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Received: 17 December 2021 | Revised: 18 July 2022 | Accepted: 9 August 2022

DOI: 10.1002/mma.8663

RESEARCH ARTICLE

WILEY

Critical exponent of Fujita type for semilinear wave equations in Friedmann–Lemaître–Robertson–Walker spacetime

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Communicated by: M. Reissig

Funding information
Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP), Grant/Award Number: 2020/08276-9; 'Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), Grant/Award Number: 304408/2020-4; FCT—Fundação para a Ciência e a Tecnologia, I.P., Grant/Award Number: UIDB/05037/2020

We consider the nonlinear massless wave equation belonging to some family of the Friedmann–Lemaître–Robertson–Walker (FLRW) spacetime. We prove the global in time small data solutions for supercritical powers in the case of decelerating expansion universe.

KEYWORDS
critical exponent, global existence, semilinear wave equations, small data solutions

MSC CLASSIFICATION
35A01, 35B33, 35L05, 35L71

1 | INTRODUCTION

In this paper, we prove the global (in time) existence of small data solutions to the Cauchy problem for the semilinear wave equation with scale-invariant damping and decreasing in time propagation speed:

$$\begin{cases} u_{tt}(t,x) - (1+t)^{-2\ell}\Delta u(t,x) + \frac{\beta}{1+t}u_t(t,x) = f(u(t,x)), & t\geq 0, x\in \mathbf{R}^n, \\ u(0,x) = u_0(x), & x\in \mathbf{R}^n, \\ u_t(0,x) = u_1(x), & x\in \mathbf{R}^n, \end{cases} \tag{1}$$

with $\ell \in (0,1)$ and $\beta > 0$. We assume that $f(u) = |u|^p$ or, more in general, f verifies the following local Lipschitz-type condition:

$$f(0) = 0, \ |f(u) - f(v)| \leq C|u-v| \left(|u|^{p-1} + |v|^{p-1}\right), \tag{2}$$

for some $p > 1$. The important information is that the nonlinearity in (1) is nonnegative, the so-called source nonlinearity. Thus, it cannot be absorbed in the definition of the energy, so it is in general a perturbation which may create blow-up in finite time.

In Mathematical Physics, there are homogeneous and isotropic cosmological models represented by the wave equation in a given spacetime. In some models, an $(n+1)$-dimensional spacetime of interest is described by the family of the

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Friedmann–Lemaître–Robertson–Walker (FLRW), which is endowed with a metric written in the form

$$ds^2 = -dt^2 + \alpha^2(t)(dx_1^2 + \dots + dx_n^2),$$

with an appropriate scale factor $\alpha(t)$ (see, e.g., previous studies[1–3]). If $\frac{d}{dt}\alpha(t) > 0$ and $\frac{d^2}{dt^2}\alpha(t) < 0$, we can say that the universe is in decelerating expansion. Our model (1) is derived from $\alpha(t) = (1+t)^\ell$ with $\ell = \frac{2}{\gamma n}$ satisfying $\frac{2}{n} < \gamma \le 2$, $n \ge 2$ (see Tsutaya and Wakasugi[4]). We also have $\alpha(t) = (1+t)^{2/3}$ for the Einstein–de Sitter four-dimensional spacetime, so that it is well known as a particular case of FLRW spacetime in a decelerating expansion universe model. Moreover, the case in which $\beta = 2$, $n = 3$, and $\ell = \frac{2}{3}$ in (1) is called the covariant massless field in the Einstein–de Sitter spacetime with non-singular (at $t = 0$) coefficients (see previous studies[5–7]).

Let us start with the state of the art in the case $\ell = 0$. If $\beta \ge \frac{5}{3}$ for $n = 1$, $\beta \ge 3$ for $n = 2$, or $\beta \ge n+2$ for $n \ge 3$, by assuming data in the energy spaces with additional regularity $L^1(R^n)$, the global (in time) existence result for (1) was proved in D'Abbicco[8] for $p > p_F(n) \doteq 1 + \frac{2}{n}$, the well-known Fujita index.[9] The exponent $p_F(n)$ is critical for this model, that is, for $p \le p_F(n)$ and suitable, arbitrarily small data, there exists no global weak solution.[10] As conjectured in D'Abbicco et al[11,12] if β becomes smaller with respect to the space dimension n, the critical exponent increase to $\max\{p_S(n+\beta), p_F(n)\}$, where p_S is the Strauss exponent for the semilinear undamped wave equation.[13,14] Ikeda and Sobajima[15] proved a blow-up result and gave the upper bound for the lifespan of solutions to (1) for $1 < p \le p_S(n+\beta)$ and $\beta \in [0, \beta_\star)$, with $\beta_\star = \frac{n^2+n+2}{n+2}$. It is worth noticing that if $\beta \in [0, \beta_\star)$, then $p_F(n) < p_S(n+\beta)$ and $p_F(n) = p_S(n+\beta_\star)$.

Recently, in D'Abbicco,[16] it is proved, in the case $\ell = 0$, that the critical exponent to (1) is equal to $\max\{p_S(n+\beta), p_F(n)\}$ for $n = 1$, and in D'Abbicco,[17] it is proved the global existence of small data solutions for $p > p_F(n)$ and $\beta \ge n$ in space dimension $2 \le n \le 5$. As far as we know, it is still an open problem to prove global existence of small data solutions for $p > p_F(n)$ in the cases $\beta_\star < \beta < n$ for $n \ge 3$ and for $p > p_S(n+\beta)$ for $0 < \beta < \beta_\star$ for $n \ge 2$.

For $\ell \in [0,1)$, $\beta \ge 0$ and $n \ge 2$, let $p_S(n,\ell,\beta)$ be the positive root of the quadratic equation:

$$\left(n - 1 + \frac{\beta - \ell}{1-\ell}\right)p^2 - \left(n + 1 + \frac{\beta + 3\ell}{1-\ell}\right)p - 2 = 0.$$

Recently, in Palmieri[18] and independently in Tsutaya and Wakasugi,[3,4] the authors have proved blow-up in a finite time and upper estimates of the lifespan for energy solutions or classical solutions to (1), respectively, for

$$1 < p \le \max\{p_F(n(1-\ell)), p_S(n,\ell,\beta)\}.$$

A blow-up result for energy solutions in the case $\beta = 2$ and $\ell \in (0,1)$ in (1) was first proved in Galstian and Yagdjian.[5]

It is worth noticing that if $p_F(n(1-\ell)) = p_S(n,\ell,\beta_c(n,\ell))$, where

$$\beta_c(n,\ell) \doteq \ell + (1-\ell)\left(n + 1 - \frac{2}{p_F(n(1-\ell))}\right) = \frac{n^2(1-\ell)^2 + n(1-\ell)(1+2\ell) + 2}{2 + n(1-\ell)}.$$

In particular, if $\beta \ge \beta_c(n,\ell)$, then $p_S(n,\ell,\beta) \le p_F(n(1-\ell))$.

If the assumption that the initial data $u_1 \in L^1(R^n)$ is removed and we only assume that $u_1 \in L^2(R^n)$, that is, $u_1 \notin L^{2-\delta}(R^n)$ for all $\delta \in (0,1]$, then the critical exponent to (1) is modified into $1 + \frac{4}{n(1-\ell)}$ for a lower thresholds required for β (see Ebert and Marques[19]). For the classical damped wave equation, this phenomenon has been investigated in Ikehata and Ohta.[20]

Wirth[21] proposed a classification of non-effective and effective dissipation, respectively, for the damped wave equation:

$$u_{tt}(t,x) - \Delta u(t,x) + b(t)u_t(t,x) = 0,\ b > 0. \tag{3}$$

The dissipation produced by $b(t)u_t$ may be classified as non-effective[22] if $\lim_{t\to\infty} tb(t) = 0$ and the solutions to (3) have the same asymptotic properties of the free wave equation $u_{tt} - \Delta u = 0$ and as effective[23] if $\lim_{t\to\infty} tb(t) = \infty$ and the solutions inherit some properties related to the parabolic equation $b(t)u_t - \Delta u = 0$. In the limit case $\lim_{t\to\infty} tb(t) = \mu \in (0,\infty)$, the asymptotic behavior of solutions to (3) changes according to the size of the constant $\mu > 0$ (see Wirth[24]).

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Bui and Reissig[25] generalize the classification of non-effective and effective dissipation, respectively, for the damped wave equation with time-dependent increasing speed of propagation

$$u_{tt}(t,x) - a^2(t)\Delta u(t,x) + b(t)u_t(t,x) = 0.$$

The authors derived sharp estimates for solutions to the Cauchy problem and, in the case of effective dissipation, that os,

$$b(t)\frac{A(t)}{a(t)} \to \infty, \, as \ t \to \infty, \qquad A(t) = 1 + \int_0^t a(\tau)\,d\tau,$$

derived global (in time) existence results for the semilinear problem with power nonlinearities. A similar classification was introduced in Ebert and Reissig[26] in the case $a \in L^1$, but in this case, new effects are observed, for instance, different from the case $a \notin L^1$, additional regularity $L^1(R^n)$ for initial data brings no additional decay for solutions.

A natural generalization for the model (1) is to consider a positive and decreasing speed of propagation $a(t)$, with $a \notin L^1$. But in this paper, we restrict ourselves to the case in which a is a irrational function, since it includes interesting models by itself.

The main goal in this paper is to prove, under the assumption of small initial data in $L^1(\mathbf{R}^n) \cap H^{k-1}(\mathbf{R}^n)$, $k \geq 1$, the global (in time) existence of solutions to the Cauchy problem:

$$\begin{cases} u_{tt}(t,x) - (1+t)^{-2\ell}\Delta u(t,x) + \frac{\beta}{1+t}u_t(t,x) = f(u(t,x)), & t \geq 0, x \in \mathbf{R}^n, \\ u(0,x) = 0, & x \in \mathbf{R}^n, \\ u_t(0,x) = u_1(x), & x \in \mathbf{R}^n, \end{cases} \tag{4}$$

with $\ell \in (0,1)$, $\beta \geq \beta_c(n,\ell)$ and f satisfying (2) for supercritical powers $p > p_F(n(1-\ell))$.

We are assuming $u(0,x) = 0$ in (4) for brevity, but it is not difficult, under the assumption of additional regularity, to extend the results to non-zero first initial data (see Section 6.2). Moreover, the blow-up of energy solutions obtained in Palmieri[18] for $1 < p \leq p_F(n(1-\ell))$ are still valid in the case $u_0 \doteq 0$.

Combining the obtained results in this paper with the blow-up results derived in Palmieri,[18] we conclude that $p_F(n,\ell) = 1 + \frac{2}{n(1-\ell)}$ is the critical exponent for the global in time existence of solutions for $\beta \geq \beta_c(n,\ell)$.

As far as we know, it is still a open problem to prove global existence of small data solutions to (4) for $p > p_S(n,\ell,\beta)$ and $0 < \beta \leq \beta_c(n,\ell)$. It is expected that a similar approach to those used for the semilinear free wave equation, namely, $u_{tt} - \Delta u = |u|^p, p > 1$, may be appropriate to decrease values of β and to overcame some gaps that appear in this paper, for instance, in the case of the covariant massless field in the Einstein–de Sitter model (see Remark 9).

2 | MAIN RESULTS

Before stating the main theorems, let us introduce a suitable notion of weak solution to the Cauchy problem (1). First, let us define it in the case $f(u) \doteq 0$:

Definition 1. We say that $u \in L^1_{loc}([0,\infty) \times \mathbb{R}^n)$ is a weak solution to (1) with $f(u) \doteq 0$ if, for any test functions $\psi \in C_c^\infty([0,\infty))$ and $\varphi \in C_c^\infty(\mathbb{R}^n)$, the following equality holds true:

$$\begin{aligned} &\psi(0)\int_{\mathbb{R}^n} u_1(x)\varphi(x)\,dx + (\beta\psi(0) - \psi'(0))\int_{\mathbb{R}^n} u_0(x)\varphi(x)\,dx \\ &= \int_0^\infty \int_{\mathbb{R}^n} u(t,x)\left(\partial_t^2 - (1+t)^{-2\ell}\Delta - \frac{\beta}{1+t}\partial_t + \frac{\beta}{(1+t)^2}\right)\psi(t)\varphi(x)\,dx\,dt. \end{aligned} \tag{5}$$

Indeed, if u is a smooth solution to (1) with $f(u) \doteq 0$, then (5) follows integrating by parts.

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Hence, by Duhamel's principle, we say that $u \in L^p_{loc}([0,\infty)\times\mathbb{R}^n)$, with $p>1$, is a weak solution to (1) if, and only if, it fulfills the integral equation

$$u(t,x) = u^0(t,x) + \int_0^t K(t,s,x) *_{(x)} f(u(s,x))\, ds, \qquad \text{in } L^p_{loc}([0,\infty)\times\mathbb{R}^n),$$

where $u^0(t,x)$ is the solution to (1) with $f(u) \doteq 0$ and $K(t,s,x)$ is a fundamental solution to (15), that is, $K(t,s,x) *_{(x)} g_2(s,x)$ is the solution to the linear Cauchy problem (15) with $g_1 \equiv 0$.

To simplify the writing, from now, we consider

$$p_c(n,\ell) \doteq p_F(n(1-\ell)) = 1 + \frac{2}{n(1-\ell)}. \tag{6}$$

In the next theorems, due to the fact that $p_c(n,\ell)\to\infty$ as $\ell\to 1$, the choice of the spaces of solutions is related to fixed ranges for $\ell\in[0,1)$ and the space dimensions $n\geq 2$. To state our first result, let us define the following parameters:

$$\bar{q} \doteq \frac{2(np_c(n,\ell)-1)}{n+1}, \qquad q_\sharp \doteq \frac{2(n+1)}{n-1}. \tag{7}$$

Theorem 1. *Let ℓ be such that*

$$\begin{cases} 0 \leq \ell < 1 - \frac{n-1}{2n}, & \text{if } 2 \leq n \leq 5, \\ 1 - \frac{2(n+1)}{n(n-3)} \leq \ell < 1 - \frac{n-1}{2n}, & \text{if } 6 \leq n \leq 8, \end{cases}$$

and

$$\beta \geq \ell + (n+1)(1-\ell) - \frac{2}{\bar{q}}(1-\ell),$$

with $\bar{q}\in[p_c(n,\ell), q_\sharp]$, where $p_c(n,\ell)$, $q_\sharp$ and $\bar{q}$ are given by (6) and (7). If

$$p_c(n,\ell) < p \leq \frac{4p_c(n,\ell)}{n+3} + 1,$$

then there exists $\delta>0$ such that for any initial data

$$u_1 \in \mathcal{D} = L^1(\mathbf{R}^n)\cap L^2(\mathbf{R}^n), \qquad ||u_1||_{\mathcal{D}} \leq \delta,$$

*there exists a unique weak solution $u\in C([0,\infty), L^{p_c}(\mathbf{R}^n)\cap L^{q_\sharp}(\mathbf{R}^n))$ to (4). Moreover, the solution satisfies the following estimates for $p_c\leq q\leq q_\sharp$: If $\beta > \ell + (n+1)(1-\ell) - \frac{2}{q}(1-\ell)$, then**

$$||u(t,\cdot)||_{L^q} \lesssim (1+t)^{-n\left(1-\frac{1}{q}\right)(1-\ell)}||u_1||_{\mathcal{D}}, \tag{8}$$

whereas if $\ell + (n+1)(1-\ell) - \frac{2}{\bar{q}}(1-\ell) \leq \beta \leq \ell + (n+1)(1-\ell) - \frac{2}{q}(1-\ell)$, then for any $\varepsilon>0$,

$$||u(t,\cdot)||_{L^q} \lesssim (1+t)^{\left[\varepsilon-(n-1)\left(\frac{1}{2}-\frac{1}{q}\right)\right](1-\ell)-\frac{\beta-\ell}{2(1-\ell)}}||u_1||_{\mathcal{D}}. \tag{9}$$

Remark 1. One of the crucial property in the proof of Theorem 1 is that $r(q)p_c(n,\ell) < q_\sharp$, for all $p_c(n,\ell)\leq q\leq q_\sharp$, with

$$\frac{1}{r(q)} \doteq \frac{1}{2n} + \frac{1}{2} + \frac{1}{nq}. \tag{10}$$

* Let $f,g : \Omega\subset\mathbf{R}^n\to\mathbf{R}$ be two functions. From now on, we use the notation $f\lesssim g$ if there exists a constant $C>0$ such that $f(y)\leq Cg(y)$ for all $y\in\Omega$.

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This condition is satisfied under some condition on ℓ, namely,

$$r(q_\sharp)p_c(n,\ell) < q_\sharp \Leftrightarrow \ell < 1 - \frac{4}{q_\sharp(n+1) - 2(n-1)} = 1 - \frac{n-1}{2n}.$$

Since $r(q) \leq r(q_\sharp)$ for all $p_c(n,\ell) \leq q \leq q_\sharp$, we also have

$$r(q)p_c(n,\ell) < q_\sharp \Leftrightarrow \ell < 1 - \frac{n-1}{2n}.$$

In particular, it implies the existence of $\bar{q}$ satisfying $\bar{q} < q_\sharp$. For instance, for $\ell \in \left[0, \frac{3}{4}\right)$ if $n = 2$, and for $\ell \in \left[0, \frac{2}{3}\right)$ if $n = 3$.

Remark 2. By using that

$$r(q_\sharp)p_c(n,\ell) < q_\sharp \Longleftrightarrow p_c(n,\ell)\left(1 - \frac{r(q_\sharp)}{q_\sharp}\right) + 1 = \frac{4p_c(n,\ell)}{n+3} + 1 < \frac{q_\sharp}{r(q_\sharp)} = \frac{n+3}{n-1},$$

with $r(q)$ given by (10), we conclude that the upper bound for p in Theorem 1 satisfies

$$p \leq \min\left\{p_c(n,\ell)\left(1 - \frac{r(q_\sharp)}{q_\sharp}\right) + 1, \frac{q_\sharp}{r(q_\sharp)}\right\} = \frac{4p_c(n,\ell)}{n+3} + 1.$$

Remark 3. Taking into account that $L^1 - L^q$ linear estimates in Corollary 2 of D'Abbicco[17] hold only for $\frac{2(n-1)}{n+1} \leq q \leq q_\sharp$, in the proof of Theorem 1, we have to assume

$$p_c(n,\ell) \geq \frac{2(n-1)}{n+1}.$$

Hence, a restriction from below in ℓ is also needed, namely, $\ell \geq 1 - \frac{2(n+1)}{n(n-3)}$, $n \geq 4$. This condition is true for $\ell = 0$ under the assumption $2 \leq n \leq 5$ in Theorem 1.

Remark 4. Let $\frac{n}{r(q)} = \frac{1}{2} + \frac{n}{2} + \frac{1}{q}$. Condition (7) means that $\bar{q}$ is defined by $p_c(n,\ell)r(\bar{q}) = \bar{q}$. In particular, thanks to $(n-1)p_c(n,\ell) \geq 1$ for $n \geq 2$ it holds

$$\begin{aligned}
\frac{1}{\bar{q}} - \frac{n}{p_c(n,\ell)r(q_\sharp)} + \frac{n-1}{q_\sharp} &= \frac{n}{p_c(n,\ell)r(\bar{q})} - \frac{n-1}{\bar{q}} - \frac{n}{p_c(n,\ell)r(q_\sharp)} + \frac{n-1}{q_\sharp} \\
&= \frac{1}{p_c(n,\ell)}\left(\frac{n}{r(\bar{q})} - \frac{n}{r(q_\sharp)}\right) - (n-1)\left(\frac{1}{\bar{q}} - \frac{1}{q_\sharp}\right) \\
&= \left(\frac{1}{p_c(n,\ell)} - n + 1\right)\left(\frac{1}{\bar{q}} - \frac{1}{q_\sharp}\right) \leq 0.
\end{aligned}$$

In the case $\bar{q} = p_c(n,\ell)$, Theorem 1 yields the threshold value

$$\beta \geq \ell + (1-\ell)\left(n + 1 - \frac{2}{p_c(n,\ell)}\right) = \beta_c(n,\ell).$$

Example 2.1. If $\ell = 0$, then $p_c(n,0) = 1 + \frac{2}{n}$ and $\bar{q} = 2$, so we have to assume $\beta \geq n + 1 - \frac{2}{\bar{q}} = n$. In particular for $\ell = 0$ and $n = 2$, this condition coincides with the threshold value $\beta \geq \beta_c(2,0) = 2$ and the conclusions of Theorem 1 hold for all $2 < p \leq 2 + \frac{3}{5}$.

If $\ell = \frac{1}{2}$ and $n = 3$, then $p_c(n,0) = \frac{7}{3}$ and $\bar{q} = 3$, so we have to assume $\beta \geq 2 + \frac{1}{6}$. Hence, the conclusions of Theorem 1 hold for all $\frac{7}{3} < p \leq \frac{23}{9}$.

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In the following results, the novelty is to use higher regularity $H^k(\mathbf{R}^n)$, $k > \frac{n}{2}$, in order to consider larger values on the parameter ℓ and to relax the condition in the upper bound for p in Theorem 1. In particular, it is possible to include for $n = 3$ the speed of propagation $a(t) = (1+t)^{-\frac{2}{3}}$ that appears in the well-known Einstein–de Sitter model for decelerating expanding universe.[6]

In Theorems 2 and 3, we restrict our analysis to the case of small values for β, whereas the simple case of large values for β is treated in Theorem 4.

Theorem 2. *Let $\ell \in \left(1-\frac{2}{n}, \frac{2}{n}\right)$ for $n = 2, 3$ or $\ell = \frac{2}{3}$ for $n = 3$. If $\ell + n(1-\ell)(1+\ell) \le \beta < 2 - \ell + n(1-\ell)(1+\ell)$ and $p > p_c(n, \ell)$, with $p_c(n,\ell)$ given by (6), then there exists $\delta > 0$ such that for any initial data*

$$u_1 \in \mathcal{D} = H^{k-1}(\mathbf{R}^n) \cap L^1(\mathbf{R}^n), \qquad ||u_1||_{\mathcal{D}} \le \delta,$$

with $k \doteq 1 + \frac{n\ell}{2}$, there exists a unique weak solution $u \in C([0,\infty), H^k(\mathbf{R}^n))$ to (4), which satisfies the following estimates:

$$||u(t,\cdot)||_{L^2} \lesssim (1+t)^{\frac{n}{2}(\ell-1)}||u_1||_{\mathcal{D}},$$
$$||u(t,\cdot)||_{\dot{H}^{k-1}} \lesssim h(t)||u_1||_{\mathcal{D}},$$

with

$$h(t) = \begin{cases} (1+t)^{(\ell-1)\left(\frac{n}{2}+k-1\right)}, & \ell + n(1-\ell)(1+\ell) < \beta < 2-\ell+n(1-\ell)(1+\ell), \\ (1+t)^{\frac{\ell-\beta}{2}}(\ln(e+t))^{\frac{1}{2}}, & \beta = \ell + n(1-\ell)(1+\ell), \end{cases}$$

and

$$||u(t,\cdot)||_{\dot{H}^k} \lesssim (1+t)^{\frac{\ell-\beta}{2}}||u_1||_{\mathcal{D}}.$$

Remark 5. We point out that

$$\ell > 1 - \frac{2}{n} \iff k = 1 + \frac{n\ell}{2} > \frac{n}{2} \iff p_c(n,\ell) > 2,$$

for $n \ge 2$. Moreover, $k \le p_c(n,\ell)$ iff $\ell(1-\ell)n^2 \le 4$, in particular, this is true if $\ell \in \left(1-\frac{2}{n}, 1\right)$ for $n = 2, 3$.

Example 2.2. If $\ell = \frac{2}{3}$, the conclusion of Theorem 2 holds for $n = 2, 3$ with $\beta \ge \frac{1}{3}\left(2 + \frac{5n}{3}\right)$.

In the following result, we may consider the case $\ell \in \left(\frac{2}{3}, 1\right)$ for $n = 3$, by looking for solutions with additional regularity $H^{\kappa(r_1)-1, r_2}(\mathbf{R}^3)$, with $\kappa(r_1) = 3\left(\frac{1}{2} - \frac{1}{r_1}\right)$ and r_1, r_2 satisfying

$$r_1 > \frac{2(3\ell-1)}{1-\ell}, \qquad 2 < r_2 < \frac{6}{2\kappa(r_1)-1}. \tag{11}$$

Theorem 3. *Let $n = 3$, $\ell \in \left(\frac{2}{3}, 1\right)$, r_1, r_2 satisfying (11) with $\kappa(r_1) = 3\left(\frac{1}{2} - \frac{1}{r_1}\right)$. If $\ell + 6(1-\ell)\left(1 - \frac{1}{r_1}\right) \le \beta < 2 - \ell + 3(1-\ell)(1+\ell)$ and*

$$p_c(3,\ell) < p < 1 + \frac{r_1(2-\kappa(r_1))}{3}, \tag{12}$$

with $p_c(3,\ell)$ given by (6), then there exists $\delta > 0$ such that for any initial data

$$u_1 \in \mathcal{D} = H^{\kappa(r_1)-1}(\mathbf{R}^3) \cap L^1(\mathbf{R}^3), \qquad ||u_1||_{\mathcal{D}} \le \delta,$$

there exists a unique weak solution $u \in C([0,\infty), H^{\kappa(r_1)}(\mathbf{R}^3) \cap \dot{H}^{\kappa(r_1)-1, r_2}(\mathbf{R}^3))$ to (4), which satisfies the following estimates:

$$||u(t,\cdot)||_{\dot{H}^{j\kappa(r_1)}} \lesssim (1+t)^{(\ell-1)\left(\frac{n}{2}+j\kappa(r_1)\right)}||u_1||_{\mathcal{D}}, \ j = 0, 1;$$

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and

$$\|u(t,\cdot)\|_{\dot{H}^{\kappa(r_1)-1,r_2}} \lesssim (1+t)^{(\ell-1)\left(n\left(1-\frac{1}{r_2}\right)+\kappa(r_1)-1\right)}||u_1||_D.$$

Remark 6. We remark that (12) is not empty due to

$$r_1 > \frac{2(3\ell-1)}{1-\ell} \iff r_1(2-\kappa(r_1)) > \frac{2}{1-\ell}.$$

Since we are interested into consider small values of β, we take the smallest possible value for r_1.

Remark 7. From $r_1 > \frac{2(3\ell-1)}{1-\ell}$, it follows that $\kappa(r_1) > \frac{6\ell-3}{3\ell-1}$ and for $\ell \in \left(\frac{2}{3},1\right)$, we have

$$\frac{6\ell-3}{3\ell-1} > \frac{3\ell}{2} \iff 3\ell^2 - 5\ell + 2 < 0.$$

Therefore, for $\ell \in \left(\frac{2}{3},1\right)$, it holds that $\kappa(r_1) > k-1$, with k given by Theorem 2, in particular,

$$\ell + 6(1-\ell)\left(1-\frac{1}{r_1}\right) = \ell + 3(1-\ell) + 2(1-\ell)\kappa(r_1) > \ell + 3(1-\ell)(1+\ell).$$

In the next result, we also consider higher space dimension, but due to the technique, some additional lower bound for p and β come into play:

Theorem 4. *Let $\ell \in \left(1-\frac{2}{n},1\right)$ for $n \geq 2$ and $k \doteq 1 + \frac{n\ell}{2}$. If $\beta \geq 2-\ell+n(1-\ell)(1+\ell)$ and $p > \max\{p_c(n,\ell),k\}$, with $p_c(n,\ell)$ given by (6), then there exists $\delta > 0$ such that for any initial data*

$$u_1 \in \mathcal{D} = H^{k-1}(\mathbf{R}^n) \cap L^1(\mathbf{R}^n), \qquad ||u_1||_D \leq \delta,$$

there exists a unique weak solution $u \in C([0,\infty), H^k(\mathbf{R}^n))$ to (4), which satisfies the following estimates:

$$||u(t,\cdot)||_{L^2} \lesssim (1+t)^{\frac{n}{2}(\ell-1)}||u_1||_D; \tag{13}$$

and

$$||u(t,\cdot)||_{\dot{H}^k} \lesssim ||u_1||_D \begin{cases} (1+t)^{(\ell-1)\left(\frac{n}{2}+k\right)}, & \beta > \ell + n(1-\ell) + 2k(1-\ell), \\ (1+t)^{\frac{\ell-\beta}{2}}(\ln(e+t))^{\frac{1}{2}}, & \beta = \ell + n(1-\ell) + 2k(1-\ell). \end{cases} \tag{14}$$

3 | REPRESENTATION OF SOLUTIONS TO THE LINEAR CAUCHY PROBLEM

Let $s \geq 0$ be a parameter. Motivated by Duhamel's principle, we need to solve a family of parameter-dependent linear $(f(u) = 0)$ Cauchy problems corresponding to (1):

$$\begin{cases} u_{tt}(t,s,x) - (1+t)^{-2\ell}\Delta u(t,s,x) + \frac{\beta}{1+t}u_t(t,s,x) = 0, \ t \geq s, \\ u(s,s,x) = g_1(s,x), \\ u_t(s,s,x) = g_2(s,x). \end{cases} \tag{15}$$

We begin by applying Fourier transform to the solution of the problem (15). We denote the partial Fourier transform of a tempered distribution or of a function $u : \mathbf{R}_0^+ \times \mathbf{R^n} \to \mathbf{C}$ with respect to x, by $\hat{u} = \mathscr{F}u$ or $\hat{u}(t,s,\cdot) = \mathscr{F}u(t,s,\cdot)$. The notation $\mathscr{F}^{-1}$ denotes the inverse Fourier transform, in the appropriate sense.

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Following as in Ebert and Reissig[26] and Taniguchi and Tozaki,[27] we make the change of variables $\tau = \frac{(1+t)^{1-\ell}}{1-\ell}|\xi|$ and $v_s(\tau) \doteq \hat{u}(t,s,\xi)$. If u is the solution of (15), then v_s satisfies

$$\begin{cases} v_s''(\tau) + \frac{\beta-\ell}{(1-\ell)\tau} v_s'(\tau) + v_s(\tau) = 0, \\ v_s\left(\frac{(1+s)^{1-\ell}|\xi|}{1-\ell}\right) = \hat{g_1}(s,\xi), \\ v_s'\left(\frac{(1+s)^{1-\ell}|\xi|}{1-\ell}\right) = \frac{\hat{g_2}(s,\xi)}{|\xi|}. \end{cases} \tag{16}$$

Moreover, if we are looking for a solution in the product form $v_s(\tau) = \tau^\rho w_s(\tau)$, then $w_s(\tau)$ is a solution of the Bessel's differential equation of order $\pm\rho$:

$$\tau^2 w_s''(\tau) + \tau w_s'(\tau) + (\tau^2 - \rho^2) w_s(\tau) = 0, \tag{17}$$

where $\rho = \frac{1-\beta}{2(1-\ell)}$. We will use the set of Hankel functions, $\{H_\rho^+(\tau), H_\rho^-(\tau)\}$ to write the general solution of the ODE (17). First, according to Wirth,[24] we introduce an auxiliary function

$$\psi_{j,\gamma,\delta}(t,s,\xi) = |\xi|^j \begin{vmatrix} H_\gamma^-\left(\frac{(1+s)^{1-\ell}|\xi|}{1-\ell}\right) & H_{\gamma+\delta}^-\left(\frac{(1+t)^{1-\ell}|\xi|}{1-\ell}\right) \\ H_\gamma^+\left(\frac{(1+s)^{1-\ell}|\xi|}{1-\ell}\right) & H_{\gamma+\delta}^+\left(\frac{(1+t)^{1-\ell}|\xi|}{1-\ell}\right) \end{vmatrix}, \tag{18}$$

where j, γ, δ, s are real parameters. Since $H_\gamma^\pm = J_\gamma \pm i Y_\gamma$, we can rewrite it in the form

$$\psi_{j,\gamma,\delta}(t,s,\xi) = 2i|\xi|^j \begin{vmatrix} J_\gamma\left(\frac{(1+s)^{1-\ell}|\xi|}{1-\ell}\right) & J_{\gamma+\delta}\left(\frac{(1+t)^{1-\ell}|\xi|}{1-\ell}\right) \\ Y_\gamma\left(\frac{(1+s)^{1-\ell}|\xi|}{1-\ell}\right) & Y_{\gamma+\delta}\left(\frac{(1+t)^{1-\ell}|\xi|}{1-\ell}\right) \end{vmatrix}, \tag{19}$$

if $\gamma, \gamma+\delta \in \mathbf{Z}$, or

$$\psi_{j,\gamma,\delta}(t,s,\xi) = 2i\csc(\gamma\pi)|\xi|^j \begin{vmatrix} J_{-\gamma}\left(\frac{(1+s)^{1-\ell}|\xi|}{1-\ell}\right) & J_{-\gamma-\delta}\left(\frac{(1+t)^{1-\ell}|\xi|}{1-\ell}\right) \\ (-1)^\delta J_\gamma\left(\frac{(1+s)^{1-\ell}|\xi|}{1-\ell}\right) & J_{\gamma+\delta}\left(\frac{(1+t)^{1-\ell}|\xi|}{1-\ell}\right) \end{vmatrix}, \tag{20}$$

if $\gamma, \gamma+\delta \notin \mathbf{Z}$, where J_γ, Y_γ denote the Bessel functions of the first and second kind, respectively. We then determine the Fourier multipliers and the first-order partial derivatives with respect to t to represent $\hat{u}$ in the explicit form.

Lemma 1 (see Ebert and Marques[19]). *Let u be the solution of (15). Then, the partial Fourier transform of u with respect to x, $\hat{u}$, is represented by*

$$\hat{u}(t,s,\xi) = m_0(t,s,\xi)\hat{g_1}(s,\xi) + m_1(t,s,\xi)\hat{g_2}(s,\xi), \tag{21}$$

with Fourier multipliers and the first-order partial derivatives with respect to t given by

$$\partial_t^j m_k = \frac{(-1)^k \pi i}{4(1-\ell)}(1+s)^{1+(\beta-1)/2}(1+t)^{(1-\beta)/2-j\ell}\psi_{1+j-k,\rho+k-1,1-j-k}, \tag{22}$$

where $\rho = \frac{1-\beta}{2(1-\ell)}$, $k, j = 0, 1$.

4 | ESTIMATES FOR SOLUTIONS TO THE LINEAR CAUCHY PROBLEM (15)

In order to obtain an estimate of (21), we have to distinguish between large and small τ values. We divide the extended phase space $\mathbf{R}_0^+ \times \mathbf{R}_0^+ \times \mathbf{R}^+$ into three zones. We define the zone of high frequencies

$$Z_1 = \{(t,s,|\xi|) : 0 \le s \le t \wedge |\xi| \ge (1+s)^{\ell-1}\},$$

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and the zones of low frequencies

$$Z_2 = \{(t,s,|\xi|) : (1+t)^{\ell-1} \le |\xi| \le (1+s)^{\ell-1}\},$$
$$Z_3 = \{(t,s,|\xi|) : 0 \le s \le t \wedge |\xi| \le (1+t)^{\ell-1}\},$$

separated by the boundary $\{(t,s,|\xi|) : 0 \le s \le t \wedge (1+t)^{1-\ell}|\xi| = 1\}$.

Given a cut-off function $\chi \in C^\infty(\mathbb{R}^n)$ satisfying $\begin{cases} 1 & \text{if } r \le \frac{1}{2} \\ 0 & \text{if } r \ge 1 \end{cases}$, we define

$$\chi_1(s,\xi) = 1 - \chi((1+s)^{1-\ell}|\xi|),$$
$$\chi_2(t,s,\xi) = \chi((1+s)^{1-\ell}|\xi|)\left(1 - \chi((1+t)^{1-\ell}|\xi|)\right),$$
$$\chi_3(t,s,\xi) = \chi((1+s)^{1-\ell}|\xi|)\chi((1+t)^{1-\ell}|\xi|),$$

such that $\chi_1 + \chi_2 + \chi_3 = 1$.

Lemma 2. *Let $\ell \in (0,1)$, $\gamma \neq 0$, and $k \ge 0$. It holds*

$$|\xi|^k|\psi_{0,\gamma,0}(t,s,\xi)| \lesssim \begin{cases} |\xi|^{k-1}(1+s)^{(\ell-1)/2}(1+t)^{(\ell-1)/2} & \text{if } (t,s,\xi) \in Z_1, \\ |\xi|^{k-|\gamma|-1/2}(1+s)^{(\ell-1)|\gamma|}(1+t)^{(\ell-1)/2} & \text{if } (t,s,\xi) \in Z_2, \\ |\xi|^k(1+s)^{(\ell-1)|\gamma|}(1+t)^{(1-\ell)|\gamma|} & \text{if } (t,s,\xi) \in Z_3, \end{cases}$$

for all $s \ge 0$ and $t \ge s$.

Proof. For any $N \in (0,1)$, the following properties hold:

$$|H_\gamma^{\pm}(\tau)| \lesssim \tau^{-\frac{1}{2}}, \ \tau \in [N,\infty); \tag{23}$$
$$|H_\gamma^{\pm}(\tau)| \lesssim \tau^{-|\gamma|}, \ \tau \in (0,N), \ \gamma \neq 0; \tag{24}$$
$$|J_\gamma(\tau)| \lesssim \tau^{\gamma}, \ \tau \in (0,N); \tag{25}$$
$$|Y_\gamma(\tau)| \lesssim \tau^{-\gamma}, \ \tau \in (0,N), \ \gamma \neq 0. \tag{26}$$

For details about these asymptotic formulas, we refer to Watson[28] (Sections 3.52, 10.6, and 7.2).

First, in the zone Z_1, it is enough to use the representation (18) involving the Hankel functions and its asymptotic expansion for large arguments (23). Then, taking into account the definition of the zone Z_2, we use again (18) and both the asymptotic expansions (23) and (24), for large $(1+t)^{1-\ell}|\xi|$ and small $(1+s)^{1-\ell}|\xi|$ arguments, respectively. Finally, in the zone Z_3, we have to consider separately the representations (19) and (20) according to the cases in which γ is integer or not. If $\gamma \notin \mathbf{Z}$, we use the asymptotic expansion (25) of the Bessel functions of the first kind for small arguments, whereas if $\gamma \in \mathbf{Z}$ we use (25) and (26). □

Proposition 1. *Let $n \ge 2$, $q \ge 2$ and $\ell \in (0,1)$. Assume that*

$$g_2(s,\cdot) \in \begin{cases} L^1(\mathbf{R^n}) \cap L^m(\mathbf{R^n}) & \text{if } 0 \le k < 1, \\ L^1(\mathbf{R^n}) \cap \dot{H}^{k-1}(\mathbf{R}^n) & \text{if } k \ge 1, \end{cases}$$

with $m \in [1,2]$ such that

$$m = m(k,n,q) > \frac{nq}{n+q(1-k)}, \ k \in [0,1).$$

The solution u of the problem (15), with $g_1 \equiv 0$, satisfies the following a priori estimates.

- *For $k \in [0,1)$ and $2 \le q < \frac{nm}{[n-m+mk]_+}$:*
  - *(i) If $1 < \beta \le \ell + 2n(1-\ell)\left(1-\frac{1}{q}\right) + 2k(1-\ell)$, then*

$$\||D|^k u(t,s,\cdot)\|_{L^q} \lesssim (1+t)^{\frac{\ell-\beta}{2}}(1+s)^{1+\frac{\beta-\ell}{2}+(\ell-1)\left(n\left(1-\frac{1}{q}\right)+k\right)}\phi(g_2)(t,s). \tag{27}$$

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(ii) If $\beta > \ell + 2n(1-\ell)\left(1-\frac{1}{q}\right) + 2k(1-\ell)$, *then*

$$|||D|^k u(t,s,\cdot)||_{L^q} \lesssim (1+s)(1+t)^{(\ell-1)\left(n\left(1-\frac{1}{q}\right)+k\right)}\phi(g_2)(t,s), \tag{28}$$

where $\phi(g_2)(t,s) = d_q(t,s)||g_2(s,\cdot)||_{L^1} + (1+s)^{n(1-\ell)\left(1-\frac{1}{m}\right)}||g_2(s,\cdot)||_{L^m}$ *with*

$$d_q(t,s) = \begin{cases} \left(\ln\left(\frac{e+t}{e+s}\right)\right)^{1-\frac{1}{q}} & \text{if } \beta = \ell + 2n(1-\ell)\left(1-\frac{1}{q}\right) + 2k(1-\ell), \\ 1 & \text{otherwise.} \end{cases}$$

- *For* $k \geq 1$*:*

(i) If $1 < \beta \leq \ell + n(1-\ell) + 2k(1-\ell)$, *then*

$$||u(t,s,\cdot)||_{\dot{H}^k} \lesssim (1+t)^{\frac{\ell-\beta}{2}}(1+s)^{1+\frac{\beta-\ell}{2}+(\ell-1)\left(\frac{n}{2}+k\right)}\eta(g_2)(t,s). \tag{29}$$

(ii) If $\beta > \ell + n(1-\ell) + 2k(1-\ell)$, *then*

$$||u(t,s,\cdot)||_{\dot{H}^k} \lesssim (1+s)(1+t)^{(\ell-1)\left(\frac{n}{2}+k\right)}\eta(g_2)(t,s), \tag{30}$$

where $\eta(g_2)(t,s) = d_2(t,s)||g_2(s,\cdot)||_{L^1} + (1+s)^{(1-\ell)\left(\frac{n}{2}+k-1\right)}\|g_2(s,\cdot)\|_{\dot{H}^{k-1}}$ *with*

$$d_2(t,s) = \begin{cases} \left(\ln\left(\frac{e+t}{e+s}\right)\right)^{\frac{1}{2}} & \text{if } \beta = \ell + n(1-\ell) + 2k(1-\ell), \\ 1 & \text{otherwise.} \end{cases}$$

Proof. By using the representation of solution to (15) given by Lemma 1 and applying Lemma 2, with $\gamma = \rho = \frac{1-\beta}{2(1-\ell)} < 0$ for $\beta > 1$, we find out the estimates for the solution in every zone.

Considerations in Z_3: In the zone Z_3, by Lemma 2, we may estimate

$$|\xi|^k|m_1(t,s,\xi)| \lesssim |\xi|^k(1+s).$$

By using Haussdorff–Young inequality and Hölder inequality, setting

$$\frac{1}{r} = 1 - \frac{1}{q},$$

for $q \geq 2$, one may estimate

$$\begin{aligned} \|\mathscr{F}^{-1}(\chi_3(t,s,\cdot)|\xi|^k m_1(t,s,\cdot)) * g_2(s,\cdot)\|_{L^q} &\lesssim \|\chi_3(t,s,\cdot)|\xi|^k m_1(t,s,\cdot)\hat{g}_2(s,\cdot)\|_{L^{q'}} \\ &\lesssim \|\chi_3(t,s,\cdot)|\xi|^k m_1(t,s,\cdot)\|_{L^r}\|\hat{g}_2(s,\cdot)\|_{L^\infty} \\ &\lesssim (1+s)(1+t)^{(\ell-1)\left(n\left(1-\frac{1}{q}\right)+k\right)}\|g_2(s,\cdot)\|_{L^1} \end{aligned}$$

thanks to

$$\|\chi_3(t,s,\cdot)|\xi|^k\|^r_{L^r(Z_3)} = \int_{\xi\in Z_3}|\xi|^{rk}\,d\xi \lesssim (1+t)^{(kr+n)(\ell-1)}.$$

However, if $\beta < \ell + 2n(1-\ell)\left(1-\frac{1}{q}\right) + 2k(1-\ell)$, by using that

$$(1+s)(1+t)^{(\ell-1)\left(n\left(1-\frac{1}{q}\right)+k\right)} \leq (1+t)^{\frac{\ell-\beta}{2}}(1+s)^{1+\frac{\beta-\ell}{2}+(\ell-1)\left(n\left(1-\frac{1}{q}\right)+k\right)}, \tag{31}$$

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we obtain

$$\|\mathscr{F}^{-1}(\chi_3(t,s,\cdot)|\xi|^k m_1(t,s,\cdot)*g_2(s,\cdot)\|_{L^q} \lesssim (1+t)^{\frac{\ell-\beta}{2}}(1+s)^{1+\frac{\beta-\ell}{2}+(\ell-1)\left(n\left(1-\frac{1}{q}\right)+k\right)}\|g_2(s,\cdot)\|_{L^1}.$$

Considerations in Z_2: In the zone Z_2, by Lemma 2, we may estimate

$$|\xi|^k|m_1(t,s,\xi)| \lesssim |\xi|^{k-\alpha}(1+s)(1+t)^{(\ell-\beta)/2},$$

where $\alpha = \frac{\beta-\ell}{2(1-\ell)}$. Setting

$$\frac{1}{r} = 1-\frac{1}{q},$$

for $q \geq 2$ then thanks to

$$\|\chi_2(t,s,\cdot)|\xi|^{k-\alpha}\|^r_{L^r(Z_2)} = \int_{\xi\in Z_2}|\xi|^{(k-\alpha)r}d\xi$$
$$\lesssim \begin{cases}(1+s)^{(\ell-1)(n+r(k-\alpha))} & \text{if } \alpha < k+n\left(1-\frac{1}{q}\right)\\ \ln\left(\frac{e+t}{e+s}\right) & \text{if } \alpha = k+n\left(1-\frac{1}{q}\right)\\ (1+t)^{(\ell-1)(n+r(k-\alpha))} & \text{if } \alpha > k+n\left(1-\frac{1}{q}\right),\end{cases}$$

one may estimate

$$\|\mathscr{F}^{-1}(\chi_2(t,s,\cdot)|\xi|^k m_1(t,s,\cdot))*g_2(s,\cdot)\|_{L^q} \lesssim \|chi_2(t,s,\cdot)|\xi|^k m_1(t,s,\cdot)\hat{g}_2(s,\cdot)\|_{L^{q'}}$$
$$\lesssim \|\chi_2(t,s,\cdot)|\xi|^k m_1(t,s,\cdot)\|_{L^r}\|\hat{g}_2(s,\cdot)\|_{L^\infty} \lesssim (1+s)\|g_2(s,\cdot)\|_{L^1}$$
$$\times\begin{cases}(1+t)^{(\ell-\beta)/2}(1+s)^{(\ell-1)\left(n\left(1-\frac{1}{q}\right)+k\right)+(\beta-\ell)/2} & \text{if } 1<\beta<\ell+(1-\ell)\left[2n\left(1-\frac{1}{q}\right)+2k\right]\\ (1+t)^{(\ell-\beta)/2}\left(\ln\left(\frac{e+t}{e+s}\right)\right)^{1-\frac{1}{q}} & \text{if } \beta=\ell+(1-\ell)\left[2n\left(1-\frac{1}{q}\right)+2k\right]\\ (1+t)^{(\ell-1)\left(n\left(1-\frac{1}{q}\right)+k\right)} & \text{if } \beta>\ell+(1-\ell)\left[2n\left(1-\frac{1}{q}\right)+2k\right].\end{cases}$$

Considerations in Z_1: In the zone Z_1 by Lemma 2, we may estimate

$$|\xi|^k|m_1(t,s,\xi)| \lesssim |\xi|^{k-1}(1+s)^{(\beta+\ell)/2}(1+t)^{(\ell-\beta)/2}.$$

By using Haussdorff–Young inequality and Hölder inequality, setting

$$\frac{1}{r} = \frac{1}{q'}-\frac{1}{m'} = \frac{1}{m}-\frac{1}{q},\ m\in[1,2),$$

for $2 \leq q < \frac{nm}{(n-m+mk)_+}$ and $k \in [0,1)$, one may estimate

$$\|\mathscr{F}^{-1}(\chi_1(s,\cdot)|\xi|^k m_1(t,s,\cdot))*g_2(s,\cdot)\|_{L^q} \lesssim \|\chi_1(s,\cdot)|\xi|^k m_1(t,s,\cdot)\hat{g}_2(s,\cdot)\|_{L^{q'}}$$
$$\lesssim \|\chi_1(s,\cdot)|\xi|^k m_1(t,s,\cdot)\|_{L^r}\|\hat{g}_2(s,\cdot)\|_{L^{m'}}$$
$$\lesssim (1+t)^{\frac{\ell-\beta}{2}}(1+s)^{\frac{\ell+\beta}{2}}(1+s)^{\left(n\left(\frac{1}{m}-\frac{1}{q}\right)+k-1\right)(\ell-1)}\|g_2(s,\cdot)\|_{L^m},$$

thanks to

$$\|\chi_1(s,\cdot)|\xi|^{k-1}\|^r_{L^r(Z_1)} = \int_{\xi\in Z_1}|\xi|^{r(k-1)}d\xi \lesssim (1+s)^{(n+r(k-1))(\ell-1)},\ r(k-1)+n<0.$$

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For $k \geq 1$, it is clear that

$$\|\mathcal{F}^{-1}(\chi_1(s,\cdot)|\xi|^k m_1(t,s,\cdot)) * g_2(s,\cdot)\|_{L^2} \lesssim (1+s)^{\frac{\ell+\beta}{2}}(1+t)^{\frac{\ell-\beta}{2}}\|g_2(s,\cdot)\|_{\dot{H}^{k-1}}.$$

However, if $\beta \geq \ell + 2n(1-\ell)\left(1-\frac{1}{q}\right) + 2k(1-\ell)$, by using that

$$(1+s)^{\frac{\ell+\beta}{2}}(1+t)^{\frac{\ell-\beta}{2}} \lesssim (1+t)^{(\ell-1)\left(n\left(1-\frac{1}{q}\right)+k\right)}(1+s)^{\ell+(1-\ell)\left(n\left(1-\frac{1}{q}\right)+k\right)}, \tag{32}$$

for $2 \leq q < \frac{nm}{(n-m+mk)_+}$ and $k \in [0,1)$, we get

$$\|\mathcal{F}^{-1}(\chi_1(s,\cdot)|\xi|^k m_1(t,s,\cdot)) * g_2(s,\cdot)\|_{L^q} \lesssim (1+t)^{(\ell-1)\left(n\left(1-\frac{1}{q}\right)+k\right)}(1+s)^{1+n(1-\ell)\left(1-\frac{1}{m}\right)}\|g_2(s,\cdot)\|_{L^m},$$

whereas for $k \geq 1$ and $q = 2$, we get

$$\|\mathcal{F}^{-1}(\chi_1(s,\cdot)|\xi|^k m_1(t,s,\cdot)) * g_2(s,\cdot)\|_{L^2} \lesssim (1+t)^{(\ell-1)\left(\frac{n}{2}+k\right)}(1+s)^{\ell+(1-\ell)\left(\frac{n}{2}+k\right)}\|g_2(s,\cdot)\|_{\dot{H}^{k-1}}.$$

We conclude that the estimates in the zones Z_2 and Z_3 are the same with exception in the case $\beta = \ell + (1-\ell)\left[2n\left(1-\frac{1}{q}\right)+2k\right]$ in which a logarithmic term appears. The desired estimates follow by gluing the estimates in the zones Z_1, Z_2, and Z_3, by taking into account the range of $\beta > 1$ and the regularity of the initial data at zone Z_1. □

5 | GLOBAL EXISTENCE RESULTS

By Duhamel's principle, a function $u \in X$, where X is a suitable space, is a solution to (4) if, and only if, it satisfies the equality

$$u(t,x) = K(t,0,x) *_{(x)} u_1(x) + \int_0^t K(t,s,x) *_{(x)} f(u(s,x))\, ds, \qquad \text{in } X, \tag{33}$$

where $K(t,s,x)$ is the fundamental solution to (15), that is, $K(t,s,x) *_{(x)} g_2(s,x)$ is the solution to the linear Cauchy problem (15) with $g_1 \equiv 0$, being $K(t,s,x) = \mathcal{F}^{-1}(m_1)(t,s,x)$, that is,

$$K(t,s,x) = -\frac{\pi i}{4(1-\ell)}(1+s)^{1+(\beta-1)/2}(1+t)^{(1-\beta)/2}\mathcal{F}^{-1}\left(\psi_{0,\rho,0}\right)(t,s,x).$$

The proof of our global existence results is based on the following scheme: We define an appropriate data function space $\mathcal{D}$ and an evolution space for solutions X equipped with a norm relate to the estimates of solutions to the linear Cauchy problem (15), with $s = 0$, such that

$$\|K(t,0,x) *_{(x)} u_1(x)\|_X \leq C\|u_1\|_{\mathcal{D}}.$$

For $u \in X$, we define the operator P by

$$P : u \in X \to Pu(t,x) := K(t,0,x) *_{(x)} u_1(x) + Fu(t,x),$$

with

$$Fu(t,x) \doteq \int_0^t K(t,s,x) *_{(x)} f(u(s,x))\, ds,$$

then we prove the estimates

$$\|Pu\|_X \leq C\|u_1\|_{\mathcal{D}} + C_1(t)\|u\|_X^p,$$
$$\|Pu - Pv\|_X \leq C_2(t)\|u-v\|_X\left(\|u\|_X^{p-1} + \|v\|_X^{p-1}\right).$$

The estimates for the image Pu allow us to apply Banach's fixed point theorem. In this way, we get simultaneously a unique solution to $Pu = u$ locally in time for large data and globally in time for small data.[29] To prove the local (in time) existence, we use that $C_1(t), C_2(t)$ tend to zero as t goes to zero, whereas to prove the global (in time) existence, we use $C_1(t) \le C$ and $C_2(t) \le C$ for all $t \ge 0$.

The key point to prove Theorems 2–4 is to apply the derived estimates in Proposition 1. Since we are interested into prove a critical exponent of Fujita type and we are going to look for solutions in the evolution space $C([0,\infty), H^k(\mathbf{R}^n))$, with $k \doteq \frac{n\ell}{2}+1$, it is expected that the results are easily obtained for large values of β with respect to the parameters ℓ, k, n, since the long-time behavior of solutions in Proposition 1 are the same of the solutions to the Cauchy problem for the parabolic equation $v_t - (1+t)^{1-2\ell} \triangle v = 0$. This is the reason why we will start with the proof of Theorem 4, and then, we will explain how to get results for the challenging problem of smaller values of β.

5.1 | Proof of Theorem 4

Proof of Theorem 4. For $T > 0$, we define the space

$$X(T) \doteq C([0,T], H^k(\mathbf{R}^n)),\ \ k \doteq \frac{n\ell}{2}+1,$$

equipped with the norm

$$\|u\|_{X(T)} \doteq \sup_{t\in[0,T]} (1+t)^{(1-\ell)\frac{n}{2}} \left(\|u(t,\cdot)\|_{L^2} + g(t)(1+t)^{(1-\ell)k} \|u(t,\cdot)\|_{\dot{H}^k} \right),$$

where

$$g(t) = \begin{cases} 1 & \bar{k} > k, \\ (\ln(e+t))^{-\frac{1}{2}} & \bar{k} = k, \end{cases}$$

and $\bar{k} \doteq \frac{\beta-\ell}{2(1-\ell)} - \frac{n}{2}$. We have to prove the global existence in time of the solution u assuming that there exists $\delta > 0$ such that

$$u_1 \in \mathcal{D} \doteq H^{k-1}(\mathbf{R}^n) \cap L^1(\mathbf{R}^n), \qquad ||u_1||_{\mathcal{D}} \le \delta.$$

Applying (28) and (30) of Proposition 1, $K(t,0,x)*_{(x)}u_1(x) \in X(T)$, and it satisfies

$$\|K(t,0,x)*_{(x)}u_1(x)\|_{X(T)} \le C\|u_1\|_{\mathcal{D}}.$$

It remains to show the estimates

$$\|Fu\|_{X(T)} \le C\|u\|^p_{X(T)}, \tag{34}$$

$$\|Fu - Fv\|_{X(T)} \le C\|u-v\|_{X(T)} \left(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)} \right). \tag{35}$$

Let us begin by prove (34). Taking into account the definition of the norm in the function space $X(T)$, we split the proof accordingly to size of β:

Let $\bar{k} > k$, that is, $\beta > \ell + n(1-\ell) + 2k(1-\ell)$. Applying (28) and (30) of Proposition 1, we have

$$\|Fu(t,\cdot)\|_{L^2} \lesssim \int_0^t (1+s)(1+t)^{(\ell-1)\frac{n}{2}} \left(\||u(s,\cdot)|^p\|_{L^1} + (1+s)^{(1-\ell)n\left(1-\frac{1}{m}\right)} \||u(s,\cdot)|^p\|_{L^m} \right) ds,$$

where $\frac{2n}{n+2} < m \le 2$ and

$$\|Fu(t,\cdot)\|_{\dot{H}^k} \lesssim \int_0^t (1+s)(1+t)^{(\ell-1)\left(\frac{n}{2}+k\right)} \left(\||u(s,\cdot)|^p\|_{L^1} + (1+s)^{(1-\ell)\left(\frac{n}{2}+k-1\right)} \||u(s,\cdot)|^p\|_{\dot{H}^{k-1}} \right) ds.$$

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First, we use Gagliardo–Nirenberg inequality

$$\|u(s,\cdot)\|_{L^q} \lesssim \|u(s,\cdot)\|_{L^2}^{1-\theta(q)} \|u(s,\cdot)\|_{\dot{H}^k}^{\theta(q)}, \ \theta(q) = \frac{n}{k}\left(\frac{1}{2}-\frac{1}{q}\right),$$

for $q = jp$ with $j = 1, m$. We point out that $\theta(jp) < 1$ for all $p > 1$ provided that $\ell > 1 - \frac{2}{n}$ and $\theta(jp) > 0$ for $p > p_c(n,\ell) > 2$. Since $u \in X(T)$, we may estimate

$$\begin{aligned}\||u(s,\cdot)|^p\|_{L^{\frac{q}{p}}} &= \|u(s,\cdot)\|_{L^q}^p \lesssim \|u(s,\cdot)\|_{\dot{H}^k}^{p\theta(q)} \|u(s,\cdot)\|_{L^2}^{(1-\theta(q))p} \\ &\lesssim (1+s)^{(\ell-1)\left(\frac{n}{2}+k\right)\theta p + (\ell-1)\frac{n(1-\theta)p}{2}} \|u\|_{X(T)}^p \lesssim (1+s)^{n(\ell-1)\left(p-\frac{p}{q}\right)} \|u\|_{X(T)}^p,\end{aligned}$$

$q = p$ and $q = mp$ with $\frac{2n}{n+2} < m \le 2$, for $p > p_c(n,\ell)$. Therefore, we obtain

$$\begin{aligned}\|Fu(t,\cdot)\|_{L^2} &\lesssim (1+t)^{(\ell-1)\frac{n}{2}} \int_0^t (1+s)^{1+n(\ell-1)(p-1)} ds \|u\|_{X(T)}^p \\ &\quad + (1+t)^{(\ell-1)\frac{n}{2}} \int_0^t (1+s)^{1+(1-\ell)n\left(1-\frac{1}{m}\right)+n(\ell-1)\left(p-\frac{1}{m}\right)} ds \|u\|_{X(T)}^p \\ &\lesssim (1+t)^{(\ell-1)\frac{n}{2}} \|u\|_{X(T)}^p,\end{aligned}$$

for $p > 1 + \frac{2}{n(1-\ell)}$.

Then, in order to estimate $\|Fu(t,\cdot)\|_{\dot{H}^k}$, we may use that $H^k(\mathbf{R}^\mathbf{n})$, with $k > \frac{n}{2}$, is imbedded into $L^\infty(\mathbf{R}^\mathbf{n})$. Indeed, thanks to Corollary A.1, for $p > \max\{1, k-1\}$, we may estimate

$$\||u(s,\cdot)|^p\|_{\dot{H}^{k-1}} \le C\|u(s,\cdot)\|_{\dot{H}^{k-1}} \|u(s,\cdot)\|_{L^\infty}^{p-1}.$$

Since $u \in X(T)$, we have

$$\|u(s,\cdot)\|_{\dot{H}^{k-1}} \lesssim (1+s)^{(\ell-1)\left(\frac{n}{2}+k-1\right)} \|u\|_{X(T)},$$

and thanks to Lemma A.1, for $\tilde{k} < \frac{n}{2} < k$, it follows

$$\|u(s,\cdot)\|_{L^\infty} \lesssim \|u(s,\cdot)\|_{\dot{H}^{\tilde{k}}} + \|u(s,\cdot)\|_{\dot{H}^k} \lesssim (1+s)^{(\ell-1)\left(\frac{n}{2}+\tilde{k}\right)} \|u\|_{X(T)}.$$

If we choose $\tilde{k} = \frac{n}{2} - \varepsilon_0$, with ε_0 sufficiently small, then

$$\||u(s,\cdot)|^p\|_{\dot{H}^{k-1}} \lesssim (1+s)^{(\ell-1)\left(\frac{n}{2}+k-1\right)+(\ell-1)(n-\varepsilon_0)(p-1)} \|u\|_{X(T)}^p,$$

hence,

$$\begin{aligned}\|Fu(t,\cdot)\|_{\dot{H}^k} &\lesssim (1+t)^{(\ell-1)\left(\frac{n}{2}+k\right)} \int_0^t (1+s)^{1+n(\ell-1)(p-1)} ds \|u\|_{X(T)}^p \\ &\quad + (1+t)^{(\ell-1)\left(\frac{n}{2}+k\right)} \int_0^t (1+s)^{1+(n-\varepsilon_0)(\ell-1)(p-1)} ds \|u\|_{X(T)}^p \\ &\lesssim (1+t)^{(\ell-1)\left(\frac{n}{2}+k\right)} \|u\|_{X(T)}^p,\end{aligned}$$

for $p > 1 + \frac{2}{n(1-\ell)}$.

The case $\bar{k} = k$, that is, $\beta = \ell + n(1-\ell) + 2k(1-\ell)$.

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In this case, one may conclude that

$$\|Fu(t,\cdot)\|_{L^2} \lesssim (1+t)^{\frac{n}{2}(\ell-1)}\|u\|^p_{X(T)},$$

and

$$\|Fu(t,\cdot)\|_{\dot{H}^{\bar{k}}} \lesssim (1+t)^{(\ell-1)\left(\frac{n}{2}+\bar{k}\right)}(\ln(e+t))^{\frac{1}{2}}\|u\|^p_{X(T)},$$

for $p > 1 + \frac{2}{n(1-\ell)}$.

Finally, let us discuss the proof of (35) only in the case $\beta > \ell + n(1-\ell) + 2k(1-\ell)$. Applying (28) of Proposition 1 for $q = 2$, we have

$$\begin{aligned}\|Fu(t,\cdot) - Fv(t,\cdot)\|_{L^2} &\lesssim (1+t)^{(\ell-1)\frac{n}{2}}\int_0^t (1+s)\|(f(u)-f(v))(s,\cdot)\|_{L^1}ds\\ &\quad + (1+t)^{(\ell-1)\frac{n}{2}}\int_0^t (1+s)^{1+(1-\ell)n\left(1-\frac{1}{m}\right)}\|(f(u)-f(v))(s,\cdot)\|_{L^m}ds.\end{aligned}$$

Here, we may take $m \in [1,2]$ such that $m > \frac{2n}{n+2}$.

By using (2) and Hölder inequality, we find that

$$\begin{aligned}\|(f(u)-f(v))(s,\cdot)\|_{L^\alpha} &\leq C_1\|(u-v)(|u|^{p-1}+|v|^{p-1})(s,\cdot)\|_{L^\alpha}\\ &\leq C_1\|(u-v)(s,\cdot)\|_{L^{p\alpha}}\left(\|u(s,\cdot)\|^{p-1}_{L^{p\alpha}} + \|v(s,\cdot)\|^{p-1}_{L^{p\alpha}}\right)\\ &\leq C_2(1+s)^{n(\ell-1)\left(p-\frac{1}{\alpha}\right)}\|u-v\|_{X(T)}\left(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\right),\end{aligned} \tag{36}$$

for any $1 \leq \alpha \leq m$. Therefore,

$$\begin{aligned}\|Fu(t,\cdot) - Fv(t,\cdot)\|_{L^2} &\lesssim (1+t)^{(\ell-1)\frac{n}{2}}\int_0^t (1+s)^{1+n(\ell-1)(p-1)}ds\|u-v\|_{X(T)}\left(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\right)\\ &\quad + (1+t)^{(\ell-1)\frac{n}{2}}\|u-v\|_{X(T)}\left(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\right),\end{aligned}$$

for $p > 1 + \frac{2}{n(1-\ell)}$.

Applying (30) of Proposition 1, we have

$$\begin{aligned}\|Fu(t,\cdot) - Fv(t,\cdot)\|_{\dot{H}^k} &\lesssim (1+t)^{(\ell-1)\left(\frac{n}{2}+k\right)}\int_0^t (1+s)\|(f(u)-f(v))(s,\cdot)\|_{L^1}\,ds\\ &\quad + (1+t)^{(\ell-1)\left(\frac{n}{2}+k\right)}\int_0^t (1+s)^{1+(1-\ell)\left(\frac{n}{2}+k-1\right)}\|(f(u)-f(v))(s,\cdot)\|_{\dot{H}^{k-1}}\,ds.\end{aligned}$$

From now, we assume that $f(u) = |u|^p$, without lose of generality. In order to estimate $\|(f(u)-f(v))(s,\cdot)\|_{\dot{H}^{k-1}}$, we use

$$|u(s,x)|^p - |v(s,x)|^p = p\int_0^1 |v+\tau(u-v)|^{p-2}(v+\tau(u-v))(s,x)d\tau(u-v)(s,x).$$

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Hence, applying Proposition A.3 gives

$$\||u(s,x)|^p - |v(s,x)|^p\|_{\dot{H}^{k-1}} \lesssim \|(u-v)(s,\cdot)\|_{\dot{H}^{k-1}} \int_0^1 \||v+\tau(u-v)|^{p-2}(v+\tau(u-v))(s,\cdot)\|_\infty \, d\tau$$
$$+ \|(u-v)(s,\cdot)\|_\infty \int_0^1 \||v+\tau(u-v)|^{p-2}(v+\tau(u-v))(s,\cdot)\|_{\dot{H}^{k-1}} \, d\tau.$$

Now, since $u, v \in X(T)$, we have

$$\|(u-v)(s,\cdot)\|_{\dot{H}^{k-1}} \lesssim (1+s)^{(\ell-1)\left(\frac{n}{2}+k-1\right)} \|u-v\|_{X(T)}.$$

Applying Lemma A.1, for $\tilde{k} < \frac{n}{2} < k$, it follows

$$\|(u-v)(s,\cdot)\|_\infty \lesssim \|(u-v)(s,\cdot)\|_{\dot{H}^{\tilde{k}}} + \|(u-v)(s,\cdot)\|_{\dot{H}^k} \lesssim (1+s)^{(\ell-1)\left(\frac{n}{2}+\tilde{k}\right)} \|u-v\|_{X(T)},$$

and

$$\||v+\tau(u-v)|^{p-2}(v+\tau(u-v))(s,\cdot)\|_\infty \lesssim (1+s)^{(\ell-1)\left(\frac{n}{2}+\tilde{k}\right)(p-1)} \left(\|u\|_{X(T)}^{p-1} + \|v\|_{X(T)}^{p-1}\right),$$

with $\tilde{k} = \frac{n}{2} - \varepsilon_0$ and ε_0 sufficiently small.

For $p > k$, Corollary A.1 implies

$$\||v+\tau(u-v)|^{p-2}(v+\tau(u-v))(s,\cdot)\|_{\dot{H}^{k-1}} \le C\|(v+\tau(u-v))(s,\cdot)\|_{\dot{H}^{k-1}} \|(v+\tau(u-v))(s,\cdot)\|_{L^\infty}^{p-2}$$
$$\lesssim (1+s)^{(\ell-1)\left(\frac{n}{2}+k-1\right)} (1+s)^{(\ell-1)\left(\frac{n}{2}+\tilde{k}\right)(p-2)} \left(\|u\|_{X(T)}^{p-1} + \|v\|_{X(T)}^{p-1}\right).$$

Therefore,

$$\|Fu(t,\cdot) - Fv(t,\cdot)\|_{\dot{H}^k} \lesssim (1+t)^{(\ell-1)\left(\frac{n}{2}+k\right)} \int_0^t (1+s)^{1+n(\ell-1)(p-1)} ds \|u-v\|_{X(T)} \left(\|u\|_{X(T)}^{p-1} + \|v\|_{X(T)}^{p-1}\right)$$
$$+ (1+t)^{(\ell-1)\left(\frac{n}{2}+k\right)} \int_0^t (1+s)^{(\ell-1)(n-\varepsilon_0)(p-1)} ds \|u-v\|_{X(T)} \left(\|u\|_{X(T)}^{p-1} + \|v\|_{X(T)}^{p-1}\right)$$
$$\le (1+t)^{(\ell-1)\left(\frac{n}{2}+k\right)} \|u-v\|_{X(T)} \left(\|u\|_{X(T)}^{p-1} + \|v\|_{X(T)}^{p-1}\right),$$

for $p > 1 + \frac{2}{n(1-\ell)}$. □

5.2 | Proof of Theorem 2

Proof of Theorem 2. Let $n = 2$ or $n = 3$ and $k \doteq \frac{n\ell}{2} + 1$ such that $\frac{n}{2} < k \le 2$. For $T > 0$, we define the space

$$X(T) \doteq C([0,T], H^k(\mathbf{R}^n)),$$

equipped with the norm

$$\|u\|_{X(T)} \doteq \sup_{t\in[0,T]} \left((1+t)^{(1-\ell)\frac{n}{2}} \|u(t,\cdot)\|_{L^2} + h(t)\|u(t,\cdot)\|_{\dot{H}^{k-1}} + (1+t)^{\frac{\beta-\ell}{2}} \|u(t,\cdot)\|_{\dot{H}^k} \right),$$

where

$$h(t) = \begin{cases} (1+t)^{(1-\ell)\left(\frac{n}{2}+k-1\right)}, & k-1 < \bar{k} < k, \\ (1+t)^{\frac{\beta-\ell}{2}} (\ln(e+t))^{-\frac{1}{2}}, & \bar{k} = k-1. \end{cases}$$

and $\bar{k} \doteq \frac{\beta-\ell}{2(1-\ell)} - \frac{n}{2}$. In the following, we only prove (34). Let $k-1 < \bar{k} < k$, that is,

$$\ell + n(1-\ell) + 2(k-1)(1-\ell) < \beta < \ell + n(1-\ell) + 2k(1-\ell).$$

Applying (28) of Proposition 1, we have

$$\|Fu(t,\cdot)\|_{L^2} \lesssim \int_0^t (1+s)(1+t)^{(\ell-1)\frac{n}{2}} \left(\||u(s,\cdot)|^p\|_{L^1} + (1+s)^{(1-\ell)n\left(1-\frac{1}{m}\right)} \||u(s,\cdot)|^p\|_{L^m} \right) ds,$$

with $\frac{2n}{n+2} < m \leq 2$. We will use now the fractional Sobolev embedding (for instance, see Bahouri et al.[30]):

$$\|u(s,\cdot)\|_{L^q} \lesssim \|u(s,\cdot)\|_{\dot{H}^{\kappa(q)}}, \; \kappa(q) = n\left(\frac{1}{2} - \frac{1}{q}\right), \; 2 \leq q < \infty,$$

by taking $q = jp$ with $j = 1, m$. We consider three possibilities for $\kappa_j = \kappa(jp) = n\left(\frac{1}{2} - \frac{1}{jp}\right)$, $j = 1, m$, which satisfying $\kappa_1 < \kappa_m < k$ for $\ell > 1 - \frac{2}{n}$. In the first one, suppose that $k_m \leq k-1$ with $\kappa_m = \kappa(mp) = n\left(\frac{1}{2} - \frac{1}{mp}\right)$, we may estimate

$$\begin{aligned} \||u(s,\cdot)|^p\|_{L^j} &= \|u(s,\cdot)\|^p_{L^{jp}} \lesssim \|u(s,\cdot)\|^p_{\dot{H}^{\kappa_j}} \\ &\lesssim (1+s)^{n(\ell-1)\left(p-\frac{1}{j}\right)} \|u\|^p_{X(T)}, \; j = 1, m, \end{aligned}$$

hence, as in the proof of Theorem 4, we conclude

$$\|Fu(t,\cdot)\|_{L^2} \lesssim (1+t)^{\frac{n}{2}(\ell-1)} \|u\|^p_{X(T)},$$

for $p > 1 + \frac{2}{n(1-\ell)}$. In the second one, suppose that $\kappa_1 \leq k-1 < k_m$ we may estimate

$$\begin{aligned} \||u(s,\cdot)|^p\|_{L^1} &= \|u(s,\cdot)\|^p_{L^p} \lesssim \|u(s,\cdot)\|^p_{\dot{H}^{\kappa_1}} \\ &\lesssim (1+s)^{p(\ell-1)\left(\frac{n}{2}+\kappa_1\right)} \|u\|^p_{X(T)}, \end{aligned}$$

with $\kappa_1 = \kappa(p) = n\left(\frac{1}{2} - \frac{1}{p}\right)$, whereas

$$\begin{aligned} \||u(s,\cdot)|^p\|_{L^m} &= \|u(s,\cdot)\|^p_{L^{mp}} \lesssim \|u(s,\cdot)\|^p_{\dot{H}^{\kappa_m}} \\ &\lesssim \|u(s,\cdot)\|^{(1-\theta)p}_{\dot{H}^{k-1}} \|u(s,\cdot)\|^{p\theta}_{\dot{H}^k} \lesssim (1+s)^{p(\ell-1)\left(\frac{n}{2}+k-1\right)} \|u\|^p_{X(T)}, \end{aligned}$$

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with $\theta = k_m - k + 1$ since $k-1 < k_m < k$. Therefore, if $m > \frac{2n}{n+2}$ is chosen sufficiently small, we conclude that

$$\begin{aligned}\|Fu(t,\cdot)\|_{L^2} &\lesssim (1+t)^{\frac{n}{2}(\ell-1)}\int_0^t (1+s)^{1+n(\ell-1)(p-1)}ds\|u\|^p_{X(T)}\\ &+ (1+t)^{\frac{n}{2}(\ell-1)}\int_0^t (1+s)^{1+(1-\ell)n\left(1-\frac{1}{m}\right)+p(\ell-1)\left(\frac{n}{2}+k-1\right)}ds\|u\|^p_{X(T)}\\ &\lesssim (1+t)^{\frac{n}{2}(\ell-1)}\|u\|^p_{X(T)},\end{aligned}$$

for

$$p > 1 + \frac{2}{n(1-\ell)} > \frac{1}{1+\ell} + \frac{2}{n(1-\ell)}.$$

In the last one, suppose that $k-1 < \kappa_1 < k_m < k$, we may estimate

$$\||u(s,\cdot)|^p\|_{L^j} = \|u(s,\cdot)\|^p_{L^{jp}} \lesssim \|u(s,\cdot)\|^p_{\dot{H}^{\kappa_j}} \lesssim (1+s)^{p(\ell-1)\left(\frac{n}{2}+k-1\right)}\|u\|^p_{X(T)},\ j = 1, m,$$

and we can conclude as in the previous one that

$$\|Fu(t,\cdot)\|_{L^2} \lesssim (1+t)^{\frac{n}{2}(\ell-1)}\|u\|^p_{X(T)},$$

for $p > 1 + \frac{2}{n(1-\ell)}$.

If $\ell = \frac{2}{n}$ for $n = 3$, that is, $k = 2$, applying (30) of Proposition 1, we have

$$\|Fu(t,\cdot)\|_{\dot{H}^1} \lesssim \int_0^t (1+s)(1+t)^{(\ell-1)\left(\frac{n}{2}+1\right)}\left(\||u(s,\cdot)|^p\|_{L^1} + (1+s)^{\frac{n(1-\ell)}{2}}\||u(s,\cdot)|^p\|_{L^2}\right)ds.$$

Using the fractional Sobolev embedding, we may estimate

$$\||u(s,\cdot)|^p\|_{L^2} = \|u(s,\cdot)\|^p_{L^{2p}} \lesssim \|u(s,\cdot)\|^p_{\dot{H}^{\kappa(2p)}} \lesssim (1+s)^{p(\ell-1)\left(\frac{n}{2}+1\right)}\|u\|^p_{X(T)},$$

where $\kappa(2p) = \frac{3}{2}\left(1-\frac{1}{p}\right) > 1$ for $p > 3$, i.e., $p > p_c\left(3, \frac{2}{3}\right)$. Therefore, if $\ell = \frac{2}{n}$ and $n = 3$, we obtain

$$\begin{aligned}\|Fu(t,\cdot)\|_{\dot{H}^1} &\lesssim (1+t)^{(\ell-1)\left(\frac{n}{2}+1\right)}\int_0^t (1+s)^{1+\max\{n(\ell-1)(p-1);\,p(\ell-1)\left(\frac{n}{2}+1\right)\}}ds\|u\|^p_{X(T)}\\ &+ (1+t)^{(\ell-1)\left(\frac{n}{2}+1\right)}\int_0^t (1+s)^{1+\frac{n(1-\ell)}{2}+p(\ell-1)\left(\frac{n}{2}+1\right)}ds\|u\|^p_{X(T)}\\ &\lesssim (1+t)^{(\ell-1)\left(\frac{n}{2}+1\right)}\|u\|^p_{X(T)},\end{aligned}$$

for $p > 3$, that is, $p > p_c\left(3, \frac{2}{3}\right)$.

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However, if $\ell < \frac{2}{n}$, that is, $k < 2$ applying (28) of Proposition 1, we obtain

$$\begin{aligned}\|Fu(t,\cdot)\|_{\dot{H}^{k-1}} &\lesssim \int_0^t (1+s)(1+t)^{(\ell-1)\left(\frac{n}{2}+k-1\right)}\||u(s,\cdot)|^p\|_{L^1}\,ds\\ &\quad+\int_0^t (1+t)^{(\ell-1)\left(\frac{n}{2}+k-1\right)}(1+s)^{1+n(1-\ell)\left(1-\frac{1}{m}\right)}\||u(s,\cdot)|^p\|_{L^m}\,ds,\end{aligned}$$

where $m > \frac{2n}{n+2(2-k)}$, that is, $m > \frac{2n}{n(1-\ell)+2}$. Using the fractional Sobolev embedding, we may estimate

$$\begin{aligned}\||u(s,\cdot)|^p\|_{L^j} &= \|u(s,\cdot)\|^p_{L^{jp}} \lesssim \|u(s,\cdot)\|^p_{\dot{H}^{\kappa_j}}\\ &\lesssim (1+s)^{\max\{n(\ell-1)\left(p-\frac{1}{j}\right);\,p(\ell-1)\left(\frac{n}{2}+k-1\right)\}}\|u\|^p_{X(T)},\ j=1,m,\end{aligned}$$

for $\kappa_j < k$, where $\kappa_j = n\left(\frac{1}{2}-\frac{1}{jp}\right)$. As we seen we have to consider three possibilities for κ_j. Suppose that $\kappa_j > k-1$ (otherwise, we can prove as before), if $m > \frac{2n}{n+2}$ is chosen sufficiently small, we conclude that

$$\begin{aligned}\|Fu(t,\cdot)\|_{\dot{H}^{k-1}} &\lesssim (1+t)^{(\ell-1)\left(\frac{n}{2}+k-1\right)}\int_0^t (1+s)^{1+p(\ell-1)\left(\frac{n}{2}+k-1\right)}ds\|u\|^p_{X(T)}\\ &\quad+(1+t)^{(\ell-1)\left(\frac{n}{2}+k-1\right)}\int_0^t (1+s)^{1+n(1-\ell)\left(1-\frac{1}{m}\right)+p(\ell-1)\left(\frac{n}{2}+k-1\right)}ds\|u\|^p_{X(T)}\\ &\lesssim (1+t)^{(\ell-1)\left(\frac{n}{2}+k-1\right)}\|u\|^p_{X(T)},\end{aligned}$$

for $p > 1+\frac{2}{n(1-\ell)}$.

Moreover, for $\ell+n(1-\ell)(1+\ell) < \beta < \ell+n(1-\ell)+2k(1-\ell)$ applying (29) of Proposition 1, we have

$$\begin{aligned}\|Fu(t,\cdot)\|_{\dot{H}^{k}} &\lesssim \int_0^t (1+t)^{\frac{\ell-\beta}{2}}(1+s)^{1+\frac{\beta-\ell}{2}+(\ell-1)\left(\frac{n}{2}+k\right)}\||u(s,\cdot)|^p\|_{L^1}\,ds\\ &\quad+\int_0^t (1+t)^{\frac{\ell-\beta}{2}}(1+s)^{1+\frac{\beta-\ell}{2}+(\ell-1)\left(\frac{n}{2}+k\right)+(1-\ell)\left(\frac{n}{2}+k-1\right)}\||u(s,\cdot)|^p\|_{\dot{H}^{k-1}}\,ds.\end{aligned}$$

As before, we may estimate

$$\begin{aligned}\||u(s,\cdot)|^p\|_{L^1} &= \|u(s,\cdot)\|^p_{L^p} \lesssim \|u(s,\cdot)\|^p_{\dot{H}^{\kappa_1}}\\ &\lesssim \|u\|^p_{X(T)}\begin{cases}(1+s)^{(\ell-1)n(p-1)} & \text{if } \kappa_1 \le k-1\\ (1+s)^{p(\ell-1)\left(\frac{n}{2}+k-1\right)} & \text{if } k-1<\kappa_1<k,\end{cases}\end{aligned}$$

with $\kappa_1 = n\left(\frac{1}{2}-\frac{1}{p}\right)$ and

$$\int_0^t (1+s)^{1+\frac{\beta-\ell}{2}+(\ell-1)\left(\frac{n}{2}+k\right)}\||u(s,\cdot)|^p\|_{L^1}\,ds \lesssim \|u\|^p_{X(T)},$$

for $p > 1+\frac{2}{n(1-\ell)}$ and $\beta < \ell+n(1-\ell)+2k(1-\ell)$.

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Using Lemma A.1 for $k-1<\tilde{k}<\frac{n}{2}<k$, it follows

$$\begin{aligned}\|u(s,\cdot)\|_{L^\infty} &\lesssim \|u(s,\cdot)\|_{\dot{H}^{\tilde{k}}}+\|u(s,\cdot)\|_{\dot{H}^k}\\ &\lesssim \|u(s,\cdot)\|^{(1-\theta)}_{\dot{H}^{k-1}}\|u(s,\cdot)\|^{\theta}_{\dot{H}^k}\\ &\lesssim (1+s)^{(\ell-1)\left(\frac{n}{2}+k-1\right)+\theta\left[\frac{\ell-\beta}{2}+(1-\ell)\left(\frac{n}{2}+k-1\right)\right]}\|u\|_{X(T)},\end{aligned}$$

with $\theta=\tilde{k}-k+1$, and we may estimate

$$\begin{aligned}\||u(s,\cdot)|^p\|_{\dot{H}^{k-1}} &\lesssim \|u(s,\cdot)\|_{\dot{H}^{k-1}}\|u(s,\cdot)\|^{p-1}_{L^\infty}\\ &\lesssim (1+s)^{(\ell-1)\left(\frac{n}{2}+k-1\right)p+\theta\left[\frac{\ell-\beta}{2}+(1-\ell)\left(\frac{n}{2}+k-1\right)\right](p-1)}\|u\|^p_{X(T)}.\end{aligned}$$

If we choose $\tilde{k}=\frac{n}{2}-\varepsilon_0$, with ε_0 sufficiently small, then $\theta=\frac{n(1-\ell)}{2}-\varepsilon_0$, and we obtain

$$\int_0^t (1+s)^{\frac{\beta+\ell}{2}}\||u(s,\cdot)|^p\|_{\dot{H}^{k-1}}ds\lesssim \|u\|^p_{X(T)},$$

for $p>1+\frac{2}{n(1-\ell)}$, hence,

$$\|Fu(t,\cdot)\|_{\dot{H}^k}\lesssim (1+t)^{\frac{\ell-\beta}{2}}\|u\|^p_{X(T)}.$$

Here, we remark that $k=1+\frac{n\ell}{2}$, $\theta<1$,

$$\ell+\frac{(\beta-\ell)}{2}(1-\theta(p-1))+(\ell-1)\left(\frac{n}{2}+k-1\right)+(\theta-1)(1-\ell)\left(\frac{n}{2}+k-1\right)(p-1)<-1,$$

and $\theta(p-1)>1$ for $p>1+\frac{2}{n(1-\ell)}$.

In the case $\bar{k}=k-1$, that is, $\beta=\ell+n(1-\ell)+2(k-1)(1-\ell)$, one may conclude that

$$\begin{aligned}\|Fu(t,\cdot)\|_{L^2}&\lesssim (1+t)^{\frac{n}{2}(\ell-1)}\|u\|^p_{X(T)},\\ \|Fu(t,\cdot)\|_{\dot{H}^k}&\lesssim (1+t)^{\frac{\ell-\beta}{2}}\|u\|^p_{X(T)},\end{aligned}$$

and

$$\|Fu(t,\cdot)\|_{\dot{H}^{\bar{k}}}\lesssim (1+t)^{\frac{\ell-\beta}{2}}(\ln(e+t))^{\frac{1}{2}}\|u\|^p_{X(T)},$$

for $p>1+\frac{2}{n(1-\ell)}$. □

5.3 | The proof of Theorem 3

The proof of Theorem 3. Let $n=3$ and $\ell\in\left(\frac{2}{3},1\right)$. We consider r_1,r_2 satisfying

$$r_1>\frac{2(3\ell-1)}{1-\ell},\ 2<r_2<\frac{6}{2\kappa(r_1)-1},$$

with $\kappa(r_1)=3\left(\frac{1}{2}-\frac{1}{r_1}\right)$. If $\ell+6(1-\ell)\left(1-\frac{1}{r_1}\right)\leq\beta<2-\ell+3(1-\ell)(1+\ell)$ and $T>0$, we define the following space:

$$X(T)\doteq C([0,T],H^{\kappa(r_1)}(\mathbf{R}^3)\cap\dot{H}^{\kappa(r_1)-1,r_2}(\mathbf{R}^3)),$$

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equipped with the norm

$$\|u\|_{X(T)} \doteq \sup_{t\in[0,T]} \left((1+t)^{(1-\ell)\frac{3}{2}} \left(\|u(t,\cdot)\|_{L^2} + (1+t)^{(1-\ell)\kappa(r_1)} \|u(t,\cdot)\|_{\dot{H}^{\kappa(r_1)}} \right) \right) + \sup_{t\in[0,T]} \left((1+t)^{(1-\ell)\left(3\left(1-\frac{1}{r_2}\right)+\kappa(r_1)-1\right)} \|u(t,\cdot)\|_{\dot{H}^{\kappa(r_1)-1,r_2}} \right).$$

Taking into account the proof of Theorem 2, we can prove that

$$\|Fu(t,\cdot)\|_{L^2} \lesssim (1+t)^{\frac{3}{2}(\ell-1)} \|u\|^p_{X(T)}.$$

For

$$\beta > \ell + 6(1-\ell)\left(1-\frac{1}{r_2}\right) + 2(\kappa(r_1)-1)(1-\ell),$$

applying (28) of Proposition 1, we obtain

$$\|Fu(t,\cdot)\|_{\dot{H}^{\kappa(r_1)-1,r_2}} \lesssim \int_0^t (1+s)(1+t)^{(\ell-1)\left(3\left(1-\frac{1}{r_2}\right)+\kappa(r_1)-1\right)} \||u(s,\cdot)|^p\|_{L^1}\, ds + \int_0^t (1+t)^{(\ell-1)\left(3\left(1-\frac{1}{r_2}\right)+\kappa(r_1)-1\right)} (1+s)^{1+\frac{3(1-\ell)}{2}} \||u(s,\cdot)|^p\|_{L^2}\, ds.$$

We have

$$1 + \frac{r_1(2-\kappa(r_1))}{3} \le \frac{r_1}{2} \iff r_1 \ge 6.$$

But $r_1 > \frac{2(3\ell-1)}{1-\ell} > 6$ for $\ell > \frac{2}{3}$. Hence, the assumption that $p < 1 + \frac{r_1(2-\kappa(r_1))}{3}$ implies $\kappa(2p) \le \kappa(r_1)$ and using the fractional Sobolev embedding, we get

$$\||u(s,\cdot)|^p\|_{L^j} = \|u(s,\cdot)\|^p_{L^{jp}} \lesssim \|u(s,\cdot)\|^p_{\dot{H}^{\kappa(jp)}} \lesssim (1+s)^{3(\ell-1)\left(p-\frac{1}{j}\right)} \|u\|^p_{X(T)},\ j=1,2.$$

Hence,

$$\begin{aligned}\|Fu(t,\cdot)\|_{\dot{H}^{\kappa(r_1)-1,r_2}} &\lesssim (1+t)^{(\ell-1)\left(3\left(1-\frac{1}{r_2}\right)+\kappa(r_1)-1\right)} \int_0^t (1+s)^{1+3(\ell-1)(p-1)} ds \|u\|^p_{X(T)} \\ &\quad + (1+t)^{(\ell-1)\left(3\left(1-\frac{1}{r_2}\right)+\kappa(r_1)-1\right)} \int_0^t (1+s)^{1+\frac{3(1-\ell)}{2}+3(\ell-1)\left(p-\frac{1}{2}\right)} ds \|u\|^p_{X(T)} \\ &\lesssim (1+t)^{(\ell-1)\left(3\left(1-\frac{1}{r_2}\right)+\kappa(r_1)-1\right)} \|u\|^p_{X(T)},\end{aligned}$$

for $p > 1 + \frac{2}{3(1-\ell)}$. Then, applying again (28) of Proposition 1,

$$\|Fu(t,\cdot)\|_{\dot{H}^{\kappa(r_1)}} \lesssim \int_0^t (1+s)(1+t)^{(\ell-1)\left(\frac{3}{2}+\kappa(r_1)\right)} \||u(s,\cdot)|^p\|_{L^1}\, ds + \int_0^t (1+t)^{(\ell-1)\left(\frac{3}{2}+\kappa(r_1)\right)} (1+s)^{1+(1-\ell)\left(\frac{3}{2}+\kappa(r_1)-1\right)} \||u(s,\cdot)|^p\|_{\dot{H}^{\kappa(r_1)-1}}\, ds.$$

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Using Proposition A.2, we may estimate

$$\||u(s,\cdot)|^p\|_{\dot{H}^{\kappa(r_1)-1}} \lesssim \|u(s,\cdot)\|_{\dot{H}^{\kappa(r_1)-1,r_2}} \|u(s,\cdot)\|_{L^{r_1}}^{p-1}, \tag{37}$$

with r_1 and r_2 satisfying $\frac{p-1}{r_1} + \frac{1}{r_2} = \frac{1}{2}$. In the admissible range of r_2, we claim that p is bounded above, that is,

$$p = 1 + r_1\left(\frac{1}{2} - \frac{1}{r_2}\right) < 1 + \frac{r_1(2-\kappa(r_1))}{3}.$$

Using the fractional Sobolev embedding and the definition of $u \in X(T)$, we obtain

$$\|u(s,\cdot)\|_{L^{r_1}} \lesssim \|u(s,\cdot)\|_{\dot{H}^{\kappa(r_1)}} \lesssim (1+s)^{(\ell-1)\left(\frac{3}{2}+\kappa(r_1)\right)}\|u\|_{X(T)}; \tag{38}$$

$$\|u(s,\cdot)\|_{\dot{H}^{\kappa(r_1)-1,r_2}} \lesssim (1+s)^{(\ell-1)\left(3\left(1-\frac{1}{r_2}\right)+\kappa(r_1)-1\right)}\|u\|_{X(T)}. \tag{39}$$

Therefore, from (37), (38), and (39), it follows that

$$\||u(s,\cdot)|^p\|_{\dot{H}^{\kappa(r_1)-1}} \lesssim (1+s)^{(\ell-1)\left(3\left(1-\frac{1}{r_2}\right)+\kappa(r_1)-1\right)+(\ell-1)\left(\frac{3}{2}+\kappa(r_1)\right)(p-1)}\|u\|_{X(T)}^p,$$

so that

$$\begin{aligned}(1+s)^{1+(1-\ell)\left(\frac{3}{2}+\kappa(r_1)-1\right)}\||u(s,\cdot)|^p\|_{\dot{H}^{\kappa(r_1)-1}} &\lesssim (1+s)^{1+(\ell-1)3\left(\frac{1}{2}-\frac{1}{r_2}\right)+(\ell-1)\left(\frac{3}{2}+\kappa(r_1)\right)(p-1)}\|u\|_{X(T)}^p \\ &\lesssim (1+s)^{1+(\ell-1)\frac{3(p-1)}{r_1}+3(\ell-1)\left(1-\frac{1}{r_1}\right)(p-1)}\|u\|_{X(T)}^p.\end{aligned}$$

Finally, we conclude that

$$\begin{aligned}\|Fu(t,\cdot)\|_{\dot{H}^{\kappa(r_1)}} &\lesssim \int_0^t (1+t)^{(\ell-1)\left(\frac{3}{2}+\kappa(r_1)\right)}(1+s)^{1+3(\ell-1)(p-1)}\,ds\|u\|_{X(T)}^p \\ &\lesssim (1+t)^{(\ell-1)\left(\frac{3}{2}+\kappa(r_1)\right)}\|u\|_{X(T)}^p,\end{aligned}$$

for $1 + \frac{2}{3(1-\ell)} < p < 1 + \frac{r_1(2-\kappa(r_1))}{3}$. □

5.4 | The proof of Theorem 1

The proof of Theorem 1. By applying the change of variable

$$v(\tau,x) = u(t,x), \qquad 1+\tau = \frac{(1+t)^{1-\ell}}{1-\ell},$$

the Cauchy problem (4) takes the form

$$\begin{cases} v_{\tau\tau} - \Delta v + \frac{\mu}{1+\tau}v_\tau = g(v), & \tau \geq s, x \in \mathbf{R}^n, \\ v(s,x) = 0, & x \in \mathbf{R}^n, \\ v_\tau(s,x) = u_1(x), & x \in \mathbf{R}^n, \end{cases} \tag{40}$$

with $s = \frac{\ell}{1-\ell}$, $g(v) = [(1-\ell)(1+\tau)]^{\frac{2\ell}{1-\ell}}|v|^p$, and

$$\mu = \frac{\beta-\ell}{1-\ell}.$$

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We enunciate Corollary 2 from D'Abbicco,[17] which will be useful in the proof of the Theorem 2.1. There was introduced the following notation: For any $1 \le r \le q \le \infty$, let be

$$d(r,q) = \begin{cases} \frac{n}{r} - \frac{n-1}{2} - \frac{1}{q}, & \text{if } r \le q', \\ \frac{1}{r} + \frac{n-1}{2} - \frac{n}{q}, & \text{if } r \ge q'. \end{cases}$$

Corollary 1 (see D'Abbicco[17]). *Let $\mu \ge 2$. Let $n = 2$ and $2 < q \le q_\sharp$, or $n = 3$ and $q \in (1, q_\sharp]$ or $n \ge 4$ and $\frac{2(n-1)}{n+1} \le q \le q_\sharp$. Then, there exists $r_2 \in (1, \min\{q, q'\}]$ such that $d(r_2, q) = 1$ and the solution to (40), with an arbitrary parameter s and $g \equiv 0$, verifies the following $(L^1 \cap L^{r_2}) - L^q$ decay estimate*

$$||v(\tau,\cdot)||_{L^q} \lesssim (1+s)(1+\tau)^{-n\left(1-\frac{1}{q}\right)} \left(||u_1||_{L^1} + (1+s)^{\frac{n-1}{2}-\frac{1}{q}} ||u_1||_{L^{r_2}} \right),$$

if $\mu > n+1-\frac{2}{q}$, and for any $\varepsilon > 0$ verifies the $(L^1 \cap L^{r_2}) - L^q$ estimate

$$||v(\tau,\cdot)||_{L^q} \lesssim (1+s)^{\frac{\mu}{2}-\varepsilon}(1+\tau)^{\varepsilon-(n-1)\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{\mu}{2}} \left((1+s)^{\frac{1}{q}-\frac{n-1}{2}} ||u_1||_{L^1} + ||u_1||_{L^{r_2}} \right),$$

if $\mu \le n+1-\frac{2}{q}$.

Remark 8 The condition $q \le q_\sharp$ is equivalent to $d(q', q) = \frac{n}{q'} - \frac{n-1}{2} - \frac{1}{q} \le 1$.

It is enough to prove the global existence of small data solutions to (40). For $T > 0$, we define the space

$$X(T) \doteq C([0,T], L^{p_c}(\mathbf{R}^n) \cap L^{q_\sharp}(\mathbf{R}^n)),$$

equipped with the norm

$$\|v\|_{X(T)} \doteq \sup_{\tau\in[0,T]} \left\{ (1+\tau)^{n\left(1-\frac{1}{p_c}\right)} \|v(\tau,\cdot)\|_{L^{p_c}} + (1+\tau)^{n\left(1-\frac{1}{q_\sharp}\right)} \|v(\tau,\cdot)\|_{L^{q_\sharp}} \right\},$$

if $\mu > n+1-\frac{2}{q_\sharp}$ and

$$\|v\|_{X(T)} \doteq \sup_{\tau\in[0,T]} \left\{ (1+\tau)^{n\left(1-\frac{1}{p_c}\right)} \|v(\tau,\cdot)\|_{L^{p_c}} + (1+\tau)^{n\left(1-\frac{1}{\bar q}\right)} \|v(\tau,\cdot)\|_{L^{\bar q}} + (1+\tau)^{(n-1)\left(\frac{1}{2}-\frac{1}{q_\sharp}\right)+\frac{\mu}{2}-\varepsilon} \|v(\tau,\cdot)\|_{L^{q_\sharp}} \right\},$$

if $n+1-\frac{2}{\bar q} < \mu \le n+1-\frac{2}{q_\sharp}$ and

$$\|v\|_{X(T)} \doteq \sup_{\tau\in[0,T]} \left\{ (1+\tau)^{n\left(1-\frac{1}{p_c}\right)} \|v(\tau,\cdot)\|_{L^{p_c}} + (1+\tau)^{(n-1)\left(\frac{1}{2}-\frac{1}{\bar q}\right)+\frac{\mu}{2}-\varepsilon} \|v(\tau,\cdot)\|_{L^{\bar q}} + (1+\tau)^{(n-1)\left(\frac{1}{2}-\frac{1}{q_\sharp}\right)+\frac{\mu}{2}-\varepsilon} \|v(\tau,\cdot)\|_{L^{q_\sharp}} \right\},$$

if $\mu = n+1-\frac{2}{\bar q}$.

For any $v \in X(T)$, we define the operator P by

$$P : v \in X(T) \to Pv(\tau,x) := v^0(\tau,x) + Fv(\tau,x),$$

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where $v^0(\tau,x)$ is the solution to (40) with $g \equiv 0$ and

$$Fv(\tau,x) = \int_0^\tau K(\tau,s,x)*_{(x)}[(1-\ell)(1+s)]^{\frac{2\ell}{1-\ell}}|v|^p\,ds,$$

with $K(\tau,s,x)*_{(x)}h(v)$ is the solution to (40) with $g \equiv 0$ and $v_\tau(s,x) \equiv h(v)$. Then, we prove the estimates

$$\|Pv\|_{X(T)} \le C\|u_1\|_D + C_1(t)\|v\|^p_{X(T)},$$
$$\|Pu - Pv\|_{X(T)} \le C_2(t)\|u-v\|_{X(T)}\left(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\right).$$

By Corollary 1, if $\mu \ge 2$, then $v^0 \in X(T)$, and it satisfies

$$\|v^0\|_{X(T)} \le C\|u_1\|_D.$$

Let us prove the desire estimate for $\|Fv(\tau,\cdot)\|_{X(T)}$. One may follow the steps of the proof of $\|Fv(\tau,\cdot)\|_{X(T)}$ to conclude the Lipschitz property.

First, if $\mu > n+1-\frac{2}{q_\sharp}$ applying Corollary 1, we have

$$\|Fv(\tau,\cdot)\|_{L^q} \lesssim \int_0^\tau (1+s)^{1+\frac{2\ell}{1-\ell}}(1+\tau)^{-n\left(1-\frac{1}{q}\right)}\left(\||v(s,\cdot)|^p\|_{L^1} + (1+s)^{\frac{n-1}{2}-\frac{1}{q}}\||v(s,\cdot)|^p\|_{L^{r(q)}}\right)ds,$$

with $r(q) \in [1,2[$ given by $\frac{n}{r(q)} = \frac{1}{2}+\frac{n}{2}+\frac{1}{q}$, for all $p_c \le q \le q_\sharp$. Taking into account that $v \in X(T)$, thanks to $r(q_\sharp)p_c < q_\sharp$, we may estimate

$$\||v(s,\cdot)|^p\|_{L^{r(q)}} = \|v(s,\cdot)\|^p_{L^{r(q)p}} \lesssim (1+s)^{-n\left(1-\frac{1}{pr(q)}\right)p}\|v\|^p_{X(T)}$$
$$\lesssim (1+s)^{-n\left(p-\frac{1}{r(q)}\right)}\|v\|^p_{X(T)},$$

for $p_c \le q \le q_\sharp$ and $p_c < p \le \frac{q_\sharp}{r(q_\sharp)}$. Therefore, if $\mu > n+1-\frac{2}{q_\sharp}$, we have

$$\|Fv(\tau,\cdot)\|_{L^q} \lesssim (1+\tau)^{-n\left(1-\frac{1}{q}\right)}\int_0^\tau (1+s)^{1+\frac{2\ell}{1-\ell}-n(p-1)}ds\|v\|^p_{X(T)}$$
$$+ (1+\tau)^{-n\left(1-\frac{1}{q}\right)}\int_0^\tau (1+s)^{1+\frac{2\ell}{1-\ell}-n\left(p-\frac{1}{r(q)}\right)}(1+s)^{\frac{n-1}{2}-\frac{1}{q}}ds\|v\|^p_{X(T)}$$
$$\lesssim (1+\tau)^{-n\left(1-\frac{1}{q}\right)}\|v\|^p_{X(T)},$$

for $p_c \le q \le q_\sharp$ and $p > 1+\frac{2}{n(1-\ell)}$.

Then, if $n+1-\frac{2}{\bar{q}} < \mu \le n+1-\frac{2}{q_\sharp}$, applying again Corollary 1, we may estimate

$$\|Fv(\tau,\cdot)\|_{L^q} \lesssim (1+\tau)^{-n\left(1-\frac{1}{q}\right)}\int_0^\tau (1+s)^{1+\frac{2\ell}{1-\ell}}\left(\||v(s,\cdot)|^p\|_{L^1} + (1+s)^{\frac{n-1}{2}-\frac{1}{q}}\||v(s,\cdot)|^p\|_{L^{r(q)}}\right)ds,$$

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with $\frac{n}{r(q)}=\frac{1}{2}+\frac{n}{2}+\frac{1}{q}$, for $p_c\le q<\bar q$. We may estimate

$$\begin{aligned}\||v(s,\cdot)|^p\|_{L^p}&=\|v(s,\cdot)\|_{L^p}^p\lesssim\|v(s,\cdot)\|_{L^{p_c}}^{(1-\theta)p}\|v(s,\cdot)\|_{L^{q_\sharp}}^{\theta p}\\&\lesssim(1+s)^{-n\left(1-\frac{1}{p_c}\right)(1-\theta)p+\left(\varepsilon-(n-1)\left(\frac12-\frac{1}{q_\sharp}\right)-\frac{\mu}{2}\right)p\theta}\|v\|_{X(T)}^p\\&\lesssim(1+s)^{-n\left(1-\frac{1}{p_c}\right)p}\|v\|_{X(T)}^p,\end{aligned}$$

thanks to

$$\left[\varepsilon-(n-1)\left(\frac12-\frac{1}{q_\sharp}\right)-\frac{\mu}{2}+n\left(1-\frac{1}{p_c}\right)\right]\theta p\le\left[\varepsilon\theta+(1-n)\left(\frac{1}{p_c}-\frac{1}{p}\right)\right]p\le0,$$

for $\varepsilon>0$ sufficiently small and $\theta=\left(\frac{1}{p_c}-\frac1p\right)/\left(\frac{1}{p_c}-\frac{1}{q_\sharp}\right)$. Moreover, thanks to $r(q)p_c<r(\bar q)p_c=\bar q$, we may estimate

$$\begin{aligned}\||v(s,\cdot)|^p\|_{L^{r(q)}}&=\|v(s,\cdot)\|_{L^{r(q)p}}^p\lesssim\|v(s,\cdot)\|_{L^{r(q)p_c}}^{(1-\theta)p}\|v(s,\cdot)\|_{L^{r(\bar q)p}}^{\theta p}\\&\lesssim(1+s)^{-n\left(1-\frac{1}{r(q)p_c}\right)(1-\theta)p+\left(\varepsilon-(n-1)\left(\frac12-\frac{1}{r(\bar q)p}\right)-\frac{\mu}{2}\right)p\theta}\|v\|_{X(T)}^p,\end{aligned}$$

for all $p_c\le q<\bar q$, with $\theta=\left(\frac{1}{r(q)p_c}-\frac{1}{r(q)p}\right)/\left(\frac{1}{r(q)p_c}-\frac{1}{r(\bar q)p}\right)$. Let

$$\begin{aligned}\gamma&=-n\left(1-\frac{1}{r(q)p_c}\right)p+\left(\varepsilon-(n-1)\left(\frac12-\frac{1}{r(\bar q)p}\right)-\frac{\mu}{2}+n\left(1-\frac{1}{r(q)p_c}\right)\right)p\theta\\&\le-np+\frac{n}{r(q)}+\left(\varepsilon-\frac{(n-1)}{2}-\frac{1}{r(\bar q)p}-\frac{n+1}{2}+\frac{1}{\bar q}+n\right)p\theta\\&\le-np+\frac{n}{r(q)}+\left(\frac{1}{r(\bar q)p_c}-\frac{1}{r(\bar q)p}\right)p\theta+\varepsilon p\theta\\&\le-np+\frac{n}{r(q)}+\left(\frac{1}{r(\bar q)}\left(\frac{1}{p_c}-\frac1p\right)\right)p+\varepsilon p\theta\\&\le-np\left(1-\frac{1}{p_c}\right)-\frac{np}{p_c}+\frac{n}{r(q)}+\frac{p}{p_c}-1+\varepsilon p\theta\\&\le-np\left(1-\frac{1}{p_c}\right)-n\left(1-\frac{1}{r(q)}\right)+(n-1)\left(1-\frac{p}{p_c}\right)+\varepsilon p\theta.\end{aligned}$$

Therefore, if $n+1-\frac{2}{\bar q}<\mu\le n+1-\frac{2}{q_\sharp}$, we conclude that

$$\begin{aligned}\|Fv(t,\cdot)\|_{L^q}&\lesssim(1+\tau)^{-n\left(1-\frac1q\right)}\int_0^\tau(1+s)^{1+\frac{2\ell}{1-\ell}-n\left(1-\frac{1}{p_c}\right)p}ds\|v\|_{X(T)}^p\\&\quad+(1+\tau)^{-n\left(1-\frac1q\right)}\int_0^\tau(1+s)^{1+\frac{2\ell}{1-\ell}+\frac{n-1}{2}-\frac1q+\gamma}ds\|v\|_{X(T)}^p\\&\lesssim(1+\tau)^{-n\left(1-\frac1q\right)}\int_0^\tau(1+s)^{1+\frac{2\ell}{1-\ell}-n\left(1-\frac{1}{p_c}\right)p}ds\|v\|_{X(T)}^p\\&\quad+(1+\tau)^{-n\left(1-\frac1q\right)}\int_0^\tau(1+s)^{1+\frac{2\ell}{1-\ell}-n\left(1-\frac{1}{p_c}\right)p+\varepsilon p\theta}ds\|v\|_{X(T)}^p\\&\lesssim(1+\tau)^{-n\left(1-\frac1q\right)}\|v\|_{X(T)}^p,\end{aligned}$$

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for $p_c \le q < \bar{q}$ and

$$p > \frac{2}{n(1-\ell)}\frac{p_c}{p_c-1} = 1 + \frac{2}{n(1-\ell)}.$$

Now, if $\mu = n+1-\frac{2}{\bar{q}}$, applying again Corollary 1, we may estimate $\|Fv(\tau,\cdot)\|_{L^{p_c}}$ as before, whereas for $q = \bar{q}$ or $q = q_\sharp$,

$$\begin{aligned}\|Fv(\tau,\cdot)\|_{L^q} &\lesssim (1+\tau)^{\varepsilon-(n-1)\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{\mu}{2}}\int_0^\tau (1+s)^{\frac{2\ell}{1-\ell}+\frac{\mu}{2}-\varepsilon}\left((1+s)^{-\frac{n-1}{2}+\frac{1}{q}}\||v(s,\cdot)|^p\|_{L^1} + \||v(s,\cdot)|^p\|_{L^{r(q)}}\right)ds\\ &\lesssim (1+\tau)^{\varepsilon-(n-1)\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{\mu}{2}}\int_0^\tau (1+s)^{\frac{2\ell}{1-\ell}+1-\varepsilon}\||v(s,\cdot)|^p\|_{L^1} + (1+s)^{\frac{2\ell}{1-\ell}+\frac{\mu}{2}-\varepsilon}\||v(s,\cdot)|^p\|_{L^{r(q)}}ds,\end{aligned}$$

for any $\varepsilon > 0$, with $\frac{n}{r(q)} = \frac{1}{2}+\frac{n}{2}+\frac{1}{q}$.

Taking into account that $u \in X(T)$, as before, we may estimate

$$\||v(s,\cdot)|^p\|_{L^1} \lesssim (1+s)^{-n\left(1-\frac{1}{p_c}\right)p}\|v\|_{X(T)}^p,$$

and thanks to $r(\bar{q})p_c \le r(q_\sharp)p_c < q_\sharp$ (see Remark 1), we may estimate

$$\||v(s,\cdot)|^p\|_{L^{r(q)}} = \|v(s,\cdot)\|_{L^{pr(q)}}^p \lesssim (1+s)^{\left(\varepsilon-(n-1)\left(\frac{1}{2}-\frac{1}{pr(q)}\right)-\frac{\mu}{2}\right)p}\|v\|_{X(T)}^p,$$

for any $\varepsilon > 0$ and $p_c < p \le \frac{q_\sharp}{r(q_\sharp)}$.

We may write

$$\frac{2\ell}{1-\ell}+\frac{\mu}{2}-\varepsilon+\left(\varepsilon-(n-1)\left(\frac{1}{2}-\frac{1}{r(q)p}\right)-\frac{\mu}{2}\right)p = 1+\frac{2\ell}{1-\ell}+\varepsilon(p-1)+n-\frac{1}{r(\bar{q})}-(n-1+\mu)\frac{p}{2}+\gamma,$$

with

$$\gamma = \frac{\mu}{2}-1+n\left(\frac{1}{r(q)}-1\right)+\frac{1}{r(\bar{q})}-\frac{1}{r(q)}.$$

For $\mu = n+1-\frac{2}{\bar{q}}$ and $q = \bar{q}$, we have that $\gamma = 0$, whereas for $q = q_\sharp$, we have

$$\begin{aligned}\gamma &= \frac{\mu}{2}-1+n\left(\frac{1}{r(q_\sharp)}-1\right)+\frac{1}{r(\bar{q})}-\frac{1}{r(q_\sharp)}\\ &= \frac{(n+1)}{2}-\frac{1}{\bar{q}}-1+\frac{(1-n)}{2}+\frac{1}{q_\sharp}+\frac{1}{n}\left(\frac{1}{\bar{q}}-\frac{1}{q_\sharp}\right)\\ &= \left(\frac{1}{n}-1\right)\left(\frac{1}{\bar{q}}-\frac{1}{q_\sharp}\right) < 0.\end{aligned}$$

We conclude that for $\mu = n+1-\frac{2}{\bar{q}}$ and $q = \bar{q}$ or $q = q_\sharp$,

$$\begin{aligned}\|Fv(\tau,\cdot)\|_{L^q} &\lesssim (1+\tau)^{\varepsilon-(n-1)\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{\mu}{2}}\|v\|_{X(T)}^p\int_0^\tau (1+s)^{1+\frac{2\ell}{1-\ell}+\varepsilon(p-1)+n-\frac{1}{r(\bar{q})}-(n-1+\mu)\frac{p}{2}+\gamma}ds\\ &\lesssim (1+\tau)^{\varepsilon-(n-1)\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{\mu}{2}}\|v\|_{X(T)}^p,\end{aligned}$$

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for any $\varepsilon > 0$, $p > p_c = 1 + \frac{2}{n(1-\ell)}$, and

$$1 + \frac{2\ell}{1-\ell} + \varepsilon(p-1) + n - \frac{1}{r(\bar{q})} - (n-1+\mu)\frac{p}{2} < -1,$$

that is,

$$(n-1+\mu)\frac{p_c}{2} \geq \frac{2}{1-\ell} + n - \frac{1}{r(\bar{q})}$$

is equivalent to

$$\mu \geq n + 1 - \frac{2}{p_c r(\bar{q})} = n + 1 - \frac{2}{\bar{q}}.$$

Moreover, for $n + 1 - \frac{2}{\bar{q}} < \mu \leq n + 1 - \frac{2}{q_\sharp}$, we have

$$\begin{aligned}\|Fv(\tau,\cdot)\|_{L^{q_\sharp}} &\lesssim (1+\tau)^{\varepsilon-(n-1)\left(\frac{1}{2}-\frac{1}{q_\sharp}\right)-\frac{\mu}{2}} \int_0^\tau (1+s)^{\frac{2\ell}{1-\ell}+\frac{\mu}{2}-\varepsilon} \left((1+s)^{-\frac{n-1}{2}+\frac{1}{q_\sharp}} \||v(s,\cdot)|^p\|_{L^1} + \||v(s,\cdot)|^p\|_{L^{r(q_\sharp)}} \right) ds \\ &\lesssim (1+\tau)^{\varepsilon-(n-1)\left(\frac{1}{2}-\frac{1}{q_\sharp}\right)-\frac{\mu}{2}} \int_0^\tau (1+s)^{\frac{2\ell}{1-\ell}+1-\varepsilon} \||v(s,\cdot)|^p\|_{L^1} + (1+s)^{\frac{2\ell}{1-\ell}+\frac{\mu}{2}-\varepsilon} \||v(s,\cdot)|^p\|_{L^{r(q_\sharp)}} ds,\end{aligned}$$

for any $\varepsilon > 0$, with $\frac{n}{r(q_\sharp)} = \frac{1}{2} + \frac{n}{2} + \frac{1}{q_\sharp}$.

If $n + 1 - \frac{2}{p_c r(q_\sharp)} < \mu \leq n + 1 - \frac{2}{q_\sharp}$, we may estimate

$$\||v(s,\cdot)|^p\|_{L^{r(q_\sharp)}} \lesssim (1+s)^{\left(\varepsilon-(n-1)\left(\frac{1}{2}-\frac{1}{pr(q_\sharp)}\right)-\frac{\mu}{2}\right)p} \|v\|_{X(T)}^p,$$

hence,

$$\begin{aligned}(1+s)^{\frac{2\ell}{1-\ell}+\frac{\mu}{2}-\varepsilon} \||v(s,\cdot)|^p\|_{L^{r(q_\sharp)}} &\leq (1+s)^{\frac{2\ell}{1-\ell}+\varepsilon(p-1)-\frac{\mu(p-1)}{2}-(n-1)\left(\frac{1}{2}-\frac{1}{pr(q_\sharp)}\right)p} \\ &\leq (1+s)^{\frac{2\ell}{1-\ell}-n(p-1)+1+\varepsilon(p-1)+\gamma} \leq (1+s)^{-1},\end{aligned}$$

for $\varepsilon > 0$ sufficiently small and

$$p_c < p \leq 1 + \left(\frac{1}{r(q_\sharp)} - \frac{1}{q_\sharp}\right) p_c r(q_\sharp) = 1 + p_c - \frac{p_c r(q_\sharp)}{q_\sharp},$$

thanks to

$$\gamma = \frac{p-1}{p_c r(q_\sharp)} + \frac{1}{q_\sharp} - \frac{1}{r(q_\sharp)} \leq 0.$$

Finally, if $n + 1 - \frac{2}{\bar{q}} < \mu \leq n + 1 - \frac{2}{p_c r(q_\sharp)}$, we may estimate for $p_c < p \leq \frac{q_\sharp}{r(q_\sharp)}$

$$\begin{aligned}\||v(s,\cdot)|^p\|_{L^{r(q_\sharp)}} &= \|v(s,\cdot)\|_{L^{pr(q_\sharp)}}^p \\ &\lesssim (1+s)^{\left(\varepsilon-(n-1)\left(\frac{1}{2}-\frac{1}{pr(q_\sharp)}\right)-\frac{\mu}{2}\right)p} \|v\|_{X(T)}^p,\end{aligned}$$

for any $\varepsilon > 0$.

Now, we may write

$$\begin{aligned}&\frac{2\ell}{1-\ell} + \frac{\mu}{2} - \varepsilon + \left(\varepsilon - (n-1)\left(\frac{1}{2} - \frac{1}{r(q_\sharp)p}\right) - \frac{\mu}{2}\right)p \\ &= 1 + \frac{2\ell}{1-\ell} + \varepsilon(p-1) + n - \frac{1}{r(\bar{q})} - (n-1+\mu)\frac{p}{2} + \gamma,\end{aligned}$$

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with

$$\gamma = \frac{\mu}{2} - 1 + n\left(\frac{1}{r(q_\sharp)} - 1\right) + \frac{1}{r(\bar{q})} - \frac{1}{r(q_\sharp)}.$$

It remains to prove that $\gamma \leq 0$. Indeed, for $\mu \leq n + 1 - \frac{2}{p_c r(q_\sharp)}$ and $\frac{1}{q_\sharp} \leq \frac{1}{n-1}\left(\frac{n}{p_c r(q_\sharp)} - \frac{1}{\bar{q}}\right)$ (see Remark 4), we have

$$\begin{aligned}
\gamma &= \frac{\mu}{2} - 1 + n\left(\frac{1}{r(q_\sharp)} - 1\right) + \frac{1}{r(\bar{q})} - \frac{1}{r(q_\sharp)} \\
&= \frac{\mu}{2} - 1 + \frac{1-n}{2} + \frac{1}{q_\sharp} + \frac{1}{n}\left(\frac{1}{\bar{q}} - \frac{1}{q_\sharp}\right) \\
&\leq \frac{\mu}{2} - \frac{1+n}{2} + \frac{1}{p_c r(q_\sharp)} \leq 0.
\end{aligned}$$

Therefore, if $n + 1 - \frac{2}{\bar{q}} < \mu \leq n + 1 - \frac{2}{q_\sharp}$, we have proved that

$$\|Fv(s,\cdot)\|_{L^{q_\sharp}} \lesssim (1+\tau)^{\varepsilon - (n-1)\left(\frac{1}{2} - \frac{1}{q_\sharp}\right) - \frac{\mu}{2}} \|v\|^p_{X(T)},$$

for any $\varepsilon > 0$ sufficiently small and $p > p_c = 1 + \frac{2}{n(1-\ell)}$. □

6 | CONCLUDING REMARKS

6.1 | Einstein–de Sitter spacetime model

It is well known that the asymptotic behavior of solutions and the critical index p for the global existence of small data solutions change according to the values of β in (4). The considered equation is hyperbolic, but there exists a threshold value $\beta = \beta_c(n,\ell)$ for which the Cauchy problem have some properties related to parabolic problems for $\beta \geq \beta_c(n,\ell)$. However, it is a challenging and mainly an open problem to consider small values of β even in the case $\ell = 0$.

From now on, we explain that for $p > \frac{7}{2}$, the derived estimates in Proposition 1 may be used to get a global existence of small data solutions for the Einstein–de Sitter spacetime model with non-singular (at $t = 0$) coefficients

$$\begin{cases} u_{tt}(t,x) - (1+t)^{-\frac{4}{3}}\Delta u(t,x) + \frac{2}{1+t}u_t(t,x) = |u|^p, & t \geq 0, x \in \mathbf{R}^3, \\ u(0,x) = 0, & x \in \mathbf{R}^3, \\ u_t(0,x) = u_1(x), & x \in \mathbf{R}^3. \end{cases} \tag{41}$$

To do so, we may follow the proof of Theorem 2 with minor changes. For $T > 0$, we define the space

$$X(T) \doteq C([0,T], H^2(\mathbf{R}^n)),$$

equipped with the norm

$$\|u\|_{X(T)} \doteq \sup_{t\in[0,T]} \left(g(t)\|u(t,\cdot)\|_{\dot{H}^k}\right),$$

where

$$g(t) = \begin{cases} (1+t)^{(1-\ell)\left(\frac{n}{2}+k\right)}, & k \in [0, \frac{1}{2}), \\ (1+t)^{\frac{\beta-\ell}{2}}(\ln(e+t))^{k-1}, & k \in [\frac{1}{2}, 1), \\ (1+t)^{\frac{\beta-\ell}{2}}, & k \in [1,2], \end{cases}$$

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$\ell = \frac{2}{3}$, $\beta = 2$, and $n = 3$. Applying (27) and (28) of Proposition 1, we have

$$\|Fu(t,\cdot)\|_{\dot{H}^k} \lesssim \begin{cases} (1+t)^{(\ell-1)\left(\frac{n}{2}+k\right)} \int_0^t (1+s)G(s,u(s,\cdot))ds & k \in [0,\frac{1}{2}), \\ (1+t)^{\frac{\ell-\beta}{2}} (\ln(e+t))^{\frac{1}{2}} \int_0^t (1+s)G(s,u(s,\cdot))ds & k = \frac{1}{2}, \\ (1+t)^{\frac{\ell-\beta}{2}} \int_0^t (1+s)^{1+\frac{\beta-\ell}{2}+(\ell-1)\left(\frac{n}{2}+k\right)} G(s,u(s,\cdot))ds, & k \in (\frac{1}{2},1], \end{cases}$$

with

$$G(s,u(s,\cdot)) = \| |u(s,\cdot)|^p \|_{L^1} + (1+s)^{(1-\ell)n\left(1-\frac{1}{m}\right)} \| |u(s,\cdot)|^p \|_{L^m},$$

and $m > \frac{6}{5-2k}$. Now, we will use the fractional Sobolev embedding

$$\|u(s,\cdot)\|_{L^q} \lesssim \|u(s,\cdot)\|_{\dot{H}^{\kappa(q)}}, \ \kappa(q) = n\left(\frac{1}{2}-\frac{1}{q}\right), \ 2 \le q < \infty,$$

and for $u \in X(T)$, we obtain

$$G(s,u(s,\cdot)) \lesssim (1+s)^{(1-\ell)n\left(1-\frac{1}{m}\right)+\frac{(\ell-\beta)p}{2}} (\ln(e+s))^{p\sigma} \|u\|^p_{X(T)},$$

with $\sigma = 1-\kappa(mp)$ if $\frac{1}{2} \le \kappa(mp) < 1$ whereas $\sigma = 0$ if $\kappa(mp) \ge 1$.

If $k \in [0,\frac{1}{2})$, we may take $m = \frac{3}{2}$ and

$$\begin{aligned} \|Fu(t,\cdot)\|_{\dot{H}^k} &\lesssim (1+t)^{(\ell-1)\left(\frac{n}{2}+k\right)} \int_0^t (1+s)G(s,u(s,\cdot))ds \\ &\lesssim (1+t)^{(\ell-1)\left(\frac{n}{2}+k\right)} \int_0^t (1+s)^{1+(1-\ell)n\left(1-\frac{1}{m}\right)+\frac{(\ell-\beta)p}{2}} (\ln(e+s))^{p\sigma} ds \|u\|^p_{X(T)} \\ &\lesssim (1+t)^{(\ell-1)\left(\frac{n}{2}+k\right)} \|u\|^p_{X(T)}, \end{aligned}$$

thanks to $p > \frac{7}{2}$, that is,

$$1+(1-\ell)n\left(1-\frac{1}{m}\right)+\frac{(\ell-\beta)p}{2} < -1.$$

In the same way, for $p > \frac{7}{2}$, we conclude the estimate

$$\|Fu(t,\cdot)\|_{\dot{H}^{\frac{1}{2}}} \lesssim (1+t)^{\frac{\ell-\beta}{2}} (\ln(e+t))^{\frac{1}{2}} \|u\|^p_{X(T)}.$$

Moreover, for $k = 1$, we take $m = 2$, and we conclude that

$$\begin{aligned} \|Fu(t,\cdot)\|_{\dot{H}^1} &\lesssim (1+t)^{\frac{\ell-\beta}{2}} \int_0^t (1+s)^{1+\frac{\beta-\ell}{2}+(\ell-1)\left(\frac{n}{2}+1\right)} G(s,u(s,\cdot))ds \\ &\lesssim (1+t)^{\frac{\ell-\beta}{2}} \int_0^t (1+s)^{1+\frac{\beta-\ell}{2}+(\ell-1)\left(\frac{n}{2}+1\right)+\frac{n(1-\ell)}{2}+\frac{(\ell-\beta)p}{2}} (\ln(e+s))^{p\sigma} ds \|u\|^p_{X(T)} \\ &\lesssim (1+t)^{\frac{\ell-\beta}{2}} \|u\|^p_{X(T)}, \end{aligned}$$

for all $p > \frac{7}{2}$, that is,

$$1+\frac{\beta-\ell}{2}+(\ell-1)\left(\frac{n}{2}+1\right)+\frac{n(1-\ell)}{2}+\frac{(\ell-\beta)p}{2} < -1.$$

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By interpolation, we conclude that

$$\|Fu(t,\cdot)\|_{\dot{H}^k} \lesssim (1+t)^{\frac{\beta-\ell}{2}}(\ln(e+t))^{1-k}\|u\|^p_{X(T)},\ k\in\left(\frac{1}{2},1\right).$$

Applying (29) of Proposition 1, we have

$$\|Fu(t,\cdot)\|_{\dot{H}^2} \lesssim \int_0^t (1+t)^{\frac{\ell-\beta}{2}}(1+s)^{1+\frac{\beta-\ell}{2}+(\ell-1)\left(\frac{n}{2}+2\right)}\||u(s,\cdot)|^p\|_{L^1}\,ds$$
$$+\int_0^t (1+t)^{\frac{\ell-\beta}{2}}(1+s)^{1+\frac{\beta-\ell}{2}+(\ell-1)\left(\frac{n}{2}+2\right)+(1-\ell)\left(\frac{n}{2}+1\right)}\||u(s,\cdot)|^p\|_{\dot{H}^1}\,ds.$$

As in the Theorem 2, we may estimate

$$\||u(s,\cdot)|^p\|_{\dot{H}^1} \lesssim \|u(s,\cdot)\|_{\dot{H}^1}\|u(s,\cdot)\|^{p-1}_{L^\infty} \lesssim (1+s)^{\frac{(\ell-\beta)p}{2}}\|u\|^p_{X(T)}.$$

Hence, for $p>\frac{7}{2}$, one can also conclude that

$$\|Fu(t,\cdot)\|_{\dot{H}^2} \lesssim (1+t)^{\frac{\ell-\beta}{2}}\|u\|^p_{X(T)},$$

and by interpolation,

$$\|Fu(t,\cdot)\|_{\dot{H}^k} \lesssim (1+t)^{\frac{\ell-\beta}{2}}\|u\|^p_{X(T)},\ k\in(1,2).$$

From this analysis, we have proved (34). Similar one can derive the contraction condition (35), and the application of Banach's fixed point theorem allows us to conclude the following global existence result to the Cauchy problem (41) for all $p>\frac{7}{2}$:

Theorem 5. *If $p>\frac{7}{2}$, then there exists $\delta>0$ such that for any initial data*

$$u_1\in\mathcal{D}=H^1(\mathbf{R}^n)\cap L^1(\mathbf{R}^n),\qquad ||u_1||_{\mathcal{D}}\le\delta,$$

there exists a unique weak solution $u\in C([0,\infty),H^2(\mathbf{R}^n))$ to (41), which satisfies the following estimates:

$$||u(t,\cdot)||_{\dot{H}^k}\lesssim h(t)||u_1||_{\mathcal{D}},$$

with

$$h(t)=\begin{cases}(1+t)^{-\frac{1}{2}-\frac{k}{3}}, & k\in[0,\frac{1}{2}),\\ (1+t)^{-\frac{2}{3}}(\ln(e+t))^{1-k}, & k\in[\frac{1}{2},1),\\ (1+t)^{-\frac{2}{3}}, & k\in[1,2].\end{cases}$$

Remark 9. From Galstian and Yagdjian[5] and Palmieri,[18] we know that solutions to (41) blow-up in finite time for $1<p\le3$, even for small data. It is expected that a different approach than the one applied to prove Theorem 5 is needed to overcome the gap for $3<p\le\frac{7}{2}$.

6.2 | Non-zero first initial data

In this paper, we assumed $u(0,x)=0$ in (4), for the sake of brevity. However, thanks to Duhamel's principle (see (33)) and using the representation of solutions to (15) in Section 3, it is not difficult to extend the smallness assumption on the initial data in order to include a non-zero first initial data in the statements.

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Let us explain to the interested reader how to active the result only for Theorem 4. The key point is, under the assumption $v_0 \in \mathcal{D} = H^k(\mathbf{R}^n) \cap L^1(\mathbf{R}^n)$, to prove a priori estimates for solutions to the Cauchy problem

$$\begin{cases} v_{tt}(t,x) - (1+t)^{-2\ell}\Delta v(t,x) + \frac{\beta}{1+t} v_t(t,x) = 0, & t \geq 0, x \in \mathbf{R}^n, \\ v(0,x) = v_0(x), & x \in \mathbf{R}^n, \\ v_t(0,x) = 0, & x \in \mathbf{R}^n. \end{cases} \tag{42}$$

By Lemma 1, the solution v of (42) can be obtained from

$$\hat{v}(t,\xi) = \frac{\pi i}{4(1-\ell)}(1+t)^{(1-\beta)/2}\psi_{1,\rho-1,1}(t,0,\xi)\hat{v_0}(\xi).$$

As in the proof of Lemma 2, for $\beta > 1$, we may conclude that

$$|\xi|^k|\psi_{1,\gamma,1}(t,0,\xi)| \lesssim \begin{cases} |\xi|^k(1+t)^{(\ell-1)/2} & \text{if } (t,0,\xi) \in Z_1 \\ |\xi|^{k-|\gamma|+1/2}(1+t)^{(\ell-1)/2} & \text{if } (t,0,\xi) \in Z_2 \\ |\xi|^k(1+t)^{(1-\ell)|\gamma+1|} & \text{if } (t,0,\xi) \in Z_3, \end{cases}$$

and following as in the proof of Proposition 1, if $\beta \geq \ell + n(1-\ell) + 2k(1-\ell)$, with $k \geq 0$, then

$$||v(t,\cdot)||_{\dot{H}^k} \lesssim (1+t)^{(\ell-1)\left(\frac{n}{2}+k\right)}\eta(v_0), \tag{43}$$

where $\eta(v_0) = d_2(t)||v_0||_{L^1} + \|v_0\|_{\dot{H}^k}$ and

$$d_2(t) = \begin{cases} \ln(e+t)^{\frac{1}{2}} & \text{if } \beta = \ell + n(1-\ell) + 2k(1-\ell), \\ 1 & \text{otherwise.} \end{cases}$$

Hence, one can derive the following revisited version of Theorem 4:

Theorem 6. *Let* $\ell \in \left(1 - \frac{2}{n}, 1\right)$ *for* $n \geq 2$ *and* $k \doteq 1 + \frac{n\ell}{2}$.

If $\beta \geq 2 - \ell + n(1-\ell)(1+\ell)$ *and* $p > \max\{p_c(n,\ell), k\}$, *with* $p_c(n,\ell)$ *given by (6), then there exists* $\delta > 0$ *such that for any initial data*

$$(u_0, u_1) \in \mathcal{D} = (H^k(\mathbf{R}^n) \cap L^1(\mathbf{R}^n)) \times (H^{k-1}(\mathbf{R}^n) \cap L^1(\mathbf{R}^n)), \qquad ||(u_0,u_1)||_{\mathcal{D}} \leq \delta,$$

there exists a unique weak solution $u \in C([0,\infty), H^k(\mathbf{R}^n))$ *to (1), with non-zero first initial data, which satisfies the following estimates:*

$$||u(t,\cdot)||_{L^2} \lesssim (1+t)^{\frac{n}{2}(\ell-1)}||(u_0,u_1)||_{\mathcal{D}},$$

and

$$||u(t,\cdot)||_{\dot{H}^k} \lesssim ||(u_0,u_1)||_{\mathcal{D}} \begin{cases} (1+t)^{(\ell-1)\left(\frac{n}{2}+k\right)}, & \beta > \ell + n(1-\ell) + 2k(1-\ell), \\ (1+t)^{\frac{\ell-\beta}{2}}(\ln(e+t))^{\frac{1}{2}}, & \beta = \ell + n(1-\ell) + 2k(1-\ell). \end{cases}$$

ACKNOWLEDGEMENTS

The first author is partially supported by "Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP)" Grant 2020/08276-9 and "Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq)" Grant 304408/2020-4. The second author is partially supported by "FCT—Fundação para a Ciência e a Tecnologia, I.P." Project UIDB/05037/2020.

CONFLICT OF INTEREST

This work does not have any conflicts of interest.

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How to cite this article: Ebert MR, Marques J. Critical exponent of Fujita type for semilinear wave equations in Friedmann–Lemaître–Robertson–Walker spacetime. *Math Meth Appl Sci.* 2023;46(2):2602-2635. doi:10.1002/mma.8663

APPENDIX A

In the Appendix, we list some notations used through the paper and results of Harmonic Analysis which are important tools for proving results on the global existence of small data solutions for semilinear models with power nonlinearities. Through this paper, we use the following.

For $s \geq 0$, we denote by $|D|^s f = \mathcal{F}^{-1}(|\xi|^s \hat{f})$ and $\langle D\rangle^s f = \mathcal{F}^{-1}(\langle\xi\rangle^s \hat{f})$, with $\langle\xi\rangle^s = (1+|\xi|^2)^{\frac{s}{2}}$.

For any $q \in [1,\infty]$, we denote by $L^q(\mathbf{R}^n)$ the usual Lebesgue space over $\mathbf{R}^n$. Let $s \in \mathbb{R}$ and $1 < p < \infty$. Then,

$$H^{s,p}(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) \,:\, \|\langle D\rangle^s u\|_{L^p(\mathbb{R}^n)} = \|u\|_{H^s_p(\mathbb{R}^n)} < \infty\},$$
$$\dot{H}^{s,p}(\mathbb{R}^n) = \{u \in \mathcal{Z}'(\mathbb{R}^n) \,:\, \||D|^s u\|_{L^p(\mathbb{R}^n)} = \|u\|_{\dot{H}^s_p(\mathbb{R}^n)} < \infty\}$$

are called Bessel and Riesz potential spaces, respectively. If $p = 2$, then we use the notations $H^s(\mathbb{R}^n)$ and $\dot{H}^s(\mathbb{R}^n)$, respectively. In the definition of the Riesz potential spaces, we use the space of distributions $\mathcal{Z}'(\mathbb{R}^n)$. This space of distributions can be identified with the factor space $\mathcal{S}'/\mathcal{P}$, where $\mathcal{S}'$ denotes the dual of Schwartz space and $\mathcal{P}$ denotes the set of all polynomials.

We recall that $H^{s,q}(\mathbf{R}^n) = W^{s,q}(\mathbf{R}^n)$, the usual Sobolev space, for any $q \in (1,\infty)$ and $s \in \mathbf{N}$.

The following inequality can be found in Hajaiej et al.[31]

Proposition A.1 (Fractional Gagliardo–Nirenberg inequality). *Let $1 < p, p_0, p_1 < \infty$, $\sigma > 0$, and $s \in [0,\sigma)$. Then, it holds the following fractional Gagliardo–Nirenberg inequality for all $u \in L^{p_0}(\mathbb{R}^n) \cap \dot{H}^{\sigma,p_1}(\mathbb{R}^n)$:*

$$\|u\|_{\dot{H}^{s,p}} \lesssim \|u\|_{L^{p_0}}^{1-\theta} \|u\|_{\dot{H}^{\sigma,p_1}}^{\theta},$$

where $\theta = \theta_{s,\sigma}(p,p_0,p_1) = \frac{\frac{1}{p_0}-\frac{1}{p}+\frac{s}{n}}{\frac{1}{p_0}-\frac{1}{p_1}+\frac{\sigma}{n}}$ and $\frac{s}{\sigma} \leq \theta \leq 1$.

In the following, the symbol $\lceil s\rceil$ denotes the smallest integer greater than or equal to s. We present here two results for fractional powers (for instance, see Palmieri and Reissig[32] and Runst and Sickel[33]):

Proposition A.2. *Let $f(u) = |u|^p$ or $f(u) = |u|^{p-1}u$, with $p > \max\{1, \lceil s\rceil\}$ and $1 < r, r_1, r_2 < \infty$ satisfying*

$$\frac{1}{r} = \frac{p-1}{r_1} + \frac{1}{r_2}.$$

Then, the following estimate holds:

$$\||D|^s f(u)\|_{L^r} \leq C\|u\|_{L^{r_1}}^{p-1} \||D|^s u\|_{L^{r_2}},$$

for any $u \in L^{r_1} \cap \dot{H}^{s,r_2}$.

Corollary A.1. *Let $f(u) = |u|^p$ or $f(u) = |u|^{p-1}u$, with $p > \max\{1, s\}$ and $u \in H^{s,m} \cap L^\infty$, $1 < m < \infty$. Then, the following estimate holds:*

$$\|f(u)\|_{\dot{H}^{s,m}} \leq C\|u\|_{\dot{H}^{s,m}} \|u\|_{L^\infty}^{p-1}.$$

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We refer to D'Abbicco et al[34] for the next result:

Lemma A.1. *Let* $0 < 2s_1 < n < 2s_2$. *Then, for any function* $f \in \dot{H}^{s_1} \cap \dot{H}^{s_2}$, *one has*

$$\|f\|_\infty \lesssim \|f\|_{\dot{H}^{s_1}} + \|f\|_{\dot{H}^{s_2}}.$$

The next result combine in some sense some familiar results as Leibniz rule for the product of two function and Hölder's inequality for derivatives of fractional order (Theorem 7.6.1 in Grafakos[35]):

Proposition A.3. *Let us assume* $s > 0$ *and* $1 \le r \le \infty, 1 < p_1, p_2, q_1, q_2 \le \infty$ *satisfying the relation*

$$\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2}.$$

Then, the following fractional Leibniz rules hold:

$$\| |D|^s (uv)\|_{L^r} \lesssim \| |D|^s u\|_{L^{p_1}} \|v\|_{L^{p_2}} + \|u\|_{L^{q_1}} \| |D|^s v\|_{L^{q_2}},$$

for any $u \in \dot{H}^{s,p_1}(\mathbf{R}^n) \cap L^{q_1}(\mathbf{R}^n)$ *and* $v \in \dot{H}^{s,q_2}(\mathbf{R}^n) \cap L^{p_2}(\mathbf{R}^n)$,

$$\|\langle D\rangle^s (uv)\|_{L^r} \lesssim \|\langle D\rangle^s u\|_{L^{p_1}} \|v\|_{L^{p_2}} + \|u\|_{L^{q_1}} \|\langle D\rangle^s v\|_{L^{q_2}},$$

for any $u \in H^{s,p_1}(\mathbf{R}^n) \cap L^{q_1}(\mathbf{R}^n)$ *and* $v \in H^{s,q_2}(\mathbf{R}^n) \cap L^{p_2}(\mathbf{R}^n)$.

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