Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 06–52

ON THE ETINGOF-KAZHDAN QUANTIZATION OF NON-DEGENERATE TRIANGULAR LIE BIALGEBRAS

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ABSTRACT: Relatively to a Drinfeld associator Φ we present Etingof and Kazhdan quantization of any quasitriangular finite dimension Lie-bialgebra $(\mathfrak{a}, [,]_{\mathfrak{a}}, \varepsilon_{\mathfrak{a}} = d_c r_1)$ over a field $\mathbb{K}$ of characteristic zero. When $r_1 \in \mathfrak{a} \wedge \mathfrak{a}$ is non-degenerate we prove: a) replacing r_1 by $r_\hbar = r_1 + r_2 \cdot \hbar + \cdots \in \mathfrak{a} \wedge \mathfrak{a}[[\hbar]]$ we obtain a triangular Hopf-QUE-algebra $A_{\mathfrak{a}[[\hbar]],(\tilde{J}^{\Phi}_{r_\hbar})^{-1}}$ which is a quantization of the Lie-bialgebra $\mathfrak{a}$. We call $\tilde{J}^{\Phi}_{r_\hbar}$ an invariant star product on the Lie-bialgebra $\mathfrak{a}$. In particular and for example in the real case, $\tilde{J}^{\Phi}_{r_\hbar}$ defines an invariant deformation quantization on any Lie group G endowed with the invariant Poisson structure r_1 or the invariant symplectic structure $\beta_1 = r_1^{-1} \in \mathfrak{a}^* \otimes \mathfrak{a}^*$; b) given $r_\hbar = r_1 + r_2 \cdot \hbar + \cdots$ and $r'_\hbar = r_1 + r'_2 \cdot \hbar + \cdots \in \mathfrak{a} \wedge \mathfrak{a}[[\hbar]]$, the star products $\tilde{J}^{\Phi}_{r_\hbar}$ and $\tilde{J}^{\Phi}_{r'_\hbar}$ are (Hochschild) equivalent if and only if the invariant Chevalley (de Rham) cohomology classes of $\beta_\hbar$ and $\beta'_\hbar$ coincide. The classifying space of invariant star products on the Lie-bialgebra $\mathfrak{a}$, under equivalence, in the Etingof-Kazhdan quantization and in the previously constructed Drinfeld quantization of the Lie-bialgebra $\mathfrak{a}$, Campbell-Hausdorff group, coincide and equal $H^2(\mathfrak{a})[[\hbar]]$; c) if Φ and Φ' are two Drinfeld associators and F is an invariant star product on the Lie-bialgebra $\mathfrak{a}$ we have the following triangular-Hopf-QUE-algebras isomorphisms $A_{\mathfrak{a}[[\hbar]],F^{-1}} \simeq A_{\mathfrak{a}[[\hbar]],(\tilde{J}^{\Phi}_{r_\hbar})^{-1}} \simeq A_{A[[\hbar]],(\tilde{J}^{\Phi'}_{r'_\hbar})^{-1}}$ for some $r_\hbar$ and $r'_\hbar$ as before. Explicit proofs of the above assertions will be given in a forthcoming paper.

KEYWORDS: Quantum Groups, Quasi-Hopf algebras, Lie bialgebras, deformation algebras.

1. Some definitions

Let $\mathbb{K}$ be a field of characteristic 0 and $(\mathfrak{a}, [,]_{\mathfrak{a}})$ a Lie algebra over $\mathbb{K}$. Recall the following definitions.

A finite dimensional Lie bialgebra over $\mathbb{K}$ is a set $(\mathfrak{a}, [,]_{\mathfrak{a}}, \varepsilon_{\mathfrak{a}})$ where $\varepsilon_{\mathfrak{a}} : \mathfrak{a} \to \mathfrak{a} \otimes \mathfrak{a}$ is a 1-cocycle of $\mathfrak{a}$, with values in $\mathfrak{a} \otimes \mathfrak{a}$, with respect to the adjoint action of $\mathfrak{a}$ such that $\varepsilon^{\mathsf{t}}_{\mathfrak{a}} : \mathfrak{a}^* \otimes \mathfrak{a}^* \to \mathfrak{a}^*$ is a Lie bracket on $\mathfrak{a}^*$.

It is called quasitriangular if $\varepsilon_{\mathfrak{a}} = d_c r_1$, where d_c is the Chevalley-Eilenberg coboundary, $r_1 \in \mathfrak{a} \otimes \mathfrak{a}$ is a solution to CYBE ($[r_1, r_1] = 0$) and $(r_1)_{12} + (r_1)_{21}$ is $\mathrm{ad}_{\mathfrak{a}}$-invariant. In case r_1 is skew-symmetric, it is said to be a triangular

Received November 9, 2006.
The second author acknowledges the financial support POCI/MAT/58452/2004 and CMUC.

Lie bialgebra. Moreover if $det(r_1) \neq 0$ it is called a nondegenerate triangular Lie bialgebra.

From any Lie bialgebra $(\mathfrak{a}, [,]_{\mathfrak{a}}, \varepsilon_{\mathfrak{a}})$ we may obtain a Manin triple [5] $(\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^*, [,]_{\mathfrak{a}\oplus\mathfrak{a}^*}, \langle;\rangle_{\mathfrak{a}\oplus\mathfrak{a}^*})$, where $\langle(x;\xi);(y;\eta)\rangle_{\mathfrak{a}\oplus\mathfrak{a}^*} = \langle\xi;y\rangle + \langle\eta;x\rangle$ and

$$[(x;\xi),(y;\eta)]_{\mathfrak{a}\oplus\mathfrak{a}^*} = ([x,y]_{\mathfrak{a}} + ad^*_{\xi}y - ad^*_{\eta}x; [\xi,\eta]_{\mathfrak{a}^*} + ad^*_x\eta - ad^*_y\xi).$$

This Manin triple has a canonical quasitriangular Lie bialgebra structure defined by the (canonical) element $r = \sum_{i=1}^{n}(e_i;0)\otimes(0;e^i) \in (\mathfrak{a}\oplus\mathfrak{a}^*)\otimes(\mathfrak{a}\oplus\mathfrak{a}^*)$, where $(e_i, i = 1, \ldots, n)$ is a basis of $\mathfrak{a}$ and (e^i) its dual basis.

The set $(\mathfrak{a} \oplus \mathfrak{a}^*, [,]_{\mathfrak{a}\oplus\mathfrak{a}^*}, \varepsilon_{\mathfrak{a}\oplus\mathfrak{a}^*} = d_c r)$ is called the (quasitriangular Lie bialgebra) classical double of the Lie bialgebra $(\mathfrak{a}, [,]_{\mathfrak{a}}, \varepsilon_{\mathfrak{a}})$.

2. Etingof-Kazhdan quantization method

Etingof and Kazhdan (E-K) presented [9] a method to construct a quantization to, in particular, any finite dimensional quasitriangular Lie bialgebra. See also [19].

2.1. Quantization of the classical double.

2.1.1. *Drinfeld associator.*

Let T_n, $n > 1$, be the associative algebra with unit over a field $\mathbb{K}$ generated by the elements $\{t_{ij}, i \neq j, 1 \leq i, j \leq n\}$ with defining relations $t_{ij} = t_{ji}$, $[t_{ij}, t_{kl}] = 0$, if i, j, k, l are distinct and $[t_{ij}, t_{ik} + t_{jk}] = 0$.

Let $I_1, \ldots, I_n$ be disjoint subsets of $\{1, \ldots, m\}$.

Proposition 2.1. [10] *There exists a unique homomorphism $\tau_{I_1,\ldots,I_n} : T_n \longrightarrow T_m$ defined by $\tau_{I_1,\ldots,I_n}(t_{ij}) = \sum_{k\in I_i, l\in I_j} t_{kl}$.*

For any $X \in T_n$, we denote $\tau_{I_1,\ldots,I_n}(X)$ by $X_{I_1,\ldots,I_n}$. Let $T_n[[\hbar]]$ be the set of the formal power series in $\hbar$ with coefficients in the algebra T_n.

Definition 2.2. [8] *A* **Drinfeld associator** *is an element $\Phi \in T_3[[\hbar]]$ such that*

(i) $\Phi = 1 + O(\hbar^2)$;

(ii) $\Phi = e^{P(\hbar\, t_{12}, \hbar\, t_{23})}$, P is a Lie formal series with coefficients in $\mathbb{K}$ (we may take $\mathbb{K} = \mathbb{Q}$) ;

(iii) In $T_4[[\hbar]], \Phi$ satisfies the pentagon relation

$$\Phi_{1,2,34} \cdot \Phi_{12,3,4} = \Phi_{2,3,4} \cdot \Phi_{1,23,4} \cdot \Phi_{1,2,3};$$

(iv) If $B_{12} = e^{\frac{\hbar}{2}t_{12}} \in T_2[[\hbar]]$*, in* $T_3[[\hbar]]$*,* Φ *and* B *satisfy the hexagon relations*

$$B_{12,3} = \Phi_{3,1,2} \cdot B_{1,3} \cdot \Phi^{-1}_{1,3,2} \cdot B_{2,3} \cdot \Phi_{1,2,3}$$
$$B_{1,23} = \Phi^{-1}_{2,3,1} \cdot B_{1,3} \cdot \Phi_{2,1,3} \cdot B_{1,2} \cdot \Phi^{-1}_{1,2,3}.$$

2.1.2. *Drinfeld braided tensor category* $\mathcal{M}_{\mathfrak{g}}$*.*

Let $(\mathfrak{a}, [,]_{\mathfrak{a}}, \varepsilon_{\mathfrak{a}})$ be a Lie bialgebra and let $(\mathfrak{g} \equiv \mathfrak{a} \oplus \mathfrak{a}^*, [,]_{\mathfrak{g}}, \varepsilon_{\mathfrak{g}} = d_c r)$ be its classical double. We use the identifications $\mathfrak{a} \equiv \mathfrak{g}_+$ and $\mathfrak{a}^* \equiv \mathfrak{g}_-$. We set

$$\Omega_{12} = r_{12} + r_{21} = \sum((e_i; 0) \otimes (0; e^i) + (0; e^i) \otimes (e_i; 0)).$$

Let $\mathcal{M}_{\mathfrak{g}}$ denote the category whose objects are $\mathfrak{g}$-modules and $\mathbf{Hom}_{\mathcal{M}_{\mathfrak{g}}}(U, V) = \mathbf{Hom}_{\mathfrak{g}}(U, V)[[\hbar]]$. This category has a tensor product $\otimes$. If $U, W \in \mathbf{Ob}_{\mathcal{M}_{\mathfrak{g}}}$, $U \otimes W$ is the usual tensor product of $\mathfrak{g}$-modules.

Consider the algebra homomorphism $\Theta : T_n \longrightarrow \mathcal{U}\mathfrak{g}^{\otimes^n}$ defined through $\Theta(t_{ij}) = \Omega_{ij}$. The element $\Theta(t_{12}) = \Omega_{12} \in \mathcal{U}\mathfrak{g} \otimes \mathcal{U}\mathfrak{g}$ defines a $\mathfrak{g}$-module morphism $\Omega_{UV} : U \otimes V \longrightarrow U \otimes V$, because Ω is $ad_{\mathfrak{g}}$-invariant. The element $\Phi = \Theta(\Phi) \in \mathcal{U}\mathfrak{g}[[\hbar]]^{\otimes^3}$ is also $ad_{\mathfrak{g}}$-invariant, so it defines an element $\Phi_{VUW} \in \mathbf{Hom}_{\mathcal{M}_{\mathfrak{g}}}((V \otimes U) \otimes W, V \otimes (U \otimes W))$ for any $V, U, W \in \mathbf{Ob}_{\mathcal{M}_{\mathfrak{g}}}$.

For any $V, U \in \mathbf{Ob}_{\mathcal{M}_{\mathfrak{g}}}$ introduce the isomorphism $\beta_{VU} : V \otimes U \longrightarrow U \otimes V$ where $\beta = \sigma R^0 = \sigma \circ e^{\hbar\Omega/2}$ and σ is the usual permutation.

It follows from pentagon and hexagon relations on the associator that the morphisms Φ_{VUW} and β_{VU} define the structure of a braided monoidal category on $\mathcal{M}_{\mathfrak{g}}$ ([7, 11]).

Using the elements $\Phi \in \mathcal{U}\mathfrak{g}[[\hbar]]^{\otimes^3}$ and $R^0 = e^{\hbar\Omega/2} \in \mathcal{U}\mathfrak{g}[[\hbar]]^{\otimes^2}$ we may define (see [8]) a structure of quasi-triangular quasi-Hopf QUE algebra

$$(\mathcal{U}\mathfrak{g}[[\hbar]], \cdot, 1, \Delta_0, \epsilon_0, \Phi, S_0, \alpha = c^{-1}, \beta = 1, R^0 = e^{\frac{\hbar}{2}\Omega}) \qquad (1)$$

where $\Delta_0, \epsilon_0, S_0$, are the extensions of the corresponding maps in the universal enveloping algebra $\mathcal{U}\mathfrak{g}$, $c = \sum X_i \cdot S_0(Y_i) \cdot Z_i$ with $\Phi = \sum X_i \otimes Y_i \otimes Z_i$, see [7] prop.1.3.

2.1.3. *The functor* $\mathcal{F} : \mathcal{M}_{\mathfrak{g}} \longrightarrow \mathcal{A}$*.*

Let $\mathcal{A}$ denote the symmetric monoidal category (tensor category) [11] of topologically free $\mathbb{K}[[\hbar]]$-modules, with trivial associativity and commutativity constraints.

Let $\mathcal{F} : \mathcal{M}_{\mathfrak{g}} \longrightarrow \mathcal{A}$ be the functor given by $\mathcal{F}(V) = \mathbf{Hom}_{\mathcal{M}_{\mathfrak{g}}}(\mathcal{U}\mathfrak{g}, V)$, for $V \in \mathbf{Ob}_{\mathcal{M}_{\mathfrak{g}}}$ and if $V, U \in \mathbf{Ob}_{\mathcal{M}_{\mathfrak{g}}}$ the map $\mathcal{F} : \mathbf{Hom}_{\mathcal{M}_{\mathfrak{g}}}(V, U) \longrightarrow$

$\mathbf{Hom}_{\mathcal{A}}(\mathcal{F}(V), \mathcal{F}(U))$ is defined as $\mathcal{F}(f)(g) = f \circ g$, for $f \in \mathbf{Hom}_{\mathcal{M}_\mathfrak{g}}(V, U)$ and $g \in \mathcal{F}(V)$.

There is also a (forgetful) functor $\mathcal{G} : \mathcal{M}_\mathfrak{g} \longrightarrow \mathcal{A}$ defined in objects by $\mathcal{G}(V) = V[[\hbar]]$ and if $f \in \mathbf{Hom}_{\mathcal{M}_\mathfrak{g}}(V, W)$ then $\mathcal{G}(f) \equiv f$.

The map $\delta : \mathcal{F} \longrightarrow \mathcal{G}$ defined, for each $V \in \mathbf{Ob}_{\mathcal{M}_\mathfrak{g}}$ and $f \in \mathcal{F}(V)$, by $\delta_V(f) = f(1)$ defines a natural isomorphism from the functor $\mathcal{F}$ to $\mathcal{G}$.

2.1.4. $\mathfrak{g}$-*modules* M_+ *and* M_-.

By the Poincaré-Birkhoff-Witt theorem, the product in $\mathcal{U}\mathfrak{g} = \mathcal{U}(\mathfrak{g}_+ \oplus \mathfrak{g}_-)$ defines linear isomorphisms $\mathcal{U}\mathfrak{g}_+ \otimes \mathcal{U}\mathfrak{g}_- \longrightarrow \mathcal{U}\mathfrak{g}$ and $\mathcal{U}\mathfrak{g}_- \otimes \mathcal{U}\mathfrak{g}_+ \longrightarrow \mathcal{U}\mathfrak{g}$.

Introduce the following two induced representations, objects of $\mathcal{M}_\mathfrak{g}$

$$
\begin{aligned}
M_+ &= \mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{g}_+} W_+ = \mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{g}_+} e_+ = \mathcal{U}\mathfrak{g}_-(1 \otimes_{\mathbb{K}} e_+) = \mathcal{U}\mathfrak{g}_- 1_+ \\
M_- &= \mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{g}_-} W_- = \cdots = \mathcal{U}\mathfrak{g}_+(1 \otimes_{\mathbb{K}} e_-) = \mathcal{U}\mathfrak{g}_+ 1_-
\end{aligned}
$$

where $W_\pm$ is the trivial $\mathfrak{g}_\pm$-module and $1_\pm \otimes 1_\pm$ is $\mathfrak{g}_\pm$-invariant.

Lemma 2.3. *There exist* $\mathfrak{g}$-*module morphisms* $i_\pm : M_\pm \longrightarrow M_\pm \otimes M_\pm$ *such that* $i_\pm(1_\pm) = 1_\pm \otimes 1_\pm$.

Proposition 2.4. [9] *The assignment* $1 \in \mathcal{U}\mathfrak{g} \longrightarrow 1_+ \otimes 1_-$ *extends to an isomorphism of* $\mathfrak{g}$-*modules* $\phi : \mathcal{U}\mathfrak{g} \longrightarrow M_+ \otimes M_-$.

Proof: If $\phi(1) = 1_+ \otimes 1_-$ and ϕ is a $\mathfrak{g}$-module morphism, the construction of ϕ is unique. It is clear that ϕ preserves the standard filtration, then it defines a map *grad* (ϕ) on the associated graded objects. This map *grad* (ϕ) is bijective and then ([3] chapter III, §2, n8, corollary of theorem 1) ϕ is bijective. ∎

2.1.5. *A tensor structure on* $\mathcal{F}$.

Definition 2.5. [11] *A tensor structure on the functor* $\mathcal{F}$ *is a natural isomorphism of functors* $J : \mathcal{F}(\cdot) \otimes \mathcal{F}(\cdot) \longrightarrow \mathcal{F}(\cdot \otimes \cdot)$ *such that* $J_{VW} : \mathcal{F}(V) \otimes \mathcal{F}(W) \longrightarrow \mathcal{F}(V \otimes W)$ *satisfies*

$$
\begin{aligned}
&\mathcal{F}(\Phi_{VWU}) \circ J_{V\otimes W, U} \circ (J_{VW} \otimes 1) = J_{V, W\otimes U} \circ (1 \otimes J_{WU}),\ V, W, U \in \mathbf{Ob}_{\mathcal{M}_\mathfrak{g}} \\
&J_{VI} = J_{IV} = 1, \quad \mathcal{F}(I_{\mathcal{M}_\mathfrak{g}}) = \mathbb{K}[[\hbar]] = I_{\mathcal{A}}, V \in \mathbf{Ob}_{\mathcal{M}_\mathfrak{g}}.
\end{aligned}
$$

A functor equipped with a tensor structure is called a tensor functor.

Etingof and Kazdhan [9, 12] define the following map, for $V, W \in \mathbf{Ob}_{\mathcal{M}_\mathfrak{g}}$ and $v \in \mathcal{F}(V)$, $w \in \mathcal{F}(W)$, let $J_{VW}(v \otimes w)$ be the composition

$$\begin{aligned} J_{VW}(v\otimes w) =&(v \otimes w) \circ (\phi^{-1} \otimes \phi^{-1}) \circ \Phi^{-1}_{1,2,34} \circ (1 \otimes \Phi_{2,3,4}) \circ \\ & \circ (1 \otimes (\sigma \circ e^{\frac{\hbar}{2}\Omega}) \otimes 1) \circ (1 \otimes \Phi^{-1}_{2,3,4}) \circ \Phi_{1,2,34} \circ (i_+ \otimes i_-) \circ \phi. \end{aligned}$$

Proposition 2.6. [9] *The maps J_{VW} define a tensor structure on $\mathcal{F}$.*

2.2. The quasitriangular Hopf algebra $(End(\mathcal{F}), \cdot, id, \hat{\Delta}, \hat{\epsilon}, \hat{S}, \hat{R})$**.**

We may adapt the general reconstruction theorem in [13] (Chapter 9) to obtain [9] a quasitriangular Hopf algebra $(End(\mathcal{F}), \cdot, id, \hat{\Delta}, \hat{\epsilon}, \hat{S}, \hat{R})$, where $End(\mathcal{F})$ is the set of natural transformations from the functor $\mathcal{F}$ to itself.

To define a comultiplication $\hat{\Delta} : End(\mathcal{F}) \longrightarrow End(\mathcal{F}) \otimes End(\mathcal{F})$, we introduce first a functor $\mathcal{F}^2 : \mathcal{M}_\mathfrak{g} \times \mathcal{M}_\mathfrak{g} \longrightarrow \mathcal{A}$ defined by $\mathcal{F}^2(V, W) = \mathcal{F}(V)\hat{\otimes}\mathcal{F}(W)$, $(\mathcal{F}^2 = \otimes(\mathcal{F} \times \mathcal{F}))$.

Consider the map $\hat{\Delta} : End(\mathcal{F}) \longrightarrow End(\mathcal{F})\hat{\otimes}End(\mathcal{F}) = End(\mathcal{F}^2)$ where, if $a \in End(\mathcal{F})$, for any $V, W \in \mathbf{Ob}_{\mathcal{M}_\mathfrak{g}}$.

$$(\hat{\Delta}(a))_{V,W} = J^{-1}_{V,W} \circ a_{V\otimes W} \circ J_{V,W}. \tag{2}$$

Lemma 2.7. *If $a \in End(\mathcal{F})$, then $\hat{\Delta}(a)$ is a natural transformation from $\mathcal{F}^2$ to $\mathcal{F}^2$, i.e., $\hat{\Delta}(a) \in End(\mathcal{F}^2)$.*

Consider now the element $\hat{R} \in End(\mathcal{F})\hat{\otimes}End(\mathcal{F}) = End(\mathcal{F}^2)$ such that

$$\hat{R}_{V,W} = \sigma\circ J^{-1}_{W,V}\circ\mathcal{F}(\sigma\circ e^{\frac{\hbar}{2}\Omega_{V,W}})\circ J_{V,W} = \sigma\circ J^{-1}_{W,V}\circ\mathcal{F}(\beta_{V,W})\circ J_{V,W},\ V, W \in \mathbf{Ob}_{\mathcal{M}_\mathfrak{g}}. \tag{3}$$

Lemma 2.8. *$\hat{R}$ is a natural transformation from $\mathcal{F}^2$ to $\mathcal{F}^2$, i.e., $\hat{R} \in End(\mathcal{F}^2)$.*

Finally, let us consider the map $\hat{\epsilon} : End(\mathcal{F}) \longrightarrow \mathbb{K}[[\hbar]]$ defined as

$$\hat{\epsilon}(a) = \delta_{\mathrm{K}}\left(a_{\mathrm{K}}(\delta^{-1}_{\mathrm{K}}(1))\right) \tag{4}$$

for $a \in End(\mathcal{F})$, where δ is the natural isomorphism defined before from $\mathcal{F}$ to the forgetful functor on $\mathcal{M}_\mathfrak{g}$.

Remark 2.9. *If H is a quasi-Hopf algebra, over a commutative ring k, with antipode S, then the category of left H-modules of finite dimension is rigid*

([7, 13]). The left dual of the H-module V is $V^* = Hom_k(V,k)$ with

$$(h \cdot f)(v) = f((Sh)v), \quad v \in V, f \in V^*, h \in H.$$

and maps $ev(f \otimes v) = f(\alpha \cdot v), coev = \sum_i \beta \cdot e_i \otimes e^i$, where α, β are elements of the definition of quasi-Hopf algebra, $\{e_i\}$ is a basis of V and $\{e^i\}$ its dual basis.

Consider now the category of left H-modules (not necessarily of finite dimension), for each object V, let us call dual object of V to the element $V^* = Hom_k(V,k)$ which is also a H-module, considering the action already defined. Thus, we have a map between objects of this category. If f is a morphism of H-modules from V to W we define $f^{\mathfrak{t}} : W^* \longrightarrow V^*$ as

$$f^{\mathfrak{t}}(w^*)(v) = w^*(f(v)), \quad v \in V, w^* \in W^*.$$

It is easy to prove that $f^{\mathfrak{t}}$ is also a module morphism and $(f \circ g)^{\mathfrak{t}} = g^{\mathfrak{t}} \circ f^{\mathfrak{t}}$ for any morphisms f and g of H-modules. Then we conclude that $*$ is a contravariant functor from the category of H-modules (not necessarily of finite dimension) to itself.

We can also define a H-module morphism, for any module V,

$$\begin{aligned} \tau_V &: V^* \otimes V \longrightarrow k \\ \tau_V(v^* \otimes v) &= v^*(\alpha \cdot v), \quad v^* \in V^*, v \in V. \end{aligned} \tag{5}$$

We remark that τ_V coincides with the map ev_V defined above in case of finite dimension.

If $\mathcal{D}$ is a rigid tensor category, V an object of $\mathcal{D}$ and F is a tensor functor, then it is immediate that

$$F(V)^{*'} = F(V^*) \quad ev'_{F(V)} = F(ev_V) \circ J_{V^*,V} \quad coev'_{F(V)} = J^{-1}_{V,V^*} \circ F(coev_V)$$

are a left dual for $F(V)$. Hence according to the uniqueness of duals up to isomorphism, we have induced an isomorphism $d_V : F(V^*) \longrightarrow F(V)^*$ defined by

$$d_V = \left(ev'_{F(V)} \otimes id_{F(V)^*}\right) \circ \left(id_{F(V^*)} \otimes coev_{F(V)}\right), \tag{6}$$

with inverse $d_V^{-1} = \left(ev_{F(V)} \otimes id_{F(V^*)}\right) \circ \left(id_{F(V)^*} \otimes coev'_{F(V)}\right)$.

Consider the quasi-Hopf algebra (1), the functor $*$ is a functor from $\mathcal{M}_{\mathfrak{g}}$ to $\mathcal{M}_{\mathfrak{g}}$, and it exists also an analogous functor $*$ from $\mathcal{A}$ to $\mathcal{A}$ (generalization of the one in vector spaces, if V is a vector space and $V[[\hbar]]$ an object of $\mathcal{A}$, the dual object is $(V[[\hbar]])^*$. Then, $(\mathcal{F} \circ *)$ and $(* \circ \mathcal{F})$ are (contravariant) functors from the category $\mathcal{M}_\mathfrak{g}$ to the category $\mathcal{A}$.

For any $V \in \mathbf{Ob}_{\mathcal{M}_\mathfrak{g}}$, let us define $\xi_V : \mathcal{F}(V^*) \longrightarrow \mathcal{F}(V)^*$ as

$$\xi_V(v^*)(v) = \delta_{\mathbb{K}}(\mathcal{F}(\tau_V) \circ J_{V^*,V}(v^* \otimes v)), \quad v^* \in \mathcal{F}(V^*), v \in \mathcal{F}(V), \quad (7)$$

where δ is the natural isomorphism from $\mathcal{F}$ to the forgetful functor, and τ the map defined in (5). This map corresponds to the natural isomorphism defined by Etingof-Kazhdan in [9], where $\mathcal{F}(\mathbb{K})$ and $\mathbb{K}[[\hbar]]$ are identified.

Lemma 2.10. *ξ is a natural isomorphism from the functor $(\mathcal{F} \circ *)$ to the functor $(* \circ \mathcal{F})$.*

If $a \in End(\mathcal{F})$, let us define

$$\overline{S}(a)_V = (\xi_V^{\mathrm{t}})^{-1} \circ a_{V^*}^{\mathrm{t}} \circ \xi_V^{\mathrm{t}} \quad (8)$$

a morphism from $\mathcal{F}(V)^{**}$ to $\mathcal{F}(V)^{**}$.

Lemma 2.11. *The subspace $\mathcal{F}(V) \subset \mathcal{F}(V)^{**}$ is invariant by this morphism.*

Lemma 2.12. *For $a \in End(\mathcal{F})$, let us consider*

$$\hat{S}(a)_V = \overline{S}(a)_V \mid_{\mathcal{F}(V)} \quad (9)$$

for any $V \in \mathbf{Ob}_{\mathcal{M}_\mathfrak{g}}$. Then, $\hat{S}$ is a map $End(\mathcal{F}) \longrightarrow End(\mathcal{F})$.

We may, using the proof of this lemma, present a simpler expression to define the map $\hat{S}$. Let $a \in End(\mathcal{F})$, V an object of $\mathcal{M}_\mathfrak{g}$ and $u \in \mathcal{F}(V)$, then

$$\hat{S}(a)_V(u) = \delta_V^{-1}\left(\mu_V(Q^{-1} \cdot S_0(A) \cdot Q) \circ \delta_V(u)\right),$$

where $\mu_V : V[[\hbar]] \longrightarrow V[[\hbar]]$ is defined as $\mu_V(A)(x) = Ax$, $x \in V$ and $A = a_{\mathcal{U}\mathfrak{g}}(id_{\mathcal{U}\mathfrak{g}})(1)$.

Using the definition (8), we may see that the map $\hat{S}$ is an algebra anti-automorphism, that is, $\hat{S}(a \cdot b) = \hat{S}(b) \cdot \hat{S}(a), \quad a, b \in End(\mathcal{F})$.

In the particular case of considering only finite dimensional modules the map $\hat{S} : End(\mathcal{F}) \longrightarrow End(\mathcal{F})$ is defined, for any $a \in End(\mathcal{F})$, by

$$(\hat{S}(a)_V)^{\mathrm{t}} = d_V \circ a_{V^*} \circ d_V^{-1},$$

where d_V is the isomorphism that appears in (6), identifying $\mathcal{F}(\mathbb{K})$ and $\mathbb{K}[[\hbar]]$, and the transpose map is defined in the same section. For each $v^* \in \mathcal{F}(V^*)$, we have

$$\begin{aligned} d_V(v^*) &= (\mathcal{F}(ev_V) \circ J_{V^*,V} \otimes id_{\mathcal{F}(V)^*}) \circ (id_{\mathcal{F}(V^*)} \otimes coev_{\mathcal{F}(V)})(v^*) \\ &= (\mathcal{F}(ev_V) \circ J_{V^*,V})(v^* \otimes e_i) \otimes e^i, \quad e_i \in \mathcal{F}(V) \text{ basis }, e^i \in \mathcal{F}(V)^* \end{aligned}$$

(see chapter 9.4 in [13]). If $v \in \mathcal{F}(V)$, finally, we get

$$d_V(v^*)(v) = (\mathcal{F}(ev_V) \circ J_{V^*,V})(v^* \otimes v),$$

coinciding with the definition of ξ_V in [9]. Without using the above identification, the expression for $d_V(v^*)(v)$ would coincide with the one of $\xi_V(v^*)(v)$ defined in (7). Using the definition of transpose map and the expression of d_V, we have

$$\begin{aligned} \hat{S}(a)_V = (id_{\mathcal{F}(V)} \otimes (\mathcal{F}(ev_V) \circ J_{V^*,V})) \circ (id_{\mathcal{F}(V)} \otimes a_{V^*} \otimes id_{\mathcal{F}(V)}) \circ \\ \circ ((J^{-1}_{V,V^*} \circ \mathcal{F}(coev_V)) \otimes id_{\mathcal{F}(V)}). \end{aligned}$$

Theorem 2.13. *Relatively to the elements defined in (2), (3), (4) and (9), we have*

(a) $(\hat{\Delta} \otimes 1) \circ \hat{\Delta}(a) = (1 \otimes \hat{\Delta}) \circ \hat{\Delta}(a)$,
(b) $\hat{\Delta}(a \cdot b) = \hat{\Delta}(a) \cdot \hat{\Delta}(b), \quad a, b \in End(\mathcal{F})$.
(c) $(\hat{\Delta} \otimes 1)\hat{R} = \hat{R}_{13} \cdot \hat{R}_{23}, \quad (1 \otimes \hat{\Delta})\hat{R} = \hat{R}_{13} \cdot \hat{R}_{12}$,
(d) $\sigma(\hat{\Delta}(a)) \cdot \hat{R} = \hat{R} \cdot \hat{\Delta}(a)$,
(e) $(\hat{\epsilon} \otimes 1) \circ \hat{\Delta} = 1, \quad (1 \otimes \hat{\epsilon}) \circ \hat{\Delta} = 1$,
(f) $\cdot((\hat{S} \otimes 1) \circ \hat{\Delta}(a)) = id(\hat{\epsilon}(a)), \quad \cdot((1 \otimes \hat{S}) \circ \hat{\Delta}(a)) = id(\hat{\epsilon}(a))$.

Therefore, the set $\left(End(\mathcal{F}), \cdot, id, \hat{\Delta}, \hat{\epsilon}, \hat{S}, \hat{R}\right)$ *is a quasitriangular Hopf algebra over the ring* $\mathbb{K}[[\hbar]]$.

Theorem 2.14. [9] *The algebra* $End(\mathcal{F})$ *and the topological usual associative algebra* $\mathcal{U}\mathfrak{g}[[\hbar]]$ *are isomorphic.*

This isomorphism allows us to define on the algebra $\mathcal{U}\mathfrak{g}[[\hbar]]$ the pull-back of the quasitriangular Hopf algebra structure on $End(\mathcal{F})$: $(\mathcal{U}\mathfrak{g}[[\hbar]], \cdot, \Delta, S, R)$.

Theorem 2.15. [9] *The set* $(\mathcal{U}\mathfrak{g}[[\hbar]], \cdot, \Delta, S, R)$ *verifies*

$$\Delta(u) = J^{-1} \cdot \Delta_0(u) \cdot J, \quad S(u) = Q^{-1} \cdot S_0(u) \cdot Q, \quad R = \sigma J^{-1} \cdot e^{\frac{\hbar}{2}\Omega} \cdot J,$$

where $Q = \sum S_0(u_i) \cdot v_i$ *if* $J = \sum u_i \otimes v_i$, $u_i, v_i \in \mathcal{U}\mathfrak{g}[[\hbar]]$ *and*

$$J = (\phi^{-1} \otimes \phi^{-1}) \left(\Phi^{-1}_{1,2,34} \circ \Phi_{2,3,4} \circ \sigma_{23} \circ e^{\frac{\hbar}{2}\Omega_{23}} \circ \Phi^{-1}_{2,3,4} \circ \Phi_{1,2,34}(i_+ \otimes i_-)(\phi(1)) \right).$$

It is a quasitriangular Hopf QUE algebra and a quantization, [7, 9], *of the quasitriangular Lie bialgebra* $(\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^*, [,]_{\mathfrak{g}}, \varepsilon_{\mathfrak{g}} = d_c r)$ *where the topological algebra structure of* $\mathcal{U}\mathfrak{g}[[\hbar]]$ *is the usual one. It is obtained from the quasitriangular quasi-Hopf algebra (1) by a twist via* J^{-1}*, and we denote it as* $A_{\mathfrak{g}[[\hbar]],\Omega,J^{-1}}$.

2.3. Quantization of quasitriangular Lie bialgebras.

1)Let $(\mathfrak{a}, [,]_{\mathfrak{a}}, \varepsilon_{\mathfrak{a}} = d_c r_1)$ be a quasitriangular Lie bialgebra. Set

$$\mathfrak{g}_+ = \{(1 \otimes f) r_1 \mid \forall f \in (\mathcal{U}\mathfrak{a})^*\} \subseteq \mathcal{U}\mathfrak{a}, \quad \mathfrak{g}_- = \{(f \otimes 1) r_1 \mid \forall f \in (\mathcal{U}\mathfrak{a})^*\} \subseteq \mathcal{U}\mathfrak{a}$$

where $r_1 = r_1^{mn}\, e_m \otimes e_n$ with $\{e_m\}$ a basis of $\mathfrak{a}$. Then $(1 \otimes f)(r_1) = r_1^{mn} e_m f(e_n)$ with $f \in (\mathcal{U}\mathfrak{a})^*$. Because r_1 is a finite sum, the only elements for which $(1 \otimes f)(r_1)$ is not zero will be elements $f \in \mathfrak{a}^*$. Therefore, in this case, the above definitions are equivalent to

$$\mathfrak{g}_+ = \{(1 \otimes f) r_1 \mid \forall f \in \mathfrak{a}^*\} \subseteq \mathfrak{a}, \quad \mathfrak{g}_- = \{(f \otimes 1) r_1 \mid \forall f \in \mathfrak{a}^*\} \subseteq \mathfrak{a}.$$

Using a result from [18], $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are Lie subalgebras of $\mathfrak{a}$.

Let us define $\chi_{r_1} : \mathfrak{g}_+^* \to \mathfrak{g}_-$ by $\chi_{r_1}(g) = (g \otimes 1) r_1$. χ_{r_1} is a linear isomorphism. Using this isomorphism we may define (see [9]) a Lie algebra structure $[,]_{\overline{\mathfrak{g}}}$ on the vector space $\overline{\mathfrak{g}} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$, and also a $ad_{\overline{\mathfrak{g}}}$-invariant bilinear form $<;>_{\overline{\mathfrak{g}}}$. With these structures $(\overline{\mathfrak{g}} = \mathfrak{g}_+ \oplus \mathfrak{g}_-, [,]_{\overline{\mathfrak{g}}}, <;>_{\overline{\mathfrak{g}}})$ is a Manin triple. This Manin triple is the image by $(1 \oplus \chi_{r_1})$ of the classical double of the Lie bialgebra $(\mathfrak{g}_+, [,]_{\mathfrak{g}_+}, \varepsilon)$, $\mathfrak{g}_+ \subseteq \mathfrak{a}$, where ε is the transpose of $[,]_{\mathfrak{g}_+^*}$ (see also [19]).

Furthermore, the map $\pi : \mathfrak{g}_+ \oplus \mathfrak{g}_- \longrightarrow \mathfrak{a}$ defined as $\pi(x; y) = x + y$, for $x \in \mathfrak{g}_+$, $y \in \mathfrak{g}_-$ is a Lie algebra morphism.

Theorem 2.16. [9] *From* $\pi : \mathfrak{g}_+ \oplus \mathfrak{g}_- \longrightarrow \mathfrak{a}$*, let us define* $\tilde{\pi} : \mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_+^* \longrightarrow \mathfrak{a}$ *by* $\tilde{\pi} = \pi \circ (1 \oplus \chi_{r_1})$*. Then,* $\tilde{\pi}$ *is a Lie algebra homomorphism and*

$$(\tilde{\pi} \otimes \tilde{\pi})(r) = r_1.$$

Therefore, $\tilde{\pi}$ *can be uniquely extended to an associative algebra homomorphism* $\tilde{\pi} : \mathcal{U}\mathfrak{g} \longrightarrow \mathcal{U}\mathfrak{a}$.

2)Let $\mathcal{M}_{\mathfrak{a}}$ be the category whose objects are $\mathfrak{a}$-modules and $\mathbf{Hom}_{\mathcal{M}_{\mathfrak{a}}}(V,W) = \mathbf{Hom}_{\mathfrak{a}}(V,W)[[\hbar]]$. Let $\mathcal{M}_{\mathfrak{g}}$ be the Drinfeld category associated to $\mathfrak{g}$. We have the pullback functor $\tilde{\pi}^* : \mathcal{M}_{\mathfrak{a}} \longrightarrow \mathcal{M}_{\mathfrak{g}}$ defined by $\tilde{\pi}^*(V) = V$, as vector spaces, and if $v \in \tilde{\pi}^*(V)$, $x \cdot v = \tilde{\pi}(x) \cdot v$, $x \in \mathfrak{g}$. We have also $\tilde{\pi}^* : \mathbf{Hom}_{\mathcal{M}_{\mathfrak{a}}}(V,W) \longrightarrow \mathbf{Hom}_{\mathcal{M}_{\mathfrak{g}}}(\tilde{\pi}^*V, \tilde{\pi}^*W)$ defined by $(\tilde{\pi}^*f)(x\cdot v) = (\tilde{\pi}^*f)(\tilde{\pi}(x)\cdot v) = f(\tilde{\pi}(x)\cdot v) = \tilde{\pi}(x)\cdot f(v) = x \cdot (\tilde{\pi}^*f)(v)$ with $x \in \mathfrak{g}$, $v \in \tilde{\pi}^*V = V$.

We may define on $\mathcal{M}_{\mathfrak{a}}$ a braided monoidal structure, using the one of $\mathcal{M}_{\mathfrak{g}}$: the associativity constraint on $\mathcal{M}_{\mathfrak{a}}$ is $\tilde{\Phi} = (\tilde{\pi} \otimes \tilde{\pi} \otimes \tilde{\pi})\Phi \in \mathcal{U}\mathfrak{a}^{\otimes^3}[[\hbar]]$ where $\Phi \in \mathcal{U}\mathfrak{g}^{\otimes^3}[[\hbar]]$ is the one of $\mathcal{M}_{\mathfrak{g}}$ and the commutativity constraint on $\mathcal{M}_{\mathfrak{a}}$ is $\tilde{\beta} = (\tilde{\pi} \otimes \tilde{\pi})\beta \in \mathcal{U}\mathfrak{a}^{\otimes^2}[[\hbar]]$ where $\beta \in \mathcal{U}\mathfrak{g}^{\otimes^2}[[\hbar]]$ is the one of $\mathcal{M}_{\mathfrak{g}}$.

3)Let us consider the following functor $\tilde{\mathcal{F}} : \mathcal{M}_{\mathfrak{a}} \longrightarrow \mathcal{A}$ defined by $\tilde{\mathcal{F}}(V) = \mathbf{Hom}_{\mathcal{M}_{\mathfrak{g}}}(\mathcal{U}\mathfrak{g}, \tilde{\pi}^*V) = \mathcal{F}(\tilde{\pi}^*V)$ and if $f \in \mathbf{Hom}_{\mathcal{M}_{\mathfrak{a}}}(V,W)$, $\tilde{\mathcal{F}}(f) = \tilde{\pi}^*(f)$.

The tensor structure on $\tilde{\mathcal{F}}$ is introduced in the same way that the one on $\mathcal{F}$, that is to say, let $\tilde{J}_{VW} : \tilde{\mathcal{F}}(V) \otimes \tilde{\mathcal{F}}(W) \longrightarrow \tilde{\mathcal{F}}(V \otimes W)$ be such that, if $\tilde{v} \in \tilde{\mathcal{F}}(V)$, $\tilde{w} \in \tilde{\mathcal{F}}(W)$, then $\tilde{J}_{VW}(\tilde{v} \otimes \tilde{w}) : \mathcal{U}\mathfrak{g} \longrightarrow \tilde{\pi}^*(V \otimes W)$ is the composition

$$\begin{aligned}\tilde{J}_{VW}(\tilde{v} \otimes \tilde{w}) =& (\tilde{v} \otimes \tilde{w}) \circ (\phi^{-1} \otimes \phi^{-1}) \circ \Phi^{-1}_{1,2,34} \circ (1 \otimes \Phi_{2,3,4}) \circ \\ & \circ (1 \otimes (\sigma \circ e^{\frac{\hbar}{2}\Omega}) \otimes 1) \circ (1 \otimes \Phi^{-1}_{2,3,4}) \circ \Phi_{1,2,34} \circ (i_+ \otimes i_-) \circ \phi.\end{aligned}$$

Proposition 2.17. *$\tilde{J}$ is a tensor structure on the functor $\tilde{\mathcal{F}}$.*

The general reconstruction theorem [13] (chapter 9) can be considered in the present setting, obtaining in this way [9] a quasitriangular Hopf algebra $(End(\tilde{\mathcal{F}}), \cdot, \check{\Delta}, \check{S}, \check{R})$.

Proposition 2.18. *There exists an algebra isomorphism $\tilde{\Lambda} : \mathcal{U}\mathfrak{a}[[\hbar]] \longrightarrow End(\tilde{\mathcal{F}})$, between the topological usual associative algebra $\mathcal{U}\mathfrak{a}[[\hbar]]$ and $End(\tilde{\mathcal{F}})$.*

This isomorphism $\tilde{\Lambda}$ allows us to define on the algebra $\mathcal{U}\mathfrak{a}[[\hbar]]$ the pull-back of the quasitriangular Hopf algebra structure on $End(\tilde{\mathcal{F}})$: $(\mathcal{U}\mathfrak{a}[[\hbar]], \cdot, \tilde{\Delta}, \tilde{S}, \tilde{R})$.

Theorem 2.19. *Suppose that $(\mathfrak{a}, [,]_{\mathfrak{a}}, \varepsilon_{\mathfrak{a}} = d_c r_1)$ is a quasitriangular Lie bialgebra. Let $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_+^*$ be the classical double of the Lie bialgebra $(\mathfrak{g}_+, [,]_{\mathfrak{g}_+}, \varepsilon)$ and let $(\mathcal{U}\mathfrak{g}[[\hbar]], \cdot, \Delta, S, R)$ be the quasitriangular Hopf QUE algebra obtained* *by Etingof-Kazhdan quantization (theorem 2.15). Then, we have*

$$\tilde{\Delta}(a) = \tilde{J}_{r_1}^{-1} \cdot \Delta_{\mathfrak{a}}(a) \cdot \tilde{J}_{r_1}, \quad \textit{where } \tilde{J}_{r_1} = (\tilde{\pi} \otimes \tilde{\pi})\, J, \quad \tilde{R} = (\tilde{\pi} \otimes \tilde{\pi})\, R,$$
$$\tilde{S}(a) = \tilde{Q}^{-1} \cdot S_{\mathfrak{a}}(a) \cdot \tilde{Q}, \quad \textit{where } \tilde{Q} = \textstyle\sum S_{\mathfrak{a}}(p_i) \cdot q_i, \quad \tilde{J}_{r_1} = \textstyle\sum p_i \otimes q_i, a \in \mathcal{U}\mathfrak{a},$$

where $\tilde{\pi} = \pi \circ (1 \oplus \chi_{r_1})$ *and* $\Delta_{\mathfrak{a}}$, $S_{\mathfrak{a}}$ *are the usual ones on* $\mathcal{U}\mathfrak{a}$.

$(\mathcal{U}\mathfrak{a}[[\hbar]], \cdot, \tilde{\Delta}, \tilde{S}, \tilde{R})$ *is a quasitriangular Hopf QUE algebra and a quantization,* [7, 9], *of the quasitriangular Lie bialgebra* $(\mathfrak{a}, [,]_{\mathfrak{a}}, \varepsilon_{\mathfrak{a}} = d_c r_1)$.

Remark 2.20. $\tilde{J}_{r_1}$ *satisfies the relation*

$$\tilde{\Phi}_{1,2,3} \cdot (\Delta_{\mathfrak{a}} \otimes 1)\tilde{J}_{r_1} \cdot (\tilde{J}_{r_1} \otimes 1) = (1 \otimes \Delta_{\mathfrak{a}})\tilde{J}_{r_1} \cdot (1 \otimes \tilde{J}_{r_1}).$$

3. Quantization of nondegenerate triangular Lie bialgebras

Let $(\mathfrak{a}, [,]_{\mathfrak{a}}, \varepsilon_{\mathfrak{a}} = d_c r_1)$ be a nondegenerate triangular Lie bialgebra. In this case $r_1 \in \wedge^2(\mathfrak{a})$ and $det(r_1) \neq 0$. So, as vector spaces we have

$$\mathfrak{g}_+ = \{(1 \otimes f) r_1 \mid \forall f \in \mathfrak{a}^*\} = \mathfrak{a} = \mathfrak{g}_- = \{(f \otimes 1) r_1 \mid \forall f \in \mathfrak{a}^*\}.$$

Then the classical double of $\mathfrak{g}_+$ is the classical double of the Lie bialgebra $\mathfrak{a}$. If $\{e_i\}$ is a basis of $\mathfrak{a}$ and $\{e^i\}$ is the corresponding dual basis, then

$$(\tilde{\pi} \otimes \tilde{\pi})\Omega = (\tilde{\pi} \otimes \tilde{\pi})(r_{12} + r_{21}) = (r_1)_{12} + (r_1)_{21} = 0.$$

Proposition 3.1. *Given a nondegenerate triangular Lie bialgebra* $\mathfrak{a}$ *and a Drinfeld associator* Φ, *we have*

$$\tilde{\Phi} = (\tilde{\pi} \otimes \tilde{\pi} \otimes \tilde{\pi})\Phi = 1 \otimes 1 \otimes 1, \quad \Phi \in \mathcal{U}\mathfrak{g}^{\otimes 3}[[\hbar]]$$
$$\tilde{R} = (\tilde{\pi} \otimes \tilde{\pi})R = (\sigma \tilde{J}_{r_1}^{-1}) \cdot (1 \otimes 1) \cdot \tilde{J}_{r_1}, \quad \tilde{J}_{r_1} \in \mathcal{U}\mathfrak{a}^{\otimes 2}[[\hbar]]$$
$$(\Delta_{\mathfrak{a}} \otimes 1)\tilde{J}_{r_1} \cdot (\tilde{J}_{r_1} \otimes 1) = (1 \otimes \Delta_{\mathfrak{a}})\tilde{J}_{r_1} \cdot (1 \otimes \tilde{J}_{r_1})$$

($\tilde{J}_{r_1}$ *is called an invariant star product (ISP) on* $(\mathfrak{a}, [,]_{\mathfrak{a}}, \varepsilon_{\mathfrak{a}} = d_c r_1)$ [1, 6, 14, 15, 16]*).*

The quantization of the above nondegenerate triangular Lie bialgebra is the triangular Hopf QUE algebra $(\mathcal{U}\mathfrak{a}[[\hbar]], \cdot, \tilde{\Delta}, \tilde{S}, \tilde{R})$, *denoted by* $A_{\mathfrak{a}[[\hbar]], \tilde{J}_{r_1}^{-1}}$, *which is obtained by a twist via* $\tilde{J}_{r_1}^{-1}$ *from the usual Hopf QUE algebra* $(\mathcal{U}\mathfrak{a}[[\hbar]], \cdot, \Delta_{\mathfrak{a}}, S_{\mathfrak{a}}, R_{\mathfrak{a}} = 1)$.

Let Γ be a neighborhood of 0 in $\mathbb{R}$. Consider a family of elements $r_t = r_1 + r_2 t + r_3 t^2 + \cdots \in \mathfrak{a} \wedge \mathfrak{a}$ such that, for each $t \in \Gamma$, r_t is a nondegenerate solution of the CYBE ($[r_t, r_t] = 0$). If $(\mathfrak{a}, [,]_{\mathfrak{a}})$ is a Lie algebra over $\mathbb{C}$, the set

$(\mathfrak{a}, [,]_{\mathfrak{a}}, \varepsilon_{\mathfrak{a}} = d_c r_t)$ is a nondegenerate triangular Lie bialgebra, for each $t \in \Gamma$. The Lie algebra structure on $\mathfrak{a}^*$ is defined by

$$[f_1, f_2]_{\mathfrak{a}^*_{r_t}} = (d_c r_t)^{\mathrm{t}}(f_1 \otimes f_2), \quad f_1, f_2 \in \mathfrak{a}^*.$$

Let $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^*$ be the classical double of the Lie bialgebra $(\mathfrak{a}, [,]_{\mathfrak{a}}, \varepsilon_{\mathfrak{a}} = d_c r_t)$, that we will denote as $\mathfrak{g}_{r_t} = \mathfrak{a} \oplus \mathfrak{a}^*_{r_t}$ because the Lie algebra structure of $\mathfrak{a}^*$ is defined as

$$[\xi, \eta]_{\mathfrak{a}^*_{r_t}} = (d_c r_t)^{\mathrm{t}}(\xi \otimes \eta). \tag{10}$$

We could begin with the Lie algebra defined from $\mathfrak{a}$ by the extension of scalars $\mathbb{K} \longrightarrow \mathbb{K}[[\hbar]]$ [4, 2]. We prefer here to begin with a convergent series in $\mathfrak{a} \wedge \mathfrak{a}$, $t \in \Gamma$, $\mathbb{K} = \mathbb{R}$. When proving, for example, Proposition 3.4 that follows, we see first both members of the equalities as formal power series in t and $\hbar$. Then put $t = \hbar$. The proof doesn't depend on the field $\mathbb{R}$ and is then true for any $\mathbb{K}$. We read in [17, 16] Drinfeld Theorem 7 [6] under this optical. We don't have today enough arguments to underestimate this optical. This simple remark is to be joint to the "universality properties" of Φ in [7, 8] which are at the basis for the existence of the Quantization Functor [9, 12]. See also [20].

Lemma 3.2. [19] *The element J of theorem 2.15 will be now $J_{r_t} \in \mathcal{U}\mathfrak{g}_{r_t}^{\otimes^2}[[\hbar]]$ given by*

$$\begin{aligned} J_{r_t} &= (\phi^{-1} \otimes \phi^{-1}) \left((\Phi_t^{-1})_{1,2,34} (\Phi_t)_{2,3,4} \sigma_{23} e^{\frac{\hbar}{2}\Omega_{23}} (\Phi_t^{-1})_{2,3,4} (\Phi_t)_{1,2,34} (i_+ \otimes i_-) \phi(1) \right) \\ &= 1 \otimes 1 + \frac{1}{2} r\, \hbar + \sum_{k \geq 2} \left(r_t^{i_1 j_1} \ldots r_t^{i_{l(k)} j_{l(k)}} Q_{i_1, \ldots, i_{l(k)}, j_1, \ldots, j_{l(k)}, k} \right) \hbar^k, \end{aligned}$$

where when an ordered basis is chosen $Q_{i_1,\ldots,i_{l(k)},j_1,\ldots,j_{l(k)},k} \in \mathcal{U}\mathfrak{g} \otimes \mathcal{U}\mathfrak{g}$ is independent of r_t, but the algebra structure depends on r_t, and $r = \sum (e_i; 0) \otimes (0; e^i)$.

Applying $\tilde{\pi}_t \otimes \tilde{\pi}_t$ to J_{r_t}, we come to the following:

Proposition 3.3. [19]

$$\begin{aligned} \tilde{J}^{\Phi}_{r_\hbar} = (\tilde{J}_{r_t})_{t=\hbar} = ((\tilde{\pi}_t \otimes \tilde{\pi}_t) J_{r_t})_{t=\hbar} = 1 \otimes 1 + \frac{1}{2} r_1 \hbar + \sum_{R=2}^{\infty} \Bigg(\frac{1}{2} r_R + \\ + \sum_{i_2, j_2, \ldots i_R, j_R} \ \sum_{A_{i_2,j_2}(R), \ldots, A_{i_R j_R}(R)} (r_1^{i_2 j_2})^{A_{i_2 j_2}(R)} \ldots (r_{R-1}^{i_R j_R})^{A_{i_R j_R}(R)} \cdot \\ \cdot H_{i_2,\ldots,i_R, j_2, \ldots, j_R, A_{i_2 j_2}(R), \ldots, A_{i_R j_R}(R), R} \Bigg) \hbar^R, \end{aligned}$$

with $H_{i_2,\dots,i_R,j_2,\dots,j_R,A_{i_2j_2}(R),\dots,A_{i_Rj_R}(R),R} \in \mathcal{U}\mathfrak{a}\otimes\mathcal{U}\mathfrak{a}$ *independent of* r_k *and* k, $A_{i_aj_a}(R)\in\mathbb{N}$.

Proposition 3.4. [19] *Let* $\Phi \in T_3[[\hbar]]$ *be a Drinfeld associator. Let* $r_t = r_1 + r_2\,t + r_3\,t^2 + \cdots$ *be a nondegenerate solution of the CYBE, for each* t. *Let* $(\mathfrak{a},[,]_{\mathfrak{a}},\varepsilon_{\mathfrak{a}} = d_c r_t)$ *be the nondegenerate triangular Lie bialgebra defined by* r_t. *Let* $\left(\mathcal{U}\mathfrak{a}[[\hbar]],\cdot,\tilde{\Delta}_t,\tilde{S}_t,\tilde{R}_t\right)$ *be the quantization obtained in the Etingof-Kazhdan's theorem, (proposition 3.1), as the quantization of the above non-degenerate triangular Lie bialgebra, from the usual Hopf algebra by a twist via* $(\tilde{J}_{r_t})^{-1}$.

Consider the element $\tilde{J}^{\Phi}_{r_\hbar} = (\tilde{J}_{r_t})_{t=\hbar} \in \mathcal{U}\mathfrak{a}[[\hbar]]\hat{\otimes}\mathcal{U}\mathfrak{a}[[\hbar]]$, *then*

(a) $(\Delta_{\mathfrak{a}}\otimes 1)\tilde{J}^{\Phi}_{r_\hbar}\cdot(\tilde{J}^{\Phi}_{r_\hbar}\otimes 1) = (1\otimes\Delta_{\mathfrak{a}})\tilde{J}^{\Phi}_{r_\hbar}\cdot(1\otimes\tilde{J}^{\Phi}_{r_\hbar})$;

(b) $\tilde{J}^{\Phi}_{r_\hbar} = 1\otimes 1 + \frac{1}{2}r_1\,\hbar + 0(\hbar^2)$;

(c) Defining $\Delta(a) = (\tilde{J}^{\Phi}_{r_\hbar})^{-1}\cdot\Delta_{\mathfrak{a}}(a)\cdot\tilde{J}^{\Phi}_{r_\hbar}$, $R = (\sigma\tilde{J}^{\Phi}_{r_\hbar})^{-1}\cdot(1\otimes 1)\cdot\tilde{J}^{\Phi}_{r_\hbar}$ *and* $S(a) = Q^{-1}\cdot S_{\mathfrak{a}}(a)\cdot Q$, *where* $Q = \sum S_{\mathfrak{a}}(a_i)\cdot b_i$, $\tilde{J}^{\Phi}_{r_\hbar} = \sum a_i\otimes b_i$, $a_i, b_i \in \mathcal{U}\mathfrak{a}[[\hbar]]$, *the set* $(\mathcal{U}\mathfrak{a}[[\hbar]],\cdot,\Delta,S,R)$ *is a triangular Hopf QUE algebra obtained twisting the usual Hopf algebra* $(\mathcal{U}\mathfrak{a}[[\hbar]],\cdot,\Delta_{\mathfrak{a}},S_{\mathfrak{a}},R_{\mathfrak{a}}=1)$ *via the element* $(\tilde{J}^{\Phi}_{r_\hbar})^{-1}\in\mathcal{U}\mathfrak{a}[[\hbar]]\hat{\otimes}\mathcal{U}\mathfrak{a}[[\hbar]]$. *We write it as* $A_{\mathfrak{a}[[\hbar]],(\tilde{J}^{\Phi}_{r_\hbar})^{-1}}$.

Let $\tilde{J}^{\Phi}_{r'_\hbar}$ *be the star product determined, in the same way, by*

$$r'_t = r_1 + r_2\,t + \cdots + r_{k-1}\,t^{k-2} + (r_k + s_k)t^{k-1} + \dots$$

that also verifies $r'_t \in \wedge^2(\mathfrak{a})$, *and* r'_t *is a nondegenerate solution of Yang-Baxter equation, for each* t. *Then* $\tilde{J}^{\Phi}_{r_\hbar}$ *and* $\tilde{J}^{\Phi}_{r'_\hbar}$ *coincide up to order* $k-1$ *and*

$$(\tilde{J}^{\Phi}_{r'_\hbar})_k - (\tilde{J}^{\Phi}_{r_\hbar})_k = \frac{1}{2}s_k.$$

4. A result on triangular Hopf QUE algebras of the form $A_{\mathfrak{a}[[\hbar]],F^{-1}}$ and Drinfeld associators

Denote by $A_{\mathfrak{a}[[\hbar]],F^{-1}}$ [7] the triangular Hopf QUE algebra which is a quantization of the nondegenerate triangular Lie bialgebra $\mathfrak{a}$ and the *twist*, via $F^{-1}\in\mathcal{U}\mathfrak{a}\otimes\mathcal{U}\mathfrak{a}[[\hbar]]$, of the usual triangular Hopf QUE algebra $\mathcal{U}\mathfrak{a}[[\hbar]]$.

Theorem 4.1. [19] *Let* $\mathfrak{a}$ *be a nondegenerate triangular Lie bialgebra and* $A_{\mathfrak{a}[[\hbar]],F^{-1}}$ *a triangular Hopf QUE algebra which is a quantization of this Lie bialgebra. Then:*

(a) There exist elements $r_\hbar = r_1 + r_2\hbar + r_3\hbar^2 + \cdots \in \wedge^2(\mathfrak{a})[[\hbar]]$ *and* $E^{r_\hbar} = 1 + E_1^{r_\hbar}\hbar + \cdots + E_n^{r_\hbar}\hbar^n + \cdots \in \mathcal{U}\mathfrak{a}[[\hbar]]$*, such that*

$$F = \Delta_{\mathfrak{a}}((E^{r_\hbar})^{-1}) \cdot \tilde{J}_{r_\hbar} \cdot (E^{r_\hbar} \otimes E^{r_\hbar}),$$

i.e., F is an invariant star product equivalent to $\tilde{J}_{r_\hbar}$ ([6, 16]*).*

(b) In this case the triangular Hopf QUE algebras $A_{\mathfrak{a}[[\hbar]],F^{-1}}$ and $A_{\mathfrak{a}[[\hbar]],\tilde{J}_{r_\hbar}^{-1}}$ are isomorphic.

As a consequence, we notice $A_{\mathfrak{a}[[\hbar]],F^{-1}} \overset{isom}{\approx} A_{\mathfrak{a}[[\hbar]],(\tilde{J}^{\Phi}_{r_\hbar})^{-1}} \overset{isom}{\approx} A_{\mathfrak{a}[[\hbar]],(\tilde{J}^{\Phi'}_{r'_\hbar})^{-1}}$ when two different associators are considered.

Using Knizhnik-Zamolodchikov associator ([7, 8, 19]) we have obtained for $\tilde{J}_{r_\hbar}$, up to the term of order $\hbar^2$, exactly the same expression obtained by Drinfeld ([6] and explicitly written in [21]).

5. Invariant star products on $(\mathfrak{a}, [,]_{\mathfrak{a}}, \varepsilon_{\mathfrak{a}} = d_c r_1)$

Let Γ be a neighbourhood of 0 in $\mathbb{R}$. Consider a family of elements $r_t \in \mathfrak{a}\wedge\mathfrak{a}$ such that, for each $t \in \Gamma$, r_t is a nondegenerate solution of the CYBE. Let $\mu_{r_t} : \wedge^r(\mathfrak{a}) \longrightarrow \wedge_r(\mathfrak{a})$ be the isomorphism defined by $\left(\mu_{r_t}^{-1}(\alpha)\right)^{i_1\ldots i_r} = (r_t)^{j_1 i_1} \ldots (r_t)^{j_r i_r} \alpha_{j_1\ldots j_r}$

If G is a Lie group with Lie algebra $\mathfrak{g}$, the adjoint representation of G induces a representation on the Chevalley complex $H^*(\mathfrak{g})$ that is trivial. This is one starting point for the proof of next theorem which is similar for the Etingof-Kazhdan quantization to that for the Drinfeld quantization of bialgebra $\mathfrak{a}$. See also [16, 17] and compare with the proof given there.

Theorem 5.1. *Let $\tilde{J}_{r_\hbar} = (\tilde{J}_{r_t})_{t=\hbar}$ and $\tilde{J}_{r'_\hbar} = (\tilde{J}_{r'_t})_{t=\hbar}$ be ISP on the nondegenerate triangular Lie bialgebra $(\mathfrak{a}, [,]_{\mathfrak{a}}, \varepsilon_{\mathfrak{a}} = d_c r_1)$ which appear from E-K quantization. Suppose $\beta_t = \mu_{r_t}(r_t) = \beta_1 + \beta_2 t + \cdots$ and $\beta'_t = \mu_{r'_t}(r'_t) = \beta_1 + \beta'_2 t + \cdots$.*

Then, $\tilde{J}_{r_\hbar}$ and $\tilde{J}_{r'_\hbar}$ are equivalent ISP if, and only if, β_t and β'_t belong to the same formal cohomological class. In other words, $\tilde{J}_{r_\hbar}$ and $\tilde{J}_{r'_\hbar}$ are equivalent ISP if, and only if, there exists a formal 1-cochain $\alpha_t = \alpha_1 t + \alpha_2 t^2 + \ldots$ such that $\beta'_t = \beta_t + d_c\alpha_t$.

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