

# CLASSICAL SOLUTIONS TO PARABOLIC SYSTEMS WITH FREE BOUNDARY OF STEFAN TYPE

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ABSTRACT: Motivated by the classical model for the binary alloy solidification (crystallization) problem, we show the local in time existence and uniqueness of solutions to a parabolic system strongly coupled through free boundary conditions of Stefan type. Using a modification of the standard change of variables method and coercive estimates in a weighted Hölder space (the weight being a power of  $t$ ) we obtain solutions with maximal global regularity (having at least equal regularity for  $t > 0$  as at the initial moment).

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## 1. Introduction

The classical Stefan problem is a simplified model for the solid-liquid phase transition in a pure material taking into account the heat diffusion in each phase with latent heat at the sharp transition interface, which is supposed kept at a given constant temperature [R, M2]. The melting or the crystallization of a two component material, like a binary alloy, differs in an essential way from that free boundary model problem in several aspects. Firstly, the temperature of the mixture at the interface is not constant and depends on the relative concentration of each component, since each one having a different melting temperature determine the former through a thermodynamic phase diagram. Secondly, in a mixture of two components, one constituent is allowed to diffuse in the interior of each phase and its concentration exhibits a discontinuity across the solid-liquid interface.

Although qualitatively there is a fairly good understanding of these phenomena, there are several physical approaches to its modelling. For instance, using the theory of nonequilibrium thermodynamics a mathematical analysis has been proposed in terms of weak formulations (see [V], Chapt. V and its

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references). As in the simpler case of the Stefan problem (see [M2] and its bibliography) these generalized solutions admit the degeneracy of the interface into a mushy zone, giving place, even in a one dimensional alloy problem, to non-uniqueness results as described in [Go].

In this work we are interested in the mathematical analysis of the multidimensional case of the sharp interface model for the temperature-concentration system with a smooth free boundary arising from a phenomenological model suggested first in [R]. We shall show that, similarly to the classical two phase Stefan problem for one equation (see [M1, M2] or [S2]), the corresponding classical free boundary problem for the coupled system, under certain non-degeneracy conditions, is well-posed locally in time. As far as we are aware, this problem has been considered previously only from a numerical point of view by several authors (see [CO, BeS, AWS, SAW], for instance) without any rigorous mathematical analysis background, except a local existence result in one dimensional case in [P].

Nowadays there are several methods to obtain classical solutions to free boundary problems of parabolic type, but here we follow the approach of Solonnikov [S2]. This method uses a suitable modification of a standard change of variables, considered by Hanzawa for the one phase Stefan problem, that allows the transformation of the free boundary problem into an equivalent highly nonlinear parabolic problem in a fixed known domain. This method, and in special that modification, is very useful for keeping the solution locally in time without loss of smoothness with respect to the regularity of the initial conditions. For the transformed problem in the fixed domain, a linearization procedure combined with sharp (coercive) estimates for solutions of the linear problem permits us to obtain local (in time) solutions by the contraction mapping principle. For parabolic equations with free boundaries of two-phase Stefan and Muskat–Verigin type this method has been developed in [BS], where rigorous proofs of the classical solvability of the corresponding problems were given keeping the maximal regularity of the solutions.

An important tool, that was also used previously in [B2, B4], with particular geometries of the domains, is the use of coercive estimates for the linear parabolic problems in weighted Hölder spaces, where the weight is a power of  $t$ . This allows to reduce the initial compatibility conditions to a minimum. Although the second order parabolic theory is now well understood in space-time weighted Hölder spaces (see [L], for instance), we use here a special

class of weight introduced in [BZ] and studied in [S1]. For our system, the transformed problem is of the type of a nonlinear system of parabolic type with nonstandard transmission conditions at the initial interface, which has been treated with precise estimates for the solutions to the corresponding equations of second order in [B1, B3]. We notice that this approach can also be extended to other problems like the ones considered in [RSY], yielding improvements in the assumptions on the initial data.

In this paper after introducing the precise formulation of the free boundary problem for the coupled parabolic linear system of second order, in Section 2, we state the (local in time) existence and uniqueness results in weighted Hölder spaces, including in particular, the local solvability in classical Hölder spaces under more restrictive assumptions on the initial data. In Section 3 we reduce the problem to a nonstandard transmission problem for a nonlinear parabolic system in the fixed known domain by means of a suitable transformation of variables, that reduces the free boundary to the initial given interface, together with a translation of unknown functions in order to work with functions with zero initial conditions. In Section 4 we show the existence and uniqueness of the solution of the linearized problem, for which we state precise estimates that are based in a model transmission problem that is solved in Appendix B and in [B3]. The application of the contraction mapping principle to the nonlinear problem is done in Section 5 and it is based on explicit estimates of the inverse Jacobian matrix of the domain transformation, which are shown in Appendix A.

## 2. Statement of the problem and results

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 2$ , with a smooth boundary  $\partial\Omega$ . Suppose there is a closed surface  $\gamma(t)$  in  $\Omega$ ,  $0 \leq t \leq T$ , dividing  $\Omega$  into two subdomains  $\Omega_1(t)$  and  $\Omega_2(t)$  such that  $\partial\Omega_1(t) = \partial\Omega \cup \gamma(t)$ ,  $\partial\Omega_2(t) = \gamma(t)$ . At the initial time  $\gamma(0) = \Gamma$  and  $\Omega_j(0) = \Omega_j$ ,  $j = 1, 2$ . We assume that the subdomains  $\Omega_1$  and  $\Omega_2$  are not degenerate, for instance, by assuming the smooth initial interface  $\Gamma$  satisfies a uniform ball property from both sides.

Let

$$Q_{jT} = \{(x, t) : x \in \Omega_j(t), t \in (0, T)\}, \quad \Omega_{jT} = \Omega_j \times (0, T), \quad j = 1, 2, \\ \Omega_T = \Omega \times (0, T), \quad \Sigma_T = \partial\Omega \times [0, T], \quad \Gamma_T = \Gamma \times (0, T).$$

We consider a multidimensional two phases problem with unknown functions  $u_j(x, t)$ ,  $c_j(x, t)$ ,  $j = 1, 2$ , and a common free boundary  $\gamma(t)$  satisfying the diffusion equations for  $j = 1, 2$ , ( $\partial_t = \partial/\partial t$  and  $\Delta = \partial_{x_1}^2 + \dots + \partial_{x_n}^2$ )

$$\partial_t u_j - a_j \Delta u_j = 0 \quad \text{in } Q_{jT} , \quad (2.1)$$

$$\partial_t c_j - b_j \Delta c_j = 0 \quad \text{in } Q_{jT} , \quad (2.2)$$

with initial and boundary conditions

$$\gamma(t)|_{t=0} = \Gamma , \quad (2.3)$$

$$u_j|_{t=0} = u_{0j}(x), \quad c_j|_{t=0} = c_{0j}(x) \quad \text{in } \Omega_j, \quad j = 1, 2 , \quad (2.4)$$

$$u_1|_{\partial\Omega} = p(x, t), \quad c_1|_{\partial\Omega} = q(x, t), \quad t \in (0, T) , \quad (2.5)$$

and the following conditions on the free boundary  $\gamma(t)$

$$u_1 = u_2, \quad c_j = \sigma_j(u_j), \quad j = 1, 2 , \quad (2.6)$$

$$\lambda_1 \partial_\nu u_1 - \lambda_2 \partial_\nu u_2 = -\kappa V_\nu , \quad (2.7)$$

$$k_1 \partial_\nu c_1 - k_2 \partial_\nu c_2 = -(c_1 - c_2) V_\nu , \quad t \in (0, T) . \quad (2.8)$$

Here  $\kappa$ ,  $a_j$ ,  $b_j$ ,  $\lambda_j$ ,  $k_j$ ,  $j = 1, 2$ , are positive constants,  $\nu(x, t)$  the normal vector to  $\gamma(t)$  directed to  $\Omega_2(t)$ ,  $\partial_\nu$  denotes the normal derivative,  $V_\nu$  the normal velocity of  $\gamma(t)$  and  $\sigma_2 \geq \sigma_1$  are continuous functions given as in Figure 1.

This problem may describe the melting or the cristallization of a two component system. The domain  $\Omega_1(t)$  is occupied by the liquid phase with the temperature  $u_1(x, t)$  and the concentration of the mixture  $c_1(x, t)$ ;  $u_2(x, t)$ ,  $c_2(x, t)$  are the temperature and concentration of the mixture in the solid phase occupying  $\Omega_2(t)$ ;  $\gamma(t)$  is the free boundary separating the liquid and solid phases. The phase transition is described by the Stefan condition (2.7) with constant latent heat  $\kappa > 0$ .

On the free boundary  $\gamma(t)$  the temperature  $u$  is continuous, but the concentration  $c$  is discontinuous and determined by a phase equilibrium diagram of the type shown in Figure 1, implying the jump  $[[c]]_{\gamma(t)} = (c_2 - c_1)|_{\gamma(t)} > 0$ . The differential equation (2.8) represents the mass balance of the mixture

and may be regarded as a Stefan type condition with variable “latent heat”.

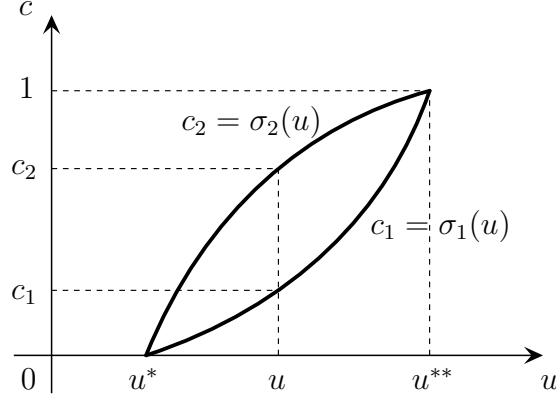


Figure 1

Here the functions  $\sigma_1(u)$ ,  $\sigma_2(u)$  are defined on the interval  $U = (u^*, u^{**})$ , with  $u^* < u^{**}$ , and their graphs represent the *liquidus* and *solidus* lines, respectively, in the temperature-concentration plane.

In the classical Stefan problem for a monocomponent material, in addition to (2.7) we have the following condition on free boundary  $\gamma(t)$ :

$$u_1 = u_2 = \theta ,$$

where  $\theta$  is the phase transition temperature, which is given by a known constant.

The Stefan problem describes physically the equilibrium process of melting or solidification of a pure substance and in each phase it holds always

$$u_1 \geq \theta \text{ in } \Omega_1(t), \quad u_2 \leq \theta \text{ in } \Omega_2(t) . \quad (2.9)$$

The phase transition process described by problem (2.1)–(2.8) is not in equilibrium, as in the Stefan problem, and the melting (crystallization) temperature  $\theta$  is a priori unknown and condition (2.9) does not hold in general. The presence of the unknown concentration  $c$  and unknown phase transition temperature on  $\gamma(t)$  makes the problem a much more complex and difficult one.

Let  $\Gamma \in C^{2+\alpha}$ ,  $N(\xi) = (N_1, \dots, N_n)$  be a unit vector field on  $\Gamma$  such that  $N(\xi) \in C^{2+\alpha}(\Gamma)$  and

$$\nu_0(\xi) \cdot N(\xi) = \nu_0(\xi) N^T(\xi) \geq d_1 > 0 \quad \forall \xi \in \Gamma , \quad (2.10)$$

where  $\nu_0(\xi)$  is the unit normal to  $\Gamma$  directed to  $\Omega_2$ .

For small  $t \leq t_0$  we can represent free boundary  $\gamma(t)$  in the form ([BS])

$$x = \xi + \rho(\xi, t) N(\xi), \quad \xi \in \Gamma, \quad t \in [0, t_0], \quad (2.11)$$

where  $\rho|_{t=0} = 0$  and  $x = \xi$  at the initial moment. In particular, if  $N(\xi) = \nu_0(\xi)$  we obtain the following equation to the free boundary:

$$x = \xi + \rho(\xi, t) \nu_0(\xi), \quad \xi \in \Gamma, \quad t \in [0, t_0],$$

which is the representation of  $\gamma(t)$  used earlier by A.M. Meirmanov [M1] and E.I. Hanzawa [H].

Let the following assumptions hold:

**A)**  $\sigma_j(u) \in C^5(\bar{U})$ ,  $j = 1, 2$ ,  $\sigma_j$  are strictly increasing functions in  $U = (u^*, u^{**})$ ,  $\sigma_j(u^*) = 0$ ,  $\sigma_j(u^{**}) = 1$ ,  $j = 1, 2$ ,  $\sigma_1(u) < \sigma_2(u) \forall u \in U$  (see Figure 1);

**B)**

$$(c_{02}(x) - c_{01}(x))|_{\Gamma} \geq d_2 > 0, \quad (2.12)$$

$$(\partial_{\nu_0} c_{0j}(x) - \sigma'_j(u_{0j}(x)) \partial_{\nu_0} u_{0j}(x))|_{\Gamma} \geq d_2, \quad j = 1, 2, \quad (2.13)$$

$$|\kappa| \leq \delta_0, \quad |\nabla u_{01}(x) - \nabla u_{02}(x)| |_{\Gamma} \leq \delta_0, \quad |\lambda_1 - \lambda_2| \leq \delta_0, \quad (2.14)$$

where  $\delta_0$  is a small positive number;

**C)**

$$\delta_j \leq c_{0j}(x) \leq 1 - \delta_j \quad \text{in } \Omega_j, \quad 0 < \delta_j < \frac{1}{2}, \quad j = 1, 2.$$

Now we formulate the local existence results for problem (2.1)–(2.8) in the weighted Hölder spaces  $C_s^\ell(\Omega_T)$ ,  $s \leq \ell$  (with a power of  $t$  as the weight), of functions  $u(x, t)$  with the norm ([BZ, S1]):

$$|u|_{s, \Omega_T}^{(\ell)} = \sup_{t \leq T} t^{\frac{\ell-s}{2}} [u]_{\Omega'_t}^{(\ell)} + \sum_{s < 2k+|j| < \ell} \sup_{t \leq T} t^{\frac{2k+|j|-s}{2}} |\partial_t^k \partial_x^j u|_{\Omega} + \begin{cases} |u|_{\Omega_T}^{(s)}, & s \geq 0, \\ 0, & s < 0, \end{cases} \quad (2.15)$$

where  $\Omega'_t = \Omega \times (\frac{t}{2}, t)$ ,  $|v|_{\Omega} = \sup_{x \in \Omega} |v|$ ,

$$[u]_{\Omega'_t}^{(\ell)} = \sum_{2k+|j|=\ell} [\partial_t^k \partial_x^j u]_{x, \Omega'_t}^{(\ell-[\ell])} + \sum_{0 < \ell-2k-|j| < 2} [\partial_t^k \partial_x^j u]_{t, \Omega'_t}^{(\frac{\ell-2k-|j|}{2})},$$

$$[v]_{x,\Omega_T}^{(\alpha)} = \sup_{(x,t),(z,t) \in \Omega_T} |v(x,t) - v(z,t)| |x - z|^{-\alpha} ,$$

$$[v]_{t,\Omega_T}^{(\alpha)} = \sup_{(x,t),(x,t_1) \in \Omega_T} |v(x,t) - v(x,t_1)| |t - t_1|^{-\alpha} , \quad \alpha \in (0, 1) ,$$

$|u|_{\Omega_T}^{(s)}$  denotes the norm of the classical Hölder space  $C_x^{s,s/2}(\overline{\Omega_T})$  ([LSU])

$$|u|_{\Omega_T}^{(s)} = \sum_{2k+|j| \leq [s]} |\partial_t^k \partial_x^j u|_{\Omega_T} + \begin{cases} 0, & s \text{ is integer,} \\ [u]_{\Omega_T}^{(s)}, & s \text{ is noninteger,} \end{cases}$$

$$|v|_{\Omega_T} = \sup_{(x,t) \in \Omega_T} |v| ;$$

and  $C_s^\ell(\Omega_T) \equiv C_x^{\ell,\ell/2}(\overline{\Omega_T})$ , if  $s = \ell$ . Similarly we define  $C_s^\ell(\Gamma_T)$ .

The solution to the problem in this space permits us to decrease the smoothness of the data and to reduce the order of their compatibility conditions up to  $[\frac{s}{2}]$ .

We write now the compatibility conditions that the data must satisfy at initial time.

The conditions of the zero order (for  $0 < s < 1$ ) reads

$$u_{01}|_{\partial\Omega} = p(x, 0) , \quad c_{01}|_{\partial\Omega} = q(x, 0) , \quad (2.16)$$

$$u_{01}|_{\Gamma} = u_{02}|_{\Gamma} , \quad c_{0j}|_{\Gamma} = \sigma_j(u_{0j})|_{\Gamma}, \quad j = 1, 2 , \quad (2.17)$$

and (for  $1 \leq s < 2$ ), with the notation  $[[\lambda \partial_{\nu_0} u_0]] = \lambda_2 \partial_{\nu_0} u_{02} - \lambda_1 \partial_{\nu_0} u_{01}$ ,

$$\frac{1}{\kappa} [[\lambda \partial_{\nu_0} u_0]]|_{\Gamma} = \frac{1}{[[c_0]]|_{\Gamma}} [[k \partial_{\nu_0} c_0]]|_{\Gamma} ; \quad (2.18)$$

the conditions of the first order (for  $2 \leq s \leq 2 + \alpha$ ) are given by:

$$a_1 \Delta u_{01}|_{\partial\Omega} = \partial_t p(x, 0) , \quad b_1 \Delta c_{01}|_{\partial\Omega} = \partial_t q(x, 0) , \quad (2.19)$$

$$\left( [[a \Delta u_0]] - \frac{1}{[[c_0]]} [[\partial_{\nu_0} u_0]] [[k \partial_{\nu_0} c_0]] \right) \Big|_{\Gamma} = 0 , \quad (2.20)$$

$$\left( b_j \Delta c_{0j} - a_j \sigma'_j(u_{0j}) \Delta u_{0j} - \frac{1}{[[c_0]]} [[k \partial_{\nu_0} c_0]] (\partial_{\nu_0} c_{0j} - \sigma'_j(u_{0j}) \partial_{\nu_0} u_{0j}) \right) \Big|_{\Gamma} = 0 , \quad (2.21)$$

for  $j = 1, 2$ .

Equations (2.16)–(2.18) are obtained from (2.5)–(2.8) with the initial conditions (2.4); differentiating the conditions (2.5),(2.6) with respect to  $t$  and

applying of (2.1), (2.2), (2.7), (2.8), (2.4) leads to the compatibility conditions (2.19)–(2.21).

**Theorem 2.1.** *Let  $\alpha \in (0, 1)$ ,  $1 < s \leq 2 + \alpha$ . Let  $\partial\Omega, \Gamma \in C^{2+\alpha}$  and the assumption A) hold.*

*Then for each functions  $u_{0j}, c_{0j} \in C^s(\bar{\Omega}_j)$ ,  $j = 1, 2$ ,  $p, q \in C_s^{2+\alpha}(\Sigma_T)$  satisfying conditions B), C) and the compatibility conditions of  $[\frac{s}{2}]$ -th order there exists  $T_0 > 0$ , such that, problem (2.1)–(2.8) has unique solutions  $u_j, c_j \in C_s^{2+\alpha}(Q_{jT_0})$ ,  $j = 1, 2$ ,  $\rho \in C_s^{2+\alpha}(\Gamma_{T_0})$ ,  $\partial_t \rho \in C_{s-1}^{1+\alpha}(\Gamma_{T_0})$  and the following estimate holds*

$$\begin{aligned} & \sum_{j=1}^2 (|u_j|_{s, Q_{jt}}^{(2+\alpha)} + |c_j|_{s, Q_{jt}}^{(2+\alpha)}) + |\rho|_{s, t}^{(2+\alpha)} + |\partial_t \rho|_{s-1, \Gamma_t}^{(1+\alpha)} \leq \\ & \leq C_1 \left( \sum_{j=1}^2 (|u_{0j}|_{\Omega_j}^{(s)} + |c_{0j}|_{\Omega_j}^{(s)}) + |p|_{s, \Sigma_t}^{(2+\alpha)} + |q|_{s, \Sigma_t}^{(2+\alpha)} \right), \text{ for } 0 < t \leq T_0. \end{aligned} \quad (2.22)$$

Putting, in particular,  $s = 2 + \alpha$  we obtain also the local solvability of the alloy free boundary problem in classical Hölder spaces [LSU] under more restrictive assumptions on the initial data.

**Theorem 2.2.** *Let  $\alpha \in (0, 1)$ ,  $\partial\Omega, \Gamma \in C^{2+\alpha}$ , and the assumption A) hold.*

*Then for any functions  $u_{0j}, c_{0j} \in C_x^{2+\alpha, 1+\alpha/2}(\bar{\Omega}_j)$ ,  $j = 1, 2$ ,  $p, q \in C_x^{2+\alpha, 1+\alpha/2}(\Sigma_T)$  satisfying B), C) and the compatibility conditions (2.16)–(2.21), there exists  $T_0 > 0$ , such that, problem (2.1)–(2.8) has unique solutions  $u_j, c_j \in C_x^{2+\alpha, 1+\alpha/2}(Q_{jT_0})$ ,  $j = 1, 2$ ,  $\rho \in C_x^{2+\alpha, 1+\alpha/2}(\Gamma_{T_0})$ ,  $\partial_t \rho \in C_x^{1+\alpha, \frac{1+\alpha}{2}}(\Gamma_{T_0})$  and the following estimate holds:*

$$\begin{aligned} & \sum_{j=1}^2 (|u_j|_{Q_{jt}}^{(2+\alpha)} + |c_j|_{Q_{jt}}^{(2+\alpha)}) + |\rho|_{\Gamma_t}^{(2+\alpha)} + |\partial_t \rho|_{\Gamma_t}^{(1+\alpha)} \leq \\ & \leq C_2 \left( \sum_{j=1}^2 (|u_{0j}|_{\Omega_j}^{(2+\alpha)} + |c_{0j}|_{\Omega_j}^{(2+\alpha)}) + |p|_{\Sigma_t}^{(2+\alpha)} + |q|_{\Sigma_t}^{(2+\alpha)} \right), \end{aligned}$$

for  $0 < t \leq T_0$ .

**Remark 2.1.** *Here  $c_j(s, t)$  represents the concentration of the mixture in the alloy in phase  $j = 1, 2$ . The condition C) guarantees that  $c_j \in (0, 1)$  for small  $t \leq T_0$ . Condition (2.12) means that the concentration of the mixture  $c_0(x)$  at the initial time is a discontinuous and increasing function across  $\Gamma$ .*



**Remark 2.2.** *Condition (2.13) is physically compatible because  $\sigma'_j(u) > 0$ ,  $\forall u \in (u^*, u^{**})$  and  $\partial_{\nu_0} u_{0j}(x)|_{\Gamma} < 0$ ,  $j = 1, 2$ , due to the decrease of the temperature from the liquid phase into the solid one.*

**Remark 2.3.** *In (2.14) we suppose that the latent heat of melting  $\kappa$  and the jump of the initial temperature  $[[[\nabla u_0]]]_{\Gamma}$ , as well as the difference of heat conductivity coefficients  $|\lambda_1 - \lambda_2|$ , are small values. These two last conditions lead to the condition  $[[[\lambda \partial_{\nu_0} u_0]]]_{\Gamma} \leq C_3 \delta_0$ . We note also that the requirements (2.14) are adjusted in the compatibility condition (2.18).*

### 3. Reduction to a problem in a fixed domain

Let  $\lambda_0 > 0$  be a sufficiently small number, such that, every point  $y \in \Omega$  being situated in a  $2\lambda_0$ -neighbourhood  $\mathcal{O}$  of  $\Gamma$  can be represented in the form

$$y = \xi + \lambda N(\xi), \quad \xi \in \Gamma, \quad |\lambda| < 2\lambda_0, \quad (3.1)$$

$y = \xi$ , if  $\lambda = 0$ . Let  $\chi(\lambda)$  be a smooth cut-off function, such that,  $\chi = 1$ ,  $|\lambda| < \lambda_0$ ,  $\chi = 0$ ,  $|\lambda| \geq 2\lambda_0$ . We define the coordinates transformation  $e_{\rho}: y \rightarrow x$  [BS], which is the modification of Hanzawa mapping [H], by the formulas:

$$\begin{aligned} x &= y + \chi(\lambda) \rho(\xi, \tau) N(\xi), & y \in \mathcal{O}, \quad \xi \in \Gamma, \\ x &= y, & y \in \Omega \setminus \mathcal{O}, \quad t = \tau, \end{aligned} \quad (3.2)$$

where  $\rho|_{\tau=0} = 0$ .

This mapping transforms the surface  $\Gamma: \lambda = 0, y = \xi$  into the free boundary

$$\gamma(t): x = \xi + \rho(\xi, t) N(\xi), \quad \xi \in \Gamma, \quad t \in [0, t_0],$$

and the given fixed domains  $\Omega_j$  into the unknown ones  $\Omega_j(t)$ ,  $j = 1, 2$ .

To every point  $y \in \mathcal{O}$  there correspond unique coordinates  $\xi = \xi(y)$ ,  $\lambda = \lambda(y)$  and inversely, every coordinates  $(\xi, \lambda)$ ,  $\xi \in \Gamma$ ,  $|\lambda| < 2\lambda_0$  determine a unique point  $y \in \mathcal{O}$ . So, formula (3.1) sets a one to one correspondence between the coordinates  $y$  and  $(\xi, \lambda)$  of every point in  $\mathcal{O}$ . Therefore, from the equation (3.1) we can express the coordinates  $(\xi, \lambda)$  via coordinates  $y$ :  $\xi = \xi(y) = (\xi_1(y), \dots, \xi_n(y))$ ,  $\lambda = \lambda(y)$ , where  $\xi_k(y), \lambda(y) \in C^{2+\alpha}(\overline{\mathcal{O}})$ ,  $k =$

$1, \dots, n$ , because  $\Gamma \in C^{2+\alpha}$ . Thus, we can write the transformation (3.2) in the form

$$\begin{aligned} x &= y + \chi(\lambda(y)) \rho(\xi(y), \tau) N(\xi(y)), & y \in \mathcal{O}, \\ x &= y, & y \in \Omega \setminus \mathcal{O}, \quad t = \tau, \end{aligned} \quad (3.3)$$

**Remark 3.1.** *We note that the coordinate transformation (3.3) with  $N \equiv \nu_0$  was used by Hanzawa [H]*

$$x = y + \chi \rho \nu_0, \quad y \in \mathcal{O}, \quad x = y, \quad y \in \Omega \setminus \mathcal{O} \quad (3.4)$$

but leads to the loss of smoothness of one unit, because the normal  $\nu_0(\xi)$  to  $\Gamma$  in (3.4) is expressed via the first partial derivatives of a function setting the surface  $\Gamma$ . The mapping (3.3) is due to Solonnikov (see also [S2] and [BS]) and permits to avoid this loss.

The Jacobian matrix of the transform (3.3) with respect to  $n$  variables  $x_1, \dots, x_n$  has the form

$$\begin{aligned} J &= \left\{ \frac{\partial x_i}{\partial y_j} \right\}_{1 \leq i, j \leq n} = \begin{bmatrix} 1 + \partial_{y_1}(N_1 \chi \rho) & \cdots & \partial_{y_n}(N_1 \chi \rho) \\ \cdots & \cdots & \cdots \\ \partial_{y_1}(N_n \chi \rho) & \cdots & 1 + \partial_{y_n}(N_n \chi \rho) \end{bmatrix} \\ &= \{\delta_{ij} + \partial_{y_j}(N_i \chi \rho)\}_{1 \leq i, j \leq n} = I + (\nabla^T(N \chi \rho))^T. \end{aligned} \quad (3.5)$$

Here  $\rho|_{t=0} = 0$ ,  $J|_{t=0} = I$ , that is, at least, for small  $t \leq t_1$  the inverse matrix  $J^{-1}$  exists. In Appendix A we prove this.

In the new coordinates  $\{y\}$  the differential operators, the normal  $\nu$  to  $\gamma(t)$  and the normal velocity of the free boundary  $\gamma(t)$  are expressed, respectively, by the formulas:

$$\nabla_x^T|_{x=y+\chi\rho N} = J^{-T} \nabla_y^T, \quad \nabla_x|_{x=y+\chi\rho N} = (J^{-T} \nabla_y^T)^T, \quad (3.6)$$

$$\begin{aligned} \partial_t - a \Delta_x|_{x=y+\chi\rho N} &= \partial_\tau - \chi \partial_\tau \rho N \nabla_x^T - a \nabla_x \nabla_x^T|_{x=y+\chi\rho N} \\ &= \partial_\tau - \chi \partial_\tau \rho N J^{-T} \nabla_y^T - a (J^{-T} \nabla_y^T)^T J^{-T} \nabla_y^T, \end{aligned} \quad (3.7)$$

$$\nu|_{x=\xi+\rho N} = \nu_0 J^{-1} |\nu_0 J^{-1}|^{-1}, \quad (3.8)$$

$$\partial_\nu|_{x=\xi+\rho N} = \nu_0 J^{-1} J^{-T} \nabla_y^T |\nu_0 J^{-1}|^{-1}, \quad (3.9)$$

$$V_\nu \Big|_{\substack{x=\xi+\rho N \\ t=\tau}} = \nu_0 J^{-1} N^T \partial_\tau \rho \Big|_{\nu_0 J^{-1}}^{-1}. \quad (3.10)$$

**Remark 3.2.** *For the sake of convenience we use again variables  $t$  and  $\xi$  instead of  $\tau$  and  $\xi(y)$ .*

We apply the coordinate mapping (3.3) in problem (2.1)–(2.8) with the help of formulas (3.7), (3.9), (3.10). Then we obtain a nonlinear problem involving the unknown function  $\rho(\xi, t)$ ,

$$\begin{aligned} u_j(y + \chi \rho N, t) &= \widehat{u}_j(y, t), \\ c_j(y + \chi \rho N, t) &= \widehat{c}_j(y, t), \end{aligned} \quad (3.11)$$

in the fixed given domains  $\Omega_j$ , for  $j = 1, 2$ :

$$\partial_t \widehat{u}_j - \chi \partial_t \rho N J^{-T} \nabla^T \widehat{u}_j - a_j (J^{-T} \nabla^T)^T J^{-T} \nabla^T \widehat{u}_j = 0 \quad \text{in } \Omega_{jT}, \quad (3.12)$$

$$\partial_t \widehat{c}_j - \chi \partial_t \rho N J^{-T} \nabla^T \widehat{c}_j - b_j (J^{-T} \nabla^T)^T J^{-T} \nabla^T \widehat{c}_j = 0 \quad \text{in } \Omega_{jT}, \quad (3.13)$$

with initial and boundary conditions

$$\rho|_{t=0} = 0 \quad \text{on } \Gamma, \quad \widehat{u}_j|_{t=0} = u_{0j}(y), \quad \widehat{c}_j|_{t=0} = c_{0j}(y) \quad \text{in } \Omega_j, \quad (3.14)$$

$$\widehat{u}_1|_{\partial\Omega} = p(y, t), \quad \widehat{c}_1|_{\partial\Omega} = q(y, t), \quad t \in (0, T), \quad (3.15)$$

and with the following transmission conditions on the initial interface

$$\widehat{u}_1 = \widehat{u}_2, \quad \widehat{c}_j = \sigma_j(\widehat{u}_j) \quad \text{on } \Gamma_T, \quad j = 1, 2, \quad (3.16)$$

$$\nu_0 J^{-1} J^{-T} (\lambda_1 \nabla^T \widehat{u}_1 - \lambda_2 \nabla^T \widehat{u}_2) = -\kappa \nu_0 J^{-1} N^T \partial_t \rho \quad \text{on } \Gamma_T, \quad (3.17)$$

$$\nu_0 J^{-1} J^{-T} (k_1 \nabla^T \widehat{c}_1 - k_2 \nabla^T \widehat{c}_2) = (\widehat{c}_1 - \widehat{c}_2) \nu_0 J^{-1} N^T \partial_t \rho \quad \text{on } \Gamma_T. \quad (3.18)$$

Although this highly nonlinear system has the advantage of being set in known domains, it will be treated in a more convenient form after a reduction to new unknowns with zero initial conditions.

We determine the auxiliary functions  $\rho_0(\xi, t)$  on  $\Gamma_T$  under the conditions

$$\rho_0|_{t=0} = 0, \quad \partial_t \rho_0|_{t=0} \equiv \partial_t \rho|_{t=0} = -\frac{[[k \partial_{\nu_0} c_0]]|_\Gamma}{\nu_0 N^T [[c_0]]|_\Gamma} \quad (3.19)$$

and  $V_j(y, t)$ ,  $Z_j(y, t)$ ,  $j = 1, 2$ , as the solutions of the Cauchy problems

$$\partial_t V_j - a_j \Delta V_j - \chi \partial_t \rho_0 N \nabla^T V_j = 0 \quad \text{in } \mathbb{R}_T^n, \quad V_j|_{t=0} = \widetilde{u}_{0j}(y), \quad (3.20)$$

$$\partial_t Z_j - b_j \Delta Z_j - \chi \partial_t \rho_0 N \nabla^T Z_j = 0 \quad \text{in } \mathbb{R}_T^n, \quad Z_j|_{t=0} = \widetilde{c}_{0j}(y), \quad (3.21)$$

where  $j = 1, 2$ , symbol “ $\sim$ ” denotes the smooth extension of a function into the entire space  $\mathbb{R}^n$  and  $\mathbb{R}_T^n = \mathbb{R}^n \times (0, T)$ .

**Lemma 3.1** ([S1, BS, B2, B3, RSY]). *Let  $1 < s \leq 2 + \alpha$ ,  $\alpha \in (0, 1)$ .*

*For arbitrary functions  $u_{0j}, c_{0j} \in C^s(\Omega_j)$ ,  $j = 1, 2$ , each one of the problems (3.19)–(3.21) has a unique solution  $\rho_0 \in C_{1+s}^{3+\alpha}(\Gamma_T)$ ,  $V_j, Z_j \in C_s^{2+\alpha}(\mathbb{R}_T^n)$ ,  $j = 1, 2$ , satisfying*

$$|\rho_0|_{s+1, \Gamma_T}^{(3+\alpha)} \leq C_1 \sum_{k=1}^2 |c_{0k}|_{\Omega_k}^{(s)}, \quad (3.22)$$

$$|V_j|_{s, \mathbb{R}_T^n}^{(2+\alpha)} \leq C_2 \left( |u_{0j}|_{\Omega_j}^{(s)} + \sum_{k=1}^2 |c_{0k}|_{\Omega_k}^{(s)} \right), \quad (3.23)$$

$$|Z_j|_{s, \mathbb{R}_T^n}^{(2+\alpha)} \leq C_3 \sum_{k=1}^2 |c_{0k}|_{\Omega_k}^{(s)}, \quad j = 1, 2. \quad (3.24)$$

Now using the functions  $\rho_0, V_j, Z_j$ ,  $j = 1, 2$ , we transform problem (3.12)–(3.18) into a more suitable form. We make use of the following substitutions:

$$\rho(\xi, t) = \rho_0(\xi, t) + \psi(\xi, t), \quad (3.25)$$

$$\widehat{u}_j(y, t) = u_j(y + \chi \rho N, t) = v_j(y, t) + V_j(y, t), \quad (3.26)$$

$$\widehat{c}_j(y, t) = c_j(y + \chi \rho N, t) = z_j(y, t) + Z_j(y, t), \quad j = 1, 2,$$

where  $\rho, v_j, z_j$  are new unknown functions satisfying zero initial conditions

$$\partial_t^k v_j|_{t=0} = 0, \quad \partial_t^k z_j|_{t=0} = 0, \quad \partial_t^{k_1} \psi|_{t=0} = 0,$$

$$k = \begin{cases} 0, & 1 < s < 2, \\ 0, 1, & 2 \leq s \leq 2 + \alpha, \end{cases} \quad k_1 = 0, 1, \quad j = 1, 2.$$

We represent the composition of the functions  $\sigma_j(v_j + V_j)$  in (3.16) as follows

$$\sigma_j(v_j + V_j) = \sigma_j(V_j) + \sigma'_j(V_j) v_j + v_j^2 \int_0^1 (1 - \lambda) \sigma''_j(V_j + \lambda v_j) d\lambda.$$

The expansion formulas (A.1), (A.2), (A.5) of Appendix A of the matrices  $J^{-1}$ ,  $J_0^{-1} = J^{-1}|_{\rho=\rho_0}$  and the change (3.25), (3.26) of unknown functions

permit us to extract linear principal terms with respect to the unknown functions, known functions and the remainder nonlinear terms. Thus, we obtain problem (3.12)–(3.18) in the form, for  $j = 1, 2$ :

$$\partial_t v_j - a_j \Delta v_j - (\partial_t \psi - a_j \Delta \psi) \chi N J_0^{-T} \nabla^T V_j = f_j(y, t) + F_j(v_j, \psi) \text{ in } \Omega_{jT}, \quad (3.27)$$

$$\partial_t z_j - b_j \Delta z_j - (\partial_t \psi - b_j \Delta \psi) \chi N J_0^{-T} \nabla^T Z_j = g_j(y, t) + G_j(z_j, \psi) \text{ in } \Omega_{jT}, \quad (3.28)$$

with zero initial data, translated Dirichlet boundary conditions

$$v_1|_{\partial\Omega} = p_1(y, t), \quad z_1|_{\partial\Omega} = q_1(y, t), \quad t \in (0, T), \quad (3.29)$$

and the corresponding transmission conditions

$$(v_1 - v_2)|_{\Gamma} = \eta_0(y, t), \quad (3.30)$$

$$(z_j - \sigma'_j(V_j) v_j)|_{\Gamma} = (\eta_j(y, t) + R_j(v_j))|_{\Gamma}, \quad j = 1, 2, \quad (3.31)$$

$$(\lambda_1 \partial_{\nu_0} v_1 - \lambda_2 \partial_{\nu_0} v_2)|_{\Gamma} = (\varphi_1(y, t) + \pi_1(v_1, v_2, \psi))|_{\Gamma}, \quad (3.32)$$

$$\begin{aligned} & \left( k_1 \partial_{\nu_0} z_1 - k_2 \partial_{\nu_0} z_2 - (Z_2 - Z_1) N J_0^{-T} \nu_0^T \partial_t \psi - \right. \\ & \left. - \nu_0 N^T [(k_1 \nabla Z_1 - k_2 \nabla Z_2) J_0^{-1} J_0^{-T} + (Z_1 - Z_2) N J_0^{-T} \partial_t \rho_0] \nabla^T \psi \right) \Big|_{\Gamma} = \\ & = (\varphi_2(y, t) + \pi_2(z_1, z_2, \psi))|_{\Gamma}, \quad t \in (0, T). \end{aligned} \quad (3.33)$$

We note that in condition (3.33) on  $\Gamma$  we have used the relation

$$\nu_0(\xi) \nabla^T \psi(\xi, t) = 0, \quad \xi \in \Gamma,$$

and in (3.32) we did not single out such term.

Thus, we have reduced the free boundary problem (2.1)–(2.8) in the unknown domains  $\Omega_1(t)$  and  $\Omega_2(t)$  to the nonlinear problem (3.27)–(3.33) in given domains  $\Omega_1, \Omega_2$  with unknown functions  $v_1, v_2, z_1, z_2, \psi$  satisfying zero initial data. In the left-hand side of equations (3.27), (3.28) and conditions on  $\partial\Omega$  and  $\Gamma$  (3.29)–(3.33) there are linear terms. The functions  $f_j, g_j, p_1, q_1, \eta_0, \eta_j, \varphi_j, j = 1, 2$ , are known;  $F_j, G_j, R_j, \pi_j, j = 1, 2$ , are nonlinear functions — the rests of the expressions in (3.12), (3.13), (3.16)–(3.18) of problem (3.12)–(3.18) after separating linear terms and known functions, which expressions (see Appendix A for notations) are given by

$$f_j = \chi \partial_t \rho_0 N J_0^{-T} \nabla^T V_j - \partial_t V_j + a_j (J_0^{-T} \nabla^T)^T J_0^{-T} \nabla^T V_j, \quad (3.34)$$

$$g_j = \chi \partial_t \rho_0 N J_0^{-T} \nabla^T Z_j - \partial_t Z_j + b_j (J_0^{-T} \nabla^T)^T J_0^{-T} \nabla^T Z_j, \quad (3.35)$$

$$\begin{aligned}
F_j &= \chi \partial_t(\rho_0 + \psi) N J^{-T} (\nabla^T v_j - J_1^{-T} J_0^{-T} \nabla^T V_j) \\
&\quad - a_j [\nabla(J_{01}^T + J_1^T) + ((J_{01}^T + J_1^T) J^{-T} \nabla^T)^T] J^{-T} \nabla^T v_j \\
&\quad - a_j (\nabla \psi) \nabla^T (\chi N J_0^T \nabla^T V_j) \\
&\quad + a_j [\nabla(J_{01}^T + J_1^T) + ((J_{01}^T + J_1^T) J^{-T} \nabla^T)^T] J^{-T} J_{11}^T J_0^{-T} \nabla^T V_j \\
&\quad - a_j [(J_0^{-T} J_1^T J^{-T} \nabla^T)^T + (J^{-T} \nabla^T)^T J^{-T} J_{12}^T] J_0^{-T} \nabla^T V_j \\
&= F(V_j, v_j, \psi; a_j) , \tag{3.36}
\end{aligned}$$

$$G_j = F(Z_j, z_j, \psi; b_j) , \tag{3.37}$$

$$p_1 = (p(y, t) - V_1(y, t))|_{\partial\Omega} , \quad q_1 = (q(y, t) - Z_1(y, t))|_{\partial\Omega} , \tag{3.38}$$

$$\eta_0 = (V_2(y, t) - V_1(y, t))|_{\partial\Omega} , \tag{3.39}$$

$$\eta_j = (-Z_j + \sigma_j(V_j))|_{\Gamma} , \tag{3.40}$$

$$R_j = v_j^2 \int_0^1 (1 - \lambda) \sigma_j''(V_j + \lambda v_j) d\lambda , \tag{3.41}$$

$$\varphi_1 = -\nu_0 J_0^{-1} [J_0^{-T} (\lambda_1 \nabla^T V_1 - \lambda_2 \nabla^T V_2) + \kappa N^T \partial_t \rho_0] |_{\Gamma} , \tag{3.42}$$

$$\varphi_2 = -\nu_0 J_0^{-1} [J_0^{-T} (k_1 \nabla^T Z_1 - k_2 \nabla^T Z_2) + (Z_1 - Z_2) N^T \partial_t \rho_0] |_{\Gamma} , \tag{3.43}$$

$$\begin{aligned}
\pi_1 &= \nu_0 (J_{01} + J_1 + J^{-1} (J_{01} + J_1)) J^{-T} (\lambda_1 \nabla^T v_1 - \lambda_2 \nabla^T v_2) \\
&\quad + \nu_0 J_0^{-1} (J_0^{-T} J_1^T + J_1 J^{-1}) J^{-T} (\lambda_1 \nabla^T V_1 - \lambda_2 \nabla^T V_2) \\
&\quad - \kappa \nu_0 (J^{-1} N^T \partial_t \psi - J_0^{-1} J_1 J^{-1} N^T \partial_t \rho_0) , \tag{3.44}
\end{aligned}$$

$$\begin{aligned}
\pi_2 &= -\nu_0 J^{-1} (z_1 - z_2) N^T \partial_t (\rho_0 + \psi) \\
&\quad + \nu_0 ((J_{01}^T + J_1^T) + J^{-1} (J_{01} + J_1)) J^{-T} (k_1 \nabla^T z_1 - k_2 \nabla^T z_2) \\
&\quad - (Z_1 - Z_2) \nu_0 J^{-1} [((J_{01} + J_1) J_{11} - J_{12}) J_0^{-1} N^T \partial_t \rho_0 - J_1 J_0^{-1} N^T \partial_t \psi] \\
&\quad - \nu_0 \mathcal{M} (k_1 \nabla^T Z_1 - k_2 \nabla^T Z_2) , \tag{3.45}
\end{aligned}$$

$$\begin{aligned}
\mathcal{M} &= J^{-1} [(J_{01} + J_1) J_{11}^T + J_{01}^T J_0^{-T} J_{11}^T - J_0^{-T} J_{12}^T] J^{-T} \\
&\quad + J^{-1} ((J_{01} + J_1) J_{11} - J_{12}) J_0^{-1} J_0^{-T} . \tag{3.46}
\end{aligned}$$

For the sake of convenience, we write problem (3.27)–(3.33) conventionally in operator form

$$\mathcal{A}[w] = h + \mathcal{N}[w] , \tag{3.47}$$

where  $w = (v_1, v_2, z_1, z_2, \psi)$ ,  $h = (f_1, f_2, g_1, g_2, p_1, q_1, \eta_0, \eta_1, \eta_2, \varphi_1, \varphi_2)$ ,  $\mathcal{A}[w]$  is a linear operator determined by the expressions in the left-hand sides of the four equations (3.27), (3.28) and seven boundary conditions (3.29)–(3.33),  $\mathcal{N}[w] = (F_1, F_2, G_1, G_2, 0, 0, 0, R_1|_\Gamma, R_2|_\Gamma, \pi_1|_\Gamma, \pi_2|_\Gamma)$  is a nonlinear operator. Moreover,

$$\mathcal{A}: \mathcal{B}(\Omega_T) \rightarrow \mathcal{H}(\Omega_T), \quad \mathcal{N}: \mathcal{B}(\Omega_T) \rightarrow \mathcal{H}(\Omega_T)$$

where

$$\mathcal{B}(\Omega_T) = \mathring{C}_s^{2+\alpha}(\Omega_T^{(1)}) \times \mathring{C}_s^{2+\alpha}(\Omega_T^{(2)}) \times \mathring{C}_s^{2+\alpha}(\Omega_T^{(1)}) \times \mathring{C}_s^{2+\alpha}(\Omega_T^{(2)}) \times \mathring{\mathcal{D}}_s^{2+\alpha}(\Gamma_T)$$

represents the space of functions  $w = (v_1, v_2, z_1, z_2, \rho)$  with the norm:

$$\|w\|_{\mathcal{B}(\Omega_T)} = \sum_{j=1}^2 (|v_j|_{s, \Omega_{jT}}^{(2+\alpha)} + |z_j|_{s, \Omega_{jT}}^{(2+\alpha)}) + |\rho|_{s, \Gamma_T}^{(2+\alpha)} + |\partial_t \rho|_{s-1, \Gamma_T}^{(1+\alpha)};$$

$$\begin{aligned} \mathcal{H}(\Omega_T) &= \mathring{C}_{s-2}^\alpha(\Omega_T^{(1)}) \times \mathring{C}_{s-2}^\alpha(\Omega_T^{(2)}) \times \mathring{C}_{s-2}^\alpha(\Omega_T^{(1)}) \times \mathring{C}_{s-2}^\alpha(\Omega_T^{(2)}) \times \mathring{C}_s^{2+\alpha}(\Sigma_T) \times \\ &\times \mathring{C}_s^{2+\alpha}(\Sigma_T) \times \mathring{C}_s^{2+\alpha}(\Gamma_T) \times \mathring{C}_s^{2+\alpha}(\Gamma_T) \times \mathring{C}_s^{2+\alpha}(\Gamma_T) \times \mathring{C}_{s-1}^{1+\alpha}(\Gamma_T) \times \mathring{C}_{s-1}^{1+\alpha}(\Gamma_T) \end{aligned}$$

denotes the space of functions  $h = (f_1, f_2, g_1, g_2, p_1, q_1, \eta_0, \eta_1, \eta_2, \varphi_1, \varphi_2)$  with the norm:

$$\begin{aligned} \|h\|_{\mathcal{H}(\Omega_T)} &= \sum_{j=1}^2 (|f_j|_{s-2, \Omega_{jT}}^{(\alpha)} + |g_j|_{s-2, \Omega_{jT}}^{(\alpha)}) + |p_1|_{s, \Sigma_T}^{(2+\alpha)} + |q_1|_{s, \Sigma_T}^{(2+\alpha)} + |\eta_0|_{s, \Gamma_T}^{(2+\alpha)} \\ &+ \sum_{j=1}^2 (|\eta_j|_{s, \Gamma_T}^{(2+\alpha)} + |\varphi_j|_{\Gamma_T}^{(\alpha)}). \end{aligned}$$

Here,  $\mathring{C}_s^\ell(\Omega_T)$  is the subset of  $C_s^\ell(\Omega_T)$  consisting of functions  $u(x, t)$ , such that,  $\partial_t^k u|_{t=0} = 0$ ,  $2k \leq s$ , if  $s \geq 0$ ,  $\mathring{C}_s^\ell(\Omega_T) = C_s^\ell(\Omega_T)$ , if  $s < 0$ ; and  $\mathring{\mathcal{D}}_s^{2+\alpha}(\Gamma_T)$ ,  $\alpha \in (0, 1)$ ,  $s \leq 2 + \alpha$ , is the space of functions  $\rho(\xi, t) \in \mathring{C}_s^{2+\alpha}(\Gamma_T)$ ,  $\partial_t \rho \in \mathring{C}_{s-1}^{1+\alpha}(\Gamma_T)$ .

We consider first the vector-function  $h$ . We see from formulas (3.34), (3.35), (3.38)–(3.40), (3.42), (3.43) they are expressed via the auxiliary functions  $V_j$ ,  $Z_j$ ,  $j = 1, 2$ ,  $\rho_0$  and the inverse matrix  $J_0^{-1}$ . In Theorem A.2 of Appendix we prove the existence of  $J_0^{-1}$  for  $t \leq t_2$ . Now we show that  $h \in \mathcal{H}(\Omega_{t_2})$ .

**Lemma 3.2.** *Let the conditions of Theorem 2.1 be fulfilled. Then the vector-function  $h = (f_1, f_2, g_1, g_2, p_1, q_1, \eta_0, \eta_1, \varphi_1, \varphi_2)$  belongs to the space  $\mathcal{H}(\Omega_{t_2})$  and satisfies the estimate:*

$$\|h\|_{\mathcal{H}(\Omega_t)} \leq C_4 \left( \sum_{j=1}^2 (|u_{0j}|_{\Omega_j}^{(s)} + |c_{0j}|_{\Omega_j}^{(s)}) + |p|_{s, \Sigma_t}^{(2+\alpha)} + |q|_{s, \Sigma_t}^{(2+\alpha)} \right), \quad t \leq t_2. \quad (3.48)$$

*Proof:* To show that  $h \in \mathcal{H}(\Omega_{t_2})$  and to obtain an estimate (3.48) we take into consideration the estimates (A.12), (A.56) and (3.22)–(3.24) of the product of functions in weighted Hölder spaces; using matrix  $J_0^{-1}$  and functions  $\rho_0, V_j, Z_j, j = 1, 2$ , respectively, we derive the required inequality (3.48).

We prove that the components — functions of the vector  $h$  satisfy zero initial conditions. The functions  $f_j, g_j, j = 1, 2$ , are equal to 0 at  $t = 0$ , if  $s \geq 0$ , by the condition  $J_0^{-1}|_{t=0} = I$  and equations (3.20), (3.21).

On the basis of the initial data (3.19)–(3.21) and compatibility conditions (2.16), (2.17) we have  $p_1|_{t=0} = 0, q_1|_{t=0} = 0, \eta_0|_{t=0} = 0, \eta_j|_{t=0} = 0, \varphi_j|_{t=0} = 0, j = 1, 2$ .

Let  $s \geq 2$ . We show that the time derivatives  $\partial_t p_1, \partial_t q_1, \partial_t \eta_0, \partial_t \eta_j, j = 1, 2$ , are also equal to zero at  $t = 0$ . We consider the functions

$$\begin{aligned} \partial_t p_1|_{t=0} &= (\partial_t p - \partial_t V_1|_{\partial\Omega})|_{t=0}, \\ \partial_t q_1|_{t=0} &= (\partial_t q - \partial_t Z_1|_{\partial\Omega})|_{t=0}. \end{aligned} \quad (3.49)$$

From equations (3.20), (3.21) we obtain

$$\partial_t V_1|_{t=0} = a_1 \Delta u_{01}, \quad \partial_t Z_1|_{t=0} = b_1 \Delta c_{01},$$

by the assumption on the initial interface  $\Gamma$  and the definition of  $\chi$ . Then, from (3.49) on the basis of the compatibility conditions (2.19) we find  $\partial_t p_1|_{t=0} = 0, \partial_t q_1|_{t=0} = 0$ .

We consider the time derivatives of the functions  $\eta_0, \eta_j (j = 1, 2)$

$$\partial_t \eta_0 = \partial_t V_2 - \partial_t V_1|_{\Gamma}, \quad \partial_t \eta_j = -\partial_t Z_j + \sigma'_j(V_j) \partial_t V_j|_{\Gamma}.$$

We substitute here  $\partial_t V_j, \partial_t Z_j, \partial_t \rho_0$  found from formulas (3.19)–(3.21)

$$\partial_t \eta_0|_{t=0} = -[a \Delta u_0]|_{\Gamma} + (\partial_N u_{01} - \partial_N u_{02}) \frac{[[k \partial_{\nu_0} c_0]]}{\nu_0 N^T [[c_0]]}|_{\Gamma}, \quad (3.50)$$

$$\begin{aligned} \partial_t \eta_j|_{t=0} &= -b_j \Delta c_{0j}|_{\Gamma} + a_j \sigma'_j(u_{0j}) \Delta u_{0j}|_{\Gamma} \\ &+ (\partial_N c_{0j} - \sigma'_j(u_{0j}) \partial_N u_{0j}) \frac{[[k \partial_{\nu_0} c_0]]}{\nu_0 N^T [[c_0]]}|_{\Gamma}, \quad j = 1, 2. \end{aligned} \quad (3.51)$$



We express the derivatives on the direction  $N$  via normal derivatives. Let  $\xi \in \Gamma$  be an arbitrary point. The vector  $N(\xi)$  may be represented in the form

$$N(\xi) = \alpha_1 \nu_0(\xi) + \alpha_2 \tau(\xi) , \quad (3.52)$$

where  $\nu_0(\xi)$  and  $\tau(\xi)$  are normal and tangential unit vectors at the point  $\xi_1$  and  $\alpha_1, \alpha_2$  denote some numbers. Multiplying both parts of the identity (3.52) by the vector  $\nu_0^T(\xi)$  we obtain  $\alpha_1 = N\nu_0^T \geq d_1$  (see (2.10)) and  $N(\xi) = N\nu_0^T \nu_0 + \alpha_2 \tau$ .

Then we can write

$$(\partial_N u_{01} - \partial_N u_{02})|_\Gamma = \nu_0 N^T (\partial_{\nu_0} u_{01} - \partial_{\nu_0} u_{02})|_\Gamma + \alpha_2 (\partial_\tau u_{01} - \partial_\tau u_{02})|_\Gamma, \quad (3.53)$$

$$\begin{aligned} (\partial_N c_{0j} - \sigma'_j(u_{0j}) \partial_N u_{0j})|_\Gamma &= \nu_0 N^T (\partial_{\nu_0} c_{0j} - \sigma'_j(u_{0j}) \partial_{\nu_0} u_{0j})|_\Gamma \\ &+ \alpha_2 (\partial_\tau c_{0j} - \sigma_j(u_{0j}) \partial_\tau u_{0j})|_\Gamma . \end{aligned} \quad (3.54)$$

We differentiate the compatibility conditions (2.17) with respect to the tangential direction  $\tau$

$$(\partial_\tau u_{01} - \partial_\tau u_{02})|_\Gamma = 0, \quad (\partial_\tau c_{0j} - \sigma'_j(u_{0j}) \partial_\tau u_{0j})|_\Gamma = 0, \quad j = 1, 2 .$$

On the basis of these identities from (3.53), (3.54) we find the formulas

$$(\partial_N u_{01} - \partial_N u_{02})|_\Gamma = \nu_0 N^T [[\partial_{\nu_0} u_0]]|_\Gamma ,$$

$$(\partial_N c_{0j} - \sigma'_j(u_{0j}) \partial_N u_{0j})|_\Gamma = \nu_0 N^T (\partial_{\nu_0} c_{0j} - \sigma'_j(u_{0j}) \partial_{\nu_0} u_{0j})|_\Gamma, \quad j = 1, 2 .$$

We substitute these expressions into (3.50), (3.51) and, by the compatibility conditions (2.20), (2.21), we finally have  $\partial_t \eta_0|_{t=0} = 0, \partial_t \eta_j|_{t=0} = 0, j = 1, 2$ . ■

## 4. The linear problem

In order to solve the nonlinear problem (3.47) we consider first the linear operator  $\mathcal{A}: \mathcal{B}(\Omega_T) \rightarrow \mathcal{H}(\Omega_T)$  and the equation

$$\mathcal{A}[w] = h$$

with unknown functions  $w = (v_1(x, t), v_2(x, t), z_1(x, t), z_2(x, t), \psi(\xi, t))$  satisfying zero initial data, so that for  $j = 1, 2$ :

$$\partial_t v_j - a_j \Delta v_j - \chi(\lambda(x)) \alpha_j(x, t) (\partial_t \psi - a_j \Delta \psi) = f_j(x, t) \quad \text{in } \Omega_{jT} , \quad (4.1)$$

$$\partial_t z_j - b_j \Delta z_j - \chi(\lambda(x)) \beta_j(x, t) (\partial_t \psi - b_j \Delta \psi) = g_j(x, t) \quad \text{in } \Omega_{jT} , \quad (4.2)$$

with Dirichlet and transmission boundary conditions for  $t > 0$ :

$$v_1|_{\partial\Omega} = p_1(x, t), \quad z_1|_{\partial\Omega} = q_1(x, t), \quad (4.3)$$

$$(v_1 - v_2)|_{\Gamma} = \eta_0(x, t), \quad (4.4)$$

$$(z_j - \gamma_j(x, t) v_j)|_{\Gamma} = \eta_j(x, t), \quad j = 1, 2, \quad (4.5)$$

$$(\lambda_1 \partial_{\nu_0} v_1 - \lambda_2 \partial_{\nu_0} v_2)|_{\Gamma} = \varphi_1(x, t), \quad (4.6)$$

$$(k_1 \partial_{\nu_0} z_1 - k_2 \partial_{\nu_0} z_2 + d(x, t) \nabla^T \psi - \kappa_1(x, t) \partial_t \psi)|_{\Gamma} = \varphi_2(x, t), \quad (4.7)$$

where  $\chi(\lambda(x))$  is a cut-off function;  $a_j, b_j, \lambda_j, k_j, j = 1, 2$ , are positive constants; and  $d = (d_1, \dots, d_n)$  is a given vector function.

Let the following assumptions hold:

a)  $\partial\Omega, \Gamma \in C^{2+\alpha}, \partial\Omega \cap \Gamma = \emptyset$ ;

b)  $\alpha_j \in C_{s-1}^{1+\alpha}(\Omega_T), \gamma_j \in C_s^{2+\alpha}(\Gamma_T), \kappa_1, d_i \in C_{s-1}^{1+\alpha}(\Gamma_T), j = 1, 2, i = 1, \dots, n$ ;

c)  $\kappa_1(x, 0)|_{\Gamma} \geq d_3 > 0$ , (4.8)

$$\beta_j(x, 0) - \gamma_j(x, 0) \alpha_j(x, 0)|_{\Gamma} \geq d_3, \quad j = 1, 2, \quad (4.9)$$

$$|\alpha_1(x, 0) - \alpha_2(x, 0)|_{\Gamma}| \leq \delta_0, \quad (4.10)$$

where  $\delta_0$  is sufficiently small value.

The conditions in c) correspond, respectively, to the conditions (2.12), (2.13) and to the second condition (2.14).

**Theorem 4.1.** *Let  $\alpha \in (0, 1), s \in (1, 2 + \alpha]$ . We assume that conditions a)-c) hold.*

*Then for every functions  $f_j, g_j \in \mathring{C}_{s-2}^{\alpha}(\Omega_{jT}), p_1, q_1 \in \mathring{C}_s^{2+\alpha}(\Sigma_T), \eta_0, \eta_j \in \mathring{C}_s^{2+\alpha}(\Gamma_T), \varphi_j \in \mathring{C}_{s-1}^{1+\alpha}(\Gamma_T), j = 1, 2$ , problem (4.1)-(4.7) has a unique solution  $v_j, z_j \in \mathring{C}_s^{2+\alpha}(\Omega_{jT}), j = 1, 2, \psi \in \mathring{C}_s^{2+\alpha}(\Gamma_T), \partial_t \psi \in \mathring{C}_{s-1}^{1+\alpha}(\Gamma_T)$ , which satisfies the estimate*

$$\begin{aligned}
 \|w\|_{\mathcal{B}(\Omega_T)} &= \sum_{j=1}^2 (|v_j|_{s,\Omega_{jT}}^{(2+\alpha)} + |z_j|_{s,\Omega_{jT}}^{(2+\alpha)}) + |\psi|_{s,\Gamma_T}^{(2+\alpha)} + |\partial_t \psi|_{s-1,\Gamma_T}^{(1+\alpha)} \\
 &\leq C_1 \left( \sum_{j=1}^2 (|f_j|_{s-2,\Omega_{jT}}^{(\alpha)} + |g_j|_{s-2,\Omega_{jT}}^{(\alpha)} + |\eta_j|_{s,\Gamma_T}^{(2+\alpha)} + |\varphi_j|_{s,\Gamma_T}^{(2+\alpha)}) \right. \\
 &\quad \left. + |p_1|_{s,\Sigma_T}^{(2+\alpha)} + |q_1|_{s,\Sigma_T}^{(2+\alpha)} + |\eta_0|_{s,\Gamma_T}^{(2+\alpha)} \right) \\
 &= C_1 \|h\|_{\mathcal{H}(\Omega_T)}.
 \end{aligned} \tag{4.11}$$

*Proof:* We rewrite the equations (4.1)<sub>2</sub>, (4.2)<sub>2</sub> in the form

$$\begin{aligned}
 \partial_t v_2 - a_2 \Delta v_2 - \chi \alpha_1(x, t) (\partial_t \psi - a_2 \Delta \psi) &= \\
 = f_2 - \chi (\alpha_1(x, t) - \alpha_2(x, t)) (\partial_t \psi - a_2 \Delta \psi) &\text{ in } \Omega_{2T}, \tag{4.12}
 \end{aligned}$$

$$\begin{aligned}
 \partial_t z_2 - b_2 \Delta z_2 - \chi [\beta_2(x, t) + \gamma_2(x, t) (\alpha_1(x, t) - \alpha_2(x, t))] (\partial_t \psi - b_2 \Delta \psi) &= \\
 = g_2 - \chi \gamma_2(x, t) (\alpha_1(x, t) - \alpha_2(x, t)) (\partial_t \psi - b_2 \Delta \psi) &\text{ in } \Omega_{2T}. \tag{4.13}
 \end{aligned}$$

■

By condition (4.10) the terms containing the difference  $\alpha_1(x, t) - \alpha_2(x, t)$  are small. We have written equations (4.1)<sub>2</sub>, (4.2)<sub>2</sub> in the form (4.12), (4.13) to obtain the model problem (B.1)–(B.7) for convenience.

We construct a regularizer to prove the solvability of problem (4.1)–(4.7) and apply Schauder's method to find the estimate (4.11). Since this is a standard procedure (see [LSU], for instance) we give only the sketch of a proof.

We cover the domain  $\Omega$  with balls  $K_{i,\delta}$  and  $K_{i,2\delta}$  of radii  $\delta$  and  $2\delta$ , respectively, and with a common center  $x^{(i)}$ . Let  $\{\zeta_i(x)\}$ ,  $\{\mu_i(x)\}$  be the sets of the smooth functions subordinated to this overlapping by the balls, such that,  $\zeta_i = 1$ , if  $|x - x^{(i)}| \leq \delta$  and  $\zeta_i = 0$ , if  $|x - x^{(i)}| \geq 2\delta$ ,  $\text{supp } \mu_i = \overline{K_{i,2\delta}}$ ,  $\sum_i \zeta_i \mu_i = 1$  and  $|\partial^m \zeta_i|, |\partial^m \mu_i| \leq C_{m,i} \delta^{-|m|}$ ,  $m = (m_1, \dots, m_n)$ . Let  $\mathcal{O} = \{x \in \Omega: x = \xi + \lambda N(\xi), \xi \in \Gamma, |\lambda| < 2\lambda_0\}$ , be the  $2\lambda_0$ -neighbourhood of  $\Gamma$ .

We define a regularizer  $\mathcal{R}$  by the formula

$$\begin{aligned}
 \mathcal{R}h &= \{\mathcal{R}_1 h, \mathcal{R}_2 h, \mathcal{R}_3 h, \mathcal{R}_4 h, \mathcal{R}_5 h\} \\
 &= \left\{ \sum_i \mu_i v_{1,i}, \sum_i \mu_i v_{2,i}, \sum_i \mu_i z_{1,i}, \sum_i \mu_i z_{2,i}, \sum_i \mu_i \psi_i \right\} \tag{4.14}
 \end{aligned}$$

where the functions  $v_{j,i}$ ,  $z_{j,i}$ ,  $\psi_i$ ,  $j = 1, 2$ , are found as follows.

Let a ball  $K_{i,\delta}$  intersect the surfaces  $\Gamma$  ( $i \in \mathcal{I}_1$ ) or  $\partial\Omega$  ( $i \in \mathcal{I}_2$ ), the point  $\xi^{(i)}$  belongs to  $\Gamma \cap K_{i,\delta}$  or  $\partial\Omega \cap K_{i,\delta}$ . We pass to the local coordinates  $\{\bar{y}\}$  with the center in  $\xi^{(i)}$  and the axis  $\bar{y}_n$  directed on the normal  $\nu_0(\xi^{(i)})$  to the surface  $\Gamma$  into  $\Omega_2$  or  $\partial\Omega$  into  $\Omega_1$ . We choose a radius  $\delta$  sufficiently small so that the boundary  $\Gamma \cap K_{i,2\delta}$  or  $\partial\Omega \cap K_{i,2\delta}$  can be expressed by the equation  $\bar{y}_n = q_i(\bar{y}')$ , where  $q_i \in C^{2+\alpha}$  and  $q_i(0) = 0$ ,  $\nabla' q_i = 0$ .

Here we set  $y = (y_1, \dots, y_n)$  and  $y' = (y_1, \dots, y_{n-1})$ .

We denote by  $\tilde{q}_i(\bar{y}')$  the extension of  $q_i(\bar{y}')$  into  $\mathbb{R}^{n-1}$  which preserves regularity. Then we “straighten” the boundary  $\bar{y}_n = \tilde{q}_i$  by the formulas  $y' = \bar{y}'$ ,  $y_n = \bar{y}_n - \tilde{q}_i$ . Let  $y = Y_i(x)$  be the transformation of the coordinates  $\{x\}$  to the coordinate  $\{y\}$  consisting of the rotation of a coordinate system  $\{\bar{y}\}$  around the point  $\xi^{(i)}$  to turn an axis  $\bar{y}_n$  on a normal  $\nu_0(\xi^{(i)})$  and “straightening” of a boundary  $\bar{y}_n = \tilde{q}_i$ . We put  $\zeta_i f_j, \zeta_i g_j|_{x=Y_i^{-1}(y)} = f_{j,i}(y, t)$ ,  $g_{j,i}(y, t)$ ;  $\zeta_i \eta_k, \zeta_i \varphi_j, \zeta_i p_1, \zeta_i q_1|_{\substack{x=Y_i^{-1}(y) \\ y_n=0}} = \eta_{k,i}(y', t)$ ,  $\varphi_{j,i}(y', t)$ ,  $p_{1,i}(y', t)$ ,  $q_{1,i}(y', t)$ ,  $j = 1, 2$ ,  $k = 0, 1, 2$ , and extend by zero the functions  $f_{j,i}$ ,  $g_{j,i}$  into  $\mathcal{S}_j$  ( $i \in \mathcal{I}_1$ ) and  $f_{1,i}$ ,  $g_{1,i}$  into  $\mathcal{S}_2$  ( $i \in \mathcal{I}_2$ ) and  $\eta_{k,i}$ ,  $\varphi_{j,i}$ ,  $p_{1,i}$ ,  $q_{1,i}$  into  $\mathbb{R}^{n-1}$  retaining the preceding notations and we set

$$\mathcal{S}_1 = \mathbb{R}_-^n, \quad \mathcal{S}_2 = \mathbb{R}_+^n \quad \text{and} \quad \mathcal{S}_{jT} = \mathcal{S}_j \times (0, T), \quad j = 1, 2 .$$

1. We define the functions  $v'_{j,i}(y, t)$ ,  $z'_{j,i}(y, t)$ ,  $j = 1, 2$ ,  $\psi'_i(y', t)$ ,  $i \in \mathcal{I}_1$ , satisfying zero initial data, as the solution to the following problem:

$$\begin{aligned} \partial_t v'_{j,i} - a_j \Delta_y v'_{j,i} - \chi(\lambda(\xi^{(i)})) \alpha_1(x^{(i)}, 0) (\partial_t \psi'_i - a_j \Delta' \psi'_i) &= f_{j,i}(y, t) \\ &\text{in } \mathcal{S}_{jT}, \quad j = 1, 2, \end{aligned} \quad (4.15)$$

$$\partial_t z'_{1,i} - b_1 \Delta z'_{1,i} - \chi(\lambda(\xi^{(i)})) \beta_1(x^{(i)}, 0) (\partial_t \psi'_i - b_1 \Delta' \psi'_i) = g_{1,i}(y, t) \text{ in } \mathcal{S}_{1T}, \quad (4.16)$$

$$\begin{aligned} \partial_t z'_{2,i} - b_2 \Delta z'_{2,i} - \chi(\lambda(\xi^{(i)})) [\beta_2(x^{(i)}, 0) + \gamma_2(x^{(i)}, 0) (\alpha_1(x^{(i)}, 0) - \alpha_2(x^{(i)}, 0))] \\ \times (\partial_t \psi'_i - b_2 \Delta' \psi'_i) = g_{2,i}(y, t) \text{ in } \mathcal{S}_{2T}, \end{aligned} \quad (4.17)$$

$$(v'_{1,i} - v'_{2,i})|_{y_n=0} = \eta_{0,i}(y', t), \quad (4.18)$$

$$(z'_{j,i} - \gamma_j(\xi^{(i)}, 0) v'_{j,i})|_{y_n=0} = \eta_{j,i}(y', t), \quad j = 1, 2, \quad (4.19)$$

$$(\lambda_1 \partial_{y_n} v'_{1,i} - \lambda_2 \partial_{y_n} v'_{2,i})|_{y_n=0} = \varphi_{1,i}(y', t), \quad (4.20)$$

$$(k_1 \partial_{y_n} z'_{1,i} - k_2 \partial_{y_n} z'_{2,i})|_{y_n=0} + d'_i \nabla'^T \psi'_i - \kappa_1(\xi^{(i)}, 0) \partial_t \psi'_i = \varphi_2(y', t) \quad (4.21)$$

where

$$\begin{aligned} \chi(\lambda(\xi^{(i)})) &= 1, \quad d'_i = (d_{i,1}, \dots, d_{i,n-1}), \\ d'_i \nabla'^T \psi'_i &= d(\xi^{(i)}, 0) \nabla_x^T \psi_i(x, t)|_{\substack{x=Y_i^{-1}(y) \\ x=\xi \in \Gamma}}. \end{aligned}$$

Here we consider the equations (4.15)<sub>2</sub>, and (4.17) in accordance with the representation of the original equations (4.1)<sub>2</sub>, (4.2)<sub>2</sub> in the form (4.12), (4.13) respectively. We have also used the standard notations  $\Delta' = \partial_{y_1}^2 + \dots + \partial_{y_{n-1}}^2$  and  $\nabla' = (\partial_{y_1}, \dots, \partial_{y_{n-1}})$ .

**2.** If  $i \in \mathcal{I}_2$ , we find the functions  $v'_{1,i}(y, t)$ ,  $z'_{1,i}(y, t)$  as the solutions to the first boundary value problems

$$\partial_t v'_{1,i} - a_1 \Delta v'_{1,i} = f_{1,i}(y, t) \quad \text{in } \mathcal{S}_{2T}, \quad v'_{1,i}|_{t=0} = 0, \quad v'_{1,i}|_{y_n=0} = 0 \quad (4.22)$$

$$\partial_t z'_{1,i} - b_1 \Delta z'_{1,i} = g_{1,i}(y, t) \quad \text{in } \mathcal{S}_{2T}, \quad z'_{1,i}|_{t=0} = 0, \quad z'_{1,i}|_{y_n=0} = 0. \quad (4.23)$$

We note that (4.15)–(4.21) is the model problem (B.1)–(B.7) studied in Appendix B. So, by Theorem B.1, it has a unique solution  $v'_{j,i}, z'_{j,i} \in \mathring{C}_s^{2+\alpha}(\mathcal{S}_{jT})$ ,  $j = 1, 2$ ,  $\psi'_i \in \mathring{C}_s^{2+\alpha}(R_T)$ ,  $\partial_t \psi \in \mathring{C}_{s-1}^{1+\alpha}(R_T)$ ,  $i \in \mathcal{I}_1$ , and it satisfies the estimate (B.9) (here  $R$  is the plane  $y_n = 0$ ). The solutions  $v'_{1,i}, z'_{1,i}$ ,  $i \in \mathcal{I}_2$ , to the problems (4.22), (4.23) exist and belong to the space  $\mathring{C}_s^{2+\alpha}(\mathcal{S}_{2T})$  [S1].

The estimates for the solutions to problem (4.15)–(4.21) and to the first boundary value problems (4.22), (4.23) in coordinates  $\{x\}$  take the form

$$\begin{aligned} & \sum_{j=1}^2 (|v_{j,i}|_{s, K_{i,T}^{(j)}}^{(2+\alpha)} + |z_{j,i}|_{s, K_{i,T}^{(j)}}^{(2+\alpha)}) + |\psi_i|_{s, \Gamma_{i,T}}^{(2+\alpha)} + |\partial_t \psi|_{s-1, \Gamma_{i,T}}^{(1+\alpha)} \leq \\ & \leq C \left[ \sum_{j=1}^2 (|\zeta_i f_j|_{s-2, K_{i,T}^{(j)}}^{(\alpha)} + |\zeta_i g_j|_{s-2, K_{i,T}^{(j)}}^{(\alpha)} + |\zeta_i \eta_j|_{s, \Gamma_{i,T}}^{(2+\alpha)} + |\zeta_i \varphi_j|_{s-1, \Gamma_{i,T}}^{(1+\alpha)} + |\zeta_i \eta_0|_{s, \Gamma_{i,T}}^{(2+\alpha)}) \right], \\ & \hspace{25em} i \in \mathcal{I}_1, \quad (4.24) \end{aligned}$$

$$|v_{1,i}|_{s,K_{i,T}^{(1)}}^{(2+\alpha)} \leq C(|\zeta_i f_1|_{s-2,K_{i,T}^{(1)}}^{(\alpha)} + |\zeta_i p_1|_{s,\Sigma_{i,T}}^{(2+\alpha)}), \quad i \in \mathcal{I}_2, \quad (4.25)$$

$$|z_{1,i}|_{s,K_{i,T}^{(1)}}^{(2+\alpha)} \leq C(|\zeta_i g_1|_{s-2,K_{i,T}^{(1)}}^{(\alpha)} + |\zeta_i p_1|_{s,\Sigma_{i,T}}^{(2+\alpha)}), \quad i \in \mathcal{I}_2,$$

where  $v'_{j,i}, z'_{j,i}|_{y=Y_i(x)} = v_{j,i}(x, t), z_{j,i}(x, t), i \in \mathcal{I}_1 \cup \mathcal{I}_2, \psi'_i|_{\substack{y=Y_i(x) \\ y_n=0}} = \psi_i(\xi, t), \xi \in \Gamma \cap K_{i,2\delta}, i \in \mathcal{I}_1$ , where  $K_{i,T}^{(j)} = (K_{i,2\delta} \cap \Omega_j) \times (0, T), j = 1, 2$ , and  $\Gamma_{i,T} = (\Gamma \cap K_{i,2\delta}) \times [0, T], \Sigma_{i,T} = (\partial\Omega \cap K_{i,2\delta}) \times [0, T]$ .

**3.** Let the ball  $K_{i,\delta}, i \in \mathcal{I}_3$ , be entirely inside  $\Omega_1$  or  $\Omega_2$ , but  $K_{i,\delta} \cap \mathcal{O} \neq \emptyset$  (here  $\mathcal{O}$  is  $2\lambda_0$ -neighbourhood of  $\Gamma$ ). We determine the functions  $v_{j,i}(x, t), z_{j,i}(x, t), i \in \mathcal{I}_3$ , as the solution of a Cauchy problem

$$\begin{aligned} \partial_t v_{j,i} - a_j \Delta v_{j,i} &= \zeta_i f_j(x, t) + \chi(\lambda(\bar{x}^{(i)})) \alpha_1(x^{(i)}, 0) (\partial_t \psi_i - a_j \Delta \psi_i) \\ &\text{in } \mathbb{R}_T^n, \quad v_{j,i}|_{t=0} = 0, \end{aligned} \quad (4.26)$$

where  $j = 1, 2, \bar{x}^{(i)} \in K_{i,\delta} \in K_{i,\delta} \cap \mathcal{O}, \mathbb{R}_T^n = \mathbb{R}^n \times (0, T)$ ;

$$\begin{aligned} \partial_t z_{1,i} - b_1 \Delta z_{1,i} &= \zeta_i g_1(x, t) + \chi(\lambda(\bar{x}^{(i)})) \beta_1(x^{(i)}, 0) (\partial_t \psi_i - b_1 \Delta \psi_i) \\ &\text{in } \mathbb{R}_T^n, \quad z_{1,i}|_{t=0} = 0, \end{aligned} \quad (4.27)$$

$$\begin{aligned} \partial_t z_{2,i} - b_2 \Delta z_{2,i} &= \\ &= \zeta_i g_2(x, t) + \chi(\lambda(\bar{x}^{(i)})) [\beta_1(x^{(i)}, 0) + \gamma_2(x^{(i)}, 0) (\alpha_1(x^{(i)}, 0) - \alpha_2(x^{(i)}, 0))] \\ &\quad \times (\partial_t \psi_i - b_2 \Delta \psi_i) \quad \text{in } \mathbb{R}_T^n, \quad z_{2,i}|_{t=0} = 0, \end{aligned} \quad (4.28)$$

here the functions  $f_j, g_j$  defined in  $\Omega_j \cap K_{i,2\delta}$  are extended to  $K_{i,2\delta}$  by preserving its regularity  $\mathring{C}_{s-2}^\alpha$  and then the products  $\zeta_i f_j, \zeta_i g_j$  are extended by zero into  $\mathbb{R}^n$  (we keep the same notations for them).

In the right-hand sides of the equations of the problems (4.26)–(4.28) there are functions  $\psi_i(\xi, t), i \in \mathcal{I}_3$ . We choose them as follows. Let  $x^0$  be an arbitrary point in  $K_{i,\delta} \cap \mathcal{O}, i \in \mathcal{I}_3$ , and  $\xi^0 \in \Gamma$  be the origin of the vector  $N(\xi^0)$ , on which the point  $x^0$  is situated. In the point  $\xi^0$  there is determined, at least, one function  $\psi_j(\xi), j \in \mathcal{I}_1$ , because each point  $\xi^0 \in \Gamma$  is contained, at least, in one ball  $K_{i,\delta}, j \in \mathcal{I}_1$ . For all  $x \in K_{i,\delta} \cap \mathcal{O}, i \in \mathcal{I}_3$ , we have the

corresponding functions  $\psi_j(\xi, t)$ ,  $j \in n(i) \subset \mathcal{I}_1$ ,  $n(i) \neq \emptyset$ . For  $i \in \mathcal{I}_3$  we set

$$\psi_i(\xi, t) = \sum_{j \in n(i)} \psi_j(\xi, t) \zeta_j(x) \left( \sum_{j \in n(i)} \zeta_j(x) \right)^{-1} \Big|_{\Gamma}$$

and put it in the right-hand sides of the equations of problems (4.26)–(4.28).

4. At last, if  $K_{i,\delta}$ ,  $i \in \mathcal{I}_4$ , is entirely inside  $\Omega_1$  or  $\Omega_2$  and  $K_{i,\delta} \cap \mathcal{O} = \emptyset$  we find the functions  $v_{j,i}(x, t)$ ,  $z_{j,i}(x, t)$ ,  $i \in \mathcal{I}_4$ ,  $j = 1, 2$ , as the solutions of the Cauchy problems

$$\partial_t v_{j,i} - a_j \Delta v_j = \zeta_i f_j(x, t) \quad \text{in } \mathbb{R}_T^n, \quad v_{j,i}|_{t=0} = 0, \quad (4.29)$$

$$\partial_t z_{j,i} - b_j \Delta z_{j,i} = \zeta_i g_j(x, t) \quad \text{in } \mathbb{R}_T^n, \quad z_{j,i}|_{t=0} = 0, \quad (4.30)$$

for  $j = 1, 2$ , the functions  $\zeta_i f_j$ ,  $\zeta_i g_j$  are extended into  $\mathbb{R}^n$  as in problems (4.26)–(4.28).

The Cauchy problems (4.26)–(4.30) have unique solutions

$$v_{j,i}, z_{j,i} \in \mathring{C}_s^{2+\alpha}(\mathbb{R}_T^n), \quad i \in \mathcal{I}_3 \cup \mathcal{I}_4, \quad j = 1, 2,$$

and the following estimates hold

$$\begin{aligned} |v_{j,i}|_{s, K_{i,T}^{(j)}}^{(2+\alpha)} &\leq C (|\zeta_i f_j|_{s-2, K_{i,T}^{(j)}}^{(\alpha)} + |\psi_i|_{s, \Gamma_{i,T}}^{(2+\alpha)}), \\ |z_{j,i}|_{s, K_{i,T}^{(j)}}^{(2+\alpha)} &\leq C (|\zeta_i g_j|_{s-2, K_{i,T}^{(j)}}^{(\alpha)} + |\psi_i|_{s, \Gamma_{i,T}}^{(2+\alpha)}), \quad i \in \mathcal{I}_3, \quad j = 1, 2, \end{aligned} \quad (4.31)$$

$$\begin{aligned} |v_{j,i}|_{s, K_{i,T}^{(j)}}^{(2+\alpha)} &\leq C |\zeta_i f_j|_{s-2, K_{i,T}^{(j)}}^{(\alpha)}, \\ |z_{j,i}|_{s, K_{i,T}^{(j)}}^{(2+\alpha)} &\leq C |\zeta_i g_j|_{s-2, K_{i,T}^{(j)}}^{(\alpha)}, \quad i \in \mathcal{I}_4, \quad j = 1, 2. \end{aligned} \quad (4.32)$$

Thus, we have constructed the regularizer  $\mathcal{R}h$  in (4.14).

We introduce the norms [LSU]

$$\{w\}_{\mathcal{B}(\Omega_T)} = \sup_i \|w\|_{\mathcal{B}(K_{i,T})}, \quad (4.33)$$

$$\{h\}_{\mathcal{H}(\Omega_T)} = \sup_i \|h\|_{\mathcal{H}(K_{i,T})}, \quad (4.34)$$

where  $K_{i,T} = (K_{i,2\delta} \cap \Omega) \times (0, T)$ .

The norms (4.33),(4.34) are equivalent to the norms  $\|w\|_{\mathcal{B}(\Omega_T)}$ ,  $\|h\|_{\mathcal{H}(\Omega_T)}$  determined in formula (4.11).

**Lemma 4.1.** *The operator  $\mathcal{R}: \mathcal{H}(\Omega_T) \rightarrow \mathcal{B}(\Omega_T)$  is bounded*

$$\{\mathcal{R}h\}_{\mathcal{H}(\Omega_T)} \leq C \{h\}_{\mathcal{H}(\Omega_T)} .$$

*This estimate holds on the basis of estimates (4.24), (4.25), (4.31), (4.32).*

**Lemma 4.2.** *For any vector  $h \in \mathcal{H}(\Omega_T)$  the following identity is fulfilled:*

$$\mathcal{A}\mathcal{R}h = h + Ph \equiv (E + P)h , \quad (4.35)$$

where  $Ph = \{P_1h, P_2h, P_3h, P_4h, 0, 0, 0, P_5h, P_6h, P_7h, P_8h\}$  is fully defined vector and  $E$  the identity operator.

*Proof:* We substitute  $\mathcal{R}_1h - \mathcal{R}_5h$  (see (4.14)) into equations and conditions of the problem  $\mathcal{A}[w] = h$  instead of the functions  $v_1, v_2, z_1, z_2, \psi$  respectively. After some computations taking into consideration that  $v_{j,i}, z_{j,i}, \psi_i$  are the solutions of the model problems (4.15)–(4.21),  $i \in \mathcal{I}_1$ , (4.22),(4.23),  $i \in \mathcal{I}_2$ , (4.26)–(4.28),  $i \in \mathcal{I}_3$ , (4.29),(4.30),  $i \in \mathcal{I}_4$ , we obtain formula (4.35), where the functions  $P_1h - P_8h$  are the rests of  $\mathcal{A}\mathcal{R}h$  after extraction of the vector  $h$ . We note also that a function  $P_2h$  contains the additional term

$$\chi(\lambda) (\alpha_1(x, t) - \alpha_2(x, t)) \sum_{i \in \mathcal{I}_1 \cup \mathcal{I}_2} [\mu_i (\partial_t \psi_i - a_2 \Delta \psi_i) - a_2 \psi_i \Delta \mu_i - 2 a_2 \nabla \psi_i \nabla^T \mu_i], \quad (4.36)$$

the term similar to (4.36) with  $\beta_2(x, t) + \gamma_2(x, t) (\alpha_1(x, t) - \alpha_2(x, t))$  and  $b_2$  instead of  $\alpha_1 - \alpha_2$  and  $a_2$  is included in  $P_4h$  by the representations of the equations (4.1)<sub>2</sub>,(4.2)<sub>2</sub> in the form (4.12),(4.13).  $\blacksquare$

**Lemma 4.3.** *Under the assumptions of Theorem 4.1 there exists  $T_1 > 0$ , such that, the operator  $\mathcal{A}$  has the right inverse bounded operator  $\mathcal{A}_r^{-1} = \mathcal{R}(E + P)^{-1}: \mathcal{H}(\Omega_{T_1}) \rightarrow \mathcal{B}(\Omega_{T_1})$ .*

*Proof:* With the help of the estimates (4.24), (4.25), (4.31), (4.32) for the functions  $v_{j,i}, z_{j,i}, j = 1, 2, \psi_i$ , we estimate the norm of  $Ph$ . Choosing the radius  $\delta$  of the balls  $K_{i,\delta}$  and  $t \leq T_1$  sufficiently small and making use of condition (4.10) we obtain the estimate

$$\{Ph\}_{\mathcal{H}(\Omega_t)} \leq \varepsilon [h]_{\mathcal{H}(\Omega_t)} \quad \forall t \leq T_1 , \quad (4.37)$$

where  $\varepsilon \in (0, 1)$  (see (4.34)).



We consider an equation  $h + Ph = h_1$ , where  $h_1 \in \mathcal{H}(\Omega_T)$ . By estimate (4.37) it has a unique solution  $h \in \mathcal{H}(\Omega_{T_1})$  and  $\{h\}_{\mathcal{H}(\Omega_{T_1})} \leq \frac{1}{1-\varepsilon} \{h_1\}_{\mathcal{H}(\Omega_{T_1})}$  for every vector  $h_1 \in \mathcal{H}(\Omega_{T_1})$ , that is, the inverse operator  $(E + P)^{-1}$  exists and is bounded on the space  $\mathcal{H}(\Omega_{T_1})$ . Substituting  $h = (E + P)^{-1}h_1$  into the left-hand side of the identity (4.35) and taking into account that  $h + Ph = h_1$ , we obtain the identity  $\mathcal{A}\mathcal{R}(E + P)^{-1}h_1 = h_1$  for  $\forall h_1 \in \mathcal{H}(\Omega_{T_1})$  or  $\mathcal{A}\mathcal{R}(E + P)^{-1} = E$ .

From here it follows that an operator  $\mathcal{A}$  has the inverse right bounded operator  $\mathcal{A}_r^{-1} = \mathcal{R}(E + P)^{-1}$  determined in the whole space  $\mathcal{H}(\Omega_{T_1})$ .  $\blacksquare$

Therefore, problem  $\mathcal{A}[w] = h$  has a solution  $w = (v_1, v_2, z_1, z_2, \psi) \in \mathcal{B}(\Omega_{T_1})$  for every vector  $h \in \mathcal{H}(\Omega_{T_1})$ .

To find the estimate (4.11) to the solution we multiply both parts of the equations (4.1), (4.12), (4.2), (4.13) and conditions (4.3)–(4.7) by a cut-off function  $\zeta_i(x)$ . After some computations for the functions  $\zeta_i v_j$ ,  $\zeta_i z_j$ ,  $j = 1, 2$ ,  $\zeta_i \psi$ , we obtain a model transmission problem of the type (B.1)–(B.7), or the first boundary value problem, or Cauchy problem in connection with the position of the ball  $K_{i,\delta}$ ,  $i \in \mathcal{I} = \bigcup_{k=1}^4 \mathcal{I}_k$ . In the right-hand sides of the equations and transmission conditions of these model problems we shall have given functions multiplied by  $\zeta_i$  and the operators similar to  $P_1 h - P_8 h$ , depending on  $w$ . We write the estimates for the solutions to these model problems. Evaluating the so obtained operators, choosing radius  $\delta$  of balls and  $t \leq T_2$  sufficiently small, applying the estimate (4.10) and taking into account that  $\zeta_i(x) = 1$  in  $K_{i,\delta}$ , we can achieve the estimate

$$\|w\|_{\mathcal{B}(\omega_{i,t})} \leq C \|h\|_{\mathcal{H}(\omega_{i,t})} + \varepsilon \|w\|_{\mathcal{B}(\omega_{i,t})},$$

$t \leq T_2$ , where  $\varepsilon \in (0, 1)$ ,  $i \in \mathcal{I}$ ,  $\omega_{i,t} = (\Omega \cap K_{i,\delta}) \times (0, t)$ . From here we find

$$\begin{aligned} \{w\}_{\mathcal{B}(\Omega_t)} &= \sup_{i \in \mathcal{I}} \|w\|_{\mathcal{B}(\omega_{i,t})} \\ &\leq C(1 - \varepsilon)^{-1} \sup_{i \in \mathcal{I}} \|h\|_{\mathcal{H}(\omega_{i,t})} \\ &= C(1 - \varepsilon)^{-1} \{h\}_{\mathcal{H}(\Omega_t)}, \quad t \leq T_2. \end{aligned}$$

By the equivalence of the norms (4.33), (4.34) to the norms  $\|w\|_{\mathcal{B}(\Omega_t)}$  and  $\|h\|_{\mathcal{H}(\Omega_t)}$  we derive the estimate (4.11) for  $t \leq T_2$ .

Thus, we have proven the Theorem for  $t \leq \min(T_1, T_2)$ . We extend the solution into segment  $[0, T]$  as, for example, in [BS] and obtain Theorem 4.1 for  $t \leq T$ .  $\blacksquare$

## 5. Proof of Theorem 2.1

We have transformed the free boundary problem (2.1)–(2.8) into the non-linear problem (3.27)–(3.33) in fixed given domains:

$$\mathcal{A}[w] = h + \mathcal{N}[w] , \quad (5.1)$$

where  $w = (v_1, v_2, z_1, z_2, \psi)$  is unknown and  $h$  is a given vector function.

In Theorem 4.1 we have proved the existence and uniqueness of the solution to the linear problem  $\mathcal{A}[w] = h$  defined by (4.1)–(4.7) in the space  $\mathcal{B}(\Omega_T)$  and established the estimate (4.11)

$$\|w\|_{\mathcal{B}(\Omega_T)} \leq C_1 \|h\|_{\mathcal{H}(\Omega_T)} \quad \forall h \in \mathcal{H}(\Omega_T) ,$$

that is the linear operator  $\mathcal{A}$  has the inverse bounded operator  $\mathcal{A}^{-1}$  in  $\mathcal{H}(\Omega_T)$ .

Under the assumptions of Theorem 2.1 all the conditions of Theorem 4.1 hold, so we can write problem (5.1) in the form

$$w = \mathcal{A}^{-1}[h + \mathcal{N}[w]] \quad (5.2)$$

and apply the estimate (4.11)

$$\|w\|_{\mathcal{B}(\Omega_t)} \leq C_1 (\|h\|_{\mathcal{H}(\Omega_t)} + \|\mathcal{N}[w]\|_{\mathcal{H}(\Omega_t)}) , \quad t \leq T ,$$

where  $\mathcal{A}^{-1}$  is the inverse operator.

Let  $K(M)$  be the closed ball in the space  $\mathcal{B}(\Omega_T)$ :

$$K(M) = \{w \in \mathcal{B}(\Omega_{T_0}) : \|w\|_{\mathcal{B}(\Omega_{T_0})} \leq M\} ,$$

where  $M = C_1 \|h\|_{\mathcal{H}(\Omega_{T_0})} (1 - q)^{-1}$ ,  $q \in (0, 1)$ .

We shall prove that the nonlinear operator  $\mathcal{A}^{-1}[h + \mathcal{N}[w]]$  acts from  $K(M)$  into  $K(M)$  and is a contraction for small  $t \leq T_0$ . For that we shall estimate the following norms:

$$I_1 = \|\mathcal{A}^{-1}[h + \mathcal{N}[w]]\|_{\mathcal{B}(\Omega_t)} \leq C_1 (\|h\|_{\mathcal{H}(\Omega_t)} + \|\mathcal{N}[w]\|_{\mathcal{H}(\Omega_t)}) , \quad (5.3)$$

$$\begin{aligned} I_2 &= \|\mathcal{A}^{-1}[h + \mathcal{N}[w]] - \mathcal{A}^{-1}[h + \mathcal{N}[\tilde{w}]]\|_{\mathcal{B}(\Omega_t)} \\ &\equiv \|\mathcal{A}^{-1}[\mathcal{N}[w] - \mathcal{N}[\tilde{w}]]\|_{\mathcal{B}(\Omega_t)} \leq C_1 \|\mathcal{N}[w] - \mathcal{N}[\tilde{w}]\|_{\mathcal{H}(\Omega_t)} \end{aligned} \quad (5.4)$$

$$\forall w, \tilde{w} \in K(M), \quad t \leq T_0 .$$

We rewrite in details the norms in the right-hand sides of the inequalities (5.3),(5.4) substituting there the corresponding functions from the right-hand

sides of the equations and conditions of the problem (3.27)–(3.33)

$$I_1 \leq C_1 \|h\|_{\mathcal{H}(\Omega_t)} + C_1 \sum_{j=1}^2 (|F_j|_{s-2, \Omega_{jt}}^{(\alpha)} + |G_j|_{s-2, \Omega_{jt}}^{(\alpha)} + |R_j|_{s, \Gamma_t}^{(2+\alpha)} + |P_j|_{s-1, \Gamma_t}^{(1+\alpha)}), \quad (5.5)$$

$$\begin{aligned} I_2 \leq C_1 \sum_{j=1}^2 & \left( |F_j(v_j, \psi) - F_j(\tilde{v}_j, \tilde{\psi})|_{s-2, \Omega_{jt}}^{(\alpha)} + |G_j(z_j, \psi) - G_j(\tilde{z}_j, \tilde{\psi})|_{s-2, \Omega_{jt}}^{(\alpha)} \right. \\ & + |R_j(v_j) - R_j(\tilde{v}_j)|_{s, \Gamma_t}^{(2+\alpha)} + |P_1(v_1, v_2, \psi) - P_1(\tilde{v}_1, \tilde{v}_2, \tilde{\psi})|_{s-1, \Gamma_t}^{(1+\alpha)} \\ & \left. + |P_2(z_1, z_2, \psi) - P_2(\tilde{z}_1, \tilde{z}_2, \tilde{\psi})|_{s-1, \Gamma_t}^{(1+\alpha)} \right) \end{aligned} \quad (5.6)$$

$\forall w, \tilde{w} \in K(M)$ , where the functions  $F_j, G_j, R_j, P_j, j = 1, 2$ , are defined by formulas (3.36), (3.37), (3.41), (3.44), (3.45).

To estimate the norms in (5.5), (5.6) we apply the estimates (A.7)–(A.12) for the norms of the functions and their products, the estimates (A.13)–(A.16), (A.19)–(A.29), (A.38)–(A.43), (A.44), (A.56) and (A.58), (A.59) for the matrices  $J_{01}, J_1 = J_{11} + J_{12}, J_{01}J_1, J^{-1}, J_0^{-1}$  and  $J^{-1}(\psi) - J^{-1}(\tilde{\psi})$  respectively, then we obtain

$$\begin{aligned} |F_j|_{s-2, \Omega_{jt}}^{(\alpha)} & \leq \\ & \leq C \left[ \left( t^{\frac{1}{2}} + \begin{cases} t^{\frac{s}{2}}, & 1 < s < 2 \\ t, & 2 \leq s \leq 2 + \alpha \end{cases} + t^{\frac{s}{2}} |\partial_t \psi|_{s-1, \Gamma_t}^{(1+\alpha)} + t^{\frac{s-2}{2}} |\psi|_{s, \Gamma_t}^{(2+\alpha)} \right) |v_j|_{s, \Omega_t}^{(2+\alpha)} \right. \\ & \quad \left. + \left( t^{\frac{s-1}{2}} + t^{\frac{s}{2}} + t^{\frac{1+s}{2}} + \begin{cases} t^{\frac{s}{2}}, & 1 < s < 2 \\ t, & 2 \leq s \leq 2 + \alpha \end{cases} \right) |\partial_t \psi|_{s-1, \Gamma_t}^{(1+\alpha)} \right] \end{aligned} \quad (5.7)$$

the estimate for the function  $G_j$  is like (5.7) with  $z_j$  instead of  $v_j$ ;

$$|R_j|_{s, \Gamma_t}^{(2+\alpha)} \leq C t^{\frac{s}{2}} (|v_j|_{s, \Omega_t}^{(2+\alpha)})^2, \quad j = 1, 2, \quad (5.8)$$

$$\begin{aligned} |P_1|_{s-1, \Gamma_t}^{(1+\alpha)} & \leq C \left[ \left( \begin{cases} t^{\frac{s}{2}}, & 1 < s < 2 \\ t, & 2 \leq s \leq 2 + \alpha \end{cases} + t^{\frac{s-1}{2}} |\psi|_{s, \Gamma_t}^{(2+\alpha)} \right) \sum_{k=1}^2 |v_k|_{s, \Omega_t}^{(2+\alpha)} \right. \\ & \quad \left. + \delta_0 (|\psi|_{s, \Gamma_t}^{(2+\alpha)} + |\partial_t \psi|_{s-1, \Gamma_t}^{(1+\alpha)}) \right] \end{aligned} \quad (5.9)$$

$$\begin{aligned}
|P_2|_{s-1, \Gamma_t}^{(1+\alpha)} &\leq \\
&\leq C \left[ \left( \left( t^{\frac{s}{2}} + \begin{cases} t^{\frac{s}{2}}, & 1 < s < 2 \\ t, & 2 \leq s \leq 2+\alpha \end{cases} \right) |\psi|_{s, \Gamma_t}^{(2+\alpha)} + t^{\frac{s}{2}} |\partial_t \psi|_{s-1, \Gamma_t}^{(1+\alpha)} \right) \sum_{k=1}^2 |z_k|_{s, \Omega_t}^{(2+\alpha)} \right. \\
&\quad \left. + \left( \begin{cases} t^{\frac{s}{2}}, & 1 < s < 2 \\ t, & 2 \leq s \leq 2+\alpha \end{cases} + t + t^{\frac{s-1}{2}} (|\psi|_{s, \Gamma_t}^{(2+\alpha)} + |\partial_t \psi|_{s-1, \Gamma_t}^{(1+\alpha)}) |\psi|_{s, \Gamma_t}^{(2+\alpha)} \right) \right], \quad (5.10)
\end{aligned}$$

the estimates for the norms of the differences of the functions  $F_j, G_j, R_j, P_j$ ,  $j = 1, 2$ , in (5.6) are analogous to the estimates (5.7)–(5.10), they contain the norms  $|v_j - \tilde{v}_j|_{s, \Omega_t}^{(2+\alpha)}$ ,  $|z_j - \tilde{z}_j|_{s, \Omega_t}^{(2+\alpha)}$ ,  $|\psi - \tilde{\psi}|_{s, \Gamma_t}^{(2+\alpha)}$ ,  $|\partial_t \psi - \partial_t \tilde{\psi}|_{s-1, \Gamma_t}^{(1+\alpha)}$ . Here  $C > 0$  denotes some constant independent of  $t \in (0, T)$ .

We substitute those estimates into inequalities (5.5), (5.6), then we derive

$$I_1 \leq C_1 \|h\|_{\mathcal{H}(\Omega_t)} + r_1(t) \|w\|_{\mathcal{B}(\Omega_t)} \quad \text{and} \quad I_2 \leq r_2(t) \|w - \tilde{w}\|_{\mathcal{B}(\Omega_t)}, \quad (5.11)$$

where

$$r_j(t) = C_{1+j} \left( \delta_0 + t^{\frac{1}{2}} + \left( t^{\frac{s-1}{2}} + t^{\frac{s}{2}} + t^{\frac{1+s}{2}} + \begin{cases} t^{\frac{s}{2}}, & 1 < s < 2 \\ t, & 2 \leq s \leq 2 + \alpha \end{cases} \right) M \right),$$

$j = 1, 2$ , and

$$\|w\|_{\mathcal{B}(\Omega_t)}, \|\tilde{w}\|_{\mathcal{B}(\Omega_t)} \leq M = C_1 \|h\|_{\mathcal{H}(\Omega_{T_0})} (1 - q)^{-1}, \quad q \in (0, 1), \quad \delta_0 \ll 1.$$

We find  $T_3 > 0$  from the inequalities

$$r_j(t) \leq q, \quad q \in (0, 1), \quad j = 1, 2.$$

Then, from (5.11), we obtain

$$I_1 \leq C_1 \|h\|_{\mathcal{H}(\Omega_{T_0})} + qM = M, \quad I_2 \leq q \|w - \tilde{w}\|_{\mathcal{B}(\Omega_t)} \quad \forall t \leq T_0,$$

where  $T_0 = \min(t_0, t_1, t_2, t_3, T_3)$ . We remind that the parametrization of the free boundary by the equation (2.11) is valid for  $t \leq t_0$ ; in Theorems A.1–A.3 we prove the existence of the inverse matrices  $J^{-1}$  and  $J_0^{-1}$  for  $t \leq t_1$  and  $t \leq t_2$  respectively and the estimates of the difference  $J^{-1}(\psi) - J^{-1}(\tilde{\psi})$  for  $t \leq t_3$ .

Thus, we have

$$\|\mathcal{A}^{-1}[h + \mathcal{N}[w]]\|_{\mathcal{B}(\Omega_t)} \leq M,$$

$$\|\mathcal{A}^{-1}[h + \mathcal{N}[w]] - \mathcal{A}^{-1}[h + \mathcal{N}[\tilde{w}]]\|_{\mathcal{B}(\Omega_t)} \leq q \|w - \tilde{w}\|_{\mathcal{B}(\Omega_t)}$$

for all  $w, \tilde{w} \in K(M)$ ,  $t \leq T_0$ .

Whence it follows that the operator  $\mathcal{A}^{-1}[h + \mathcal{N}[w]]$  is a contraction from the closed ball  $K(M)$  into itself. Therefore problem (5.1) (i.e. (3.27)–(3.33)) has a unique solution  $w = (v_1, v_2, z_1, z_2, \psi) \in \mathcal{B}(\Omega_{T_0})$  and it satisfies the estimate

$$\|w\|_{\mathcal{B}(\Omega_t)} = \|\mathcal{A}^{-1}[h + \mathcal{N}[w]]\|_{\mathcal{B}(\Omega_t)} \leq C_1(1 - q)^{-1} \|h\|_{\mathcal{H}(\Omega_t)} \quad \forall t \leq T_0. \quad (5.12)$$

Now we return to the problem (2.1)–(2.8). From formulas (3.11), (3.25), (3.26) we have

$$\begin{aligned} \rho &= \rho_0 + \psi, \\ u_j(x, t) &= v_j(x - N\chi\rho, t) + V_j(x - N\chi\rho, t), \\ c_j(x, t) &= z_j(x - N\chi\rho, t) + Z_j(x - N\chi\rho, t), \quad j = 1, 2, \end{aligned} \quad (5.13)$$

where the auxiliary functions  $V_j, Z_j$  and  $\rho_0$  belong to  $C_s^{2+\alpha}(\mathbb{R}_T^n)$  and  $C_{1+s}^{3+\alpha}(\Gamma_T)$ , respectively, and  $v_j, z_j \in \mathring{C}_s^{2+\alpha}(\Omega_{jT_0})$ ,  $j = 1, 2$ ,  $\psi \in \mathring{C}_s^{2+\alpha}(\Gamma_{T_0})$ ,  $\partial_t\psi \in \mathring{C}_{s-1}^{1+\alpha}(\Gamma_{T_0})$ . Then the compositions of the functions  $v_j, z_j, V_j, Z_j$  with  $\rho$  belong to  $C_s^{2+\alpha}(Q_{jT_0})$ ,  $j = 1, 2$ .

From formulas (5.13) we derive

$$\begin{aligned} &\sum_{j=1}^2 (|u_j|_{s, Q_{jt}}^{(2+\alpha)} + |c_j|_{s, Q_{jt}}^{(2+\alpha)}) + |\rho|_{s, \Gamma_t}^{(2+\alpha)} + |\partial_t\rho|_{s, \Gamma_t}^{(2+\alpha)} \leq \\ &\leq C_8 \left( \|w\|_{\mathcal{B}(\Omega_t)} + \sum_{j=1}^2 (|V_j|_{s, Q_{jt}}^{(2+\alpha)} + |Z_j|_{s, Q_{jt}}^{(2+\alpha)} + |\rho_0|_{1+s, \Gamma_t}^{(3+\alpha)}) \right), \quad t \leq T_0, \end{aligned} \quad (5.14)$$

where  $w = (v_1, v_2, z_1, z_2, \psi)$  satisfies the estimate (5.12) with

$$h = (f_1, f_2, g_1, g_2, p_1, q_1, \eta_0, \eta_1, \eta_2, \varphi_1, \varphi_2).$$

For this vector  $h$  we may use the estimates (3.46) of Lemmas 3.1 and 3.2 to complete the proof of Theorem 2.1.  $\blacksquare$

## Appendix A. Expansions and estimates of the inverse Jacobian matrix $J^{-1}$

We consider a Jacobian matrix (3.5)  $J$  of the transformation of coordinates (3.3)

$$J = \{\delta_{ij} + \partial_{y_j}(N_i \xi(\rho_0 + \psi))\}_{1 \leq i, j \leq n} = I + J_{01} + J_1,$$

with

$$J_{01} = \{\partial_{y_j}(N_i \chi \rho_0)\}_{1 \leq i, j \leq n} = (\nabla^T N \chi \rho_0)^T$$

and

$$J_1 = \{\partial_{y_j}(N_i \chi \psi)\}_{1 \leq i, j \leq n} ,$$

where  $N(\xi) = (N_1, \dots, N_n)$ ,  $\chi(\lambda(y))$ ,  $N_i \in C^{1,\alpha}$ ,  $i = 1, \dots, n$ ,  $\xi \in \Gamma$ .

We denote

$$J_0 = I + J_{01} , \quad J_1 = J_{11} + J_{12} ,$$

$$J_{11} = \{N_i \chi \partial_{y_j} \psi\}_{1 \leq i, j \leq n} = N^T \chi \nabla \psi , \quad J_{11}^T = (\nabla^T \psi) N \chi ,$$

$$J_{12} = \psi \{\partial_{y_j}(N_i \chi)\}_{1 \leq i, j \leq n} = \psi (\nabla^T (N \chi))^T .$$

We note that  $J|_{t=0} = I$ ,  $J_0|_{t=0} = I$ , because  $\rho_0|_{t=0} = 0$ ,  $\psi|_{t=0} = 0$ . That is for the small  $t$  the inverse matrices  $J^{-1}$ ,  $J_0^{-1}$  exist (see Theorems A.1, A.2 below).

Now we write the expansion formulas of the matrices  $J^{-1}$ ,  $J_0^{-1}$  by linear algebra straightforward computations.

**Lemma A.1.** *For  $t \leq t_1$  the matrix  $J^{-1}$  can be represented in the form*

$$J^{-1} = J_0^{-1} - J_0^{-1} J_1 J^{-1} = J_0^{-1} - J^{-1} J_1 J_0^{-1} , \quad (\text{A.1})$$

$$J^{-1} = I - B J^{-1} , \quad J^{-1} = I - J^{-1} B , \quad (\text{A.2})$$

$$J^{-1} = I - B + B^2 J^{-1} = I - B + J^{-1} B^2 , \quad (\text{A.3})$$

$$J^{-1} = (I - B) (I - B^2)^{-1} = (I - B^2)^{-1} (I - B) , \quad (\text{A.4})$$

where  $B = J_{01} + J_1$ . ■

**Lemma A.2.** *For  $t \leq t_2$  the matrix  $J_0^{-1}$  can be represented in the form*

$$J_0^{-1} = I - J_{01} J_0^{-1} , \quad J_0^{-1} = I - J_0^{-1} J_{01} , \quad (\text{A.5})$$

$$J_0^{-1} = (I - J_{01}^2)^{-1} (I - J_{01}) = (I - J_{01}) (I - J_{01}^2)^{-1} . \quad \blacksquare \quad (\text{A.6})$$

To estimate the norms of the functions in the weighted Hölder spaces we shall use the following results. We denote by  $C$  different positive constants.

**Lemma A.3** (Imbedding Theorem [BZ]). *If  $q(x, t) \in C_s^\ell(\Omega_T)$ ,  $\ell$  a positive number,  $s \leq \ell$ , and  $k = 2i + |j| \leq [\ell]$ , then  $\partial_t^i \partial_x^j q \in C_{s-k}^{\ell-k}(\Omega_T)$  and*

$$|\partial_t^i \partial_x^j q|_{s-k, \Omega_T}^{(\ell-k)} \leq |q|_{s, \Omega_T}^{(\ell)} . \quad (\text{A.7})$$

**Lemma A.4** ([B2]). *Let  $\ell$  be positive noninteger,  $r$  a nonnegative number,  $s \leq \ell$ ,  $s+r \geq 0$ ,  $f_1(x, t) \in \mathring{C}_{s+r}^{\ell+r}(\Omega_T)$ ,  $f_2(x, t) \in \mathring{C}_s^\ell(\Omega_T)$ ,  $q_1(x, t) \in C_{s+r}^{\ell+r}(\Omega_T)$ ,  $q_2(x, t) \in C_s^\ell(\Omega_T)$ , then for  $t \leq T$  the following estimates hold:*

$$|f_1|_{s, \Omega_t}^{(\ell)} \leq C t^{\frac{r}{2}} |f_1|_{s+r, \Omega_t}^{(\ell+r)}, \quad (\text{A.8})$$

$$|f_1 q_2|_{s, \Omega_t}^{(\ell)} \leq C |f_1|_{s+r, \Omega_t}^{(\ell+r)} \left[ \sup_{\tau \leq t} \tau^{\frac{r}{2}} |q_2|_{\Omega} + t^{\frac{s+r}{2}} |q_2|_{s, \Omega_t}^{(\ell)} \right], \quad (\text{A.9})$$

$$|f_1 f_2|_{s, \Omega_t}^{(\ell)} \leq C t^{\frac{s+r}{2}} |f_1|_{s+r, \Omega_t}^{(\ell+r)} |f_2|_{s, \Omega_t}^{(\ell)}, \quad (\text{A.10})$$

$$|f_2 q_1|_{s, \Omega_t}^{(\ell)} \leq C |f_2|_{s, \Omega_t}^{(\ell)} \left[ t^{\frac{s+r}{2}} |q_1|_{s+r, \Omega_t}^{(\ell+r)} + |q_1|_{\Omega_t} \right], \quad (\text{A.11})$$

$$|q_1 q_2|_{s, \Omega_t}^{(\ell)} \leq C |q_1|_{s+r, \Omega_t}^{(\ell+r)} |q_2|_{s, \Omega_t}^{(\ell)}. \quad (\text{A.12})$$

We estimate the norms of the matrices  $J_{01}$ ,  $J_1 = J_{11} + J_{12}$ , then we prove the existence of the inverse matrices  $J^{-1}$ ,  $J_0^{-1}$  and evaluate the difference  $J^{-1}(\rho_0 + \psi) - J^{-1}(\rho_0 + \tilde{\psi})$ .

**Lemma A.5.** *Let  $\rho_0(\xi, t) \in C_{1+s}^{3+\alpha}(\Gamma_T)$ ,  $\alpha \in (0, 1)$ ,  $1 < s \leq 2 + \alpha$ ,  $\rho_0|_{t=0} = 0$  and  $|\rho_0|_{1+s, \Gamma_T}^{(3+\alpha)} \leq M_1$ ,  $M_1 > 0$ .*

*Then for the matrix  $J_{01}$  the following estimates hold for  $t \leq T$ :*

$$\|J_{01}\|_{s-2, \Gamma_t}^{(\alpha)} = n \max_{i,j} |\partial_{y_j}(N_i \chi \rho_0)|_{s-2, \Gamma_t}^{(\alpha)} \leq C M_1 \begin{cases} t, & 1 < s < 2 \\ t^{\frac{4-s}{2}}, & 2 \leq s \leq 2 + \alpha, \end{cases} \quad (\text{A.13})$$

$$\|J_{01}\|_{s-1, \Gamma_t}^{(1+\alpha)} \leq C M_1 \begin{cases} t^{\frac{1}{2}}, & 1 < s < 2 \\ t^{\frac{3-s}{2}}, & 2 \leq s \leq 2 + \alpha. \end{cases} \quad (\text{A.14})$$

*Proof:* We write the norms of the element  $\partial_{y_j}(N_i \chi \rho_0)$  in accordance to their definition (2.9), for example,

$$\begin{aligned} |\partial_{y_j}(N_i \chi \rho_0)|_{s-2, \Gamma_t}^{(\alpha)} &= \sup_{\tau \leq t} \tau^{\frac{2+\alpha-s}{2}} [\partial_{y_j}(N_i \chi \rho_0)]_{\Gamma'_\tau}^{(\alpha)} \\ &\quad + \sup_{\tau \leq t} |\partial_{y_j}(N_i \chi \rho_0)| \begin{cases} t^{\frac{2-s}{2}}, & 1 < s < 2 \\ 1, & 2 \leq s \leq 2 + \alpha \end{cases} \end{aligned}$$

and evaluate each term taking into account that  $\rho_0|_{t=0} = 0$  (here  $\Gamma'_\tau = \Gamma \times [\frac{\tau}{2}, \tau]$ ). ■

**Corollary A.1.** *Let  $f(y, t) \in \overset{\circ}{C}_{s-1}^{1+\alpha}(\Gamma_T)$ , then*

$$\|f J_{01}\|_{s-2, \Gamma_t}^{(\alpha)} \leq C M_1 |f|_{s-1, \Gamma_t}^{(1+\alpha)} \begin{cases} t^{\frac{1+s}{2}}, & 1 < s < 2 \\ t^{\frac{3}{2}}, & 2 \leq s \leq 2 + \alpha \end{cases}, \quad (\text{A.15})$$

$$\|f J_{01}\|_{s-1, \Gamma_t}^{(1+\alpha)} \leq C M_1 |f|_{s-1, \Gamma_t}^{(1+\alpha)} \begin{cases} t^{\frac{s}{2}}, & 1 < s < 2 \\ t, & 2 \leq s \leq 2 + \alpha \end{cases}, \quad (\text{A.16})$$

for  $t \leq T$ .

*Proof:* We apply estimate (A.9)

$$\|f J_{01}\|_{s-2, \Gamma_t}^{(\alpha)} \leq C |f|_{s-1, \Gamma_t}^{(1+\alpha)} \left[ \sup_{\tau \leq t} \tau^{\frac{1}{2}} n \max_{i,j} |\partial_{y_j}(N_i \chi \rho_0)|_{\Gamma} + t^{\frac{s-1}{2}} \|J_{01}\|_{s-2, \Gamma_t}^{(\alpha)} \right],$$

$$\|f J_{01}\|_{s-1, \Gamma_t}^{(1+\alpha)} \leq C |f|_{s-1, \Gamma_t}^{(1+\alpha)} \left[ \sup_{\tau \leq t} n \max_{i,j} |\partial_{y_j}(N_i \chi \rho_0)| + t^{\frac{s-1}{2}} \|J_{01}\|_{s-1, \Gamma_t}^{(1+\alpha)} \right].$$

On the basis of the estimates (A.13), (A.14) and

$$\begin{aligned} |\partial_{y_j}(N_i \chi \rho_0)| &\leq \int_0^\tau |\partial_{\tau_1} \nabla_{y_j}(N_i \chi \rho_0)| d\tau_1 \\ &\leq \sup_{\tau_1 \leq \tau} \left( |\partial_{\tau_1} \partial_{t_j}(N_i \chi \rho_0)| \begin{cases} \tau_1^{\frac{2-s}{2}}, & 1 < s < 2 \\ 1, & 2 \leq s \leq 2 + \alpha \end{cases} \right) \begin{cases} \frac{2}{s} \tau^{\frac{s}{2}}, & 1 < s < 2 \\ \tau, & 2 \leq s \leq 2 + \alpha \end{cases} \end{aligned}$$

we obtain (A.15), (A.16).

We note that these estimates can be derived with the direct evaluations of the corresponding norms or with the help of an estimate (A.10), because  $\partial_{y_j} \rho_0$  may be considered as the function belonging to  $\overset{\circ}{C}_{s-1}^{1+\alpha}(\Gamma_T)$  by the condition  $\rho_0|_{t=0} = 0$ .  $\blacksquare$

**Corollary A.2.** *For the matrix  $J_{01}^2$  the following estimates hold for  $t \leq T$ :*

$$\|J_{01}^2\|_{s-2, \Gamma_t}^{(\alpha)} \leq C M_1^2 \begin{cases} t^{\frac{2+s}{2}}, & 1 < s < 2 \\ t^{\frac{6-s}{2}}, & 2 \leq s \leq 2 + \alpha \end{cases}, \quad (\text{A.17})$$

$$\|J_{01}^2\|_{s-1, \Gamma_t}^{(1+\alpha)} \leq C M_1^2 \begin{cases} t^{\frac{1+s}{2}}, & 1 < s < 2 \\ t^{\frac{5-s}{2}}, & 2 \leq s \leq 2 + \alpha \end{cases}. \quad (\text{A.18})$$

*Proof:* These estimates are found with the help of the inequalities (A.10), (A.13), (A.14).  $\blacksquare$



**Lemma A.6.** *Let  $\psi(\xi, t) \in \mathring{C}_s^{2+\alpha}(\Gamma_T)$ ,  $\partial_t \psi \in \mathring{C}_{s-1}^{1+\alpha}(\Gamma_T)$ ,  $\alpha \in (0, 1)$ ,  $1 < s \leq 2 + \alpha$ , then for the matrix  $J_1 = J_{11} + J_{12}$  the following estimates hold for  $t \leq T$ :*

$$\|J_{11}\|_{s-2, \Gamma_t}^{(\alpha)} = n \max_{i,j} |N_i \chi \partial_{y_j} \psi|_{s-2, \Gamma_t}^{(\alpha)} \leq C t |\partial_t \psi|_{s-1, \Gamma_t}^{(1+\alpha)}, \quad (\text{A.19})$$

$$\|J_{11}\|_{s-1, \Gamma_t}^{(1+\alpha)} \leq C |\psi|_{s, \Gamma_t}^{(2+\alpha)}, \quad (\text{A.20})$$

$$\|J_{12}\|_{s-2, \Gamma_t}^{(\alpha)} = n \max |\psi \partial_{y_j} (N_i \chi)|_{s-2, \Gamma_t}^{(\alpha)} \leq C t^{\frac{3}{2}} |\partial_t \psi|_{s-1, \Gamma_t}^{(1+\alpha)}, \quad (\text{A.21})$$

$$\|J_{12}\|_{s-1, \Gamma_t}^{(1+\alpha)} \leq C t |\partial_t \psi|_{s-1, \Gamma_t}^{(1+\alpha)}. \quad (\text{A.22})$$

*Proof:* The estimates (A.19), (A.21), (A.22) are derived with direct evaluation of the corresponding norms and applying an inequality  $|\partial_y^m \psi| \leq \int_0^T |\partial_{\tau_1} \partial_y^m \psi| d\tau_1$ ,  $|m| = 0, 1$ ; an estimate (A.20) follows from (A.7). ■

**Corollary A.3.** *For the matrix  $J_1 = J_{11} + J_{12}$  the following estimates are valid for  $t \leq T$ :*

$$\|J_1\|_{s-2, \Gamma_t}^{(\alpha)} \leq C t |\partial_t \psi|_{s-1, \Gamma_t}^{(1+\alpha)}, \quad (\text{A.23})$$

$$\|J_1\|_{s-1, \Gamma_t}^{(1+\alpha)} \leq C (|\psi|_{s, \Gamma_t}^{(2+\alpha)} + |\partial_t \psi|_{s-1, \Gamma_t}^{(1+\alpha)}). \quad (\text{A.24})$$

**Corollary A.4.** *Let  $f(y, t) \in \mathring{C}_{s-2+k}^{k+\alpha}(\Gamma_T)$ ,  $k = 0, 1, 2$ , then the following estimates are valid for  $t \leq T$ :*

$$\|f J_{11}\|_{s-2, \Gamma_t}^{(\alpha)} \leq C t^{\frac{s-1}{2}} |\psi|_{s, \Gamma_t}^{(2+\alpha)} |f|_{s-2, \Gamma_t}^{(\alpha)}, \quad k = 0, \quad (\text{A.25})$$

$$\|f J_{11}\|_{s-2, \Gamma_t}^{(\alpha)} \leq C t^{\frac{s+k}{2}} |\partial_t \psi|_{s-1, \Gamma_t}^{(1+\alpha)} |f|_{s-2+k, \Gamma_t}^{(k+\alpha)}, \quad k = 1, 2, \quad (\text{A.26})$$

$$\|f J_{11}\|_{s-1, \Gamma_t}^{(1+\alpha)} \leq C t^{\frac{s+k-2}{2}} |\psi|_{s, \Gamma_t}^{(2+\alpha)} |f|_{s-2+k, \Gamma_t}^{(k+\alpha)}, \quad k = 1, 2, \quad (\text{A.27})$$

$$\|f J_{12}\|_{s-2, \Gamma_t}^{(\alpha)} \leq C t^{\frac{s+k+1}{2}} |\partial_t \psi|_{s-1, \Gamma_t}^{(1+\alpha)} |f|_{s-2+k, \Gamma_t}^{(k+\alpha)}, \quad k = 0, 1, 2, \quad (\text{A.28})$$

$$\|f J_{12}\|_{s-1, \Gamma_t}^{(1+\alpha)} \leq C t^{\frac{s+k}{2}} |\partial_t \psi|_{s-1, \Gamma_t}^{(1+\alpha)} |f|_{s-2+k, \Gamma_t}^{(k+\alpha)}, \quad k = 1, 2. \quad (\text{A.29})$$

*Proof:* These estimates are derived with the help of the inequalities (A.10) and (A.19)–(A.22). ■

**Corollary A.5.** *For  $t \leq T$  the following estimates hold:*

$$\|J_{11}^2\|_{s-2, \Gamma_t}^{(\alpha)} \leq C t^{\frac{2+s}{2}} (|\partial_t \psi|_{s-1, \Gamma_t}^{(1+\alpha)})^2, \quad (\text{A.30})$$

$$\|J_{11}^2\|_{s-1, \Gamma_t}^{(1+\alpha)} \leq C t^{\frac{s-1}{2}} (|\psi|_{s, \Gamma_t}^{(2+\alpha)})^2, \quad (\text{A.31})$$

$$\|J_{12}^2\|_{s-2, \Gamma_t}^{(\alpha)} \leq C t^{\frac{4+s}{2}} (|\partial_t \psi|_{s-1, \Gamma_t}^{(1+\alpha)})^2, \quad (\text{A.32})$$

$$\|J_{12}^2\|_{s-1, \Gamma_t}^{(1+\alpha)} \leq C t^{\frac{3+s}{2}} (|\partial_t \psi|_{s-1, \Gamma_t}^{(1+\alpha)})^2, \quad (\text{A.33})$$

$$\|J_{11} J_{12}\|_{s-2, \Gamma_t}^{(\alpha)} \leq C t^{\frac{3+s}{2}} (|\partial_t \psi|_{s-1, \Gamma_t}^{(1+\alpha)})^2, \quad (\text{A.34})$$

$$\|J_{11} J_{12}\|_{s-1, \Gamma_t}^{(1+\alpha)} \leq C t^{\frac{1+s}{2}} |\psi|_{s, \Gamma_t}^{(2+\alpha)} |\partial_t \psi|_{s-1, \Gamma_t}^{(1+\alpha)}. \quad (\text{A.35})$$

*Proof:* All inequalities can be derived with the help of direct evaluations of the corresponding norms. An estimate (A.31) follows from (A.10) and (A.20). We can find inequalities (A.32), (A.34), (A.35) applying (A.10) and estimates (A.19)–(A.22) of the matrices  $J_{11}$ ,  $J_{12}$ .  $\blacksquare$

**Corollary A.6.** *For the matrix  $J_1 = J_{11} + J_{12}$  the following estimates hold for  $t \leq T$ :*

$$\|J_1^2\|_{s-2, \Gamma_t}^{(\alpha)} \leq C t^{\frac{2+s}{2}} (|\partial_t \psi|_{s-1, \Gamma_t}^{(1+\alpha)})^2, \quad (\text{A.36})$$

$$\|J_1^2\|_{s-1, \Gamma_t}^{(1+\alpha)} \leq C t^{\frac{s-1}{2}} (|\psi|_{s, \Gamma_t}^{(2+\alpha)} + t |\partial_t \psi|_{s-1, \Gamma_t}^{(1+\alpha)})^2. \quad (\text{A.37})$$

**Corollary A.7.** *Let  $\rho_0 \in C_{1+s}^{3+\alpha}(\Gamma_T)$ ,  $\rho_0|_{t=0} = 0$ ,  $|\rho_0|_{1+s, \Gamma_T}^{(3+\alpha)} \leq M_1$ . Then for  $t \leq T$  the following estimates hold:*

$$\|J_{01} J_{11}\|_{s-2, \Gamma_t}^{(\alpha)} \leq C M_1 \begin{cases} t^{\frac{2+s}{2}}, & 1 < s < 2 \\ t^2, & 2 \leq s \leq 2 + \alpha \end{cases} |\partial_t \psi|_{s-1, \Gamma_t}^{(1+\alpha)}, \quad (\text{A.38})$$

$$\|J_{01} J_{11}\|_{s-1, \Gamma_t}^{(1+\alpha)} \leq C M_1 \begin{cases} t^{\frac{s}{2}}, & 1 < s < 2 \\ t, & 2 \leq s \leq 2 + \alpha \end{cases} |\psi|_{s, \Gamma_t}^{(2+\alpha)}, \quad (\text{A.39})$$

$$\|J_{01} J_{12}\|_{s-2, \Gamma_t}^{(\alpha)} \leq C M_1 \begin{cases} t^{\frac{3+s}{2}}, & 1 < s < 2 \\ t^{\frac{5}{2}}, & 2 \leq s \leq 2 + \alpha \end{cases} |\partial_t \psi|_{s-1, \Gamma_t}^{(1+\alpha)}, \quad (\text{A.40})$$

$$\|J_{01} J_{12}\|_{s-1, \Gamma_t}^{(1+\alpha)} \leq C M_1 \begin{cases} t^{\frac{2+s}{2}}, & 1 < s < 2 \\ t^2, & 2 \leq s \leq 2 + \alpha \end{cases} |\partial_t \psi|_{s-1, \Gamma_t}^{(1+\alpha)}, \quad (\text{A.41})$$

$$\|J_{01}J_1\|_{s-2,\Gamma_t}^{(\alpha)} \leq C M_1 (1 + t^{\frac{1}{2}}) \begin{cases} t^{\frac{2+s}{2}}, & 1 < s < 2 \\ t^2, & 2 \leq s \leq 2 + \alpha \end{cases} |\partial_t \psi|_{s-1,\Gamma_t}^{(1+\alpha)}, \quad (\text{A.42})$$

$$\|J_{01}J_1\|_{s-1,\Gamma_t}^{(1+\alpha)} \leq C M_1 \begin{cases} t^{\frac{s}{2}}, & 1 < s < 2 \\ t, & 2 \leq s \leq 2 + \alpha \end{cases} (|\psi|_{s,\Gamma_t}^{(2+\alpha)} + |\partial_t \psi|_{s-1,\Gamma_t}^{(1+\alpha)}). \quad (\text{A.43})$$

*Proof:* We consider the matrices  $J_{01}$  and  $J_{11}$ . With the help of an estimate (A.10) for the product of the functions we derive

$$\|J_{01}J_{11}\|_{s-2,\Gamma_t}^{(\alpha)} \leq C t^{\frac{s-1}{2}} \|J_{01}\|_{s-1,\Gamma_t}^{(1+\alpha)} \|J_{11}\|_{s-2,\Gamma_t}^{(\alpha)},$$

$$\|J_{01}J_{11}\|_{s-1,\Gamma_t}^{(1+\alpha)} \leq C t^{\frac{s-1}{2}} \|J_{01}\|_{s-1,\Gamma_t}^{(1+\alpha)} \|J_{11}\|_{s-1,\Gamma_t}^{(1+\alpha)}.$$

On the basis of the estimates (A.14) for  $J_{01}$  and (A.19),(A.20) for  $J_{11}$  we obtain (A.38),(A.39). The inequalities (A.40),(A.41) are derived analogously by applying the estimates (A.10) and (A.14), (A.21), (A.22) for  $J_{01}$ ,  $J_{12}$ . Formulas (A.42), (A.43) with the matrix  $J_1 = J_{11} + J_{12}$  follow from (A.38)–(A.41).  $\blacksquare$

**Remark A.1.** *All these lemmas and corollaries are applied in the proofs of the Lemma 3.1, Theorem 2.1 (Section 5) and Theorems A.1–A.3.*

Now we prove the existence of the inverse matrices  $J^{-1}$  and  $J_0^{-1}$ .

**Theorem A.1.** *Let  $\psi(\xi, t) \in \mathring{C}_s^{2+\alpha}(\Gamma_T)$ ,  $\partial_t \psi \in \mathring{C}_{s-1}^{1+\alpha}(\Gamma_T)$ ,  $\alpha \in (0, 1)$ ,  $s \in [1, 2 + \alpha]$ ,  $\rho_0(\xi, t) \in C_{1+s}^{3+\alpha}(\Gamma_T)$ ,  $\rho_0|_{t=0} = 0$  and  $|\psi|_{s,\Gamma_T}^{(2+\alpha)} + |\partial_t \psi|_{s-1,\Gamma_T}^{(1+\alpha)} \leq M$ ,  $|\rho_0|_{1+s,\Gamma_T}^{(3+\alpha)} \leq M$ ,  $M > 0$ ,  $M_1 > 0$ .*

*Then there is  $t_1 \in [0, T]$ , such that, the inverse Jacobian matrix  $J^{-1}$  exists, is bounded*

$$\|J^{-1}\|_{s-2+\nu,\Gamma_t}^{(\alpha+\nu)} \leq C(1 + M), \quad \nu = 0, 1, \quad (\text{A.44})$$

*and it can be represented in the form*

$$J^{-1} = \sum_{k=0}^{\infty} (J_{01} + J_1)^{2k} (I - (J_{01} + J_1)). \quad (\text{A.45})$$

*Proof:* We consider Jacobian matrix  $J = I + J_{01} + J_1$ . We have found formally the inverse Jacobian matrix  $J^{-1}$  in Lemma A.1

$$J^{-1} = (I - (J_{01} + J_1)^2)^{-1} (I - (J_{01} + J_1)). \quad (\text{A.46})$$

To prove the existence of  $J^{-1}$  we show the existence of an inverse matrix  $(I - (J_{01} + J_1)^2)^{-1}$ . For that we estimate the norms of the matrix  $(J_{01} + J_1)^2$ .

By the estimates (A.17), (A.18) for  $J_{01}^2$ , (A.36), (A.37) for  $J_1^2 = (J_{11} + J_{12})^2$  and (A.42), (A.43) for  $J_{01}J_1$  we obtain

$$\begin{aligned} \|(J_{01} + J_1)^2\|_{s-2, \Gamma_t}^{(\alpha)} &\leq \|J_{01}^2\|_{s-2, \Gamma_t}^{(\alpha)} + \|J_{01}J_1\|_{s-2, \Gamma_t}^{(\alpha)} + \|J_1J_{01}\|_{s-2, \Gamma_t}^{(\alpha)} + \|J_1^2\|_{s-2, \Gamma_t}^{(\alpha)} \\ &\leq \mu_1(t) , \end{aligned} \quad (\text{A.47})$$

$$\|(J_{01} + J_1)^2\|_{s-1, \Gamma_t}^{(1+\alpha)} \leq \mu_2(t) , \quad (\text{A.48})$$

where

$$\begin{aligned} \mu_1(t) &= C(M + M_1)^2 \begin{cases} t^{\frac{2+s}{2}}, & 1 < s < 2 \\ t^{\frac{2+s}{2}} + t^{\frac{6-s}{2}}, & 2 \leq s \leq 2 + \alpha , \end{cases} \\ \mu_2(t) &= C(M + M_1)^2 \left( t^{\frac{s-1}{2}} + \begin{cases} t^{\frac{s}{2}}(1 + t^{\frac{1}{2}}), & 1 < s < 2 \\ t(1 + t^{\frac{3-s}{2}}), & 2 \leq s \leq 2 + \alpha , \end{cases} \right) . \end{aligned}$$

Let  $\mathcal{D} = (J_{01} + J_1)^2$ . Applying the estimate (A.10), we find

$$\|\mathcal{D}^2\|_{s-2+\nu, \Gamma_t}^{(\alpha+\nu)} \leq C_{48} t^{\frac{s-1}{2}} \|\mathcal{D}\|_{s-2+\nu, \Gamma_t}^{(\alpha+\nu)} \|\mathcal{D}\|_{s-1, \Gamma_t}^{(1+\alpha)} , \quad \nu = 0, 1 .$$

By mathematical induction, we have

$$\|\mathcal{D}^k\|_{s-2+\nu, \Gamma_t}^{(\alpha+\nu)} \leq \|\mathcal{D}\|_{s-2+\nu, \Gamma_t}^{(\alpha+\nu)} (C t^{\frac{s-1}{2}} \|\mathcal{D}\|_{s-1, \Gamma_t}^{(1+\alpha)})^{k-1} , \quad \nu = 0, 1 .$$

Here we make use of the estimates (A.47), (A.48)

$$\|\mathcal{D}^k\|_{s-2+\nu, \Gamma_t}^{(\alpha+\nu)} \leq \mu_{1+\nu}(t) (C t^{\frac{s-1}{2}} \mu_2(t))^{k-1} , \quad \nu = 0, 1 .$$

We find  $t_1 > 0$  from the inequalities

$$C t^{\frac{s-1}{2}} \mu_2(t) \leq q, \quad \mu_{1+\nu}(t) \leq q, \quad \nu = 0, 1, \quad q \in (0, 1) ,$$

then we obtain

$$\|(J_{01} + J_1)^{2k}\|_{s-2+\nu, \Gamma_t}^{(\alpha+\nu)} \leq q^k , \quad \nu = 0, 1, \quad k = 1, 2, \dots, \quad t \leq t_1 . \quad (\text{A.49})$$

From here it follows

$$\sum_{k=0}^{\infty} \|(J_{01} + J_1)^{2k}\|_{s-2+\nu, \Gamma_t}^{(\alpha+\nu)} \leq \sum_{k=0}^{\infty} q^k = \frac{1}{1-q} , \quad \nu = 0, 1, \quad t \leq t_1 . \quad (\text{A.50})$$

On the basis of these estimates we shall have that the inverse matrix  $(I - (J_{01} + J_1)^2)^{-1}$  exists, is expressed in the form

$$(I - (J_{01} + J_1)^2)^{-1} = \sum_{k=0}^{\infty} (J_{01} + J_1)^{2k} \quad (\text{A.51})$$

and is bounded

$$\|(I - (J_{01} + J_1)^2)^{-1}\|_{s-2+\nu, \Gamma_t}^{(\alpha+\nu)} \leq \frac{1}{1-q}, \quad \nu = 0, 1, \quad t \leq t_1. \quad (\text{A.52})$$

But then the matrix in the right-hand side of the formula (A.46) exists and is bounded by the estimates (A.13), (A.14) and (A.23), (A.24) of the matrices  $J_{01}$  and  $J_1$ , respectively, and (A.52)

$$\|(I - (J_{01} + J_1)^2)^{-1} (I - (J_{01} + J_1))\|_{s-2+\nu, \Gamma_t}^{(\alpha+\nu)} \leq C(1+M), \quad \nu = 0, 1, \quad t \leq t_1. \quad (\text{A.53})$$

Now we show that the matrix

$$(I - (J_{01} + J_1)^2)^{-1} (I - J_{01} - J_1) \equiv \sum_{k=0}^{\infty} (J_{01} + J_1)^{2k} (I - J_{01} - J_1)$$

is equal to the inverse Jacobian matrix  $J^{-1}$ . For that we consider an identity

$$\begin{aligned} (I + J_{01} + J_1) \sum_{k=0}^m (J_{01} + J_1)^{2k} (I - J_{01} - J_1) &= \\ &= \sum_{k=0}^m (J_{01} + J_1)^{2k} (I - J_{01} - J_1) (I + J_{01} + J_1) \quad (\text{A.54}) \\ &\equiv I - (J_{01} + J_1)^{2m+2} \quad \forall m, \end{aligned}$$

where  $I + J_{01} + J_1 = J$ .

By the estimate (A.49) we have

$$\|(J_{01} + J_1)^{2m+2}\|_{s-2+\nu, \Gamma_t}^{(\alpha+\nu)} \leq q^{m+1} \rightarrow 0, \quad m \rightarrow \infty, \quad \nu = 0, 1, \quad q \in (0, 1). \quad (\text{A.55})$$

Moreover, the series  $\sum_{k=0}^m (J_{01} + J_1)^{2k}$  converges to the bounded matrix satisfying the estimate (A.50). So we can pass to the limit as  $m \rightarrow \infty$  in the identities (A.54) taking into account (A.55), then we obtain

$$\begin{aligned} (I + J_{01} + J_1) \sum_{k=0}^{\infty} (J_{01} + J_1)^{2k} (I - J_{01} - J_1) &= \\ &= \sum_{k=0}^{\infty} (J_{01} + J_1)^{2k} (I - J_{01} - J_1) (I + J_{01} + J_1) = I. \end{aligned}$$

These identities mean that the matrix  $\sum_{k=0}^{\infty} (J_{01} + J_1)^{2k} (I - J_{01} - J_1)$  is the right and the left inverse matrix to the matrix  $J = I + J_{01} + J_1$ , i.e. it is the inverse matrix  $J^{-1}$ .

Thus we have proved that the inverse Jacobian matrix  $J^{-1}$  exists, satisfies the estimate (A.53) and is expressed in the form (A.45), which follows from the formulas (A.46) and (A.51).  $\blacksquare$

We consider now the matrix  $J_0 = I + J_{01}$ .

**Theorem A.2.** *Let  $\rho_0 \in (\xi, t) \in C_{1+s}^{3+\alpha}(\Gamma_T)$ ,  $\alpha \in (0, 1)$ ,  $s \in [1, 2 + \alpha]$ ,  $\rho_0|_{t=0} = 0$  and  $|\rho_0|_{1+s, \Gamma_T}^{(3+\alpha)} \leq M_1$ .*

*Then there is  $t_2 > 0$  such that the inverse matrix  $J_0^{-1}$  exists, is bounded*

$$\|J_{01}^{-1}\|_{s-2+\nu, \Gamma_t}^{(\alpha+\nu)} \leq \frac{1}{1-q}, \quad \nu = 0, 1, \quad q \in (0, 1), \quad t \leq t_2, \quad (\text{A.56})$$

and can be represented in the form

$$J_0^{-1} = \sum_{k=0}^{\infty} (-1)^k J_{01}^k. \quad (\text{A.57})$$

*Proof:* To prove the existence of the inverse matrix  $J_0^{-1}$  it is sufficient to show that the series in (A.57) converges to a bounded matrix, i.e. satisfies the estimate (A.56). On the basis of the estimates (A.13), (A.14) for  $J_{01}$  we obtain

$$\|J_{01}^k\|_{s-2, \Gamma_t}^{(\alpha+\nu)} \leq \mu_{3+\nu}(t) (t^{\frac{s-1}{2}} \mu_{\nu}(t))^{k-1}, \quad k = 1, 2, \dots,$$

where

$$\mu_{3+\nu}(t) = C M_1 \begin{cases} t^{\frac{2-\nu}{2}}, & 1 < s < 2 \\ t^{\frac{4-s-\nu}{2}}, & 2 \leq s \leq 2 + \alpha \end{cases}, \quad \nu = 0, 1.$$

We choose  $t_2 > 0$  from the inequalities

$$\mu_3(t) \leq q, \quad \mu_4(t) \leq q, \quad t^{\frac{s-1}{2}} \mu_4(t) \leq q, \quad q \in (0, 1),$$

then we shall have

$$\|J_{01}^k\|_{s-2+\nu, \Gamma_t}^{(\alpha+\nu)} \leq q^k \quad \forall t \leq t_2, \quad \nu = 0, 1,$$

and

$$\sum_{k=0}^{\infty} \|J_{01}^k\|_{s-2+\nu, \Gamma_t}^{(\alpha+\nu)} \leq \sum_{k=0}^{\infty} q^k = \frac{1}{1-q}, \quad t \leq t_2.$$

That is the matrix  $\sum_{k=0}^{\infty} (-1)^k J_{01}^k$  exists, is bounded and equals  $J_0^{-1}$ .  $\blacksquare$

In Section 4 we make use of the contraction principle. For that we have to estimate the difference of the matrices  $J^{-1} - \tilde{J}^{-1}$ , where  $J = I + J_{01} + J_1$ ,  $\tilde{J} = I + J_{01} + \tilde{J}_1$ ,  $J_1 = \{\partial_{y_j}(N_i \chi \psi)\}_{1 \leq i, j \leq n}$ ,  $\tilde{J}_1 = \{\partial_{y_j}(N_i \chi \tilde{\psi})\}_{1 \leq i, j \leq n}$ .

**Theorem A.3.** *Let  $\rho_0 \in (\xi, t) \in C_{1+s}^{3+\alpha}(\Gamma_T)$ ,  $\rho_0|_{t=0} = 0$ ;  $\psi(\xi, t), \tilde{\psi}(\xi, t) \in \mathring{C}_s^{2+\alpha}(\Gamma_T)$ ,  $\partial_t \psi, \partial_t \tilde{\psi} \in \mathring{C}_{s-1}^{1+\alpha}(\Gamma_T)$ ,  $\alpha \in (0, 1)$ ,  $s \in (1, 2 + \alpha]$ , and  $|\rho_0|_{s+1, \Gamma_T}^{(3+\alpha)} \leq M_1$ ,  $|\psi|_{s, \Gamma_T}^{(2+\alpha)} + |\partial_t \psi|_{s-1, \Gamma_T}^{(1+\alpha)} \leq M$ ,  $|\tilde{\psi}|_{s-1, \Gamma_T}^{(2+\alpha)} + |\partial_t \tilde{\psi}|_{s-1, \Gamma_T}^{(1+\alpha)} \leq M$ .*

*Then there is  $t_3 > 0$  such that for  $t \leq t_3$  the following estimates hold:*

$$\|J^{-1} - \tilde{J}^{-1}\|_{s-2, \Gamma_t}^{(\alpha)} \leq C t |\partial_t \psi - \partial_t \tilde{\psi}|_{s-1, \Gamma_t}^{(1+\alpha)}, \quad (\text{A.58})$$

$$\|J^{-1} - \tilde{J}^{-1}\|_{s-1, \Gamma_t}^{(1+\alpha)} \leq C (|\psi - \tilde{\psi}|_{s, \Gamma_t}^{(2+\alpha)} + |\partial_t \psi - \partial_t \tilde{\psi}|_{s-1, \Gamma_t}^{(1+\alpha)}). \quad (\text{A.59})$$

*Proof:* We compare the difference  $J^{-1} - \tilde{J}^{-1}$  of the inverse Jacobian matrix  $J^{-1}$ , which we take in the form (A.3)

$$J^{-1} - \tilde{J}^{-1} = -(J_1 - \tilde{J}_1) + (J_{01} + J_1)^2 J^{-1} - (J_{01} + \tilde{J}_1)^2 \tilde{J}^{-1},$$

where  $J_1 - \tilde{J}_1 = \{\partial_{y_j}(N_i \chi(\psi - \tilde{\psi}))\}_{1 \leq i, j \leq n}$ .

After some computations we shall have

$$\begin{aligned} J^{-1} - \tilde{J}^{-1} &= (I - (J_{01} + \tilde{J}_1)^2)^{-1} [-(J_1 - \tilde{J}_1) + (J_{01} + \tilde{J}_1)(J_1 - \tilde{J}_1) J^{-1} \\ &\quad + (J_1 - \tilde{J}_1)(J_{01} + J_1) J^{-1}], \end{aligned} \quad (\text{A.60})$$

where the matrices  $(I - (J_{01} + \tilde{J}_1)^2)^{-1}$ ,  $J^{-1}$  exist (see Theorem A.1). We evaluate the norms of the matrices in the right-hand side of the identity (A.60) with the help of the estimates (A.12) for the product of the functions, (A.23), (A.24), (A.42), (A.43), (A.36), (A.37) for the matrices  $J_1$ ;  $J_{01} J_1$ ;  $J_1^2$  respectively and (A.52), (A.44), then we find

$$\begin{aligned} \|J^{-1} - \tilde{J}^{-1}\|_{s-2, \Gamma_t}^{(\alpha)} &\leq t (C + \mu_5(t)) |\partial_t \psi - \partial_t \tilde{\psi}|_{s-1, \Gamma_t}^{(1+\alpha)}, \\ \|J^{-1} - \tilde{J}^{-1}\|_{s-1, \Gamma_t}^{(1+\alpha)} &\leq (C + \mu_6(t)) (|\psi - \tilde{\psi}|_{s, \Gamma_t}^{(2+\alpha)} + |\partial_t \psi - \partial_t \tilde{\psi}|_{s-1, \Gamma_t}^{(1+\alpha)}), \end{aligned} \quad (\text{A.61})$$

where

$$\mu_{5+\nu}(t) = C t^{\frac{s-1}{2}} \left( M t^{\frac{1-\nu}{2}} + M_1 \begin{cases} t^{\frac{1}{2}}, & 1 < s < 2 \\ t^{\frac{3-s}{2}}, & 2 \leq s \leq 2 + \alpha \end{cases} (1+M) \right), \quad \nu = 0, 1.$$

We choose  $t_4 > 0$  from the inequalities

$$\mu_5(t) \leq 1, \quad \mu_6(t) \leq 1,$$

and then from (A.61) we obtain the estimates (A.58), (A.59) for  $t \leq t_3 = \min(t_1, t_4)$ .  $\blacksquare$

## Appendix B. The model transmission problem

Let  $\mathcal{S}_1 = \mathbb{R}_-^n$  and  $\mathcal{S}_2 = \mathbb{R}_+^n$  be half-spaces  $x_n < 0$  and  $x_n > 0$  in  $\mathbb{R}^n$  respectively,  $\mathcal{S}_{jT} = \mathcal{S}_j \times (0, T)$ ;  $R$  a plane  $x_n = 0$ ,  $R_T = R \times [0, T]$ .

In Section 4 we study the linear problem written in the form (4.1)<sub>1</sub>, (4.12), (4.2)<sub>1</sub>, (4.13), (4.3)–(4.7). The proof of the solvability of this problem is based on the following model transmission problem for the unknown functions  $u_j(x, t)$ ,  $c_j(x, t)$ ,  $j = 1, 2$ ,  $\psi(x', t)$  satisfying zero initial data:

$$\partial_t u_j - a_j \Delta u_j - \alpha_1 (\partial_t \psi - a_j \Delta' \psi) = f_j(x, t) \quad \text{in } \mathcal{S}_{jT}, \quad j = 1, 2, \quad (\text{B.1})$$

$$\partial_t c_1 - b_1 \Delta c_1 - \beta_1 (\partial_t \psi - b_1 \Delta' \psi) = g_1(x, t) \quad \text{in } \mathcal{S}_{1T}, \quad (\text{B.2})$$

$$\partial_t c_2 - b_2 \Delta c_2 - [\beta_2 + \gamma_2 (\alpha_1 - \alpha_2)] (\partial_t \psi - b_2 \Delta' \psi) = g_2(x, t) \quad \text{in } \mathcal{S}_{2T}, \quad (\text{B.3})$$

$$(u_1 - u_2)|_{x_n=0} = \eta_0(x', t), \quad (\text{B.4})$$

$$(c_j - \gamma_j u_j)|_{x_n=0} = \eta_j(x', t), \quad j = 1, 2, \quad (\text{B.5})$$

$$(\lambda_1 \partial_{x_n} u_1 - \lambda_2 \partial_{x_n} u_2)|_{x_n=0} = \varphi_1(x', t), \quad (\text{B.6})$$

$$(k_1 \partial_{x_n} c_1 - k_2 \partial_{x_n} c_2)|_{x_n=0} + d' \nabla' \psi - \kappa_1 \partial_t \psi = \varphi_2(x', t), \quad t \in (0, T), \quad (\text{B.7})$$

where all coefficients  $\kappa_1$ ,  $a_j$ ,  $b_j$ ,  $\gamma_j$ ,  $\lambda_j$ ,  $k_j$ ,  $j = 1, 2$ , are positive constants and  $d' = (d_1, \dots, d_{n-1})$ .

**Theorem B.1.** *Let  $\alpha \in (0, 1)$ ,  $s \in (1, 2 + \alpha]$ . We assume*

$$\kappa_1 > 0, \quad \mu_j = \beta_j - \alpha_j \gamma_j > 0, \quad j = 1, 2. \quad (\text{B.8})$$

*Then for every functions  $f_j, g_j \in \mathring{C}_{s-2}^\alpha(\mathcal{S}_{jT})$ ,  $\eta_0, \eta_j \in \mathring{C}_s^{2+\alpha}(R_T)$ ,  $\varphi_j \in \mathring{C}_s^{1+\alpha}(R_T)$ ,  $j = 1, 2$ , the problem (B.1)–(B.7) has a unique solution  $u_j, c_j \in \mathring{C}_s^{2+\alpha}(\mathcal{S}_{jT})$ ,  $\psi \in \mathring{C}_s^{2+\alpha}(R_T)$ ,  $\partial_t \psi \in \mathring{C}_{s-1}^{1+\alpha}(R_T)$  and it satisfies the estimate*



$$\begin{aligned}
 & \sum_{j=1} (|u_j|_{s, \mathcal{S}_{jT}}^{(2+\alpha)} + |c_j|_{s, \mathcal{S}_{jT}}^{(2+\alpha)}) + |\psi|_{s, R_T}^{(2+\alpha)} + |\partial_t \psi|_{s-1, R_T}^{(1+\alpha)} \leq \\
 & \leq C_1 \left( \sum_{j=1}^2 (|f_j|_{s-2, \mathcal{S}_{jT}}^{(\alpha)} + |g_j|_{s-2, \mathcal{S}_{jT}}^{(\alpha)} + |\eta_j|_{s, R_T}^{(2+\alpha)} + |\varphi_j|_{s-1, R_T}^{(1+\alpha)}) + |\eta_0|_{s, R_T}^{(2+\alpha)} \right). \quad (\text{B.9})
 \end{aligned}$$

*Proof:* In the equations (B.1) and conditions (B.4), (B.6) we make the change

$$u_j = v_j + \alpha_1 \psi, \quad j = 1, 2, \quad (\text{B.10})$$

and then we obtain the problem for the new unknown functions  $v_1, v_2$

$$\begin{aligned}
 & \partial_t v_1 - a_1 \Delta v_1 = f_1 \quad \text{in } \mathcal{D}_{1T}, \\
 & \partial_t v_2 - a_2 \Delta v_2 = f_2 \quad \text{in } \mathcal{D}_{2T}, \\
 & (v_1 - v_2)|_{x_n=0} = \eta_0, \\
 & (\lambda_1 \partial_{x_n} v_1 - \lambda_2 \partial_{x_n} v_2)|_{x_n=0} = \varphi_1.
 \end{aligned} \quad (\text{B.11})$$

We note that (B.11) is the transmission (or conjunction) problem [B1], its solution  $v_j$  belongs to  $\mathring{C}_s^{2+\alpha}(\mathcal{S}_{jT})$ ,  $j = 1, 2$ , and satisfies the estimate

$$\sum_{j=1}^2 |v_j|_{s, \mathcal{S}_{jT}}^{(2+\alpha)} \leq C \left( \sum_{j=1}^2 |f_j|_{s-2, \mathcal{S}_{jT}}^{(2+\alpha)} + |\eta_0|_{s, R_T}^{(2+\alpha)} + |\varphi_1|_{s-1, R_T}^{(1+\alpha)} \right). \quad (\text{B.12})$$

We have represented the equations (4.1)<sub>2</sub>, (4.2)<sub>2</sub> in the form (4.12), (4.13) to obtain problem (B.11) separated from other unknown functions.

We construct auxiliary functions  $Z_1, Z_2$  as the solutions of the first boundary value problems in half-space  $\mathcal{S}_j$

$$\begin{aligned}
 & \partial_t Z_j - b_j \Delta Z_j = g_j \quad \text{in } \mathcal{S}_{jT}, \\
 & Z_j|_{t=0} = 0, \quad Z_j|_{x_n=0} = (\eta_j + \gamma_j v_j)|_{x_n=0},
 \end{aligned} \quad (\text{B.13})$$

where  $j = 1, 2$  and known functions  $v_j$ . Each one of the problems (B.13) has a unique solution  $V_j \in \mathring{C}_s^{2+\alpha}(\mathcal{S}_{jT})$  [S1, LSU] and the following estimate holds for  $j = 1, 2$ :

$$|Z_j|_{s, \mathcal{S}_{jT}}^{(2+\alpha)} \leq C \left( |g_j|_{s-2, \mathcal{S}_{jT}}^{(\alpha)} + |\eta_j|_{s-1, R_T}^{(1+\alpha)} + \sum_{k=1}^2 |f_k|_{s-2, \mathcal{S}_{kT}}^{(\alpha)} + |\eta_0|_{s, R_T}^{(2+\alpha)} + |\varphi_1|_{s-1, R_T}^{(1+\alpha)} \right). \quad (\text{B.14})$$

After the substitution (B.10) and the following ones

$$\begin{aligned} c_1 &= z_1 + Z_1 + \beta_1 \psi, \\ c_2 &= z_2 + Z_2 + (\beta_2 + \gamma_2(\alpha_1 - \alpha_2))\psi \end{aligned} \quad (\text{B.15})$$

in the equations (B.2), (B.3) and conditions (B.5), (B.7) we obtain the problem for the functions  $z_1, z_2, \psi$  with time derivative in the transmission condition

$$\partial_t z_1 - b_1 \Delta z_1 = 0 \quad \text{in } \mathcal{D}_{1T}, \quad (\text{B.16})$$

$$\partial_t z_2 - b_2 \Delta z_2 = 0 \quad \text{in } \mathcal{D}_{2T}, \quad (\text{B.17})$$

$$z_j|_{x_n=0} + \mu_j \psi = 0, \quad j = 1, 2, \quad (\text{B.18})$$

$$(k_1 \partial_{x_n} z_1 - k_2 \partial_{x_n} z_2)|_{x_n=0} + d' \nabla'^T \psi - \kappa_1 \partial_t \psi = \varphi(x', t), \quad (\text{B.19})$$

where  $\mu_j = \beta_j - \gamma_j \alpha_j > 0, j = 1, 2, \varphi = \varphi_2 - (k_1 \partial_{x_n} Z_1 - k_2 \partial_{x_n} Z_2)|_{x_n=0} \in \mathring{C}_{s-1}^{1+\alpha}(R_T)$ ,

$$\begin{aligned} |\varphi|_{s-1, R_T}^{(1+\alpha)} &\leq |\varphi_2|_{s-1, R_T}^{(1+\alpha)} + C \left( \sum_{j=1}^2 (|f_j|_{s-2, \mathcal{S}_{jT}}^{(\alpha)} + |g_j|_{s-2, \mathcal{S}_{jT}}^{(\alpha)} + |\eta_j|_{s-1, R_T}^{(1+\alpha)}) \right. \\ &\quad \left. + |\varphi_0|_{s, R_T}^{(2+\alpha)} + |\varphi_1|_{s-1, R_T}^{(1+\alpha)} \right). \end{aligned} \quad (\text{B.20})$$

We apply Fourier transform on  $x'$  and Laplace transforms on  $t$  to the problem (B.16)–(B.19)

$$\tilde{v}(\xi', x_n, p) = \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_0^\infty e^{-pt} dt \int_{\mathbb{R}^{n-1}} v(x, t) e^{-ix'\xi'} dx'.$$

Then from the heat equations of the problem, we find the solution in the images of Fourier and Laplace transforms

$$\tilde{z}_1 = B_1 e^{r_1 x_n}, \quad x_n < 0, \quad \tilde{z}_2 = B_2 e^{-r_2 x_n}, \quad x_n > 0, \quad (\text{B.21})$$

where  $r_j = \frac{1}{b_j}(p + b_j \xi'^2)$ ,  $B_j = B_j(\xi', p)$ ,  $j = 1, 2$ , are unknown functions. From the conditions on the plane  $x_n = 0$

$$B_j + \mu_j \tilde{\psi} = 0, \quad j = 1, 2,$$

$$k_1 r_1 B_1 + k_2 r_2 B_2 - (\kappa_1 p - i d' \xi') \tilde{\psi} = \tilde{\varphi}$$

we find  $B_j, \tilde{\psi}$

$$B_j = \frac{\mu_j}{\kappa_1} \frac{1}{\zeta} \tilde{\varphi}, \quad j = 1, 2, \quad \tilde{\psi} = -\frac{1}{\kappa_1 \zeta} \tilde{\varphi},$$

where

$$\zeta = p + \frac{k_1 \mu_1}{\kappa_1} r_1 + \frac{k_2 \mu_2}{\kappa_1} r_2 - \frac{i}{\kappa_1} d' \xi',$$

$\operatorname{Re} \zeta > 0$  by conditions  $k_j > 0, \mu_j > 0, \kappa_1 > 0$ . So we can represent the fraction  $\frac{1}{\zeta}$  in the form

$$\frac{1}{\zeta} = \int_0^\infty e^{-\zeta \sigma} d\sigma.$$

Substituting the functions  $B_j$  into formulas (B.21) and applying this expression of  $\frac{1}{\zeta}$  we shall have

$$\tilde{z}_j = \frac{\mu_j}{\kappa_1} \tilde{\varphi} \int_0^\infty e^{-\zeta \sigma - r_j |x_n|} d\sigma,$$

$$\tilde{\psi} = -\frac{1}{\kappa_1} \tilde{\varphi} \int_0^\infty e^{-\zeta \sigma} d\sigma.$$

With the help of the inverse Laplace and Fourier transforms [BE] we find  $z_j$  and  $\psi$  in the closed form

$$z_j(x, t) = \frac{\mu_j}{\kappa_1} \int_0^t d\tau \int_{\mathbb{R}^{n-1}} \varphi(y', \tau) G_j(x' - y', |x_n|, t - \tau) dy', \quad j = 1, 2, \quad (\text{B.22})$$

$$\psi(x', t) = -\frac{1}{\kappa_1} \int_0^t d\tau \int_{\mathbb{R}^{n-1}} \varphi(y', \tau) G_1(x' - y', 0, t - \tau) dy', \quad (\text{B.23})$$

where

$$\begin{aligned} G_1 &= 4b_1 b_2 \int_0^t d\sigma \int_0^{t-\sigma} d\tau_1 \int_{\mathbb{R}^{n-1}} \partial x_n \Gamma_1 \left( x' - \eta' + \frac{d'}{\kappa_1} \sigma, \frac{k_1 \mu_1 \sigma}{\kappa_1} - x_n, t - \sigma - \tau_1 \right) \\ &\quad \times \partial_{y_n} \Gamma_2 \left( \eta, \frac{k_2 \mu_2 \sigma}{\kappa_1} - \eta_n, \tau_1 \right) \Big|_{\eta_n=0} d\eta', \quad x_n < 0, \end{aligned}$$

$$\begin{aligned} G_2 &= 4b_1 b_2 \int_0^t d\sigma \int_0^{t-\sigma} d\tau_1 \int_{\mathbb{R}^{n-1}} \partial \eta_n \Gamma_1 \left( \eta + \frac{d'}{\kappa_1} \sigma, \frac{k_1 \mu_1 \sigma}{\kappa_1} + \eta_n, \tau_1 \right) \Big|_{\eta_n=0} \\ &\quad \times \partial_{x_n} \Gamma_2 \left( x' - \eta', \frac{k_2 \mu_2 \sigma}{\kappa_1} + x_n, t - \sigma - \tau_1 \right) d\eta', \quad x_n > 0. \end{aligned}$$

Here  $\Gamma_j(x, t)$  is the fundamental solution of the heat equation  $\partial_t v - b_j \Delta v = 0$ :

$$\Gamma_j(x, t) = \frac{1}{(2\sqrt{\pi b_j t})^n} e^{-\frac{x^2}{4b_j t}}.$$

As it was proved in [B4] the potentials (B.22) with the kernels  $G_j(x, t)$  belong to the space  $\mathring{C}_s^{2+\alpha}(\mathcal{S}_{jT})$  and satisfy the estimate

$$|z_j|_{s, \mathcal{S}_{jT}}^{(2+\alpha)} \leq C |\varphi|_{s-1, R_T}^{(1+\alpha)}. \quad (\text{B.24})$$

The function  $\psi(x', t)$  ((B.23)) as a trace of potential  $z_j$  on the plane  $x_n = 0$  also belongs to  $\mathring{C}_s^{2+\alpha}(R_T)$  and the estimate for it holds

$$|\psi|_{s, R_T}^{(2+\alpha)} \leq C |\varphi|_{s-1, R_T}^{(1+\alpha)}. \quad (\text{B.25})$$

From the boundary condition (B.19), by the above conclusions and estimates (B.24), (B.25), we have

$$\begin{aligned} \partial_t \psi &= \frac{1}{\kappa_1} (k_1 \partial_{x_n} z_1 - k_2 \partial_{x_n} z_2 + d' \nabla' \psi - \varphi) \in \mathring{C}_{s-1}^{1+\alpha}(R_T), \\ |\partial_t \psi| &\leq C \left( \sum_{j=1}^2 |z_j|_{s, \mathcal{D}_{jT}}^{(2+\alpha)} + |\psi|_{s, R_T}^{(2+\alpha)} + |\varphi|_{s-1, R_T}^{(1+\alpha)} \right) \leq C |\varphi|_{s-1, R_T}^{(1+\alpha)}. \end{aligned} \quad (\text{B.26})$$

Recalling the substitutions (B.10), (B.15) we obtain that the functions  $u_j = v_j + \alpha_1 \psi$ ,  $j = 1, 2$ ,  $c_1 = z_1 + Z_1 + \beta_1 \psi$  ( $j = 1$ ),  $c_2 = z_1 + Z_2 + (\beta_2 + \gamma_2(\alpha_1 - \alpha_2)) \psi$  ( $j = 2$ ) belong to the space  $\mathring{C}_s^{2+\alpha}(\mathcal{S}_{jT})$ ,  $j = 1, 2$ . Applying estimates (B.12), (B.24), (B.14), (B.25), (B.26) and (B.20) of the functions  $v_j$ ,  $z_j$ ,  $\psi$ ,  $\partial_t \psi$  and  $\varphi$  respectively leads to the required estimate (B.9).  $\blacksquare$

**Remark B.1.** *The conditions on the coefficients (B.8) correspond to the conditions (4.8), (4.9) of the Theorem 4.1 respectively.*

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