



Parameterized codes over graphs

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Abstract

In this article we review known results on parameterized linear codes over graphs, introduced by Rentería et al. (Finite Fields Appl 17(1):81–104, 2011). Very little is known about their basic parameters and invariants. We review in detail the parameters dimension, regularity and minimum distance. As regards the parameter *dimension*, we explore the connection to Eulerian ideals in the ternary case and we give new combinatorial formulas.

Keywords Minimum Distance · Regularity · Dimension · Linear Code

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1 Introduction

A parameterized code over a graph is a linear code obtained by evaluating forms of fixed degree on a set of points obtained from the graph, in projective space over a finite field. They were introduced by Rentería et al. in [13] and, with some exceptions, their study is wide open. In this article we will touch upon the basic parameters and invariants of these codes, reviewing known results. Section 2 concerns the parameter *dimension* and focuses on the case of ternary linear codes, by exploring

Dedicated to Rafael Villarreal, on the occasion of his 70th birthday.

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the relation with Eulerian ideals. Theorem 2.7, which gives a combinatorial formula for the dimension of parameterized code over a graph in the ternary case, and Theorem 2.8, which gives this formula explicitly in the case of an even cycle, are both new. Section 3 is dedicated to the invariant *regularity* and Sect. 4 to the parameter *minimum distance*.

Let G be a simple graph. We assume that $V_G = \{1, 2, \dots, n\}$ and we denote $s = |E_G|$, which we always assume positive. We also fix a choice of ordering of the edges, e_1, \dots, e_s . Take K to be a field and consider the two polynomial rings $K[x_1, \dots, x_n]$ and $K[t_1, \dots, t_s]$. (It is convenient to identify E_G with the set $\{t_1, \dots, t_s\}$.) Thus we may refer to the monomial obtained by multiplying a given set of edges.) Defining a homomorphism of polynomial rings $\varphi : K[t_1, \dots, t_s] \rightarrow K[x_1, \dots, x_n]$ by

$$t_k \mapsto x_i x_j$$

if and only if t_k is the edge $\{i, j\}$, we obtain a rational map of \mathbb{P}^{n-1} to \mathbb{P}^{s-1} , which, when restricted to the projective torus

$$\mathbb{T}^{n-1} = \{(x_1, \dots, x_n) \in \mathbb{P}^{n-1} : x_i \neq 0, \text{ for all } i\},$$

is a morphism. We denote the image of \mathbb{T}^{n-1} by this morphism by X . This set is then a subset (and, moreover, subgroup) of the corresponding projective torus in \mathbb{P}^{s-1} . The set X is called the *projective algebraic toric set parameterized by the edges of G* . Assume K is finite. Then X is also finite and the number of its elements can be determined as a function of G (see Theorem 1.1, below). At this point, let us denote this number by m and let $X = \{P_1, \dots, P_m\}$ correspond to a choice of ordering. Let $d \geq 0$. Then, the *parameterized code of order d over G* , denoted by $C_X(d) \subseteq K^m$, is the image of the space of homogeneous polynomials in t_1, \dots, t_s , of degree d , by the map defined by

$$f \rightarrow \left(\frac{f(P_1)}{f_0(P_1)}, \dots, \frac{f(P_m)}{f_0(P_m)} \right) \in K^m, \tag{1.1}$$

for every $f \in K[t_1, \dots, t_s]_d$ and where $f_0 = t_1^d$.

A graph gives a sequence of linear codes:

$$C_X(0), C_X(1), \dots, C_X(d), \dots$$

all of which are subspaces of K^m . The list of dimensions of the codes in this sequence starts with 1 and is strictly increasing until it reaches m . (We will explain this in more detail in Sect. 2). From a coding theory point of view, the degree at which the dimension of $C_X(d)$ reaches m is an important parameter of this construction. We call it the *index of regularity* (or, simply the *regularity*) for reasons we will explain later. Other important invariants of the codes include their *minimum distances*, which is the minimum number of nonzero components of a vector over all non-zero vectors in the code, and their *length* (the number of components of a vector); which in this construction is m , common to all codes in the sequence. Given that $C_X(d)$ are constructed from G , the expectation is that all of these invariants are in some way related to invariants of the graph. For a general graph, not much is

known about the dimension and minimum distance of these codes. There has, however, been significant progress on the computation of the index of regularity and we will postpone a detailed account to Sect. 3. As for the parameter length, denoted above by $m = |X|$, a formula, holding for any graph, was given in [11]. To state this result, let us denote the number of connected components of G by $b_0(G)$ and let q denote the cardinality of the field.

Theorem 1.1 *If G is a bipartite graph then*

$$|X| = (q - 1)^{n-b_0(G)-1}.$$

If G is non-bipartite then

$$|X| = \begin{cases} \left(\frac{1}{2}\right)^{\gamma-1} (q - 1)^{n-b_0(G)+\gamma-1} & \text{if } q \text{ is odd,} \\ (q - 1)^{n-b_0(G)+\gamma-1} & \text{if } q \text{ is even,} \end{cases}$$

where γ is the number of non-bipartite components.

Proof See [11, Theorem 3.2]. □

2 Dimension

From now on, let us denote $S = K[t_1, \dots, t_s]$ and let $I(X) \subseteq S$ be the homogeneous vanishing ideal of $\{P_1, \dots, P_m\}$. Then $S/I(X)_d \simeq C_X(d)$ and therefore the dimension of $C_X(d)$, as $d \geq 0$, coincides with the Hilbert function of the module $S/I(X)$. Since $I(X)$ is the vanishing ideal of a set of points in projective space, we know that the Hilbert function of $S/I(X)$, and hence $\dim C_X(d)$, is strictly increasing until it reaches a constant value equal to the number of points of X .

Denote the projective torus $\mathbb{T}^{s-1} \subseteq \mathbb{P}^{s-1}$ by \mathbb{T} . As $X \subseteq \mathbb{T}$ we get

$$I(\mathbb{T}) = (t_1^{q-1} - t_s^{q-1}, \dots, t_{s-1}^{q-1} - t_s^{q-1}) \subseteq I(X). \tag{2.1}$$

From the point of view of the Hilbert Function, the easiest case is when X coincides with the projective torus $\mathbb{T} = \mathbb{T}^{s-1} \subseteq \mathbb{P}^{s-1}$ and, hence, $I(X)$ is a complete intersection. We may use the Hilbert series of $S/I(\mathbb{T})$ to obtain

$$\dim C_{\mathbb{T}}(d) = \sum_{j \geq 0} (-1)^j \binom{s-1}{j} \binom{s-1+d-(q-1)j}{s-1} \tag{2.2}$$

(see [1, 4, 14] for details). According to [14, Theorem 4.4], $X = \mathbb{T}$ is the only case in which $I(X)$ is a complete intersection. Note that the formula of Theorem 1.1 gives $X = \mathbb{T}$ if G is a tree or, more generally, a forest, or when G is a unicyclic graph with a unique odd cycle. On the opposite end of the class of bipartite graphs are the complete bipartite graphs $\mathcal{K}_{a,b}$. In this case $I(X)$ is far from being a complete intersection, but the dimension function of $C_X(d)$ is known. To state it, let $k(s, d, q)$ be the summation on the right of (2.2). Then,

$$\dim C_X(d) = k(a, d, q)k(b, d, q)$$

(see [3, Theorem 5.2]). To our knowledge, these are the only two instances in which a formula for the dimension function of parameterized codes is known.

2.1 Dimension in the case of ternary codes

When $K = \mathbb{Z}/3$, the situation is bettered by the recent results on the *Eulerian ideal* of G . This ideal, defined in [12], is the pre-image of the ideal

$$(x_i^2 - x_j^2 : 1 \leq i, j \leq n) \subseteq K[x_1, \dots, x_n]$$

by the map φ , defined at the begining of Sect. 1. By [12, Proposition 2.9], when $K = \mathbb{Z}/3$, the ideal $I(X)$ and the Eulerian ideal are the same. A set of generators which is, moreover, a Gröbner basis, is available from [8]. To state the result let us fix some notation. Given $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{N}^s$ let us denote $t_1^{\alpha_1} \dots t_s^{\alpha_s}$ by \mathbf{t}^α . We say that $\mathbf{t}^\alpha - \mathbf{t}^\beta$ is an Eulerian binomial if \mathbf{t}^α and \mathbf{t}^β are relatively prime, square-free, of the same degree, and the edges with index set $\text{supp}(\alpha) \sqcup \text{supp}(\beta) \subseteq \{1, \dots, s\}$ induce a subgraph of G with vertices of even degree; i.e., an Eulerian subgraph. We denote by \mathcal{E} the (finite) set of all Eulerian binomials and by $\mathcal{T} = \{t_i^2 - t_j^2 : 1 \leq i, j \leq s\}$.

Theorem 2.1 *Let $K = \mathbb{Z}/3$. The set of homogeneous binomials $\mathcal{T} \cup \mathcal{E}$ is a Gröbner basis of $I(X)$ with respect to the graded reverse lexicographic order in S .*

Proof See [8, Theorem 3.3]. □

In particular, $I(X)$ is generated in degree ≥ 2 . As

$$\dim C_X(d) = \dim_K(S/I(X))_d,$$

we deduce that $\dim C_X(0) = 1$ and $\dim C_X(1) = s$, regardless of G . This holds also for any parameterized code over a graph, over any finite field.

A technique that has always proved useful when trying to link the combinatorics of G with the algebra of $S/I(X)$, is to take an Artinian quotient of this graded ring. This is specially easy to produce since any monomial in S is $S/I(X)$ -regular. (Indeed since X is a subset of the projective torus $\mathbb{T} \subseteq \mathbb{P}^{s-1}$, a monomial does not vanish at any point of X .) To study the dimension function of the codes the correct Artinian quotient is $S/(I(X), t_s^2)$, where t_s is the last edge of the graph.

Definition 2.2 Given $d \geq 0$, let \mathcal{B}_d be the set of monomials of degree d that are not divisible by any leading term of a polynomial in $(I(X), t_s^2)$, with respect to the graded reverse lexicographic order in S . Extend the notation \mathcal{B}_d to negative d by setting $\mathcal{B}_d = \emptyset$ and denote the cardinality of \mathcal{B}_d by $\beta(d)$.

Since, for every $i = 1, \dots, s$, t_i^2 is a leading term of an element of $(I(X), t_s^2)$ a monomial in \mathcal{B}_d is necessarily square-free. In particular, \mathcal{B}_d is surely empty as soon

as $d > s$. As $(I(X), t_s^2)$ is generated in degrees ≥ 2 , we deduce that $\mathcal{B}_0 = \{1\}$ and $\mathcal{B}_1 = \{t_1, \dots, t_s\}$. As we show below, the elements of \mathcal{B}_d , correspond to special sets of edges of the graph. Before, let us reveal the connection with the dimension function of the family of codes $C_X(d)$, $g \geq 0$.

Proposition 2.3 *Let $K = \mathbb{Z}/3$ and $d \geq 0$. Then*

$$\dim C_X(d) = \sum_{i \geq 0} \beta(d - 2i).$$

Proof Let us use induction on d . It is clear that the formula holds for $d = 0$ and $d = 1$. Assume $d > 1$. Since t_s^2 is $S/I(X)$ -regular, the short exact sequence

$$0 \rightarrow S/I(X)[-2] \xrightarrow{t_s^2} S/I(X) \rightarrow S/(I(X), t_s^2) \rightarrow 0 \tag{2.3}$$

gives $\dim C_X(d) = \dim C_X(d - 2) + \dim_K(S/(I(X), t_s^2))_d$. By Macaulay’s Theorem, the cosets with representatives in \mathcal{B}_d form a K -basis of the vector space of $(S/(I(X), t_s^2))_d$. In other words, $\beta(d) = \dim_K(S/(I(X), t_s^2))_d$. Hence the formula follows by induction. \square

The key to get a combinatorial formula for $\dim C_X(d)$ is then the combinatorial characterization of the elements of \mathcal{B}_d . For this, we need a Gröbner basis of $(I(X), t_s^2)$, which is easily obtained from that of $I(X)$.

Proposition 2.4 *Let $K = \mathbb{Z}/3$. The set $\mathcal{T} \cup \mathcal{E} \cup \{t_s^2\}$ is a Gröbner basis of $(I(X), t_s^2)$ with respect to the graded reverse lexicographic order in S .*

Proof Since t_s^2 and the leading term of any binomial in $\mathcal{T} \cup \mathcal{E}$ are coprime, their S -polynomial reduces to zero. Since $\mathcal{T} \cup \mathcal{E}$ is a Gröbner basis, the S -polynomials of all pairs of elements of $\mathcal{T} \cup \mathcal{E}$ also reduce to zero. \square

Let us now introduce the combinatorics.

Definition 2.5 [8, Definition 4.4] $J \subseteq E_G$ is called a parity join if and only if $|J \cap E_C| \leq \frac{|E_C|}{2}$, for every Eulerian subgraph of $C \subset G$ with an even number of edges.

The terminology of parity join comes from the relation with T -joins of cardinality of fixed parity, as explained in [8]. A parity join need not use half the edges of every Eulerian subgraph. When it does use half the edges of a given Eulerian subgraph, these need not include the last edge.

Definition 2.6 Given $d \geq 0$, let \mathcal{J}_d denote the set of parity joins, $J \subseteq E_G$, of cardinality d , that contain the last edge of every Eulerian subgraph $C \subseteq G$ for which $|J \cap E_C| = \frac{|E_C|}{2}$. Let us also extend this notation by setting $\mathcal{J}_d = \emptyset$, for all $d < 0$.

The proof of the next result is an adaptation of the ideas of [8]. There, the approach privileges fixed parity T -joins.

Theorem 2.7 *Let $K = \mathbb{Z}/3$. The map $\mathcal{B}_d \rightarrow \mathcal{J}_d$ given by*

$$\mathbf{t}^\gamma \mapsto \{e_i : i \in \text{supp}(\gamma)\}$$

is well-defined and a bijection. In particular,

$$\dim C_X(d) = \sum_{i \geq 0} |\mathcal{J}_{d-2i}|.$$

Proof As $\mathbf{t}^\gamma \in \mathcal{B}_d$ is square-free, $\{e_i : i \in \text{supp}(\gamma)\}$ is a set of d edges. Let $C \subseteq G$ be any Eulerian subgraph with an even number of edges. Assume

$$|\mathcal{J}(\mathbf{t}^\gamma) \cap E_C| > \frac{|E_C|}{2}.$$

Let \mathbf{t}^α be the product of the first $\frac{|E_C|}{2}$ edges in $\mathcal{J}(\mathbf{t}^\gamma) \cap E_C$ and let \mathbf{t}^β be the product of the remaining edges of C . Then $\mathbf{t}^\alpha - \mathbf{t}^\beta$ is an Eulerian binomial and, as \mathbf{t}^β is divisible by the last edge of $\mathcal{J}(\mathbf{t}^\gamma) \cap E_C$, its leading term is \mathbf{t}^α . But then \mathbf{t}^α divides $\mathbf{t}^\gamma \in \mathcal{B}_d$, and this is a contradiction. Hence

$$|\mathcal{J}(\mathbf{t}^\gamma) \cap E_C| \leq \frac{|E_C|}{2}.$$

We deduce that $\{e_i : i \in \text{supp}(\gamma)\}$ is a parity join. Additionally, if

$$|\mathcal{J}(\mathbf{t}^\gamma) \cap E_C| = \frac{|E_C|}{2}$$

but $\{e_i : i \in \text{supp}(\gamma)\}$ does not contain the last edge of C , the same argument leads to a contradiction. Hence the map is well-defined.

It is clearly an injective map. To prove surjectivity, let $J \in \mathcal{J}_d$, let \mathbf{t}^γ be the product of the edges in J and let us show that $\mathbf{t}^\gamma \in \mathcal{B}_d$. Clearly $\text{deg}(\mathbf{t}^\gamma) = |J| = d$, so that all we need to show is that \mathbf{t}^γ is not divisible by any leading term of $(I(X), t_s^2)$. Since $\mathcal{T} \cup \mathcal{E} \cup \{t_s^2\}$ is a Gröbner basis for this ideal (Proposition 2.4) it is enough to check that \mathbf{t}^γ is not divisible by the leading term of any element of $\mathcal{T} \cup \mathcal{E} \cup \{t_s^2\}$. Since \mathbf{t}^γ is square-free, $t_i^2 \nmid \mathbf{t}^\gamma$, for all $i = 1, \dots, s$. Let $g = \mathbf{t}^\alpha - \mathbf{t}^\beta \in \mathcal{E}$, with $\text{lt}(g) = \mathbf{t}^\alpha$ (without loss of generality). Let $C \subseteq G$ be the corresponding Eulerian subgraph, i.e., the graph induced by $\{e_i : i \in \text{supp}(\alpha)\} \sqcup \{e_j : j \in \text{supp}(\beta)\} \subseteq E_G$. With a view to a contradiction, suppose that $\mathbf{t}^\alpha \mid \mathbf{t}^\gamma$. Then, as J is a parity join,

$$|J \cap E_C| = \frac{|E_C|}{2}$$

which implies that $J \cap E_C = \{e_i : i \in \text{supp}(\alpha)\}$. But if $J \in \mathcal{J}_d$ then J must contain the last edge of C which means that \mathbf{t}^α is divisible by this edge. But this is a contradiction since we are assuming that $\text{lt}(g) = \mathbf{t}^\alpha$. Hence $\mathbf{t}^\alpha \nmid \mathbf{t}^\gamma$, for the leading term of any element of \mathcal{E} . We conclude that $\mathbf{t}^\gamma \in \mathcal{B}_d$ and hence the map is also surjective. This bijection yields $|\mathcal{B}_d| = |\mathcal{J}_d|$ and the formula for $\dim C_X(d)$ follows from Proposition 2.3. □

Let us illustrate the applications of this result by considering the case when G has no Eulerian subgraphs with an even number of edges. Note that, by Theorem 2.1, $\mathcal{E} = \emptyset$ so that

$$I(X) = (T) = (t_1^2 - t_s^2, \dots, t_{s-1}^2 - t_s^2)$$

is a complete intersection and the dimension of $C_X(d)$ is given by (2.2), with $q = 3$. If G possesses no Eulerian subgraphs with even number of edges then every subset of edges is a parity join, hence

$$\mathcal{J}_d = \{J \subseteq E_G : |J| = d\}.$$

Then, by Theorem 2.7,

$$\dim C_X(d) = \sum_{i \geq 0}^k \binom{s}{d - 2i}.$$

To see that this amounts to the same as (2.2) with $q = 3$, let us manipulate the Hilbert series of $S/I(X)$, as in [14], but aiming at our formula. Since the ideal $I(X) \subseteq K[t_1, \dots, t_s]$ is a complete intersection of $s - 1$ forms of degree two, the Hilbert series of $S/I(X)$ is

$$\frac{(1 - T^2)^{s-1}}{(1 - T)^s} = \frac{(1 + T)^s}{1 - T^2} = (1 + T)^s \sum_{i \geq 0} T^{2i}.$$

Equating the coefficient of T^d ,

$$\dim C_X(d) = \dim_K S/I(X) = \sum_{i \geq 0} \binom{s}{d - 2i}.$$

We end this section by applying Theorem 2.7 to the case of an even cycle.

Theorem 2.8 *Let $K = \mathbb{Z}/3$ and let $G = C_{2\ell}$ be a cycle of length $s = 2\ell$. Then*

$$\dim C_X(d) = \begin{cases} 2^{s-2}, & \text{if } d \geq \ell - 1, \\ \sum_{i \geq 0} \binom{s}{d - 2i}, & \text{if } 0 \leq d \leq \ell - 2. \end{cases}$$

Proof Given that a parity join in G is simply a subset of $d \leq \ell$ edges, we get $\mathcal{J}_{\ell+i} = \emptyset$, for all $i > 0$. Also, an element in \mathcal{J}_ℓ must contain the edge t_s and so $|\mathcal{J}_\ell| = \binom{s-1}{\ell-1}$. For $0 \leq d \leq \ell - 1$, the elements of \mathcal{J}_d are the sets of d edges of G , without any condition. Thus $|\mathcal{J}_d| = \binom{s}{d}$. Using Theorem 2.7, if $0 \leq d \leq \ell - 1$,

$$\dim C_X(d) = \sum_{i \geq 0} |\mathcal{J}_{d-2i}| = \sum_{i \geq 0} \binom{s}{d - 2i}.$$

The sum of all binomial coefficients of lower indices of the same parity is well-known:

$$\sum_{i \geq 0} \binom{s}{\ell - 1 - 2i} + \sum_{i \geq 0} \binom{s}{\ell + 1 + 2i} = 2^{s-1}.$$

Since $s = 2\ell$ and hence $\binom{s}{\ell - 1 - 2i} = \binom{s}{\ell + 1 + 2i}$ we deduce that $\dim C_X(\ell - 1) = 2^{s-2}$. If $d = \ell$, using Pascal’s identity and the same kind of argument as above,

$$\begin{aligned} \dim C_X(\ell) &= \sum_{i \geq 1} \binom{s}{\ell - 2i} + \binom{s-1}{\ell - 1} \\ &= \sum_{i \geq 1} \binom{s-1}{\ell - 1 - 2i} + \sum_{i \geq 1} \binom{s-1}{\ell - 2i} + \binom{s-1}{\ell - 1} \\ &= \sum_{i \geq 1} \binom{s-1}{\ell - 1 - 2i} + \binom{s-1}{\ell - 1} + \sum_{i \geq 1} \binom{s-1}{\ell - 1 + 2i} \\ &= 2^{s-2}. \end{aligned}$$

Finally, if $d > \ell$, given that $|\mathcal{J}_{\ell+i}| = 0$, for all $i > 0$ and given the formula of Theorem 2.7, we deduce that $\dim C_X(d)$ is equal to either $\sum_{i \geq 0} |\mathcal{J}_{\ell-2i}|$ or to $\sum_{i \geq 0} |\mathcal{J}_{\ell-1-2i}|$, both of which are equal to 2^{s-2} . □

3 Regularity

Since $I(X)$ is the vanishing ideal of a set of m points in projective space, the Hilbert polynomial of $S/I(X)$ is constant and equal to m . In other words, there exists r such that

$$\dim C_d(X) = m \iff C_d(X) = K^m,$$

for all $d \geq r$. (From the coding theory point of view, this is where $C_d(X)$ becomes a trivial linear code.) The least r in these conditions is called the *index of regularity* of $S/I(X)$. Since any monomial is $S/I(X)$ regular, this module is 1-dimensional and Cohen–Macaulay. Hence the index of regularity coincides with the Castelnuovo–Mumford regularity of $S/I(X)$. From now on we will refer to this integer simply by the *regularity* of $S/I(X)$ and we will denote it by $\text{reg} S/I(X)$. The next table summarizes the early known results regarding this invariant.

In Table 1, \mathcal{K}_n denotes a complete graph on $n > 3$ vertices. The value for the regularity was given in [6, Remark 3]. In the case of the complete bipartite graph, the regularity was obtained in [3, Corollary 5.4] and the case of an even cycle, $G = \mathcal{C}_{2\ell}$, in [11, Theorem 6.2]. The value of regularity for a complete multipartite graph on $n = a_1 + \dots + a_r$ vertices, denoted here by $G = \mathcal{K}_{a_1, \dots, a_r}$, was given in [10, Theorem 4.3].

Table 1 Known values of $\text{reg}S/I(X)$

	$\text{reg}S/I(X)$
$X = \mathbb{T}^{s-1}$	$(s - 1)(q - 2)$
$G = \mathcal{K}_{a,b}$	$(\max\{a, b\} - 1)(q - 2)$
$G = \mathcal{K}_n, n > 3$	$\lceil (n - 1)(q - 2)/2 \rceil$
$G = \mathcal{C}_{2\ell}$	$(\ell - 1)(q - 2)$
$G = \mathcal{K}_{a_1, \dots, a_r}, r > 2$	$\max\{a_1(q - 2), \dots, a_r(q - 2), \lceil (n - 1)(q - 2)/2 \rceil\}$

3.1 Parallel compositions

A graph is a parallel composition of paths if there exist path graphs P_1, P_2, \dots, P_r such that G is obtained by identifying all the first end-points of the paths into a single vertex and all of the second end-points of the paths into another vertex. We have used *first* and *second* for the sake of clarity; we do not fix any orientation on the paths. Figure 1 illustrates this definition.

A parallel composition of paths may be bipartite or non-bipartite. The bipartite case is when the lengths of P_i have the same parity. The value of the regularity of $S/I(X)$ for a graph of this type was computed in [7].

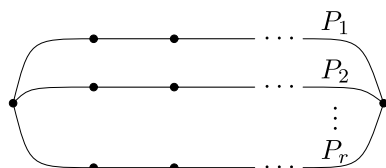
Theorem 3.1 [7, Theorems 1.1 and 1.2] *Let G be parallel composition of paths of lengths k_1, \dots, k_r , with $r \geq 2$. If G is bipartite then*

$$\text{reg}S/I(X) = \begin{cases} (\lfloor \frac{k_1}{2} \rfloor + \dots + \lfloor \frac{k_r}{2} \rfloor)(q - 2), & \text{if } k_i \text{ are odd,} \\ (\frac{k_1}{2} + \dots + \frac{k_r}{2} - 1)(q - 2), & \text{if } k_i \text{ are even.} \end{cases}$$

If G is non-bipartite then, assuming without loss of generality that k_1, \dots, k_ℓ are even and $k_{\ell+1}, \dots, k_r$ are odd,

$$\text{reg}S/I(X) = \begin{cases} (k_1 + k_2 - 1)(q - 2), & \text{if } \ell = 1, r = 2, \\ (k_1 + \lfloor \frac{k_2}{2} \rfloor + \dots + \lfloor \frac{k_r}{2} \rfloor)(q - 2), & \text{if } \ell = 1, r > 2, \\ (\frac{k_1}{2} + \dots + \frac{k_\ell}{2} + k_{\ell+1})(q - 2), & \text{if } \ell > 1, r = \ell + 1, \\ (\frac{k_1}{2} + \dots + \frac{k_\ell}{2} + \lfloor \frac{k_{\ell+1}}{2} \rfloor + \dots + \lfloor \frac{k_r}{2} \rfloor)(q - 2), & \text{if } \ell > 1, r > \ell + 1. \end{cases}$$

Fig. 1 The parallel composition of paths P_1, P_2, \dots, P_r



3.2 Nested ear decompositions

We say that G is endowed with an open ear decomposition if there exist subgraphs E_1, \dots, E_r , with E_1 a cycle and E_2, \dots, E_r paths such that, for each $i = 2, \dots, r$, the end-points of E_i are distinct and belong to $E_1 \cup \dots \cup E_{r-1}$, while all other vertices do not. The subgraphs E_1, \dots, E_r are called the *ears* of the decomposition. Given $i = 2, \dots, r$, we say that E_i determines a nest interval if both its end-points belong to the same E_j , for some $j < i$ and, in this case, we define the corresponding nest interval to be the sub-path of E_j determined by the two end-points of E_i . (If $j = 1$, we take any of the two sub-paths.) In [2], Eppstein defines the notion of nested ear decomposition by requiring that, in addition to the original assumptions, all E_i , for $i = 2, \dots, r$ determine a nest interval and, for any two nest intervals contained in a same ear E_j , either they are disjoint or one is contained in the other.

Theorem 3.2 [9, Theorem 4.4] *Assume G is bipartite and that E_1, \dots, E_r is a nested ear decomposition of G with ϵ ears of even length. Then*

$$\text{reg}S/I(X) = \frac{|V_G| + \epsilon - 3}{2}(q - 2).$$

Note that, in particular, it follows that the number of even length ears in any nested ear decomposition of a graph is constant. In the proof of Theorem 3.2, it is necessary to relax the definition of nested ear decomposition and, as a result, this theorem holds for a more general notion of ear decomposition called *weak nested ear decomposition*.

Any parallel composition of paths P_1, \dots, P_r is endowed with a nested ear decomposition, simply by setting E_1 equal to $P_1 \cup P_2$ and, if $r > 2$, by setting $E_i = P_{i+1}$, for all $i = 2, \dots, r - 1$. If the lengths of P_i are all even, then ϵ , with respect to the ear decomposition we have defined, is equal to $r - 1$. As

$$|V_G| = (\sum_{i=1}^r k_i) - r + 2$$

we get:

$$\text{reg}S/I(X) = \frac{|V_G| + \epsilon - 3}{2}(q - 2) = (\frac{k_1}{2} + \dots + \frac{k_r}{2} - 1)(q - 2),$$

which agrees with Theorem 3.1. If the lengths of the paths are all odd, the same can be verified.

3.3 Regularity in the case of ternary codes

If $K = \mathbb{Z}/3$ then, as

mentioned above, the vanishing ideal $I(X)$ coincides with the Eulerian ideal defined over $\mathbb{Z}/3$.

Theorem 3.3 [8, Theorem 4.13] *Let $K = \mathbb{Z}/3$ and G be any graph. Then $\text{reg}S/I(X)$ is equal to the maximum cardinality of a parity join minus 1.*

We end this section with a purely combinatorial result on the maximal cardinality of a parity join, which is straightforward by combining the previous theorem with the formulas for the regularity given before, with $q = 3$.

Proposition 3.4 *Denote by \mathcal{K}_n a complete graph on n vertices, $\mathcal{K}_{a,b}$ a complete bipartite graph on $n = a + b$ vertices, $\mathcal{K}_{\alpha_1, \dots, \alpha_r}$ a complete multipartite graph on $n = \alpha_1 + \dots + \alpha_r$ vertices, where $r > 2$, $\text{Pc}(k_1, \dots, k_r)$ the parallel composition of r paths of lengths k_1, \dots, k_r , and denote by $\mu(G)$ the maximal cardinality of a parity join. Let H be any bipartite graph with a nested ear decomposition having ϵ even length ears. The following holds:*

	$\mu(G)$
$G = \mathcal{K}_{a,b}$	$\max\{a, b\}$;
$G = \mathcal{K}_n, n > 3$	$\lfloor \frac{n-1}{2} \rfloor + 1$;
$G = \mathcal{K}_{\alpha_1, \dots, \alpha_r}$	$\max\{\alpha_1, \dots, \alpha_r, \lfloor \frac{n-1}{2} \rfloor\} + 1$;
$G = \text{Pc}(k_1, \dots, k_r)$ and k_i even	$\frac{k_1}{2} + \dots + \frac{k_r}{2}$;
$G = \text{Pc}(k_1, \dots, k_r)$ and k_i odd	$\lfloor \frac{k_1}{2} \rfloor + \dots + \lfloor \frac{k_r}{2} \rfloor + 1$;
$G = H$	$\frac{ V_G + \epsilon - 1}{2}$.

4 Minimum distance

We recall that the minimum distance $\delta_X(d)$ of the code $C_X(d) \subseteq K^m$ is defined as follows

$$\delta_X(d) = \min \{ \|a\|, a = (a_1, \dots, a_m) \in C_X(d), a \neq 0 \},$$

where $\|a\| = |\{i : a_i \neq 0\}|$. Clearly $1 \leq \delta_d \leq m$. The Singleton Bound (see [15], p.41) tells us that

$$\delta_X(d) \leq |X| - \dim C_X(d) + 1.$$

Since for $d \geq \text{reg } S/I(X)$, $\dim C_X(d) = |X|$, we have $\delta_X(d) = 1$, for $d \geq \text{reg } S/I(X)$. Moreover, the minimum distance is strictly decreasing until it reaches 1 ([13, 16]):

$$\begin{cases} \delta_X(d) > 1 \Rightarrow \delta_X(d) > \delta_X(d + 1) \\ \delta_X(d) = 1 \Rightarrow \delta_X(d + 1) = 1 \end{cases}.$$

The minimum distance is a very difficult parameter to calculate. In the case of evaluation codes, this calculation corresponds to counting zeros of homogeneous

polynomials. The next theorem is one of the few cases where we have an explicit formula for the minimum distance.

Theorem 4.1 [14, Theorem 3.4] *When $X = \mathbb{T}$, the projective torus in \mathbb{P}^{s-1} , and $d \geq 1$, the minimum distance of $C_X(d)$ is given by*

$$\delta_X(d) = \begin{cases} (q-1)^{s-(k+\ell)}(q-1-\ell) & \text{if } d \leq (q-2)(s-1) - 1 \\ 1 & \text{if } d \geq (q-2)(s-1) \end{cases}$$

where k and ℓ are the unique integers such that $k \geq 0$, $1 \leq \ell \leq q-2$ and $d = k(q-2) + \ell$.

Recall that we say that a linear code is maximum distance separable (MDS) if equality holds in the Singleton Bound. By the theorem above (see also [4]), if $X = \mathbb{T}$ is the projective torus in \mathbb{P}^1 and $d \geq 1$, then $C_X(d)$ is an MDS code and its minimum distance is given by

$$\delta_X(d) = \begin{cases} q-1-d & \text{if } d \leq q-3 \\ 1 & \text{if } d \geq q-2 \end{cases}.$$

As we have seen in Sect. 2, if G is a connected graph and X is the projective algebraic toric set parameterized by the edges of G , then $X = \mathbb{T}$ if and only if G is a tree or G is a unicyclic graph with a unique odd cycle. If G is a forest, we also have $X = \mathbb{T}$. Hence, for these graphs, $\delta_X(d)$ is known.

In the case G is a complete bipartite graph, the minimum distance of $C_X(d)$ is also known; it can be obtained from Theorem 4.1 together with the following result:

Theorem 4.2 [3, Theorem 5.5] *Let $G = \mathcal{K}_{a,b}$ be the complete bipartite graph with $a + b$ vertices, let X be the projective algebraic toric set parameterized by the edges of G , and let X_1 and X_2 be the projective tori in \mathbb{P}^{a-1} and \mathbb{P}^{b-1} respectively. Then*

$$\delta_X(d) = \delta_{X_1}(d)\delta_{X_2}(d).$$

Example 4.3 Let $G = \mathcal{K}_{2,3}$ be the complete bipartite graph with $2+3$ vertices, let X be the projective algebraic toric set parameterized by the edges of G , and let X_1 and X_2 be the projective tori in \mathbb{P}^1 and \mathbb{P}^2 respectively. For $q = 5$ and $d = 3$,

$$\delta_{X_1}(3) = 1 \quad \delta_{X_2}(3) = 4,$$

and therefore, by Theorem 4.2, $\delta_X(3) = 4$. This shows that $C_X(3)$ is not an MDS code, since the Singleton bound in this case is

$$\delta_X(3) \leq 64 - 40 + 1 = 25.$$

For the general case of a connected bipartite graph, the following bounds hold:

Theorem 4.4 *Let $G = \mathcal{K}_{a,b}$ be a connected bipartite graph with $a + b$ vertices, and let X be the projective algebraic toric set parameterized by the edges of G . If X_1 , X_2 and X_3 are the projective tori in \mathbb{P}^{a-1} , \mathbb{P}^{b-1} and \mathbb{P}^{a+b-2} respectively, then*

$$\delta_{X_1}(d)\delta_{X_2}(d) \leq \delta_X(d) \leq \delta_{X_3}(d).$$

These bounds can be explained using Lemma 4.5 below, knowing that a connected bipartite graph, G , contains a spanning tree and is contained in a complete bipartite graph with the same partition as G .

Lemma 4.5 [17, Lemma 3.5] *Suppose G is a subgraph of G' , and X and X' are the projective algebraic toric sets parameterized by the respective edges. If $|X| = |X'|$, then*

$$\delta_{X'}(d) \leq \delta_X(d).$$

Example 4.6 *If G is an hexagon ($n = s = 6$) and X is the projective algebraic toric set parameterized by the edges of G , the bounds of Theorem 4.4 for $q = 5$ and $d = 1$ are*

$$144 \leq \delta_X(1) \leq 192.$$

This is a better result than the Singleton bound, which in this case is

$$\delta_X(1) \leq 256 - 6 + 1 = 251.$$

We end by stating a result for a connected non-bipartite graph (see [5, 6]).

Theorem 4.7 [5, Corollary 3.12] *Let G be a connected non-bipartite graph and let X be the projective algebraic toric set parameterized by the edges of G . If X' is the projective torus in $\mathbb{P}^{|V_G|-1}$, then*

$$\delta_{X'}(2d) \leq \delta_X(d).$$

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