dR_n^p < t < 0} \int_{K_{c_0 R_n} \times \{t\}} (\theta - \bar{k}_n)_-^p \xi_n^p + \frac{1}{d}\left(\frac{2}{\omega}\right)^{p(2-p)} 2^{(n+3)p(2-p)} \iint_{Q_n} |\nabla(\theta - \bar{k}_n)_- \xi_n|^p$$

$$\leq C\left(\frac{\omega}{2}\right)^p \frac{2^{(n+2)p}}{dR^p} 2^{(n+3)p(2-p)} \left\{\left(\frac{\omega}{2}\right)^{p(p-2)} c_0^{-p} + \left(\frac{\omega}{2}\right)^{p(p-2)+(1-p)} d^{-1}\right\} \iint_{Q_n} \chi_{[\theta < k_n]}$$

$$= C\left(\frac{\omega}{2}\right)^p \frac{2^{(3-p)pn}}{dR^p} 2^{p(2+3(2-p))} \left\{\left(\frac{2}{B}\right)^{p(2-p)} + \frac{2}{\omega}\right\} \iint_{Q_n} \chi_{[\theta < k_n]}$$

$$\leq \frac{C}{\omega}\left(\frac{\omega}{2}\right)^p \frac{2^{(3-p)pn}}{dR^p} 2^{p(2+3(2-p))} \left\{\left(\frac{2}{B}\right)^{p(2-p)} + 2\right\} \iint_{Q_n} \chi_{[\theta < k_n]}$$

since $\omega \leq 1$.

Consider the change of variables

$$y = \frac{x}{c_0} \ , \qquad z = \frac{t}{d}$$

and define the new functions

$$\bar{\theta}(y, z) = \theta(x, t) \ , \qquad \bar{\xi}_n(y, z) = \xi_n(x, t) \ .$$

Denoting the new variables again by (x,t) and defining

$$A_n = \int_{-R_n^p}^{0} |A_n(t)|\ dt\ , \qquad A_n(t) = \left\{ x \in K_{R_n}\ :\ \bar\theta(x,t) < k_n \right\}\ ,$$

the above inequality reads

$$\sup_{-R_n^p<t<0} \int_{K_{R_n}} (\bar\theta - \bar k_n)_-^p \bar\xi_n^p + \left(\frac{2}{\omega}\right)^{p(2-p)} 2^{(n+3)p(2-p)} \iint_{Q(R_n^p,R_n)} \left|\nabla(\bar\theta - \bar k_n)_- \bar\xi_n\right|^p$$
$$\leq \frac{C}{\omega}\left(\frac{\omega}{2}\right)^p \frac{2^{(3-p)pn}}{R^p} 2^{p(2+3(2-p))} \left\{ \left(\frac{2}{B}\right)^{p(2-p)} + 2 \right\} A_n\ .$$

Recalling once again that $\omega \leq 1$, and the choice of ξ_n, we conclude that

$$\left\|(\bar\theta - \bar k_n)_-\right\|^p_{V^p(Q(R^p_{n+1},R_{n+1}))} \leq \frac{C}{\omega}\left(\frac{\omega}{2}\right)^p \frac{2^{(3-p)pn}}{R^p} 2^{p(2+3(2-p))} \left\{ \left(\frac{2}{B}\right)^{p(2-p)} + 2 \right\} A_n\ .$$

Now, on the one hand, we have

$$\iint_{Q(R^p_{n+1},R_{n+1})} (\bar\theta - \bar k_n)_-^p \geq (\bar k_n - k_{n+1})^p A_{n+1} = \left(\frac{\omega}{2}\right)^p \frac{1}{2^{p(n+3)}} A_{n+1}$$

and, on the other hand, using Corollary 3.1 of Chapter I of [7],

$$\iint_{Q(R^p_{n+1},R_{n+1})} (\bar\theta - \bar k_n)_-^p$$
$$\leq C \left|[\bar\theta < \bar k_n] \cap Q(R^p_{n+1},R_{n+1})\right|^{\frac{p}{N+p}} \ \left\|(\bar\theta - \bar k_n)_-\right\|^p_{V^p(Q(R^p_{n+1},R_{n+1}))}$$
$$\leq C A_n^{\frac{p}{N+p}} \left\|(\bar\theta - \bar k_n)_-\right\|^p_{V^p(Q(R^p_{n+1},R_{n+1}))}$$

Combining these two inequalities with the previous one, we get

$$A_{n+1} \leq \frac{C}{\omega} \frac{2^{p(5+3(2-p))}}{R^p} \left\{ \left(\frac{2}{B}\right)^{p(2-p)} + 2 \right\} 2^{(4-p)pn} A_n^{1+\frac{p}{N+p}}\ .$$

Defining

$$Y_n = \frac{A_n}{|Q(R_n^p,R_n)|}$$

and noting that

$$\frac{|Q(R_n^p,R_n)|^{1+\frac{p}{N+p}}}{|Q(R^p_{n+1},R_{n+1})|} \leq 2^{N+2p} R^p$$

we arrive at

$$Y_{n+1} \le \frac{C}{\omega}\left\{\left(\frac{2}{B}\right)^{p(2-p)} + 2\right\} 2^{(4-p)pn} Y_n^{1+\frac{p}{N+p}} \ .$$

Using a lemma on the fast geometric convergence of sequences (see [7, page 12]), we conclude that if

$$Y_0 \le \left(\frac{C}{\omega}\left\{\left(\frac{2}{B}\right)^{p(2-p)} + 2\right\}\right)^{\frac{-(N+p)}{p}} 2^{-p(4-p)\left(\frac{N+p}{p}\right)^2}$$

then $Y_n \to 0$ as $n \to \infty$. Since

$$\left\{\left(\frac{2}{B}\right)^{p(2-p)} + 2\right\}^{\frac{N+p}{p}} \le 2^{\frac{N}{p}}\left\{\left(\frac{2}{B}\right)^{(2-p)(N+p)} + 2^{\frac{N+p}{p}}\right\} \le 2^{\frac{N}{p}}\left\{1 + 2^{\frac{N+p}{p}}\right\} \ ,$$

if we take

$$Y_0 \le C^{-\frac{N+p}{p}} 2^{-p(4-p)\left(\frac{N+p}{p}\right)^2} \omega^{\frac{N+p}{p}} 2^{-\frac{N}{p}}\left\{1 + 2^{\frac{N+p}{p}}\right\}^{-1} = C\omega^{\frac{N+p}{p}} \equiv \nu_0 \qquad (21)$$

we obtain $Y_n \to 0$ as $n \to \infty$, which implies that $A_n \to 0$ as $n \to \infty$. But

$$Y_0 = \frac{\displaystyle\int_{-R^p}^{0} \left|x \in K_R \ : \ \bar{\theta}(x,t) < \mu^- + \frac{\omega}{2}\right| \ dt}{|Q(R^p, R)|} \ ,$$

so (21) is our hypothesis (19). Since $R_n \searrow \frac{R}{2}$ and $k_n \searrow \mu^- + \frac{\omega}{4}$, having $A_n \to 0$ as $n \to \infty$, means that

$$\left|(x,t) \in Q\left(\left(\frac{R}{2}\right)^p, \frac{R}{2}\right) \ : \ \bar{\theta}(x,t) < \mu^- + \frac{\omega}{4}\right| = 0$$

that is, going back to the original variables and function,

$$\theta(x,t) \ge \mu^- + \frac{\omega}{4} \ , \qquad \text{a.e. } \ (x,t) \in Q\left(d\left(\frac{R}{2}\right)^p, c_0\frac{R}{2}\right) \ .$$

■

Consider now the time level $-t_* = \bar{t} - d(\frac{R}{2})^p$. From the conclusion of Lemma 1, we have

$$\theta(x,-t_*) \ge \mu^- + \frac{\omega}{4} \ , \qquad \text{a.e. } \ x \in K_{c_0\frac{R}{2}} \ .$$

We will use this time level as an initial condition to bring the information up to $t = 0$, and therefore to obtain an analogous inequality in a full cylinder of the type $Q(\tau, c_0\rho)$. A first step in this direction is given by the following result.

Lemma 2. *Assume that (17) holds for some $\bar{t} \in \mathcal{I}(\omega)$ and for all $\bar{x} \in K_{\mathcal{R}(\omega)}$. Given $\nu_1 \in (0,1)$, there exists $1 < s_1 \in \mathbf{N}$, depending only upon the data and A, such that, if $B \geq 2^{s_1}$, then*

$$\left| x \in K_{c_0 \frac{R}{4}} \ : \ \theta(x,t) < \mu^- + \frac{\omega}{2^{s_1}} \right| < \nu_1 \left| K_{c_0 \frac{R}{4}} \right| \ , \qquad \forall t \in (-t_*, 0) \ .$$

Proof. Consider the cylinder $Q(t_*, c_0\frac{R}{2})$ and write the logarithmic estimate (11) over this cylinder, for the function $(\theta - k)_-$, with $k = \mu^- + \frac{\omega}{4}$ and $c = \frac{\omega}{2^{n+2}}$ (n to be chosen). Defining

$$k - \theta \leq H^-_{\theta,k} := \operatorname*{ess\,sup}_{Q(t_*, c_0\frac{R}{2})} (\theta - k)_- \leq \frac{\omega}{4} < \frac{\omega}{2}$$

(that we may assume strictly positive, otherwise there is nothing to prove), we then choose n sufficiently large so that $0 < c < H^-_{\theta,k}$. Then the logarithmic function $\psi^- \equiv \psi^-(\theta)$, introduced in (10), is well defined and satisfies the estimates

$$\psi^- \leq \ln(2^n) = n \ln 2 \ , \qquad \text{because} \quad \frac{H^-_{\theta,k}}{H^-_{\theta,k} + \theta - k + c} \leq \frac{\frac{\omega}{4}}{\frac{\omega}{2^{n+2}}} = 2^n;$$

$$\begin{aligned} |(\psi^-)'|^{2-p} &= \left(\frac{1}{H^-_{\theta,k} + \theta - k + c} \right)^{2-p} \\ &= \left(H^-_{\theta,k} + \theta - k + c \right)^{(p-1)(2-p)} \left(\frac{1}{H^-_{\theta,k} + \theta - k + c} \right)^{p(2-p)} \\ &\leq \left(\frac{\omega}{2} \right)^{(p-1)(2-p)} \left(\frac{1}{c} \right)^{p(2-p)} = d^{-1} \left(\frac{2^{n+2}}{\omega} \right)^{p(2-p)} \ , \end{aligned}$$

since $c \leq H^-_{\theta,k} + \theta - k + c \leq H^-_{\theta,k} < \frac{\omega}{2}$ and $0 < p - 1 < 1$.

Take as a cutoff function $x \rightarrow \xi(x)$ (independent of t), defined in $K_{c_0\frac{R}{2}}$, and satisfying

$$\begin{cases} 0 \leq \xi \leq 1 \quad \text{in} \quad K_{c_0\frac{R}{2}} \\ \xi \equiv 1 \quad \text{in} \quad K_{c_0\frac{R}{4}} \\ |\nabla \xi| \leq \frac{4}{c_0 R} \ . \end{cases}$$

The logarithmic estimate takes the form

$$\sup_{-t_*<t<0}\int_{K_{c_0\frac{R}{2}}}(\psi^-)^2\xi^p \le C\int_{-t_*}^{0}\int_{K_{c_0\frac{R}{2}}}\psi^-|(\psi^-)'|^{2-p}|\nabla\xi|^p$$

$$\le Cn\ln 2\, d^{-1}\left(\frac{2^{n+2}}{\omega}\right)^{p(2-p)}\frac{4^p}{c_0^pR^p}\left|K_{c_0\frac{R}{2}}\right| t_*\ ,$$

since $\theta(x,-t_*)\ge k$ in the cube $K_{c_0\frac{R}{2}}$ which implies that $\psi^-(x,-t_*)=0$, for $x\in K_{c_0\frac{R}{2}}$. Recalling that $t_*<a_0R^p$, and taking $B\ge 2^{n+2}$, we get

$$\sup_{-t_*<t<0}\int_{K_{c_0\frac{R}{2}}}(\psi^-)^2\xi^p \le CA^{(p-1)(2-p)}n\left|K_{c_0\frac{R}{4}}\right|\ .$$

We estimate from below the left hand side of the above inequality integrating over the smaller set

$$\left\{x\in K_{c_0\frac{R}{4}}\ :\ \theta(x,t)<\mu^-+\frac{\omega}{2^{n+2}}\right\}\subset K_{c_0\frac{R}{2}}\ .$$

In this set, $\xi\equiv 1$ and

$$\psi^-=\ln\left(\frac{H^-_{\theta,k}}{H^-_{\theta,k}+\theta-k+c}\right)\ge\ln\left(\frac{H^-_{\theta,k}-\frac{\omega}{4}+\frac{\omega}{4}}{H^-_{\theta,k}-\frac{\omega}{4}+\frac{\omega}{2^{n+1}}}\right)\ge(n-1)\ln 2$$

since $H^-_{\theta,k}-\frac{\omega}{4}\le 0$. Then, for all $t\in(-t_*,0)$,

$$\left|x\in K_{c_0\frac{R}{4}}\ :\ \theta(x,t)<\mu^-+\frac{\omega}{2^{n+2}}\right|\le CA^{(p-1)(2-p)}\frac{n}{(n-1)^2}\left|K_{c_0\frac{R}{4}}\right|\ .$$

Choosing $n>1+\frac{2C}{\nu_1}A^{(p-1)(2-p)}$ we get $CA^{(p-1)(2-p)}\frac{n}{(n-1)^2}<\nu_1$ and the result is proved for the choice $s_1=n+2$.

■

Remark 4. *Note that the dependency of s_1 on ω, if it occurs, occurs through A; it will be shown that A can be determined independently of ω. Observe also that the ϵ dependency appearing in the logarithmic estimate was overcome.*

We are now in position to prove the main result of this section, namely

Proposition 5. *There exist constants $\nu_0\in(0,1)$, depending upon the data and ω, and $1<s_1\in\mathbf{N}$, depending only upon the data and A, such that, if*

(17) holds for some $\bar{t} \in \mathcal{I}(\omega)$ *and for all* $\bar{x} \in K_{\mathcal{R}(\omega)}$*, then*

$$\theta(x,t) \geq \mu^- + \frac{\omega}{2^{s_1+1}} , \qquad \text{a.e.} \quad (x,t) \in Q\left(t_*, c_0\frac{R}{8}\right) . \tag{22}$$

Proof. Consider the decreasing sequence of real numbers

$$R_n = \frac{R}{8} + \frac{R}{2^{n+3}} , \qquad n = 0, 1, \dots$$

and construct the family of nested and shrinking cylinders $Q_n = Q(t_*, c_0 R_n)$, where t_* is given as before. Write the energy estimates (8) for the functions $(\theta - k_n)_-$ over Q_n, with

$$k_n = \mu^- + \frac{\omega}{2^{s_1+1}} + \frac{\omega}{2^{s_1+1+n}} ,$$

and choosing piecewise smooth cutoff functions $\xi_n(x)$ defined in $K_{c_0 R_n}$ and satisfying, for $n = 0, 1, 2, \dots$,

$$\begin{cases} 0 \leq \xi_n \leq 1 \quad \text{in} \quad K_{c_0 R_n} \\ \xi_n \equiv 1 \quad \text{in} \quad K_{c_0 R_{n+1}} \\ |\nabla \xi_n| \leq \frac{2^{n+4}}{c_0 R} . \end{cases}$$

Since, for all $n = 0, 1, \dots$,

$$\theta(x, -t_*) \geq \mu^- + \frac{\omega}{4} > k_n , \qquad \text{for} \quad x \in K_{c_0 \frac{R}{2}} \supset K_{c_0 R_n}$$

we have

$$\int_{K_{c_0 R_n}} (\theta(\cdot, -t_*) - k_n)_- \, \xi_n^p = 0 ,$$

and consequently the energy estimates read

$$\sup_{-t_* < t < 0} \int_{K_{c_0 R_n}} (\theta - k_n)_-^2 \xi_n^p + \iint_{Q_n} |\nabla (\theta - k_n)_- \xi_n|^p$$

$$\leq C \iint_{Q_n} (\theta - k_n)_-^p |\nabla \xi_n|^p \leq C \left(\frac{\omega}{2^{s_1}}\right)^p \frac{2^{(n+4)p}}{c_0^p R^p} \iint_{Q_n} \chi_{[\theta < k_n]} .$$

Reasoning as in the proof of Lemma 1, we estimate from below the left hand side in the following way: letting

$$k_{n+1} < \bar{k}_n \equiv \frac{k_{n+1} + k_n}{2} < k_n$$

then, since $1 < p < 2$, $(\theta - k_n)_- \le \frac{\omega}{2^{s_1}}$ and $t_* \le a_0 R^p$, we have

$$(\theta - k_n)_-^2 \ge \left(\frac{\omega}{2^{s_1}}\right)^{(1-p)(2-p)} \frac{(\frac{R}{2})^p}{t_*}\frac{t_*}{(\frac{R}{2})^p} \left(\frac{\omega}{2^{s_1}}\right)^{p(2-p)} 2^{(n+3)p(p-2)}(\theta - \bar{k}_n)_-^p$$

$$\ge 2^{-p}\left(\frac{\omega}{2^{s_1}}\right)^{(1-p)(2-p)} \left(\frac{\omega}{A}\right)^{(p-1)(2-p)} \left(\frac{\omega}{2^{s_1}}\right)^{p(2-p)} 2^{(n+3)p(p-2)}\frac{t_*}{(\frac{R}{2})^p}(\theta - \bar{k}_n)_-^p$$

$$\ge 2^{-p}\left(\frac{2^{s_1}}{A}\right)^{(p-1)(2-p)} \left(\frac{\omega}{B}\right)^{p(2-p)} 2^{(n+3)p(p-2)}\frac{t_*}{(\frac{R}{2})^p}(\theta - \bar{k}_n)_-^p$$

$$\ge c_0^{-p} 2^{(n+3)p(p-2)}\frac{t_*}{(\frac{R}{2})^p}(\theta - \bar{k}_n)_-^p$$

if we choose $B \ge 2^{s_1}$ and s_1 such that

$$s_1 > \log_2 A + \frac{p}{(p-1)(2-p)}.$$

Therefore the above integral inequality takes the form

$$\sup_{-t_*<t<0}\int_{K_{c_0R_n}} (\theta - \bar{k}_n)_-^p \xi_n^p + c_0^p 2^{(n+3)p(2-p)}\frac{(\frac{R}{2})^p}{t_*}\iint_{Q_n} \left|\nabla(\theta - \bar{k}_n)_-\xi_n\right|^p$$

$$\le C\left(\frac{\omega}{2^{s_1}}\right)^p \frac{2^{(3-p)pn}}{(\frac{R}{2})^p} 2^{3p(3-p)}\frac{(\frac{R}{2})^p}{t_*}\iint_{Q_n} \chi_{[\theta<k_n]} \ .$$

Introducing the change of variables

$$y = \frac{x}{c_0}\ , \qquad z = \left(\frac{R}{2}\right)^p \frac{t}{t_*}$$

and defining the new functions

$$\bar{\theta}(y,z) = \theta(x,t)\ , \qquad \bar{\xi}_n(y) = \xi_n(x)\ ,$$

we write the inequality in the new variables (again denoted by (x,t)), obtaining

$$\sup_{-(\frac{R}{2})^p<t<0}\int_{K_{R_n}} (\bar{\theta} - \bar{k}_n)_-^p \bar{\xi}_n^p + c_0^p 2^{(n+3)p(2-p)}\iint_{Q((\frac{R}{2})^p,R_n)} \left|\nabla(\bar{\theta} - \bar{k}_n)_-\bar{\xi}_n\right|^p$$

$$\le C\left(\frac{\omega}{2^{s_1}}\right)^p \frac{2^{(3-p)pn}}{(\frac{R}{2})^p} 2^{3p(3-p)}\iint_{Q((\frac{R}{2})^p,R_n)} \chi_{[\bar{\theta}<k_n]} \ .$$

Defining

$$A_n = \int_{-(\frac{R}{2})^p}^{0} |A_n(t)|\ dt\ , \qquad A_n(t) = \left\{ x \in K_{R_n}\ :\ \bar{\theta}(x,t) < k_n \right\}$$

and recalling the definition of c_0 and that $\omega \le 1$, we arrive at

$$\left\| \bar{\theta} - \bar{k}_n)_- \right\|^p_{V^p(Q((\frac{R}{2})^p, R_{n+1}))} \le C \left(\frac{\omega}{2^{s_1}} \right)^p \frac{2^{(3-p)pn}}{(\frac{R}{2})^p} 2^{3p(3-p)} A_n\ .$$

Now, since

$$\left(\frac{\omega}{2^{s_1}} \right)^p \frac{1}{2^{(n+3)p}} A_{n+1} = (\bar{k}_n - k_{n+1})^p A_{n+1}$$

$$\le \iint_{Q((\frac{R}{2})^p, R_{n+1})} (\bar{\theta} - \bar{k}_n)_-^p \le C A_n^{\frac{p}{N+p}} \left\| \bar{\theta} - \bar{k}_n)_- \right\|^p_{V^p(Q((\frac{R}{2})^p, R_{n+1}))}\ ,$$

we get

$$A_{n+1} \le C \frac{2^{(4-p)pn}}{(\frac{R}{2})^p} 2^{3p(4-p)} A_n^{1+\frac{p}{N+p}}\ .$$

Define $Y_n = \frac{A_n}{|Q((\frac{R}{2})^p, R_n)|}$. Due to the fact that

$$\frac{|Q((\frac{R}{2})^p, R_n)|^{1+\frac{p}{N+p}}}{|Q((\frac{R}{2})^p, R_{n+1})|} \le 2^{N+p} \left(\frac{R}{2} \right)^p$$

we get the algebraic inequality

$$Y_{n+1} \le C 2^{(4-p)pn} Y_n^{1+\frac{p}{N+p}}$$

so, as before, if $Y_0 \le C^{-\frac{N+p}{p}} 2^{-p(4-p)(\frac{N+p}{p})^2} \equiv \nu_1$ then $Y_n \to 0$ as $n \to \infty$. Using Lemma 2 for this value of ν_1 we conclude that there exists $1 < s_1 \in \mathbf{N}$ such that

$$\left| x \in K_{c_0 \frac{R}{4}}\ :\ \theta < \mu^- + \frac{\omega}{2^{s_1}} \right| < \nu_1 \left| K_{c_0 \frac{R}{4}} \right|\ , \qquad \forall t \in (-t_*, 0)\ .$$

To conclude, note that

$$Y_0 = \frac{\displaystyle\int_{-(\frac{R}{2})^p}^{0} \left| x \in K_{\frac{R}{4}}\ :\ \bar{\theta}(x,t) < \mu^- + \frac{\omega}{2^{s_1}} \right|\ dt}{|Q((\frac{R}{2})^p, \frac{R}{4})|}$$

$$= \frac{(\frac{R}{2})^p}{t_*} \frac{\int_{-t_*}^{0} \left| x \in K_{c_0 \frac{R}{4}} \ : \ \theta(x,t) < \mu^- + \frac{\omega}{2^{s_1}} \right| \ dt}{(\frac{R}{2})^p |K_{c_0 \frac{R}{4}}|} \leq \nu_1$$

so Y_n , $A_n \to 0$ as $n \to \infty$. Since $R_n \searrow \frac{R}{8}$ and $k_n \searrow \mu^- + \frac{\omega}{2^{s_1+1}}$, we obtain

$$\theta(x,t) \geq \mu^- + \frac{\omega}{2^{s_1+1}} \ , \qquad \text{a.e.} \quad (x,t) \in Q\left(t_*, c_0 \frac{R}{8}\right) \ .$$

■

Corollary 1. *Assume that (17) holds for some $\bar{t} \in \mathcal{I}(\omega)$ and for all $\bar{x} \in K_{\mathcal{R}(\omega)}$. Then there exist constants $\nu_0 \in (0,1)$, depending only upon the data and ω, and $\sigma_0 \in (0,1)$, depending only upon the data and A, such that*

$$\underset{Q(d(\frac{R}{2})^p, c_0 \frac{R}{8})}{ess\ osc} \ \theta \leq \sigma_0 \omega \ . \tag{23}$$

Proof. The proof is trivial, recalling that

$$-t_* = \bar{t} - d\left(\frac{R}{2}\right)^p < -d\left(\frac{R}{2}\right)^p$$

from which follows $Q\left(t_*, c_0 \frac{R}{8}\right) \supset Q\left(d(\frac{R}{2})^p, c_0 \frac{R}{8}\right)$.

■

5. The second alternative

If **the first alternative** does not hold then **the second alternative** is in force. We will show that, in this case, we can achieve a conclusion similar to (23). Note that the constant ν_0 has already been determined and is given by (21).

Lemma 3. *Fix $\bar{t} \in \mathcal{I}(\omega)$ and $\bar{x} \in K_{\mathcal{R}(\omega)}$ for which (18) holds. There exists a time level*

$$t_0 \in \left[\bar{t} - dR^p, \bar{t} - \frac{\nu_0}{2} dR^p\right] \tag{24}$$

such that

$$\left| x \in [\bar{x} + K_{d_* R}] \ : \ \theta(x,t_0) > \mu^+ - \frac{\omega}{2} \right| \leq \left(\frac{1-\nu_0}{1-\frac{\nu_0}{2}}\right) |K_{d_* R}| \ . \tag{25}$$

Proof. Suppose not. Then, for all $t \in [\bar{t} - dR^p, \bar{t} - \frac{\nu_0}{2} dR^p]$,

$$\begin{aligned}
&\left| (x,t) \in [(\bar{x},\bar{t}) + Q(dR^p, d_* R)] \ : \ \theta(x,t) > \mu^+ - \frac{\omega}{2} \right| \\
&\geq \int_{\bar{t}-dR^p}^{\bar{t}-\frac{\nu_0}{2}dR^p} \left| x \in [\bar{x} + K_{d_* R}] \ : \ \theta(x,\tau) > \mu^+ - \frac{\omega}{2} \right| \, \mathrm{d}\tau \\
&> \left(\frac{1-\nu_0}{1-\frac{\nu_0}{2}} \right) |K_{d_* R}| \left(1 - \frac{\nu_0}{2} \right) dR^p = (1-\nu_0) \, |Q(dR^p, d_* R)|
\end{aligned}$$

which contradicts (18).

■

This result tells us that, at the time level t_0, the portion of the cube $[\bar{x} + K_{d_* R}]$ where $\theta(x)$ is near its supremum is small. In what follows we prove that the same happens at all time levels near the top of the cylinder $[(\bar{x},\bar{t}) + Q(dR^p, d_* R)]$.

Lemma 4. *There exists $1 < s_2 \in \mathbf{N}$, depending only upon the data and ω, such that*

$$\left| x \in [\bar{x} + K_{d_* R}] \ : \ \theta(x,t) > \mu^+ - \frac{\omega}{2^{s_2}} \right| < \left(1 - \left(\frac{\nu_0}{2} \right)^2 \right) |K_{d_* R}| \ , \qquad (26)$$

for all $t \in (t_0, \bar{t})$.

Proof. Assume, without loss of generality, that $\bar{x} \equiv 0$. Consider the logarithmic estimate (12) written over the cylinder $K_{d_* R} \times (t_0, \bar{t})$, for the function $(\theta - k)_+$, with $k = \mu^+ - \frac{\omega}{2}$ and $c = \frac{\omega}{2^{n_*+1}}$ (n_* to be chosen). Assuming that the number n_* has been chosen, we determine the length of the cube $K_{d_* R}$ by choosing

$$d_* = \left(\frac{\omega}{2^{n_*+1}} \right)^{p-2} \ .$$

In the definition of ψ^+ take

$$\theta - k \leq H^+_{\theta,k} := \operatorname*{ess\,sup}_{K_{d_* R} \times (t_0,\bar{t})} \ (\theta - k)_+ \leq \frac{\omega}{2} \ .$$

Assume that $H^+_{\theta,k} > 0$ and then choose n_* large enough so that $0 < c = \frac{\omega}{2^{n_*+1}} < H^+_{\theta,k}$. Since $H^+_{\theta,k} - \theta + k + c > 0$, the logarithmic function ψ^+ is well defined and satisfies the estimates

$$\psi^+ \leq n_* \ln 2 \ , \quad \text{since} \quad \frac{H^+_{\theta,k}}{H^+_{\theta,k} - \theta + k + c} \leq \frac{\frac{\omega}{2}}{\frac{\omega}{2^{n_*+1}}} = 2^{n_*};$$

$$\left[\left(\psi^+\right)'\right]^{2-p} = \left(H^+_{\theta,k} - \theta + k + c\right)^{(p-1)(2-p)} \left(\frac{1}{H^+_{\theta,k} - \theta + k + c}\right)^{p(2-p)}$$
$$\leq d^{-1}\left(\frac{2^{n_*+1}}{\omega}\right)^{p(2-p)} = d^{-1} d_*^p\ ,$$

for the non trivial case $\theta > k + c$.

Take as a cutoff function $x \to \xi(x)$, defined in $K_{d_* R}$, and satisfying

$$\begin{cases} 0 \leq \xi \leq 1 \text{ in } K_{d_* R} \\ \xi \equiv 1 \text{ in } K_{(1-\sigma)d_* R}\ , \text{ for some } \sigma \in (0,1) \\ |\nabla \xi| \leq (\sigma d_* R)^{-1}\ . \end{cases}$$

The logarithmic estimates read

$$\sup_{t_0<t<\bar{t}} \int_{K_{d_* R}\times\{t\}} \left(\psi^+\right)^2 \xi^p \leq \int_{K_{d_* R}\times\{t_0\}} \left(\psi^+\right)^2 \xi^p$$
$$+C\int_{t_0}^{\bar{t}}\int_{K_{d_* R}} \psi^+ \left[\left(\psi^+\right)'\right]^{2-p} |\nabla\xi|^p$$

and therefore

$$\begin{aligned} \sup_{t_0<t<\bar{t}} \int_{K_{d_* R}\times\{t\}} \left(\psi^+\right)^2 \xi^p \ &\leq\ n_*^2(\ln 2)^2\, |x \in K_{d_* R}\ :\ \theta(x,t_0) > k + c| \\ &\quad + \frac{C}{\sigma^p d_*^p R^p}\, n_* \ln 2\, d^{-1} d_*^p\, |K_{d_* R}|\, (\bar{t} - t_0) \\ &\leq\ \left\{ n_*^2(\ln 2)^2 \left(\frac{1-\nu_0}{1-\frac{\nu_0}{2}}\right) + C\frac{n_*}{\sigma^p}\right\} |K_{d_* R}|\ , \end{aligned}$$

using Lemma 3, the estimates for ψ presented above, and the fact that $\bar{t} - t_0 \leq dR^p$. In order to bound the left hand side from below, we integrate over the smaller set

$$S = \left\{ x \in K_{(1-\sigma)d_* R}\ :\ \theta(x,t) > \mu^+ - \frac{\omega}{2^{n_*+1}} \right\} \subset K_{(1-\sigma)d_* R}$$

where $\xi \equiv 1$ and $\psi^+ \geq (n_* - 1)\ln 2$, since

$$\frac{H^+_{\theta,k}}{H^+_{\theta,k} - \frac{\omega}{2} + \frac{\omega}{2^{n_*}}} = \frac{H^+_{\theta,k} - \frac{\omega}{2} + \frac{\omega}{2}}{H^+_{\theta,k} - \frac{\omega}{2} + \frac{\omega}{2^{n_*}}} \geq 2^{n_*-1}\ , \quad \text{because} \quad H^+_{\theta,k} - \frac{\omega}{2} \leq 0\ .$$

We obtain

$$|S| \le \left\{ \left(\frac{n_*}{n_*-1} \right)^2 \left(\frac{1-\nu_0}{1-\frac{\nu_0}{2}} \right) + \frac{C}{n_* \sigma^p} \right\} |K_{d_*R}|.$$

Consequently, for all $t \in (t_0, \bar{t})$, we have

$$\begin{aligned} \left| x \in K_{d_*R} : \theta(x,t) > \mu^+ - \frac{\omega}{2^{n_*+1}} \right| &\le |S| + \left| K_{d_*R} \setminus K_{(1-\sigma)d_*R} \right| \\ &\le |S| + N\sigma |K_{d_*R}| \\ &\le \left\{ \left(\frac{n_*}{n_*-1} \right)^2 \left(\frac{1-\nu_0}{1-\frac{\nu_0}{2}} \right) + \frac{C}{n_* \sigma^p} + N\sigma \right\} |K_{d_*R}|. \end{aligned}$$

Choose σ so small that $N\sigma \le \frac{3}{8}\nu_0^2$ and then n_* so large that

$$\frac{C}{n_* \sigma^p} \le \frac{3}{8}\nu_0^2 \qquad \text{and} \qquad \left(\frac{n_*}{n_*-1} \right)^2 \le \left(1 - \frac{\nu_0}{2} \right) (1+\nu_0) \equiv \beta \ .$$

With these choices we obtain

$$\left| x \in K_{d_*R} \ : \ \theta(x,t) > \mu^+ - \frac{\omega}{2^{n_*+1}} \right| \le \left(1 - \left(\frac{\nu_0}{2} \right)^2 \right) |K_{d_*R}| \ , \quad \forall t \in (t_0, \bar{t})$$

and the result follows with $s_2 = n_* + 1$.

■

Remark 5. *From the choice $\sigma \le \frac{3}{8N}\nu_0^2$, we see that, in order to satisfy the first condition, it suffices to choose the number n_* such that*

$$n_* \ge C \nu_0^{-2(p+1)} \ ,$$

where C is a constant depending only upon the data. From the second condition on n_* we get

$$n_* \ge \frac{\beta + \sqrt{\beta}}{\beta - 1} \ ,$$

and, since $\beta > 1$ (and assuming, without loss of generality, that $\nu_0 \in (0, \frac{1}{2})$),

$$\frac{\beta + \sqrt{\beta}}{\beta - 1} \le \frac{2\beta}{\beta - 1} = \frac{4}{\nu_0(1-\nu_0)} + 2 \le \frac{4}{\nu_0^2} + 2$$

and it suffices to choose

$$n_* \ge \frac{4}{\nu_0^2} + 2 \ .$$

So n_* is to be chosen such that

$$n_* \geq \max\left\{C\nu_0^{-2(p+1)}, \frac{4}{\nu_0^2} + 2\right\} \ .$$

Recalling that $\nu_0 = C\omega^{\frac{N+p}{p}}$, we choose

$$n_* \geq C\omega^{-\alpha} \ , \qquad \alpha = \frac{2(p+1)(N+p)}{p}$$

and n_* depends upon the data and ω.

Remark 6. *This result determines the value s_2 and consequently d_*, which represents the size of the subcubes $[\bar{x} + K_{d_*R}]$ making up the partition of the full cube K_{c_0R}; it has a double scope: we determine a level and a cylinder such that the measure of the set where θ is above such a level can be made small on that particular cylinder.*

Remark 7. *In the proof we assumed that $H_k^+ > 0$. If $H_k^+ = 0$ we argue as in the next section, with $\bar{\theta} = (\theta - \mu^+)\frac{1}{\omega}$.*

Since for the different values of $\bar{t}$ we may get cylinders with different axes, we start by expanding the *negativity* to a complete cylinder in space and then use the fact that $\bar{t}$ is an arbitrary element of $\mathcal{I}(\omega)$ to get a reduction of the oscillation in a smaller cylinder centered at the origin. In order to do so, and since the location of $\bar{x}$ within the cube $K_{\mathcal{R}(\omega)}$ is only known qualitatively, we will assume that $\bar{x}$ is the centre of the larger cube $[\bar{x}+K_{8c_0R}]$ which we assume to be contained in $K_{R^{\frac{1}{2}}}$. Indeed, if that is not the case, there is nothing to prove since

$$\omega \leq C(B,p)\, R^{\frac{1}{2(2-p)}} \ .$$

We then work within the cylinder

$$[\bar{x} + K_{8c_0R}] \times (t_0, \bar{t})$$

which is mapped into $Q_4 \equiv K_4 \times (-4^p, 0)$ through the change of variables

$$y = \frac{x - \bar{x}}{2c_0R} \ , \qquad z = 4^p\left(\frac{t - \bar{t}}{\bar{t} - t_0}\right) \ .$$

With this same mapping, the cube $[\bar{x} + K_{d_*R}]$ is mapped into K_{h_0}, where $h_0 = \frac{1}{2}\left(\frac{2^{s_2}}{B}\right)^{2-p} < 1$. Define the new function

$$\bar{\theta} = (\theta - \mu^+)\left(\frac{2^{n_*}}{\omega}\right)$$

and observe that $\bar\theta \le 0$ and that, for the new variables and function, (26) is now written in the form

$$\left| y \in K_{h_0} \ : \ \bar\theta(y,z) < -\frac{1}{2} \right| > \left(\frac{\nu_0}{2}\right)^2 |K_{h_0}| \ , \quad \forall z \in (-4^p, 0) \ . \tag{27}$$

5.1. An equation in dimensionless form. The new function satisfies, in the sense of the distributions, an equation similar to that satisfied by θ, namely (denoting again the new variables by (x,t))

$$\partial_t \tilde\gamma_\epsilon(\bar\theta) - C \operatorname{div}\left(|\nabla\bar\theta|^{p-2}\nabla\bar\theta\right) = 0 \qquad \text{in} \quad \mathcal{D}'(Q_4) \tag{28}$$

where $\tilde\gamma_\epsilon$ is such that $\tilde\gamma_\epsilon{}'(\bar\theta) = \gamma_\epsilon'(\theta)$ and

$$C = \frac{1}{2^{3p}} \omega^{(p-1)(2-p)} \left(\frac{2^{n_*}}{B^p}\right)^{2-p} \frac{(\bar t - t_0)}{R^p} \ .$$

Since $t_0 \in \left[\bar t - dR^p, \bar t - \frac{\nu_0}{2} dR^p\right]$,

$$C \in \left[\frac{1}{2^{3p}} \left(\frac{2^{n_*+p-1}}{B^p}\right)^{2-p} \frac{\nu_0}{2}, \frac{1}{2^{3p}} \left(\frac{2^{n_*+p-1}}{B^p}\right)^{2-p} \right] \ .$$

In order to simplify the calculations we will assume, for the time being, that

$$\partial_t \tilde\gamma_\epsilon\left(\bar\theta\right) \in C\left(-4^p, 0; L^1(K_4)\right) \tag{29}$$

and will remove this assumption later. The weak formulation of (28) is then given by

$$\int_{K_4} \partial_t \tilde\gamma_\epsilon(\bar\theta)\varphi + C \int_{K_4} \left|\nabla\bar\theta\right|^{p-2} \nabla\bar\theta \cdot \nabla\varphi = 0 \ ,$$

for all $t \in (-4^p, 0)$ and for all $\varphi \in C(Q_4) \cap C(-4^p, 0; W_0^{1,p}(K_4))$.

Due to the fact that the equation is weakly parabolic and $\bar\theta$ is a solution, $(\bar\theta - k)_+$ is a sub-solution and then, for all admissible test functions $\varphi \ge 0$,

$$\int_{K_4} \partial_t \tilde\gamma_\epsilon \left[(\bar\theta - k)_+\right] \varphi + C \int_{K_4} \left|\nabla(\bar\theta - k)_+\right|^{p-2} \nabla(\bar\theta - k)_+ \cdot \nabla\varphi \le 0 \ ,$$

for all $t \in (-4^p, 0)$. In this inequality we take

$$\varphi = \frac{\xi^p}{\left(-k(1-\delta) - (\bar\theta - k)_+\right)^{p-1}}$$

where k and $\delta \in (-1,0)$ are to be chosen and $\xi(x,t) = \xi_1(x)\xi_2(t)$ is a piecewise smooth cutoff function defined in Q_4 and satisfying

$$\begin{cases} 0 \leq \xi \leq 1 \ \text{ in } Q_4 \ ; \qquad \xi \equiv 1 \ \text{ in } Q_2 \ ; \qquad \xi \equiv 0 \ \text{ on } \partial_p Q_4 \ ; \\ \\ \qquad |\nabla \xi_1| \leq 1 \ ; \qquad\quad 0 \leq \partial_t \xi_2 \leq 1 \ ; \end{cases}$$

and the property that the sets $\{x \in K_4 \ : \ \xi_1(x) > -k\}$ are convex for all $k \in (-1,0)$. Set

$$\psi_k(\bar\theta) = \ln\left(\frac{-k(1-\delta)}{-k(1-\delta) - (\bar\theta - k)_+}\right) \geq 0 \tag{30}$$

and

$$\phi_k(\bar\theta) = \int_0^{(\bar\theta - k)_+} \frac{1}{(-k(1-\delta) - s)^{p-1}}\, \mathrm{d}s \tag{31}$$

and observe that

1. for $k, \delta \in (-1,0)$, the functions φ, ψ_k and ϕ_k are well defined since
$$-k(1-\delta) - (\bar\theta - k)_+ = \begin{cases} -k + k\delta & \text{if } \ \bar\theta \leq k \\ \\ k\delta - \bar\theta & \text{if } \ \bar\theta > k \end{cases} \geq 0 \ ;$$
2. the graph γ has a jump at $\theta = 0$ so the corresponding graph $\tilde\gamma$ has a jump at $\bar\theta = -\mu^+\left(\frac{2^{n_*}}{\omega}\right) < 0$. If we choose n_* such that $-\mu^+\left(\frac{2^{n_*}}{\omega}\right) < -1$, that is,
$$n_* > \log_2\left(\frac{\omega}{\mu^+}\right)$$
then, for $k \in (-1,0)$, the function $(\bar\theta - k)_+$ is above the singularity in time and $\tilde\gamma_\epsilon{}' \equiv 1$.

Therefore, for $n_* > \max\left\{C\omega^{-\alpha}, \log_2\left(\frac{\omega}{\mu^+}\right)\right\}$ and $k, \delta \in (-1,0)$, the weak formulation presented above reads

$$\int_{K_4} \partial_t(\bar\theta - k)_+ \varphi + C\int_{K_4} \left|\nabla(\bar\theta - k)_+\right|^{p-2} \nabla(\bar\theta - k)_+ \cdot \nabla\varphi \leq 0 \ , \tag{32}$$

for all $t \in (-4^p, 0)$.

The first term of (32) can be estimated from below as follows

$$\int_{K_4} \partial_t(\bar\theta - k)_+ \varphi \ = \ \int_{K_4} \partial_t \phi_k(\bar\theta)\xi^p$$

$$
\begin{aligned}
&= \frac{d}{dt}\int_{K_4}\phi_k(\bar\theta)\xi^p - p\int_{K_4}\phi_k(\bar\theta)\xi^{p-1}\xi_t\\
&\geq \frac{d}{dt}\int_{K_4}\phi_k(\bar\theta)\xi^p - \frac{C_1}{2-p}
\end{aligned}
$$

using the conditions on ξ and the fact that

$$
\begin{aligned}
\phi_k(\bar\theta) &= \frac{1}{2-p}\left\{(-k(1-\delta))^{2-p} - (-k(1-\delta)-(\bar\theta-k)_+)^{2-p}\right\}\\
&< \frac{1}{2-p}(-k(1-\delta))^{2-p} < \frac{2^{2-p}}{2-p}\ , \quad k,\delta\in(-1,0)\ .
\end{aligned}
$$

For the second term we have

$$
\begin{aligned}
&\int_{K_4}\left|\nabla(\bar\theta-k)_+\right|^{p-2}\nabla(\bar\theta-k)_+\cdot\nabla\varphi\\
&= p\int_{K_4}\left(\frac{\xi}{-k(1-\delta)-(\bar\theta-k)_+}\right)^{p-1}\left|\nabla(\bar\theta-k)_+\right|^{p-2}\nabla(\bar\theta-k)_+\cdot\nabla\xi\\
&\quad +(p-1)\int_{K_4}\left(\frac{|\nabla(\bar\theta-k)_+|}{-k(1-\delta)-(\bar\theta-k)_+}\right)^p\xi^p\ .
\end{aligned}
$$

Recalling the definition of ψ_k, we have

$$
\nabla\psi_k(\bar\theta) = \frac{\nabla(\bar\theta-k)_+}{-k(1-\delta)-(\bar\theta-k)_+}
$$

and, using Young's inequality (ρ is to be chosen), we get from the identity above

$$
\begin{aligned}
&\int_{K_4}\left|\nabla(\bar\theta-k)_+\right|^{p-2}\nabla(\bar\theta-k)_+\cdot\nabla\varphi\\
&\geq (p-1)\left[1-\rho^{\frac{-p}{p-1}}\right]\int_{K_4}\left|\nabla\psi_k(\bar\theta)\right|^p\xi^p - \rho^p\int_{K_4}|\nabla\xi|^p\ .
\end{aligned}
$$

Taking $\rho = 2^{\frac{p-1}{p}}$, recalling the conditions on ξ and the bounds on C, we finally get

$$
\frac{d}{dt}\int_{K_4}\phi_k(\bar\theta)\xi^p + \tilde C_0\int_{K_4}\left|\nabla\psi_k(\bar\theta)\right|^p\xi^p \leq \frac{\tilde C_1}{2-p} \tag{33}
$$

where $\tilde C_0 = \tilde C_0(p,N,n_*,B,\omega,data)$ and $\tilde C_1 = \tilde C_1(p,N,n_*,B)$. Note that, for all $t\in(-4^p,0)$,

$$
\left|[\psi_k(\bar\theta)=0]\cap[\xi=1]\right| = \left|[\bar\theta\leq k]\cap K_2\right| \geq \left|[\bar\theta<k]\cap K_{h_0}\right| > \tilde\nu_0
$$

using (27) for $k = -\frac{1}{2}$ and $\tilde{\nu_0} = \left(\frac{\nu_0}{2}\right)^2 \left(\frac{2^{s_2}}{B}\right)^{N(2-p)} > 0$. We can then apply Proposition 2.1 of Chapter I of [7] to conclude that

$$\int_{K_4} \left|\nabla\psi_k(\bar{\theta})\right|^p \xi^p \geq C \int_{K_4} \psi_k^p(\bar{\theta})\xi^p \ , \qquad C = C(N, p, \tilde{\nu_0})$$

and consequently

Lemma 5. *There exist constants C_0 and C_1, that can be determined a priori only in terms of, respectively, p, N, ω, n_*, B and the data and p, N, n_*, B such that*

$$\frac{d}{dt}\int_{K_4} \phi_k(\bar{\theta})\xi^p + C_0 \int_{K_4} \psi_k^p(\bar{\theta})\xi^p \leq C_1 \ . \tag{34}$$

This integral inequality will be used to prove an auxiliary proposition which is an important tool in obtaining the sought expansion of "negativity". Introduce the quantities

$$Y_n := \sup_{-4^p \leq t \leq 0} \int_{K_4 \cap [\bar{\theta} > -|\delta|^n]} \xi^p \ , \qquad n = 0, 1, \ldots \tag{35}$$

Proposition 6. *The number ν being fixed, we can find numbers δ, σ, depending only upon the data, ω and ν, and independent of ϵ, such that, for $n = 0, 1, 2, \ldots$, either*

$$Y_n \leq \nu$$

or

$$Y_{n+1} \leq \max\{\nu, \sigma Y_n\} \ .$$

Proof. Take $k = -|\delta|^n$ in (34), where $\delta \in (-1, 0)$ is to be chosen. From (35), it follows that for every $\rho \in (0, 1)$ there exists $t_0 \in (-4^p, 0)$ such that

$$Y_{n+1} - \rho \leq \int_{K_4 \cap [\bar{\theta} > -|\delta|^{n+1}]} \xi^p(\cdot, t_0) \ , \qquad n = 0, 1, 2, \ldots \tag{36}$$

After fixing $n \in \mathbf{N}$ and $t_0 \in (-4^p, 0)$, one of the following two situations holds: either

$$\frac{d}{dt}\left(\int_{K_4} \phi_{-|\delta|^n}(\bar{\theta})\xi^p\right)(t_0) \geq 0 \tag{37}$$

or

$$\frac{d}{dt}\left(\int_{K_4} \phi_{-|\delta|^n}(\bar{\theta})\xi^p\right)(t_0) < 0 \ . \tag{38}$$

In either case we may assume that $Y_n > \nu$ (otherwise the result is trivial).

Assume that (37) holds. Then

$$\int_{K_4} \psi^p_{-|\delta|^n}(\bar\theta)\xi^p(\cdot, t_0) \le C \ ,$$

for a positive constant C, independent of ϵ. Perform an integration over the smaller set

$$\left\{x \in K_4 \ : \ \bar\theta(x, t_0) > -|\delta|^{n+1}\right\} \ ,$$

where

$$\psi^p_{-|\delta|^n}(\bar\theta(x,t_0)) = \ln^p\left(\frac{|\delta|^n(1-\delta)}{-|\delta|^n\delta - \bar\theta(x,t_0)}\right) \ge \ln^p\left(\frac{1-\delta}{-2\delta}\right) \ .$$

Then

$$\int_{K_4\cap[\bar\theta(.,t_0)>-|\delta|^{n+1}]} \xi^p(\cdot,t_0) \le C\ln^{-p}\left(\frac{1-\delta}{-2\delta}\right) \ .$$

Using (36) and taking, without loss of generality, $\rho \in (0, \frac{\nu}{2})$, we obtain

$$Y_{n+1} \le \frac{\nu}{2} + C\ln^{-p}\left(\frac{1-\delta}{-2\delta}\right) \ .$$

We take $|\delta|$ so small that

$$C\ln^{-p}\left(\frac{1-\delta}{-2\delta}\right) \le \frac{\nu}{2} \ ,$$

that is

$$|\delta| = -\delta \le \frac{1}{2\exp\left(\frac{2C}{\nu}\right)^{\frac{1}{p}} - 1} \in (0,1)$$

and the proposition is proved assuming that (37) holds.

Now assume that (38) holds and define

$$t_* := \sup\left\{t \in (-4^p, t_0) \ : \ \frac{d}{dt}\left(\int_{K_4} \phi_{-|\delta|^n}(\bar\theta)\xi^p\right) \ge 0\right\} \ .$$

Then

$$\begin{aligned}
\int_{K_4} \phi_{-|\delta|^n}(\bar\theta)\xi^p(\cdot,t_0) &\le \int_{K_4} \phi_{-|\delta|^n}(\bar\theta)\xi^p(\cdot,t_*) \\
&= \int_{K_4}\left(\int_0^{|\delta|^n} \chi_{[(\bar\theta+|\delta|^n)_+>s]}\frac{\mathrm{d}s}{(|\delta|^n(1-\delta)-s)^{p-1}}\right)\xi^p(\cdot,t_*)
\end{aligned}$$

$$= \int_{K_4} \left(\int_0^1 \chi_{[(\bar\theta+|\delta|^n)_+ > s|\delta|^n]} \frac{|\delta|^{n(2-p)}}{(1-\delta-s)^{p-1}} \, \mathrm{d}s \right) \xi^p(\cdot, t_*)$$

$$= \int_0^1 \frac{|\delta|^{n(2-p)}}{(1-\delta-s)^{p-1}} \left(\int_{K_4 \cap [(\bar\theta+|\delta|^n)_+ > s|\delta|^n]} \xi^p(\cdot, t_*) \right) \, \mathrm{d}s \ .$$

We estimate from above the integral in brackets, for $s \in [0,1]$. On the one hand we have, using the definition of Y_n,

$$\int_{K_4 \cap [(\bar\theta+|\delta|^n)_+ > s|\delta|^n]} \xi^p(\cdot, t_*) \le \int_{K_4 \cap [\bar\theta > -|\delta|^n]} \xi^p(\cdot, t_*) \le Y_n \ .$$

On the other hand, from the definition of t_*, we first get

$$\int_{K_4} \psi^p_{-|\delta|^n}(\bar\theta) \xi^p(\cdot, t_*) \le C$$

and then, integrating over the smaller set

$$K_4 \cap [(\bar\theta + |\delta|^n)_+ > s|\delta|^n] \ ,$$

we obtain

$$\int_{K_4 \cap [(\bar\theta+|\delta|^n)_+ > s|\delta|^n]} \xi^p(\cdot, t_*) \le C \ln^{-p} \left(\frac{1-\delta}{1-\delta-s} \right)$$

since, in this set,

$$\psi^p_{-|\delta|^n}(\bar\theta) \ge \ln^p \left(\frac{1-\delta}{1-\delta-s} \right) \ .$$

Then, for all $s \in [0,1]$

$$\int_{K_4 \cap [(\bar\theta+|\delta|^n)_+ > s|\delta|^n]} \xi^p(\cdot, t_*) \le \min \left\{ Y_n, C \ln^{-p} \left(\frac{1-\delta}{1-\delta-s} \right) \right\} \ .$$

Let s_* be such that $Y_n = C \ln^{-p} \left(\frac{1-\delta}{1-\delta-s_*} \right)$, *i.e.*,

$$s_* = \frac{\exp\left(\frac{C}{Y_n}\right)^{\frac{1}{p}} - 1}{\exp\left(\frac{C}{Y_n}\right)^{\frac{1}{p}}} (1-\delta) \ .$$

For $0 \le s < s_*$

$$C \ln^{-p} \left(\frac{1-\delta}{1-\delta-s} \right) > C \ln^{-p} \left(\frac{1-\delta}{1-\delta-s_*} \right) = Y_n$$

and for $s_* \le s \le 1$

$$C \ln^{-p}\left(\frac{1-\delta}{1-\delta-s}\right) \le C \ln^{-p}\left(\frac{1-\delta}{1-\delta-s_*}\right) = Y_n \ .$$

Then

$$\int_0^1 \frac{|\delta|^{n(2-p)}}{(1-\delta-s)^{p-1}} \left(\int_{K_4\cap[(\bar\theta+|\delta|^n)_+ > s|\delta|^n]} \xi^p(\cdot,t_*)\right) \, ds$$

$$\le \int_0^{s_*} \frac{|\delta|^{n(2-p)}}{(1-\delta-s)^{p-1}} Y_n \, ds + \int_{s_*}^1 \frac{|\delta|^{n(2-p)}}{(1-\delta-s)^{p-1}} C \ln^{-p}\left(\frac{1-\delta}{1-\delta-s}\right) \, ds$$

$$= |\delta|^{n(2-p)} Y_n \left[\int_0^1 \frac{1}{(1-\delta-s)^{p-1}} \, ds \right.$$

$$\left. - \int_{s_*}^1 \frac{1}{(1-\delta-s)^{p-1}} \left\{1 - \frac{C}{Y_n} \ln^{-p}\left(\frac{1-\delta}{1-\delta-s}\right)\right\} \, ds \right] \ .$$

Our next goal is to obtain an estimate from below, independent of Y_n, for the second integral on the right hand side of this inequality. We start by noting that,

$$s_* > \sigma_0(1-\delta) \ , \qquad \sigma_0 \equiv \frac{\exp\left(\frac{C}{\nu}\right)^{\frac{1}{p}} - 1}{\exp\left(\frac{C}{\nu}\right)^{\frac{1}{p}}}$$

since we are assuming that $Y_n > \nu$, and for $s_* \le s \le 1$

$$0 \le 1 - \frac{C}{Y_n} \ln^{-p}\left(\frac{1-\delta}{1-\delta-s}\right) \ .$$

Therefore

$$\int_{s_*}^1 \frac{1}{(1-\delta-s)^{p-1}} \left\{1 - \frac{C}{Y_n} \ln^{-p}\left(\frac{1-\delta}{1-\delta-s}\right)\right\} \, ds$$

$$\ge \int_{\sigma_0(1-\delta)}^1 \frac{1}{(1-\delta-s)^{p-1}} \left\{1 - \frac{C}{Y_n} \ln^{-p}\left(\frac{1-\delta}{1-\delta-s}\right)\right\} \, ds$$

$$\ge \int_{\sigma_0(1-\delta)}^1 \frac{1}{(1-\delta-s)^{p-1}} \left\{1 - \frac{C}{\nu} \ln^{-p}\left(\frac{1-\delta}{1-\delta-s}\right)\right\} \, ds \ .$$

We obtain

$$\int_{K_4} \phi_{-|\delta|^n}(\bar\theta)\xi^p(\cdot,t_0) \le |\delta|^{n(2-p)} Y_n \left[\int_0^1 \frac{1}{(1-\delta-s)^{p-1}} \, ds \right.$$

$$- \int_{\sigma_0(1-\delta)}^{1} \frac{1}{(1-\delta-s)^{p-1}} \left\{ 1 - \frac{C}{\nu} \ln^{-p} \left(\frac{1-\delta}{1-\delta-s} \right) \right\} \, ds \Bigg]$$

$$= |\delta|^{n(2-p)} Y_n \left[\int_0^{1-|\delta|} \frac{1}{(1-\delta-s)^{p-1}} \, ds - \left\{ - \int_{1-|\delta|}^{1} \frac{1}{(1-\delta-s)^{p-1}} \, ds \right.\right.$$

$$\left.\left. + \int_{\sigma_0(1-\delta)}^{1} \frac{1}{(1-\delta-s)^{p-1}} \left\{ 1 - \frac{C}{\nu} \ln^{-p} \left(\frac{1-\delta}{1-\delta-s} \right) \right\} \, ds \right\} \right]$$

$$= |\delta|^{n(2-p)} Y_n [1 - f(\delta)] \int_0^{1-|\delta|} \frac{1}{(1-\delta-s)^{p-1}} \, ds$$

where

$$f(\delta) \int_0^{1-|\delta|} \frac{1}{(1-\delta-s)^{p-1}} \, ds \equiv - \int_{1-|\delta|}^{1} \frac{1}{(1-\delta-s)^{p-1}} \, ds$$

$$+ \int_{\sigma_0(1-\delta)}^{1} \frac{1}{(1-\delta-s)^{p-1}} \left\{ 1 - \frac{C}{\nu} \ln^{-p} \left(\frac{1-\delta}{1-\delta-s} \right) \right\} \, ds \, .$$

In order to obtain a lower bound to $f(\delta)$ note that

(i) for $\sigma_0(1-\delta) \le s \le 1$, $1 - \frac{C}{\nu} \ln^{-p} \left(\frac{1-\delta}{1-\delta-s} \right) \ge 0$;

(ii) $\sigma_0 \le \sigma_1 \equiv \dfrac{\exp\left(\frac{2C}{\nu}\right)^{\frac{1}{p}} - 1}{\exp\left(\frac{2C}{\nu}\right)^{\frac{1}{p}}}$;

(iii) for $\sigma_1(1-\delta) \le s \le 1$, $1 - \frac{C}{\nu} \ln^{-p} \left(\frac{1-\delta}{1-\delta-s} \right) \ge \frac{1}{2}$.

Then

$$f(\delta) \int_0^{1-|\delta|} \frac{1}{(1-\delta-s)^{p-1}} \, ds$$

$$\ge - \int_{1-|\delta|}^{1} \frac{1}{(1-\delta-s)^{p-1}} \, ds + \frac{1}{2} \int_{\sigma_1(1-\delta)}^{1} \frac{1}{(1-\delta-s)^{p-1}} \, ds$$

$$= \frac{1}{2-p} \left(\frac{1}{2}(-\delta)^{2-p} + \frac{1}{2}(1-\delta)^{2-p}(1-\sigma_1)^{2-p} - (-2\delta)^{2-p} \right)$$

$$\ge \frac{1}{2-p} \left(\frac{1}{2}(1-\delta)^{2-p}(1-\sigma_1)^{2-p} - (-2\delta)^{2-p} \right)$$

and consequently

$$f(\delta) \geq \frac{1}{2}(1-\sigma_1)^{2-p} - \left(\frac{-2\delta}{1-\delta}\right)^{2-p} .$$

Choosing $\delta \in (-1,0)$ such that

$$\left(\frac{-2\delta}{1-\delta}\right)^{2-p} = \frac{1}{4}(1-\sigma_1)^{2-p}$$

we get

$$f(\delta) \geq \frac{1}{4}(1-\sigma_1)^{2-p}$$

and then, for $\sigma \equiv 1 - \frac{1}{4}(1-\sigma_1)^{2-p} \in (0,1)$,

$$\int_{K_4} \phi_{-|\delta|^n}(\bar\theta)\xi^p(\cdot,t_0) \leq \sigma Y_n \int_0^{1-|\delta|} \frac{|\delta|^{n(2-p)}}{(1-\delta-s)^{p-1}}\, ds .$$

To estimate from below the integral over K_4, we integrate over the smaller set $K_4 \cap [\bar\theta(\cdot,t_0) > -|\delta|^{n+1}]$ to get, using (36),

$$\int_{K_4} \phi_{-|\delta|^n}(\bar\theta)\xi^p(\cdot,t_0) \geq \int_{K_4\cap[\bar\theta(.,t_0)>-|\delta|^{n+1}]} \phi_{-|\delta|^n}(\bar\theta)\xi^p(\cdot,t_0)$$

$$= \int_{K_4\cap[\bar\theta(.,t_0)>-|\delta|^{n+1}]} \left(\int_0^{|\delta|^n+\bar\theta} \frac{1}{(|\delta|^n(1-\delta)-s)^{p-1}}\, ds\right) \xi^p(\cdot,t_0)$$

$$\geq \left(\int_{K_4\cap[\bar\theta(\cdot,t_0)>-|\delta|^{n+1}]} \xi^p(\cdot,t_0)\right)\left(\int_0^{1-|\delta|} \frac{|\delta|^{n(2-p)}}{(1-\delta-s)^{p-1}}\, ds\right)$$

$$\geq (Y_{n+1}-\rho)\int_0^{1-|\delta|} \frac{|\delta|^{n(2-p)}}{(1-\delta-s)^{p-1}}\, ds .$$

This and the previous estimate yield, since ρ is arbitrary in $(0,\frac{\nu}{2})$,

$$Y_{n+1} \leq \sigma Y_n ,$$

which completes the proof assuming that (38) holds.

We now remove assumption (29). If (29) does not hold we have to make use of the discrete time derivative in order to obtain the weak formulation of (28). This means that, for all $t \in [-4^p+h, 0]$, and $h>0$ we have

$$\int_{t-h}^{t}\int_{K_4} \partial_t(\bar\theta)\varphi + C\int_{t-h}^{t}\int_{K_4} |\nabla\bar\theta|^{p-2}\nabla\bar\theta\cdot\nabla\varphi = 0 ,$$

for all admissible testing functions φ, where C is a positive constant independent of ϵ. Being $(\bar{\theta}-k)_+$, with $k \in (-1,0)$, a sub-solution of (28) and taking φ as before, we obtain, for all $t \in [-4^p+h, 0]$ and $h>0$,

$$C_2 h \geq \int_{K_4\times\{t\}} \phi_k(\bar{\theta})\xi^p - \int_{K_4\times\{t-h\}} \phi_k(\bar{\theta})\xi^p + C_1 \int_{t-h}^{t}\int_{K_4} \psi_k^p(\bar{\theta})\xi^p \ .$$

Dividing by h and letting $h \to 0$ we get an integral inequality similar to (34):

$$\left(\frac{d}{d\tau}\right)^{-} \int_{K_4} \phi_k(\bar{\theta})\xi^p + C_1 \int_{K_4} \psi_k^p(\bar{\theta})\xi^p \leq C_2$$

where

$$\left(\frac{d}{d\tau}\right)^{-} \int_{K_4} \phi_k(\bar{\theta})\xi^p := \limsup_{h\to 0} \frac{1}{h}\left[\int_{K_4\times\{t\}} \phi_k(\bar{\theta})\xi^p - \int_{K_4\times\{t-h\}} \phi_k(\bar{\theta})\xi^p\right] \ .$$

Define the set

$$S := \left\{ t \in (-4^p, 0) \ : \ \left(\frac{d}{d\tau}\right)^{-} \int_{K_4} \phi_k(\bar{\theta})\xi^p \geq 0 \right\}$$

and let t_0 be given as in (36). If $t_0 \in S$, we proceed as in (37). If $t_0 \notin S$, take

$$\bar{t} = \sup\{t \in (-4^p, t_0) \ : \ t \in S\} \leq t_0 \ .$$

If $\bar{t} = t_0$, consider a sequence $(t_n)_n$, $t_n \in S$, such that $t_n \to t_0$. Since $t_n \in S$ we get

$$\int_{K_4\times\{t_n\}} \psi_k^p(\bar{\theta})\xi^p \leq C \ .$$

Then

$$\int_{K_4\times\{t_0\}} \psi_k^p(\bar{\theta})\xi^p \leq C$$

and we proceed as in (37). If $\bar{t} < t_0$ we have

$$\begin{cases} \displaystyle\int_{K_4\times\{\bar{t}\}} \psi_k^p(\bar{\theta})\xi^p \leq C \\[2ex] \displaystyle\int_{K_4} \phi_k(\bar{\theta})\xi^p(x,t_0) \leq \int_{K_4} \phi_k(\bar{\theta})\xi^p(x,\bar{t}) \end{cases}$$

and we work as in (38).

■

5.2. The expansion of "negativity". From Proposition 6, we get by iteration

$$Y_n \leq \max\{\nu, \sigma^n Y_0\} \;, \qquad n = 1, 2, \ldots$$

and since

$$Y_0 = \sup_{-4^p \leq t \leq 0} \int_{K_4 \cap [\bar{\theta}(.,t) > -1]} \xi^p(\cdot, t) \leq |K_4|$$

we obtain

$$Y_n \leq \max\{\nu, \sigma^n |K_4|\} = \max\{\nu, \sigma^n 2^N |K_2|\} \;, \qquad n = 1, 2, \ldots$$

Take $n = n_0 \in \mathbf{N}$ so large that $\sigma^{n_0} 2^N \leq \nu$. Then

$$Y_{n_0} \leq \max\{\nu, \sigma^{n_0} 2^N |K_2|\} \leq \max\{\nu, \nu |K_2|\} = \nu |K_2| \;.$$

Recalling the definition of Y_n, as well as the choice of ξ, we obtain

$$Y_{n_0} \geq \sup_{-4^p \leq t \leq 0} \int_{K_2 \cap [\bar{\theta}(.,t) > -|\delta|^{n_0}]} \xi^p(\cdot, t) \geq \left| x \in K_2 \;:\; \bar{\theta}(x,t) > -|\delta|^{n_0} \right| \;,$$

for all $t \in [-2^p, 0]$, and therefore

$$\left| x \in K_2 \;:\; \bar{\theta}(x,t) > -|\delta|^{n_0} \right| \leq \nu |K_2| \;, \quad \forall t \in [-2^p, 0] \;.$$

We have just proven the crucial result towards the expansion of "negativity" to a "full" cylinder in space, namely

Lemma 6. *Given $\nu \in (0,1)$, there exists $\delta^* \in (0,1)$, depending only upon the data, ν and ω, such that*

$$\left| x \in K_2 \;:\; \bar{\theta}(x,t) > -\delta^* \right| \leq \nu |K_2| \;, \qquad \forall t \in [-2^p, 0] \;. \tag{39}$$

To prove the main result of this section, we need to make use of another auxiliary result, which proof is a trivial modification to sub-solutions of the proof of Lemma 4.1 of [7, Ch. 4]. Indeed, being *above the singularity* in time, we are dealing only with powers p and 2 in the energy estimates. The proof is included here for the sake of completeness.

Lemma 7. *There exists $\tilde{\nu}$, depending upon the data, N and p, and independent of ω and ϵ, such that if θ is a sub-solution of (5) in $[(\bar{x}, \bar{t}) + Q_R(m_1, m_2)]$ satisfying*

$$\operatorname*{ess\,osc}_{[(\bar{x},\bar{t}) + Q_R(m_1,m_2)]} \theta \leq \omega$$

and

$$\left|(x,t) \in [(\bar{x},\bar{t}) + Q_R(m_1,m_2)] \ : \ \theta(x,t) > \mu^+ - \frac{\omega}{2^m}\right| \leq \tilde{\nu}\, |Q_R(m_1,m_2)|$$

then

$$\theta(x,t) \leq \mu^+ - \frac{\omega}{2^{m+1}} \ , \qquad \forall (x,y) \in \Big[(\bar{x},\bar{t}) + Q_{\frac{R}{2}}(m_1,m_2)\Big] \ ,$$

where $m = m_1 + m_2$, $m_1, m_2 \geq 0$ *and*

$$Q_R(m_1,m_2) = K_{d_1 R} \times \Big(-2^{m_2(p-2)} R^p, 0\Big) \ , \qquad d_1 = \Big(\frac{\omega}{2^{m_1}}\Big)^{\frac{p-2}{p}} \ .$$

Proof. Assume that $(\bar{x},\bar{t}) = (0,0)$ and construct the decreasing sequence of numbers

$$R_n = \frac{R}{2} + \frac{R}{2^{n+1}} \ , \qquad n = 0,1,2,\dots$$

and the families of nested and shrinking cubes and cylinders

$$K_n = K_{d_1 R_n} \ , \qquad Q_n = Q_{R_n}(m_1,m_2) \ .$$

Consider smooth cutoff functions $0 \leq \xi_n \leq 1$, defined in Q_n, and satisfying the following set of assumptions

$$\begin{cases} \xi_n \equiv 1 \quad \text{in} \quad Q_{n+1}, \qquad \xi_n \equiv 0 \quad \text{on} \quad \partial Q_n \\[2mm] |\nabla \xi_n| \leq \frac{2^{n+2}}{R} (\frac{\omega}{2^{m_1}})^{\frac{2-p}{p}}, \ \ 0 < \partial_t \xi_n \leq 2^{(2-p)m_2} \frac{2^{p(n+2)}}{R^p} \end{cases}$$

and write the local energy estimate (9) for the functions $(\theta - k_n)_+$, with

$$k_n = \mu^+ - \frac{\omega}{2^{m+1}} - \frac{\omega}{2^{m+1+n}} \ , \qquad n = 0,1,2,\dots$$

over the cylinders Q_n, with $\xi = \xi_n$. Noting that, when $(\theta - k_n)_+$ is not zero,

$$(\theta - k_n)_+ \leq \frac{\omega}{2^m} \ ,$$

the right hand side of the inequality is estimated from above by

$$C \frac{2^{np}}{R^p} 2^{(2-p)m_2} \left(\frac{\omega}{2^m}\right)^2 \iint_{Q_n} \chi_{[(\theta - k_n)_+ > 0]} \ .$$

To estimate from below the left hand side introduce the level

$$k_n < \bar{k_n} \equiv \frac{k_n + k_{n+1}}{2} < k_{n+1} \ .$$

Then, for all $t \in (-2^{m_2(p-2)} R_n^p, 0)$, we have

$$\begin{aligned} \int_{K_n} (\theta - k_n)_+^2 \xi_n^p &= \int_{K_n} (\theta - k_n)_+^{2-p} (\theta - k_n)_-^p \xi_n^p \\ &\geq \left(\frac{\omega}{2^{m+n+3}}\right)^{2-p} \int_{K_n} (\theta - \bar{k_n})_+^p \xi_n^p \end{aligned}$$

and

$$\iint_{Q_n} |\nabla(\theta - k_n)_+|^p \xi_n^p \geq \iint_{Q_n} \left|\nabla(\theta - \bar{k_n})_+\right|^p \xi_n^p \ .$$

Collecting results, and dividing by $(\frac{\omega}{2^{m+n+3}})^{2-p}$, we obtain the estimate

$$\sup_{-2^{m_2(p-2)} R_n^p < t < 0} \int_{K_n} (\theta - \bar{k_n})_+^p \xi_n^p + \left(\frac{\omega}{2^{m+n+3}}\right)^{p-2} \iint_{Q_n} \left|\nabla(\theta - \bar{k_n})_+\right|^p \xi_n^p$$

$$\leq C \frac{2^{np+(2-p)m_2}}{R^p} \left(\frac{\omega}{2^m}\right)^2 \left(\frac{\omega}{2^{m+n+3}}\right)^{p-2} \iint_{Q_n} \chi_{[(\theta - k_n)_+ > 0]} \ .$$

Introduce the change of variables

$$y = \left(\frac{\omega}{2^{m_1}}\right)^{\frac{2-p}{p}} x \ , \qquad z = 2^{(2-p)m_2} t \ ,$$

which maps the cylinder Q_n into $\overline{Q}_n := K_{R_n} \times (-R_n^p, 0)$, and define new functions

$$\bar{\theta}(y,z) := \theta(x,t) \ , \qquad \bar{\xi_n}(y,z) := \xi_n(x,t) \ .$$

Then we get (denoting the new variables again by (x,t))

$$\sup_{-R_n^p < t < 0} \int_{K_{R_n}} (\bar{\theta} - \bar{k_n})_+^p \bar{\xi_n}^p + \left(\frac{\omega}{2^{m+n+3}}\right)^{p-2} 2^{m_2(p-2)} \iint_{\overline{Q}_n} \left|\nabla(\bar{\theta} - \bar{k_n})_+\right|^p \bar{\xi_n}^p$$

$$\leq C \frac{2^{2n}}{R^p} \left(\frac{\omega}{2^m}\right)^p \iint_{\overline{Q}_n} \chi_{[(\bar{\theta} - k_n)_+ > 0]} \ .$$

Note that

$$\left(\frac{\omega}{2^{m+n+3}}\right)^{p-2} 2^{m_2(p-2)} = \left(\frac{2^{m_1+n+3}}{\omega}\right)^{2-p} \geq 1$$

since $0 < 2-p < 1$, $\omega \leq 1$, and $m_1 \geq 0$. Recalling the V^p norm and defining

$$A_n := \left|(x,t) \in \overline{Q}_n \ : \ \bar{\theta}(x,t) > k_n\right| \ ,$$

we obtain from the inequality above,

$$\left\|(\bar{\theta} - \bar{k_n})_+ \bar{\xi_n}\right\|_{V^p(\overline{Q}_{n+1})}^p \leq C \frac{2^{2n}}{R^p} \left(\frac{\omega}{2^m}\right)^p A_n \ ,$$

since $\bar{\xi}_n \equiv 1$ in $\overline{Q}_{n+1}$. Now, on the one hand we have

$$\left\|(\bar{\theta}-\bar{k_n})_+\bar{\xi_n}\right\|^p_{p,(\overline{Q}_{n+1})} = \iint_{\overline{Q}_{n+1}} (\bar{\theta}-\bar{k_n})^p_+ \geq \iint_{\overline{Q}_{n+1}\cap[\bar{\theta}>k_{n+1}]} (\bar{\theta}-\bar{k_n})^p$$
$$\geq (k_{n+1}-\bar{k_n})^p \iint_{\overline{Q}_{n+1}} \chi_{[\bar{\theta}>k_{n+1}]} = \left(\frac{\omega}{2^{m+n+3}}\right)^p A_{n+1}\ ,$$

and on the other hand, using [7, Corollary 3.1; p. 9], we have

$$\left\|(\bar{\theta}-\bar{k_n})_+\right\|^p_{p,(\overline{Q}_{n+1})} \leq C\left|[(\bar{\theta}-\bar{k_n})_+>0]\right|^{\frac{p}{N+p}} \left\|(\bar{\theta}-\bar{k_n})_+\bar{\xi_n}\right\|^p_{V^p(\overline{Q}_{n+1})}$$
$$\leq C A_n^{\frac{p}{N+p}} \left\|(\bar{\theta}-\bar{k_n})_+\right\|^p_{V^p(\overline{Q}_{n+1})}\ .$$

Then

$$A_{n+1} \leq C\frac{2^{n(2+p)}}{R^p} A_n^{1+\frac{p}{N+p}}$$

and, dividing this inequality by $|\overline{Q}_{n+1}|$, and taking $Y_n := \frac{A_n}{|\overline{Q}_n|}$,

$$Y_{n+1} \leq \frac{C4^{pn}}{R^p} Y_n^{1+\frac{p}{N+p}} \frac{|\overline{Q}_n|^{1+\frac{p}{N+p}}}{|\overline{Q}_{n+1}|}\ .$$

Since for $n=0,1,2,\ldots$

$$\frac{|\overline{Q}_n|^{1+\frac{p}{N+p}}}{|\overline{Q}_{n+1}|} = \frac{[(2R_n)^N R_n^p]^{1+\frac{p}{N+p}}}{[(2R_{n+1})^N R_{n+1}^p]} = 2^{\frac{Np}{N+p}}\left(\frac{R_n}{R_{n+1}}\right)^{N+p} R_n^p \leq 2^{N+2p}R^p\ ,$$

we arrive at

$$Y_{n+1} \leq C4^{pn} Y_n^{1+\frac{p}{N+p}}\ .$$

If we take

$$\tilde{\nu} = C^{\frac{-1}{\alpha}} 4^{\frac{-p}{\alpha^2}}\ ,$$

where $\alpha = \frac{p}{N+p}$, we can apply a lemma on the fast geometric convergence of sequences and conclude that $Y_n \to 0$ when $n \to \infty$. But this means that $A_n \to 0$ and since

$$R_n \searrow \frac{R}{2}\ , \qquad k_n \nearrow \mu^+ - \frac{\omega}{2^{m+1}}$$

this implies that

$$\theta(x,t) \leq \mu^+ - \frac{\omega}{2^{m+1}}\ , \qquad (x,t) \in \left[(\bar{x},\bar{t}) + Q_{\frac{R}{2}}(m_1,m_2)\right]$$

and the proof is complete.

■

We are now in a position to prove the main result of this section.

Proposition 7. *Assume* **the second alternative** *holds. There exists $s_3 > s_2$ such that*

$$\theta(x,t) \le \mu^+ - \frac{\omega}{2^{s_3}} , \qquad \text{a.e. } (x,t) \in Q\left(\frac{a_0}{2}\left(\frac{R}{4}\right)^p, c_0 R\right) . \tag{40}$$

Proof. Note that we are done if we prove that (40) holds in the cylinder

$$\left[(\bar{x},0) + Q\left(\frac{a_0}{2}\left(\frac{R}{4}\right)^p, 2c_0R\right)\right] \supseteq Q\left(\frac{a_0}{2}\left(\frac{R}{4}\right)^p, c_0R\right) ,$$

independently of the location of $\bar{x}$ in $K_{\mathcal{R}(\omega)}$. We will prove that there exists $s_4 > 1$ such that

$$\bar{\theta}(x,t) \le -\frac{1}{2^{s_4}} , \qquad \text{a.e. } (x,t) \in Q_1 \equiv K_1 \times (-1,0) ,$$

and then use the fact that $\bar{t}$ is an arbitrary element of $\mathcal{I}(\omega)$ to get (40) for the cylinder $\left[(\bar{x},0) + Q\left(\frac{a_0}{2}\left(\frac{R}{4}\right)^p, 2c_0R\right)\right]$ and a proper choice of A.

In Lemma 6, take $\nu = \tilde{\nu}$ from Lemma 7 and determine the corresponding $\delta^* = \delta^*(\tilde{\nu})$. Then use Lemma 7 for $R = 2$, $\mu^+ = 0$, $\omega = 1$, $m_1 = 0$ and m_2 such that $2^{-m_2} = \delta^*(\tilde{\nu})$, over the cylinders

$$\left[(0,\bar{t}) + K_2 \times (-2^{m_2(p-2)}2^p, 0)\right] \equiv [(0,\bar{t}) + Q_2(0,m_2)]$$

as long as they are contained in Q_2, that is, for $\bar{t}$ satisfying

$$-2^p + 2^{m_2(p-2)}2^p \le \bar{t} \le 0 .$$

Then

$$\left|(x,t) \in [(0,\bar{t}) + Q_2(0,m_2)] \ : \ \bar{\theta}(x,t) > -\frac{1}{2^{m_2}}\right| \le \tilde{\nu}\, |Q_2(0,m_2)|$$

for each one of the cylinders $[(0,\bar{t}) + Q_2(0,m_2)]$ (since (39) holds for all $t \in [-2^p, 0]$). Therefore we conclude that

$$\bar{\theta}(x,t) \le -\frac{1}{2^{m_2+1}} , \quad \text{a.e. } (x,t) \in [(0,\bar{t}) + Q_1(0,m_2)] = K_1 \times (\bar{t} - 2^{m_2(p-2)}, \bar{t})$$

for all $\bar{t} \in [-2^p + 2^{m_2(p-2)}2^p, 0]$. Since $-2^p + 2^{m_2(p-2)}2^p - 2^{m_2(p-2)} < -1$, we get

$$\bar{\theta}(x,t) \le -\frac{1}{2^{m_2+1}} , \qquad \text{a.e. } (x,t) \in Q_1 .$$

Returning to the initial variables and function we arrive at

$$\theta(x,t) \le \mu^+ - \frac{\omega}{2^{m_2+s_2}} \ , \qquad \text{a.e.} \ \ (x,t) \in [\bar{x} + K_{2c_0R}] \times \left(\bar{t} - \frac{\bar{t}-t_0}{4^p}, \bar{t}\right) \ .$$

Since $t_0 \in \left[\bar{t} - dR^p, \bar{t} - \frac{\nu_0}{2} dR^p\right]$, we have

$$\bar{t} - \frac{\bar{t}-t_0}{4^p} \in \left[\bar{t} - d\left(\frac{R}{4}\right)^p, \bar{t} - \frac{\nu_0}{2} d\left(\frac{R}{4}\right)^p\right]$$

and then

$$\theta(x,t) \le \mu^+ - \frac{\omega}{2^{m_2+s_2}} \ , \qquad \text{a.e.} \ \ (x,t) \in K_{c_0R} \times \left(\bar{t} - \frac{\nu_0}{2} d\left(\frac{R}{4}\right)^p, \bar{t}\right)$$

since $[\bar{x} + K_{2c_0R}] \supset K_{c_0R}$, independently of the location of $\bar{x} \in K_{\mathcal{R}(\omega)}$.

Now we just have to make use of the arbitrary choice of $\bar{t}$ in (16) to conclude that

$$\theta(x,t) \le \mu^+ - \frac{\omega}{2^{m_2+s_2}} \ , \quad \text{a.e.} \ (x,t) \in K_{c_0R} \times \left(-(a_0-d)R^p - \frac{\nu_0}{2} d\left(\frac{R}{4}\right)^p, 0\right) \ .$$

Take A such that

$$\left(\frac{A}{2}\right)^{(p-1)(2-p)} = \frac{a_0}{d} \ge 2 \ . \tag{41}$$

Then

$$\frac{a_0}{d} \ge \frac{2^{2p+1} - \nu_0}{2^{2p+1} - 1} \Leftrightarrow -(a_0-d)R^p - \frac{\nu_0}{2} d\left(\frac{R}{4}\right)^p \le -\frac{a_0}{2}\left(\frac{R}{4}\right)^p$$

and consequently

$$\theta(x,t) \le \mu^+ - \frac{\omega}{2^{s_3}} \ , \quad \text{a.e.} \ \ (x,t) \in Q\left(\frac{a_0}{2}\left(\frac{R}{4}\right)^p, c_0R\right)$$

taking $s_3 = m_2 + s_2$.

■

An immediate consequence of Proposition 7 is

Corollary 2. *Assume* **the second alternative** *holds. Then there exists* $\sigma_1 \in (0,1)$*, depending only upon the data and* ω*, such that*

$$\operatorname*{ess\,osc}_{Q\left(\frac{a_0}{2}(\frac{R}{4})^p, c_0R\right)} \theta \le \sigma_1 \omega \ . \tag{42}$$

Remark 8. *Note that we have only imposed two conditions on A: $A > 2$ and (41). So we can take*

$$A = 2^{1+\frac{1}{(p-1)(2-p)}} > 2$$

and conclude that A is independent of ω.

Remark 9. *As far as B is concerned, we have*

$$B > 2^{s_2} \qquad \text{and} \qquad B \geq 2^{s_1} \ ,$$

where, recalling the choice of A, s_1 satisfies

$$s_1 > \log_2 A + \frac{p}{(p-1)(2-p)} = 1 + \frac{p+1}{(p-1)(2-p)}$$

and

$$s_1 > 3 + \frac{2C}{\nu_1} A^{(p-1)(2-p)} = 3 + \frac{C}{\nu_1} 2^{2+(p-1)(2-p)} \ ,$$

and

$$s_2 > \max\left\{1 + C\omega^{-\alpha}, 1 + \log_2\left(\frac{\omega}{\mu^+}\right)\right\} \ .$$

Summarizing:

$$A = 2^{1+\frac{1}{(p-1)(2-p)}} > 2$$

$$s_1 > \max\left\{3 + \tfrac{C}{\nu_1} 2^{2+(p-1)(2-p)}, 1 + \tfrac{p+1}{(p-1)(2-p)}\right\}$$

$$n_* > \max\left\{C\omega^{-\alpha}, \log_2\left(\tfrac{\omega}{\mu^+}\right)\right\} \ , \quad \alpha = \tfrac{2(p+1)(N+p)}{p}$$

$$B > \max\left\{2^{n_*+1}, 2^{s_1}\right\} \ .$$

6. The main result

As a consequence of the alternative, one of the Corollaries 1 or 2 must hold and, consequently, we obtain

Lemma 8. *There exists a constant $\sigma = \sigma(\omega) \in (0,1)$, depending only upon the data and ω, such that*

$$\operatorname*{ess\,osc}_{Q\left(d(\frac{R}{8})^p, c_0\frac{R}{8}\right)} \theta \leq \sigma(\omega)\omega \ .$$

Proof. Let $\sigma = \max\{\sigma_0, \sigma_1\}$. Since $\frac{R}{8} < \frac{R}{4} < R$ and (41) is in force

$$d\left(\frac{R}{8}\right)^p \leq \frac{a_0}{2}\left(\frac{R}{4}\right)^p$$

and the result follows.

∎

We next define recursively two sequences of real positive numbers. Let

$$\omega_1 = \sigma(\omega)\omega \qquad \text{and} \qquad R_1 = \frac{R}{C(\omega_1)}\ ,$$

where

$$C(\omega_1) = \left(\frac{A}{2}\right)^{\frac{(p-1)(2-p)}{p}} 8\sigma(\omega)^{\frac{(1-p)(2-p)}{p}} B(\omega_1)^{2-p}\sigma(\omega)^{\frac{p-2}{p}} > 8\ .$$

Defining

$$Q_1 = (a_1 R_1^p, c_1 R_1)\ ; \qquad \text{with} \quad a_1 = \left(\frac{\omega_1}{A}\right)^{(1-p)(2-p)} \quad , \quad c_1 = \left(\frac{\omega_1}{B(\omega_1)}\right)^{p-2}$$

and noting that

$$\begin{aligned} a_1 R_1^p &= \left(\frac{\omega_1}{A}\right)^{(1-p)(2-p)} \frac{R^p}{C(\omega_1)^p} \\ &= \left(\frac{\omega}{2}\right)^{(1-p)(2-p)} \left(\frac{R}{8}\right)^p \frac{\sigma(\omega)^{2-p}}{B(\omega_1)^{p(2-p)}} \\ &\leq d\left(\frac{R}{8}\right)^p \end{aligned}$$

and

$$\begin{aligned} c_1 R_1 &= \left(\frac{\omega_1}{B(\omega_1)}\right)^{p-2} \frac{R}{C(\omega_1)} \\ &= \left(\frac{\omega}{B}\right)^{p-2} \left(\frac{R}{8}\right) \frac{1}{B^{2-p}\left(\frac{A}{2}\right)^{\frac{(p-1)(2-p)}{p}}} \\ &\leq c_0 \frac{R}{8} \end{aligned}$$

we get

$$Q_1 \subset Q\left(d\left(\frac{R}{8}\right)^p, c_0\frac{R}{8}\right)$$

and, consequently,

$$\operatorname*{ess\,osc}_{Q_1} \theta \leq \operatorname*{ess\,osc}_{Q\left(d\left(\frac{R}{8}\right)^p, c_0\frac{R}{8}\right)} \theta \leq \sigma(\omega)\omega = \omega_1 \ .$$

The process can now be repeated starting from Q_1 since (15) holds in this cylinder. We then define the following recursive sequences of real positive numbers

$$\begin{cases} \omega_0 = \omega \\ \\ \omega_{n+1} = \sigma(\omega_n)\omega_n \end{cases} \quad \text{and} \quad \begin{cases} R_0 = R \\ \\ R_{n+1} = \frac{R_n}{C(\omega_{n+1})} \end{cases}$$

and construct the family of nested and shrinking cylinders

$$Q_n = (a_n R_n^p, c_n R_n) \ ; \qquad \text{with} \quad a_n = \left(\frac{\omega_n}{A}\right)^{(1-p)(2-p)} \quad , \quad c_n = \left(\frac{\omega_n}{B(\omega_n)}\right)^{p-2} \ ,$$

where $n = 0, 1, \ldots$ The next result is now standard (see, for example, [18] for a proof) and yields as a consequence the proof of Theorem 1.

Theorem 2. *The sequences $(\omega_n)_n$ and $(R_n)_n$ are decreasing sequences converging to zero. Moreover, for every $n = 0, 1, \ldots$*

$$Q_{n+1} \subset Q_n \qquad \text{and} \qquad \operatorname*{ess\,osc}_{Q_n} \theta \leq \omega_n \ .$$

Acknowledgments. The authors would like to thank E. DiBenedetto and V. Vespri for several enlightening conversations.

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